FACTORING N-CYCLES AND COUNTING MAPS OF GIVEN GENUS

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Abstract

We present an explicit expression for the number of decompositions of an $n-$cycle as a product of any two permutations of cycle types given by partitions λ and μ. The same expression is also counting the number of unicellular rooted bicolored maps on an orientable surface of genus g with vertex degree distribution given by λ and μ. We use character theory and the group algebra of the symmetric group to develop our expression. The key argument is the construction of a bijection involving the character formula at one end and our final expression at the other end. Some applications to the enumeration of maps of given genus and number of edges are given.

1 Introduction

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a finite non-increasing sequence of positive integers λ_i such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$. The non-zero terms of λ are called the *parts* of λ and the number k of parts is the length of λ, denoted $\ell(\lambda)$. We also write $\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$ when α_i parts of λ are equal to i $(i = 1 \ldots n)$. When the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$, we call n the *weight* of λ and we write $\lambda \vdash n$ or $|\lambda| = n$.

The conjugacy classes C_λ of the symmetric group S_n are indexed by partitions of n which are called the cycle types of the permutations $\sigma \in C_\lambda$. Let $\mathbb{Q}[S_n]$ be the group algebra of the symmetric group over the rational numbers $\mathbb{Q}$ and let $C[S_n]$ be the center of this group algebra. The formal sum of the permutations in a conjugacy class C_λ belongs to $C[S_n]$ and we denote it K_λ. We abuse notation and also call the elements K_λ conjugacy classes. The set $\{K_\lambda\}_{\lambda \vdash n}$ of these formal sums forms a linear basis for the center $C[S_n]$. Thus a product of two conjugacy classes $K_\lambda * K_\mu$ in $C[S_n]$ can be decomposed in the basis $\{K_\lambda\}_{\lambda \vdash n}$ and the coefficients of the decomposition are non-negative integers called *connection coefficients* or *structure constants* of $C[S_n]$:

$$K_\lambda * K_\mu = \sum_{\gamma \vdash n} c^{\gamma}_{\lambda,\mu} K_\gamma \tag{1}$$

Equivalently, the number $c^{\gamma}_{\lambda,\mu}$ counts the number of solutions $(\sigma, \rho) \in C_\lambda \times C_\mu$ of the equation $\sigma\rho = \pi$ where π is any fixed permutation of C_γ.

Efforts for computing them have been made by several authors, beginning in the late 1970's with D. W. Walkup [18], Bertram and Wei [2], G. Boccara [3] and also R. P. Stanley

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[17], J. Katriel [13], Jones [12] and D. M. Jackson [10]. All these authors compute special values of $c^{\gamma}_{\lambda,\mu}$ and most of their explicit results concern the case $\gamma = n$, i.e. at least one permutation is an n-cycle, and restricted values of λ and μ. Our main result (Theorem 2.1) is an explicit expression for $c^{n}_{\lambda,\mu}$, valid for any partitions λ and μ of n.

From now on, we shall be interested only with this case $\gamma = n$ and thus we can define the *genus* $g(\lambda,\mu)$ of a pair of partitions of weight n by the equation

$$\ell(\lambda) + \ell(\mu) \quad = \quad n + 1 - 2g(\lambda,\mu). \tag{2}$$

We shall assume that $g(\lambda,\mu)$ is a non negative integer, since a straightforward decomposition into transpositions proves that otherwise $c^{n}_{\lambda,\mu} = 0$.

Our main result (Theorem 2.1) is the explicit computation of a family $(S_{k,g}(X))_{k,g\geqslant 0}$ of symmetric polynomials of degree $2g$ such that, for any partitions λ, μ of n of genus $g = g(\lambda,\mu)$,

$$c^{n}_{\lambda\mu} = \frac{n}{\prod_i \alpha_i! \prod_i \beta_i! 2^{2g}} \sum_{g_1+g_2=g} S_{\ell(\lambda),g_1}(\lambda) S_{\ell(\mu),g_2}(\mu).$$

In a previous result ([8], Theorem 3.1), the simple expression for the coefficients $c^{n}_{\lambda,\mu}$ when $g(\lambda,\mu) = 0$ was derived. This coefficient was later interpreted combinatorially by Goulden and Jackson [6] as the number of unicellular rooted bicolored maps with n edges on a surface of genus zero, the vertices of each color having degree distribution given by λ and μ respectively. Our main result (Theorem 2.1) is the extension of this result to maps on a surface of arbitrary genus i.e. to pairs of partitions (λ,μ) with arbitrary $g(\lambda,\mu)$. Except for much more restricted values of (λ,μ,γ), this is, as far as we know, the first general expression for connection coefficients which do not belong to the so-called minimal transitive case $g = 0$.

In Section 2, we state our main result (Theorem 2.1) and outline the successive steps of the proof. Unlike Goulden and Jackson's proof of the special case $g = 0$, ours follows from the general character theoretic expression and is not constructive. However crucial use is made of a new bijective interpretation during the course of the proof. In Section 3, we give the proof of Theorem 2.1.

In Section 4, we give some special cases of Theorem 2.1 and we recall the relation between pairs of permutations and maps on oriented surfaces and restate our result in this context. From this point of view, our result generalizes a formula of Walsh and Lehman ([19] Formula 9) for some monochromatic maps to the corresponding bicolored maps.

From their formula, Walsh and Lehman were able to sum over all types of monochromatic unicellular maps with given genus and number of vertices to obtain a formula for the number of maps with given number of vertices. Several equivalent expressions for this coarser enumeration were obtained by Harer and Zagier [9] in connection with the Euler characteristic of the moduli spaces of algebraic curves. Independently, Jackson [10] used the character theoretic approach to obtain the same result. His approach was simplified by Zagier [20] and extended by Jackson and Visentin [11]. Following Zagier, Adrianov [1] obtained recursive expressions which are analogue to Harer-Zagier Formulae for bicolored unicellular maps. In Section 4, using Theorem 2.1, we are able to deduce explicit expressions for the number of these maps. Although equivalent expressions were already obtained for these coaser enumerations (in [1] and [11]), our expressions are of a different nature: they involve summations of positive contributions, with number of terms bounded by a function of g and independent of n. In particular for fixed genus, asymptotic results are derived when n goes to infinity.

2 The main result: a general expression for $c^n_{\lambda,\mu}$

A *composition* p of n is a finite sequence of non-negative integers summing up to n and we denote by $p \models n$. Following [15], given a partition λ, let $z_\lambda = \prod_i \alpha_i! i^{\alpha_i}$ for a partition $\lambda = 1^{\alpha_1} \dots n^{\alpha_n}$.

Theorem 2.1 *Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_m)$ be any two partitions of n with $g(\lambda, \mu) = g$. Then*

$$c^n_{\lambda\mu} = \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{g_1+g_2=g} (\ell + 2g_1 - 1)!(m + 2g_2 - 1)! \sum_{\substack{(i_1,\dots,i_\ell)\models g_1 \\ (j_1,\dots,j_m)\models g_2}} \prod_k \binom{\lambda_k}{2i_k + 1} \prod_k \binom{\mu_k}{2j_k + 1},$$

or equivalently, with $\lambda = 1^{\alpha_1} \dots n^{\alpha_n}$ and $\mu = 1^{\beta_1} \dots n^{\beta_n}$,

$$c^n_{\lambda\mu} = \frac{n}{\prod \alpha_i! \prod \beta_i! 2^{2g}} \sum_{g_1+g_2=g} S_{\ell,g_1}(\lambda) S_{m,g_2}(\mu),$$

where

$$S_{k,g}(x_1, \dots, x_k) = (k + 2g - 1)! \sum_{(p_1,\dots,p_k)\models g} \prod_i \frac{1}{2p_i + 1} \binom{x_i - 1}{2p_i}$$

is a symmetric polynomial of degree $2g$ in the x_i.

This formula has several interesting properties:

- The coefficient $c^n_{\lambda,\mu}$ is expressed as a sum of positive contributions, thus giving easily an idea of its order of magnitude. This is used in section 4 to obtain asymptotic results for maps.
- The product form of the formula is rather surprising: it suggests in particular that the two partitions contribute independently to the genus.
- The number of terms in the summation is

$$n(\ell, m, g) = \sum_{g_1+g_2=g} \binom{\ell + g_1 - 1}{g_1} \binom{m + g_2 - 1}{g_2}$$

  and hence is a polynomial in ℓ and m of total degree g.

These properties are meant to be compared with those of the known expression which is readily obtained from character theoretic point of view (Equations (4) and (5) in Section 3.1). Indeed, this latter expression involves the evaluations $\chi_\lambda^{1^r(n-r)}$ of some characters of the symmetric group which can be exponential with respect to the number of parts of λ. Moreover in this expression, an asymptotic evaluation of $c^n_{\lambda\mu}$ is made very difficult by the occurrence of alternating signs.

Survey of the proof. In order to help the reader we provide a short overview of the successive steps in the proof.

The first step is based on the character theoretic approach which is exposed in [7] or [10] (among many other references). Using explicit expressions for characters of the symmetric group, a first formula (Formula (4)) is given in Section 3.1.

Then we produce a sequence of bijections establishing a correspondence between two sets that are respectively the combinatorial models of Formula (4) and of our final expression: In Section 3.2 our first combinatorial model, "quasi-painted diagram", is introduced and the evaluation of some characters are given as a weighted summations over the corresponding set of objects. In Section 3.3 we use a bijection to replace quasi-painted diagrams by properly "painted diagrams" and we rewrite Formula (4) as a weighted summation over some "painted diagram matchings" (Formula (8)). The introduction in Section 3.4 of "connected components" of diagram matchings allows to set apart the diagram matching from its painting and to show that the weight depends only on the painting (Formula (9)). This is used in Section 3.5 to apply a sign reversing involution. As expected, the fix-points yield positive contributions (Formula (10)). These contributions count "colorings" of the diagram matchings as explained in Section 3.6. In Section 3.7, we show that colored diagrams are enumerated by the formula of Theorem 2.1, thus completing the proof.

3 Proof of the main theorem

3.1 The character theoretic expression

Detailed proofs for this section can be found in [7], [10], [12], [15] or [17], so we only give all ingredients for the sake of completeness.

Characters χ^λ of irreducible representations of S_n are class functions and can therefore be seen as elements of the center $\mathcal{C}[S_n]$:

$$\chi^\lambda = \sum_{\sigma\in S_n} \chi^\lambda(\sigma)\sigma \in \mathcal{C}[S_n]\,.$$

The set $\{\chi^\lambda\}_{\lambda\vdash n}$ forms a linear basis and a set of fundamental idempotents (up to a constant) for $\mathcal{C}[S_n]$. This observation and the fact that the transition matrix between the bases $\{K_\lambda\}_{\lambda\vdash n}$ and $\{\chi^\lambda\}_{\lambda\vdash n}$ is the character table, allows to develop the Frobenius formula:

$$c^n_{\lambda\mu} = \frac{|C_\lambda||C_\mu|}{n!}\sum_{\nu\vdash n}\frac{1}{f^\nu}\chi^\nu_\lambda\chi^\nu_\mu\chi^\nu_{(n)} \tag{3}$$

where χ^ν_λ is the character χ^ν evaluated at the conjugacy class C_λ and f^ν is the dimension of the representation indexed by ν. The Murnaghan-Nakayama rule computes recursively the values of the characters χ^ν_λ. We obtain from this rule the value

$$\chi^\nu_{(n)} = \begin{cases} (-1)^r & \text{if } \nu = 1^r(n-r),\\ 0 & \text{otherwise.}\end{cases}$$

of a character χ^ν evaluated at a full cycle, i.e. at an element of the conjugacy class $C_{(n)}$. From the hook formula, we also have $f^{1^r(n-r)} = \binom{n-1}{r}$. Using these two observations, we rewrite Formula (3) as

$$c^n_{\lambda\mu} = \frac{n}{z_\lambda z_\mu}\sum_{r=0}^{n-1}(-1)^r r!(n-1-r)!\chi^{1^r(n-r)}_\lambda\chi^{1^r(n-r)}_\mu. \tag{4}$$

The following formula is well known (see Goupil [7], Stanley [17] or Littlewood [14] p. 139) and can be derived from the Murnaghan-Nakayama rule: let $\lambda = 1^{\alpha_1}\ldots n^{\alpha_n}$ and ℓ be the smallest index such that $\alpha_\ell \neq 0$, then

$$\chi_\lambda^{1^r(n-r)} \quad = \quad (-1)^r[T^r](1+T+\ldots+T^{\ell-1})(1-T^\ell)^{\alpha_\ell-1}\prod_{i\geqslant \ell+1}(1-T^i)^{\alpha_i}. \tag{5}$$

Our first goal is to give a combinatorial interpretation of this formula.

3.2 Painted diagrams: a combinatorial model

Let λ be a partition of n and write $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ or $\lambda = \ell^{\alpha_\ell}\ldots n^{\alpha_n}$ with $\alpha_\ell \neq 0$. The *Ferrers's diagram* of shape λ is a planar representation of λ made of rows positioned on top of each other with the ith row, starting from the bottom, having λ_i cells. In the sequel λ shall denote indistinctly a partition or its Ferrers's diagram, hence we shall refer to lines and cells of λ, meaning lines and cells of the Ferrers's diagram of shape λ. Since the top line $\lambda_{\ell(\lambda)} = \ell$ and its cells are treated differently in our combinatorial model, we shall refer to them as the *top* line and cells of λ. We define $\lambda^\bullet = (\ell-1)\ell^{\alpha_\ell-1}\ldots n^{\alpha_n}$ and $\lambda^* = \ell^{\alpha_\ell-1}\ldots n^{\alpha_n}$. In particular when $\ell = 1$, $\lambda^\bullet = \lambda^*$. A *dotted diagram* of shape λ is a diagram of shape λ in which the rightmost cell of the top line contains a bullet. In particular, if λ is dotted then the empty cells of λ form a diagram of shape $\lambda^\bullet$.

A *painted diagram* λ_{RG} of shape λ is a Ferrers's diagram of shape λ in which to each cell is assigned one of two colors, say red and green, in such a way that all cells in any given line have the same color.

Let $\mathcal{P}_r(\lambda)$ denote the set of painted diagrams of shape λ with r red cells and $\mathcal{P}(\lambda) = \cup_{0\leqslant r\leqslant|\lambda|}\mathcal{P}_r(\lambda)$.

A *quasi-painted diagram* λ_{RG} of shape λ is a dotted Ferrers's diagram of shape λ in which to each cell, except the dotted one, is assigned one of two colors, say red and green, in such a way that:

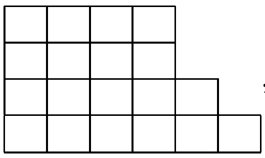
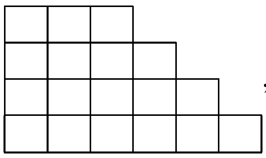
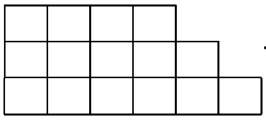
- the subdiagram λ^* is painted,
- in the top line of λ (which contains the dotted cell), red cells, if any, appear on the left of green cells, if any.

Let $\mathcal{QP}_r(\lambda)$ denote the set of quasi-painted diagrams of shape λ with r red cells.

Given a quasi-painted diagram λ_{RG}, we shall denote by λ_R the induced red diagram, λ_G the induced green diagram and λ^*_{RG} the induced painted diagram of shape λ^*. Then $|\lambda_R|$ is the number of red cells of λ_{RG}. We shall also denote by λ^*_R and λ^*_G the diagrams induced by the coloring of λ^*_{RG}. The following equalities are straightforward:

$$\begin{aligned}
|\lambda| &= |\lambda^\bullet| + 1 = |\lambda_R| + |\lambda_G| + 1,\\
\ell(\lambda) &= \ell(\lambda^*) + 1 = \ell(\lambda^*_R) + \ell(\lambda^*_G) + 1.
\end{aligned}$$

Example. Let $\lambda = (6, 5, 4, 4)$. Then,

$\lambda =$  , $\lambda^\bullet =$  , $\lambda^* =$  .

Let now λ_{RG} be the following quasi-painted diagram, and λ^*_{RG} the induced painting of λ^*:

$$\lambda_{RG} = \begin{array}{cccccc} R & R & G & \bullet & & \\ G & G & G & G & & \\ R & R & R & R & R & \\ R & R & R & R & R & R \end{array}, \qquad \lambda^*_{RG} = \begin{array}{cccccc} G & G & G & G & & \\ R & R & R & R & R & \\ R & R & R & R & R & R \end{array}.$$

We have also the following induced diagrams:

$$\lambda_{R} = \begin{array}{cccccc} R & R & & & & \\ R & R & R & R & R & \\ R & R & R & R & R & R \end{array}, \qquad \lambda_{G} = \begin{array}{cccc} G & & & \\ G & G & G & G \end{array}.$$

$\diamond$

With these notations and standard arguments on generating functions, Formula (5) is rewritten as

$$\chi_\lambda^{1^r(n-r)} = (-1)^r \sum_{\lambda_{RG} \in \mathcal{QP}_r(\lambda)} (-1)^{\ell(\lambda^*_R)}. \tag{6}$$

3.3 From quasi-painted diagrams to painted diagram matchings

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and any non negative integer ℓ such that $1 \leqslant \ell \leqslant \lambda_k - 1$, define

$$\lambda^\ell = (\lambda_1, \ldots, \lambda_{k-1}, \ell, \underbrace{1, \ldots, 1}_{\lambda_k - 1 - \ell \text{ times}})$$

and

$$\lambda^0 = (\lambda_1, \ldots, \lambda_{k-1}, \underbrace{1, \ldots, 1}_{\lambda_k - 1 \text{ times}}).$$

In particular, if $\lambda_k = 1$ then $\lambda^0 = \lambda^*$.

Let $\{R, G\}^k$ denote the set of words of length k on the alphabet $\{R, G\}$ and $S(R^k G^h)$ the set of words with k letters R and h letters G. Let also $R\{R, V\}^i$ denote the set of words of length $i + 1$ starting with an R.

Lemma 3.1 *Let λ be a partition of n, $\ell_0 = \lambda_{\ell(\lambda)}$. There exists a bijection*

$$\mathcal{QP}_r(\lambda) \times \{R, G\}^{\ell_0} \simeq \bigcup_{\ell=0}^{\ell_0 - 1} \big(\mathcal{P}_r(\lambda^\ell) \times \{R, G\}^{\ell + \delta_{\ell,0}}\big).$$

Proof. We shall use the following fact: Let k and h be non negative integers. Then the application $f_{k,h} \colon w \mapsto (u, v)$ where u is the maximal left factor of w such that $uv = w$, $|u|_R \leqslant k$ and $|u|_G \leqslant h$ defines a bijection

$$\{R, G\}^{k+h+1} \simeq \big(\bigcup_{i=0}^{k} S(R^i G^h) \times G\{R, G\}^{k-i}\big) \cup \big(\bigcup_{j=0}^{h} S(R^k G^j) \times R\{R, G\}^{h-j}\big).$$

Indeed the application $g\colon (u,v)\mapsto uv$ is clearly the inverse of $f_{k,h}$. Then we check that the image of $\{R,G\}^{k+h+1}$ is as described.

Example. Let $k=5$, $h=4$, then the image of the word $GGRGRGGGRG$ is the factorization $(GGRGRG, GGRG)$, since the left factor would be extended by a letter G whereas it already contains $q=4$ of them. Similarly $f_{5,4}(RRRGRGGRGG)=(RRRGRGGRG, G)$. $\diamond$

Now let λ be a partition of n, $\ell_0=\lambda_{\ell(\lambda)}$. Let also $\lambda_{RG}\in\mathcal{QP}(\lambda)$ and $w\in\{R,G\}^{\ell_0}$. The image of (λ_{RG},w) is defined as follows: let k and h be respectively the number of letters R and G in the top line of λ_{RG} and let $(u,v)=f_{k,h}(w)$ and $\ell=|v|-1$. If $\ell=0$ let $v'=v$ otherwise let $v=xv'$ where x is the first letter of v (so that u contains as many letters x as the top line of λ_{RG}). Let finally λ^ℓ_{RG} be the painted diagram of shape λ^ℓ in which:

- Lines of λ^* are painted according to λ_{RG}.
- The line of length ℓ of the hook of λ^ℓ has the opposite color than x.
- The $\ell_0-1-\ell$ lines of length 1 of the hook of λ^ℓ are painted according to the word u (which has length $\ell_0-1-\ell$) written from top to bottom.

The image of (λ_{RG},w) is then defined to be (λ^ℓ_{RG},v'), which satisfies $|v'|=\ell+\delta_{\ell,0}$ and this is clearly a bijection preserving the respective numbers of red and green cells in the diagrams. $\square$

According to the latter lemma, Formula (6) can be rewritten as

$$\chi_\lambda^{1^r(n-r)}=(-1)^r\sum_{\ell=0}^{\ell_0-1}2^{\ell+\delta_{\ell,0}-\ell_0}\sum_{\lambda^\ell_{RG}\in\mathcal{P}_r(\lambda^\ell)}(-1)^{\ell(\lambda^*_R)}\tag{7}$$

where $\ell_0=\lambda_{\ell(\lambda)}$.

We now consider the interpretation of Equation (4):

$$\begin{aligned}c^n_{\lambda\mu}&=\frac{n}{z_\lambda z_\mu}\sum_{r=0}^{n-1}(-1)^r r!(n-1-r)!\chi_\lambda^{1^r(n-r)}\chi_\mu^{1^r(n-r)},\\&=\frac{n}{z_\lambda z_\mu 2^{\ell_0+m_0}}\sum_{\substack{0\leqslant\ell\leqslant\ell_0-1\\0\leqslant m\leqslant m_0-1}}2^{\ell+\delta_{\ell,0}+m+\delta_{m,0}}\sum_{r=0}^{n-1}\sum_{\substack{\lambda^\ell_{RG}\in\mathcal{P}_r(\lambda^\ell)\\\mu^m_{RG}\in\mathcal{P}_r(\mu^m)}}(-1)^{r-\ell(\lambda^*_R)-\ell(\mu^*_R)}r!(n-1-r)!.\end{aligned}$$

Since $r!$ is the number of bijections between red cells of λ^ℓ_{RG} and red cells of μ^m_{RG}, and $(n-1-r)!$ is the number of bijections between green cells of those painted diagrams, we are led to consider bijections $\phi:\lambda^\ell_{RG}\to\mu^m_{RG}$ which preserve colors.

A *Ferrers's diagram matching* of type (λ,μ) is a bijection from cells of λ onto cells of μ. Let $\mathcal{M}(\lambda,\mu)$ denote the set of Ferrers's diagram matchings of type (λ,μ).

A *painted diagram matching* ϕ_{RG} of type (λ,μ) is a triple $(\lambda_{RG},\mu_{RG},\phi)$ where $\lambda_{RG}\in\mathcal{P}(\lambda)$, $\mu_{RG}\in\mathcal{P}(\mu)$ and $\phi\in\mathcal{M}(\lambda,\mu)$ such that $\phi(\lambda_R)=\mu_R$. Let $\mathcal{PM}(\lambda,\mu)$ denote the set of painted diagram matchings of type (λ,μ).

For ϕ_{RG} in $\mathcal{PM}(\lambda^\ell,\mu^m)$, let $\varepsilon(\phi_{RG})=|\lambda^\ell_R|-\ell(\lambda^*_R)-\ell(\mu^*_R)$. Then, using these definitions, we obtain

$$c^n_{\lambda\mu}=\frac{n}{z_\lambda z_\mu 2^{\ell_0+m_0}}\sum_{\substack{0\leqslant\ell\leqslant\ell_0-1\\0\leqslant m\leqslant m_0-1}}2^{\ell+\delta_{\ell,0}+m+\delta_{m,0}}\sum_{\phi_{RG}\in\mathcal{PM}(\lambda^\ell,\mu^m)}(-1)^{\varepsilon(\phi_{RG})}.$$

For any pair (λ,μ) of partitions of the same weight with $g(\lambda,\mu)=g$, let $h(\lambda,\mu)=|\lambda|-\ell(\lambda)-\ell(\mu)$. Then, by definition of λ^ℓ and μ^m and since $h(\lambda,\mu)=2g-1$, we obtain $h(\lambda^\ell,\mu^m)=2g-(\ell_0+m_0)+(\ell+\delta_{\ell,0}+m+\delta_{m,0})$ and

$$c^n_{\lambda\mu}=\frac{n}{z_\lambda z_\mu 2^{2g}}\sum_{\substack{0\leqslant\ell\leqslant\ell_0-1\\0\leqslant m\leqslant m_0-1}}2^{h(\lambda^\ell,\mu^m)}\sum_{\phi_{RG}\in\mathcal{PM}(\lambda^\ell,\mu^m)}(-1)^{\varepsilon(\phi_{RG})}. \tag{8}$$

3.4 Connected components of painted diagram matchings

Let $\phi:\lambda^\ell\to\mu^m$ be a diagram matching. Let $\sim_p$ be the following relation on cells of $\lambda^\ell\cup\mu^m$:

$$\begin{array}{lll} x\sim_p y & \text{if} & x\in\lambda^\ell, y\in\mu^m \text{ and } y=\phi(x),\\ & \text{or} & x\in\mu^m, y\in\lambda^\ell \text{ and } x=\phi(y),\\ & \text{or} & x\in\lambda^\ell, y\in\lambda^\ell \text{ with } x \text{ and } y \text{ in the same line},\\ & \text{or} & x\in\mu^m, y\in\mu^m \text{ with } x \text{ and } y \text{ in the same line}.\end{array}$$

The transitive closure $\sim$ of $\sim_p$ is an equivalence relation. Each equivalence class contains an equal number of cells of λ^ℓ and μ^m because ϕ is stable on them. Hence equivalence classes induce decompositions of λ^ℓ, μ^m and ϕ and we refer to the factors as the *connected components* of λ^ℓ, μ^m and ϕ. Let $c(\phi)$ denote the number of connected components of type λ, μ and ϕ and let $\phi=\phi^{(1)}+\phi^{(2)}+\cdots+\phi^{(c(\phi))}$ where $\phi^{(i)}:\lambda^{(i)}\to\mu^{(i)}$ is the restriction of ϕ to the ith equivalence class of $\sim$.

Our interest in connected components lies in the fact that all cells in a connected component have the same color in a painting compatible with ϕ. Hence colors can be given to components rather than to cells. Indeed as soon as colors are given to components, the bijection ϕ_{RG} induced from ϕ is stable on colors and λ^ℓ_{RG} is painted as well as μ^ℓ_{RG}. For a given diagram matching ϕ, the associated painted diagram matchings ϕ_{RG} are therefore in one-to-one correspondence with subsets of $[c(\phi)]=\{1,\dots,c(\phi)\}$, giving

Lemma 3.2 *There exists a bijection*

$$\mathcal{PM}(\lambda^\ell,\mu^m)\simeq\{(\phi,R)\mid\phi\in\mathcal{M}(\lambda^\ell,\mu^m),R\subset[c(\phi)]\}.$$

Let ϕ be in $\mathcal{M}(\lambda^\ell,\mu^m)$ and for all $i\in[c(\phi)]$, let

$$h^{(i)}=|\lambda^{(i)}|-\ell(\lambda^{*(i)})-\ell(\mu^{*(i)}),$$

where $\ell(\lambda^{*(i)})$ denotes the number of lines in the component $\phi^{(i)}$ which also belong to λ^* (i.e. cells of the top hook of λ^ℓ are ignored).

If (ϕ,R) is the image of a painted diagram matching ϕ_{RG} of type (λ^ℓ,μ^m) via Lemma 3.2, then we have

$$\sum_{i\in R}h^{(i)}=|\lambda^\ell_R|-\ell(\lambda^*_R)-\ell(\mu^*_R)=\varepsilon(\phi_{RG}).$$

Hence summation (8) becomes

$$c^n_{\lambda\mu}=\frac{n}{z_\lambda z_\mu 2^{2g}}\sum_{\substack{0\leqslant\ell\leqslant\ell_0-1\\0\leqslant m\leqslant m_0-1}}2^{h(\lambda^\ell,\mu^m)}\sum_{\phi\in\mathcal{M}(\lambda^\ell,\mu^m)}\sum_{R\subset[c(\phi)]}(-1)^{\sum_{i\in R}h^{(i)}}. \tag{9}$$

3.5 Sign reversing involution

Lemma 3.3 *Let ϕ be a diagram matching of type (λ^ℓ, μ^m), then*

$$\sum_{R \subset [c(\phi)]} (-1)^{\sum_{i \in R} h^{(i)}} = \begin{cases} 0 & \textit{if } \exists i \mid h^{(i)} \equiv 1 \pmod 2, \\ 2^{c(\phi)} & \textit{otherwise.} \end{cases}$$

Proof. Suppose there exists i such that $h^{(i)}$ is odd and take i_0 the least such index. Then the involution

$$R \mapsto \begin{cases} R \setminus \{i_0\} & \text{if } i_0 \in R, \\ R \cup \{i_0\} & \text{if } i_0 \notin R \end{cases}$$

on subsets of $[c(\phi)]$ changes the sign in the summation. Therefore all terms cancel.

Otherwise all $h^{(i)}$ are even, hence $\sum_{i \in R} h^{(i)}$ is even and the sum is equal to the number of subsets of $[c(\phi)]$ which is $2^{c(\phi)}$. □

According to Lemma 3.3, negative terms cancel in the summation (9), and we have

Proposition 3.1

$$c^n_{\lambda\mu} = \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{\substack{0 \leqslant \ell \leqslant \ell_0 - 1 \\ 0 \leqslant m \leqslant m_0 - 1}} \sum_{\phi} 2^{h(\lambda^\ell, \mu^m) + c(\phi)}, \tag{10}$$

where the second summation ranges over all $\phi \in \mathcal{M}(\lambda^\ell, \mu^m)$ such that each connected component $\phi^{(i)}$ has an even $h^{(i)}$.

3.6 Colored diagram matchings

We introduce another type of coloring:

A diagram is *colored* if one of two colors, say black and white, is given to each cell.

A *parity function* $\mathcal{C}$ on a couple (λ, μ) of partitions is a map from the set of lines of λ and μ into $\{0, 1\}$. Let $\mathrm{even}(\mathcal{C})$ be the number of *even* lines in λ and μ according to $\mathcal{C}$ (i.e. a line L is even if $\mathcal{C}(L) = 0$; $\mathrm{even}(\mathcal{C}) = |\mathcal{C}^{-1}(\{0\})|$).

A diagram matching ϕ of $\mathcal{M}(\lambda, \mu)$ is *colored* if λ and μ are colored and ϕ respects the colors (i.e. maps black cells onto black cells).

A colored diagram matching ϕ of $\mathcal{M}(\lambda, \mu)$ is *$\mathcal{C}$-correct* if:

- the number of black cells in each line L of λ has same parity as $\mathcal{C}(L)$,
- the number of white cells in each line L of μ has same parity as $\mathcal{C}(L)$.

Example. In order to draw a diagram matching we put numbers in cells so that matched cells share the same number. The following example is a $\mathcal{C}$-correct colored diagram matching with respect to be the parity function $\mathcal{C}$ whose values are written in front of each line.

1	13_B						0	5_B					
1	12_B						1	3_W					
0	11_W					$\mapsto$	0	11_W	6_W				
1	6_W	7_W	8_B	9_W	10_W		0	1_B	13_B	2_B	4_B		
0	1_B	2_B	3_W	4_B	5_B		1	12_B	7_W	10_W	9_W	8_B	

◇

Theorem 3.1 *The number of $\mathcal{C}$-correct colorings of a diagram matching ϕ of type (λ,μ) with exactly one connected component is*

$$\begin{array}{cl} 2^{h(\lambda,\mu)+1} & \text{if } h(\lambda,\mu) \equiv \mathrm{even}(\mathcal{C}) \pmod 2, \\ 0 & \text{otherwise,} \end{array}$$

where $h(\lambda,\mu) = |\lambda| - \ell(\lambda) - \ell(\mu)$.

Proof. The proof is by induction on $|\lambda|$.

First suppose that $|\lambda| = 1$. There is only one possible diagram matching ϕ which maps the only cell of λ onto the only cell of μ. In this case, $\mathcal{C}$ completely determines the colors of a $\mathcal{C}$-correct colored matching and since the two cells must have the same color because of ϕ, there is a solution only if $\mathrm{even}(\mathcal{C}) = 1$. This is in accordance with $h(\lambda,\mu) = -1$ and the result holds.

Now suppose that the theorem is proved for all shapes λ and μ with $|\lambda| < n$. Let λ, μ, ϕ and $\mathcal{C}$ be given with $|\lambda| = |\mu| = n$. Consider the multigraph $\mathcal{G}(\phi)$ whose vertices are the lines and whose edges are the fibers of ϕ (i.e. for each x in λ and y in μ such that $y = \phi(x)$ add an edge between corresponding lines). This graph is connected since its connected components are exactly those of ϕ. There are two cases:

First case: *there exists at least one line of length* 1 *in* λ *or* μ. Suppose without loss of generality that there is such a line in λ and let i_0 be the lowest such line (i_0 denote here indifferently the line or the cell). Since ϕ is connected, $j_0 = \phi(i_0)$ is not alone in its line L_0 (except if $|\lambda| = 1$ but this case has already been treated). The color of i_0 is imposed by $\mathcal{C}(i_0)$ and j_0 has the same color of i_0 because of ϕ.

Let $\lambda' = \lambda \setminus \{i_0\}$, $\mu' = \mu \setminus \{j_0\}$, ϕ' the corresponding restriction of ϕ and let $\mathcal{C}'$ be the parity function defined by:

$$\mathcal{C}' : L \mapsto \left\{ \begin{array}{cl} \mathcal{C}(L) & \text{if } L \neq L_0, \\ \mathcal{C}(L_0) & \text{if } L = L_0 \text{ and } \mathcal{C}(i_0) = 1, \\ \mathcal{C}(L_0) + 1 \mod 2 & \text{if } L = L_0 \text{ and } \mathcal{C}(i_0) = 0. \end{array} \right.$$

With these definitions, a $\mathcal{C}$-correct coloring of ϕ induces a $\mathcal{C}'$-correct coloring of ϕ' and conversely there is a unique way to extend a $\mathcal{C}'$-correct coloring of ϕ' into a $\mathcal{C}$-correct coloring of ϕ.

Now the diagram matching ϕ' has exactly one connected component because we have only suppressed a leaf of the graph $\mathcal{G}(\phi)$ to obtain $\mathcal{G}(\phi')$. Since $|\lambda'| = n - 1 < n$, by induction hypothesis, the number of $\mathcal{C}'$-correct colorings of ϕ' is $2^{h(\lambda',\mu')+1}$ if $h(\lambda',\mu') \equiv \mathrm{even}(\mathcal{C}') \pmod 2$, and 0 otherwise. But $h(\lambda',\mu') = h(\lambda,\mu)$ and $\mathrm{even}(\mathcal{C}') \equiv \mathrm{even}(\mathcal{C}) \pmod 2$ by construction. The result is therefore proved in this case.

Second case: *all lines in* λ *and* μ *have more than one cell.* Hence the graph $\mathcal{G}(\phi)$ has no vertex of degree 1 (no leaf). Such a graph contains at least a simple cycle and therefore at least an edge which is not a bridge (i.e. if this edge is suppressed, the induced subgraph is still connected). Let $(x_0, y_0) \in \lambda \times \mu$ be the unique edge of such type whose cell x_0 is the lowest rightmost possible. (Recall that by definition of edges, $y_0 = \phi(x_0)$.)

The cells x_0 and y_0 are either both black or both white in any C-correct coloring of ϕ. For any choice of color we now construct restrictions ϕ' and $\mathcal{C}'$ such that $\mathcal{C}$-correct colorings of ϕ with x_0 of the chosen color are in one-to-one correspondence with $\mathcal{C}'$-correct colorings of ϕ' and we count the latter:

Let $\lambda' = \lambda \setminus \{x_0\}$, $\mu' = \mu \setminus \{y_0\}$ and ϕ' be the corresponding restriction of ϕ, which is still connected since the edge (x_0, y_0) is not a bridge of $\mathcal{G}(\phi)$. If (x_0, y_0) is black, the parity of the line L_{x_0} containing x_0 must be adjusted. In this case, we define:

$$\mathcal{C}' : L \mapsto \begin{cases} \mathcal{C}(L) & \text{if } L \neq L_{x_0}, \\ \mathcal{C}(L_0) + 1 \mod 2 & \text{if } L = L_{x_0}. \end{cases}$$

Otherwise (x_0, y_0) is white and we define $\mathcal{C}'$ by adjusting instead the parity of the line L_{y_0} containing y_0.

In both cases, $|\lambda'| = n - 1 < n$ so that induction hypothesis can be applied. Moreover in both case the number of $\mathcal{C}'$-correct colorings of ϕ' is $2^{h(\lambda',\mu')+1}$ or 0. The number of $\mathcal{C}$-correct colorings of ϕ is therefore $2 \cdot 2^{h(\lambda',\mu')+1}$ if $h(\lambda', \mu') \equiv \mathrm{even}(\mathcal{C}') \pmod 2$ and 0 otherwise. Since $h(\lambda, \mu) = h(\lambda', \mu') + 1$ and the parity of exactly one line has changed, the theorem is proved. □

3.7 Direct enumeration of colored diagram matchings

Let $\mathcal{C}_{\ell,m}$ be the parity function defined on the couple (λ^ℓ, μ^m) which is odd on every line of λ^* and μ^* and even on the lines of the top hooks of λ^ℓ and μ^m.

Let ϕ be a diagram matching of type (λ^ℓ, μ^m), then all $\mathcal{C}_{\ell,m}$-correct colorings of ϕ may be constructed component by component. Hence according to Theorem 3.1, there exists some $\mathcal{C}_{\ell,m}$-correct colorings of ϕ if and only if each connected component $(\lambda^{(i)}, \mu^{(i)})$ of (λ^ℓ, μ^m) satisfies $h(\lambda^{(i)}, \mu^{(i)}) \equiv \mathrm{even}(\mathcal{C}^{(i)}) \pmod 2$ where $\mathcal{C}^{(i)}$ is the restriction of $\mathcal{C}_{\ell,m}$ to the connected component $(\lambda^{(i)}, \mu^{(i)})$. But since odd lines are precisely lines of λ^* and μ^*, $\mathrm{even}(\mathcal{C}^{(i)}) = \ell(\lambda^{(i)}) - \ell(\lambda^{*(i)}) + \ell(\mu^{(i)}) - \ell(\mu^{*(i)})$ and $h(\lambda^{(i)}, \mu^{(i)}) - \mathrm{even}(\mathcal{C}^{(i)}) \equiv h^{(i)} \pmod 2$. Hence the diagram matchings having $\mathcal{C}_{\ell,m}$-correct colorings are precisely those which are involved in summation (10). Moreover the number of $\mathcal{C}_{\ell,m}$-correct colorings of these ϕ is, according to Theorem 3.1, $\prod_i 2^{h(\lambda^{(i)},\mu^{(i)})+1} = 2^{h(\lambda^\ell,\mu^m)+c(\phi)}$ and corresponds to the weight of each ϕ in summation (10).

Let $\mathcal{CM}(\lambda^\ell, \mu^m)$ denote the set of $\mathcal{C}_{\ell,m}$-correct colored diagram matchings of type (λ^ℓ, μ^m). According to the previous discussion, summation (10) reads

$$c^n_{\lambda\mu} = \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{\substack{0 \leqslant \ell \leqslant \ell_0 - 1 \\ 0 \leqslant m \leqslant m_0 - 1}} \mathrm{card}\big(\mathcal{CM}(\lambda^\ell, \mu^m)\big).$$

Now all $\mathcal{C}_{\ell,m}$-correct diagrams of type (λ^ℓ, μ^m) can be constructed as follows: choose the numbers p of black cells and q of white cells $(p + q = n - 1)$. Since, according to $\mathcal{C}_{\ell,m}$, there are exactly $\ell(\lambda) - 1$ odd lines in λ, we have $p \geqslant \ell(\lambda) - 1$ and similarly $q \geqslant \ell(\mu) - 1$. Then choose the odd number $2i_k + 1$ of black cells in each line λ_k, $k \in [1, \ell(\lambda) - 1]$ of λ^*, and the even number $2i_{\ell(\lambda)}$ of black cells in the horizontal part (of length ℓ) of the top hook of λ. We have $(2i_1 + 1) + \cdots + (2i_{\ell(\lambda)} + 1) = p + 1$ hence $p = 2(i_1 + \cdots + i_{\ell(\lambda)}) + \ell(\lambda) - 1$ and similarly $q = 2(j_1 + \cdots + j_{\ell(\mu)}) + \ell(\mu) - 1$. Let $g_1 = (p - \ell(\lambda) + 1)/2$ and $g_2 = (q - \ell(\mu) + 1)/2$ then $g_1 + g_2 = g(\lambda, \mu)$ and g_1 and g_2 are non negative integers. Choose similarly $j_1 + \cdots + j_{\ell(\mu)} = g_2$. Then in each line λ_k of λ^* the black cells can be placed in $\binom{\lambda_k}{2i_k+1}$ ways for $k \in [1, \ell(\lambda) - 1]$ and in $\binom{\ell}{2i_{\ell(\lambda)}}$ ways in the horizontal line of length ℓ of the hook. The same holds for white cells in μ^m. To conclude we need to define the matching ϕ. This has to be done independently on the two colors, hence in $p!q!$ ways, that is in $(\ell(\lambda) - 1 + 2g_1)!(\ell(\mu) - 1 + 2g_2)!$ ways.

Therefore $\mathrm{card}\big(\mathcal{CM}(\lambda^\ell,\mu^m)\big)$ is equal to

$$\sum_{\substack{g_1+g_2=g(\lambda,\mu)\\ g_1\geqslant 0, g_2\geqslant 0}} (\ell(\lambda)-1+2g_1)!(\ell(\mu)-1+2g_2)!$$
$$\sum_{\substack{i_1+\cdots+i_{\ell(\lambda)}=g_1\\ j_1+\cdots+j_{\ell(\mu)}=g_2}} \binom{\ell}{2i_{\ell(\lambda)}}\binom{m}{2j_{\ell(\mu)}}\prod_{k=1}^{\ell(\lambda)-1}\binom{\lambda_k}{2i_k+1}\prod_{k=1}^{\ell(\mu)-1}\binom{\mu_k}{2j_k+1}.$$

Since $\ell_0=\lambda_{\ell(\lambda)}$ and $\sum_{\ell=0}^{\ell_0-1}\binom{\ell}{2i}=\binom{\ell_0}{2i+1}$, summation of the cardinals on ℓ and m yields,

$$c^n_{\lambda\mu}=\frac{n}{z_\lambda z_\mu 2^{2g}}\sum_{g_1+g_2=g}(\ell(\lambda)-1+2g_1)!(\ell(\mu)-1+2g_2)!$$
$$\sum_{\substack{i_1+\cdots+i_{\ell(\lambda)}=g_1\\ j_1+\cdots+j_{\ell(\mu)}=g_2}}\prod_{k=1}^{\ell(\lambda)}\binom{\lambda_k}{2i_k+1}\prod_{k=1}^{\ell(\mu)}\binom{\mu_k}{2j_k+1},$$

and Theorem 2.1 is proved. □

After this proof, we kindly encourage the reader to pause and enjoy a lighter activity such as bird watching or gardening before pursuing the reading.

4 Applications

4.1 Special cases

It was observed by several authors that in some cases, for a given diagram λ, distinct diagrams μ yields the same value of $c^\mu_{\lambda,n}$. For a given diagram λ we introduce the set of diagrams μ that *force λ to connect*, i.e. such that all diagram matchings ϕ in all $\mathcal{M}(\lambda^\ell,\mu^m)$ have exactly one connected component. Then

Corollary 4.1 *If μ forces λ to connect and $g(\lambda,\mu)\in\mathbb{N}$, then $c^\mu_{\lambda,n}=\frac{2(n-1)!}{z_\lambda}$, and this expression does not depend on the particular shape μ.*

Proof. We apply Proposition 3.1. Since there is only one connected component, all diagrams belong to the summation domain and

$$c^n_{\lambda\mu} \;=\; \frac{n!}{z_\lambda z_\mu 2^{2g}}\sum_{\substack{0\leqslant\ell\leqslant\ell_0-1\\ 0\leqslant m\leqslant m_0-1}}2^{h(\lambda^\ell,\mu^m)+1} \;=\; \frac{2(n!)}{z_\lambda z_\mu}.$$

□

For instance when λ is a $(n-1)$-cycle, all odd μ are forcing connectivity and we recover the well known formula (see [16] for an elementary constructive proof)

$$c^\mu_{1(n-1),n}=2(n-2)!\quad\forall\mu \text{ odd}.$$

More generally, for a given shape λ, one can classify all essentially distinct diagram matchings that can be constructed on the diagrams λ^ℓ and compute the contribution of each of these essential configurations. The previous case is the simplest case since there is only one essential configuration, which consists of a unique connected component.

Suppose now $\lambda = 1^k(n-k)$ with $k > 0$ and μ contains a part of length 1. Then $\lambda^0 = 1^{k-1}(n-k)$ and the essential configuration with a unique connected component is still the only relevant one. Indeed in other configurations, except for the connected component which contains the part $(n-k)$ of λ, all connected components contain only parts of length 1 in λ, and therefore only one part in μ. But for such a component, $h^{(i)} = -1$ and these diagrams do not contribute to the summation. The contribution of diagrams with only one connected component is $2^{h(\lambda^0,\mu^0)+1} = 2^{2g+1}$ and the number of such diagrams is readily obtained by an exclusion/inclusion argument. If μ has no part of length 1 the proof is more complicated but the final formula is the same: For $k > 0$ and $\mu = 1^{\beta_1}\dots n^{\beta_n}$ a partition of n of the same parity as k,

$$c^{\mu}_{1^k(n-k),n} = \frac{2(n-k-1)!}{k!}\sum_{h=0}^{k-1}(k-1)_h(n-\beta_1-h)_{k-1-h}\left(\sum_{\substack{\nu\vdash h\\ \nu=2^{\gamma_1}\dots h^{\gamma_h}}}(-1)^{\ell(\nu)}\prod_{i=2}^{h}\binom{\beta_i}{\gamma_i}\right).$$

We recover here the result of [2] that $c^{\mu}_{1^k(n-k),n}$ depends only on $\beta_1,\dots,\beta_{k-1}$, n and k and obtain a combinatorial interpretation for the coefficients. A constructive proof was given for case $k=2$ in [4] and this explicit formula was known for case $k = 1,2,3,4$ (see [2] and [12]) :

$$\begin{aligned}
c^{\mu}_{1(n-1),n} &= 2(n-2)!,\\
c^{\mu}_{1^2(n-2),n} &= (n-3)!(n-\beta_1),\\
c^{\mu}_{1^3(n-3),n} &= \frac{(n-4)!}{3}[(n-\beta_1)(n-\beta_1-1)-2\beta_2],\\
c^{\mu}_{1^4(n-4),n} &= \frac{(n-5)!}{4\cdot 3}[(n-\beta_1)_3-6(n-\beta_1-2)\beta_2-6\beta_3],\\
c^{\mu}_{1^5(n-5),n} &= \frac{(n-6)!}{5\cdot 4\cdot 3}[(n-\beta_1)_4+12\beta_2(\beta_2-1)\\
&\qquad -12(n-\beta_1-2)(n-\beta_1-3)\beta_2-24(n-\beta_1-3)\beta_3-24\beta_4],
\end{aligned}$$

4.2 Applications to the enumeration of maps

An account of the relations between pairs of permutations and maps on oriented surfaces can be found e.g. in [5]. But in order to have a self contained exposition, we shall briefly sketch the definitions of maps: a *map* $(\mathcal{S},\mathcal{G})$ on a compact oriented surface $\mathcal{S}$ without boundary is a graph $\mathcal{G}$ together with an embedding of $\mathcal{G}$ into $\mathcal{S}$ such that connected components of the complement $\mathcal{S}\backslash\mathcal{G}$ of the embedding of $\mathcal{G}$ in $\mathcal{S}$, called the faces of the map, are homeomorphic to discs. Multiple edges are allowed and our maps are rooted, i.e. one edge of $\mathcal{G}$ is distinguished. Two maps $(\mathcal{S},\mathcal{G})$ and $(\mathcal{S}',\mathcal{G}')$ are isomorphic if there exists an orientation-preserving homeomorphism $f:\mathcal{S}\to\mathcal{S}'$ such that $f(\mathcal{G}) = \mathcal{G}'$. A map is *bicolored* if its vertices are colored in black or white so that each edge is incident to one vertex of each color. A map is *unicellular* if it has one face. The *type* of a bicolored unicellular map $\mathcal{M}$ with n edges is a pair of partitions (λ,μ) whose parts give the respective degrees of black and white vertices of $\mathcal{M}$.

A relation between products of conjugacy classes and enumeration of maps is presented in the following proposition (see [5] for more details):

Proposition 4.1 *Bicolored unicellular maps of type (λ,μ) are maps on a compact orientable surface of genus g which satisfy*

$$g = g(\lambda,\mu)\,.$$

Moreover, the number $\mathrm{Bi}(\lambda,\mu)$ *of bicolored unicellular maps of type (λ,μ) with n edges is the number of pairs $(\sigma,\tau)\in C_\lambda\times C_\mu$ such that $\sigma\tau=(1,2,\dots,n)$, which is also the coefficient $c^n_{\lambda\mu}$:*

$$\mathrm{Bi}(\lambda,\mu) = c^n_{\lambda\mu}.$$

In particular when $g(\lambda,\mu)=0$, we are interested in unicellular maps on the sphere which are plane trees as remarked by Goulden and Jackson [6].

We observe that if we fix a partition λ of n and the parameter g and if we sum over all conjugacy classes indexed by partitions μ satisfying $\ell(\mu)=m=n+1-2g-\ell(\lambda)$, then the coefficient

$$\mathrm{Bi}(\lambda,m,n) = \sum_{\substack{\mu\vdash n\\ \ell(\mu)=m}} \mathrm{Bi}(\lambda,\mu,n) = \sum_{\substack{\mu\vdash n\\ \ell(\mu)=m}} c^n_{\lambda,\mu}. \tag{11}$$

counts the number of bicolored unicellular rooted maps on a surface of genus g with black degree distribution given by λ and no restriction on white degree distribution other than the number m of white vertices. Similarly, if we sum over all conjugacy classes with partition λ of a fixed length (or rank), then the coefficient

$$\mathrm{Bi}(\ell,m,n) = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=\ell}} \mathrm{Bi}(\lambda,m,n) = \sum_{\substack{\lambda\vdash n,\,\ell(\lambda)=\ell\\ \mu\vdash n,\,\ell(\mu)=m}} c^n_{\lambda,\mu} \tag{12}$$

counts the number $\mathrm{Bi}(\ell,m,n)$ of unicellular bicolored rooted maps of genus g with ℓ black vertices and $m=n+1-2g-\ell$ white vertices. These maps were recently considered by Adrianov([1]). Finally, if we sum the product in (12) over all possible lengths of λ, we obtain

$$\mathrm{Bi}(n,g) = \sum_{\substack{\ell\geqslant 1,\,m\geqslant 1\\ \ell+m=n+1-2g}} \mathrm{Bi}(\ell,m,n) \tag{13}$$

which is the number $\mathrm{Bi}(n,g)$ of unicellular bicolored rooted maps of genus g with n edges. The purpose of this section is to give explicit expressions for the sums of coefficients in (11), (12) and (13). These explicit expressions are derived from Theorem 2.1 using the following lemma, which already appears in [19], and performing usual algebraic manipulations on summations. Therefore we state the results without giving the rather cumbersome calculations.

Lemma 4.1 *[19] Let n,m be two positive integers and g a non negative integer with $\eta\vdash g$ such that $m+2g\leqslant n$, then*

$$\sum_{\substack{\mu\vdash n\\ \ell(\mu)=m}} \frac{1}{z_\mu} \sum_{\substack{q\models g\\ \ell(q)=m\\ \lambda(q)=\eta}} \prod_{k=1}^{m} \binom{\mu_k}{2q_k+1} = \frac{1}{m!}\binom{n-1}{m+2g-1} \prod_{k=1}^{\ell(\eta)} \frac{1}{2\eta_k+1} \sum_{\substack{q\models g\\ \ell(q)=m\\ \lambda(q)=\eta}} 1.$$

A straightforward consequence of this lemma and Theorem 2.1 is the following result.

Theorem 4.1 *Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and m, g two non negative integers such that $\ell + m = n + 1 - 2g$. Then*

$$\mathrm{Bi}(\lambda, m, n) = \frac{n!}{z_\lambda 2^{2g} m!} \sum_{g_1+g_2=g} \left(\sum_{i_1+\cdots+i_\ell=g_1} \prod_k \binom{\lambda_k}{2i_k+1} \right) P_{g_2}(m)$$

where $P_g(x)$ is a polynomial in x of degree g defined by:

$$P_g(x) = \sum_{\substack{\gamma \vdash g \\ \gamma = 1^{c_1} \dots g^{c_g}}} (x)_{\ell(\gamma)} \frac{1}{\prod_i c_i!(2i+1)^{c_i}}, \tag{14}$$

with $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ the descending factorial.

If we set $\mu = 2^{n/2}$, all white vertices have degree two and can be seen as centers of edges linking black vertices to black vertices so that the white vertices can be forgotten and we obtain the monochromatic unicellular maps counted by Walsh and Lehman [19]. Therefore, let us call $\mathrm{Mono}(n, g)$ the number of monochromatic maps of genus g with n edges and use Theorem 4.1 to obtain

Corollary 4.2 *[19] Let n be a positive integer and g be non negative integer, then*

$$\mathrm{Mono}(n, g) = \mathrm{Bi}((2^n), n+1-2g, 2n) = \frac{(2n)!}{2^{2g} n!(n+1-2g)!} P_g(n+1-2g).$$

Moreover, if g is fixed,

$$\mathrm{Mono}(n, g) \sim_{n\to\infty} \frac{n^{3(g-\frac{1}{2})} 4^n}{\sqrt{\pi} g! 12^g}.$$

Using once again Lemma 4.1 to sum over all λ in Theorem 4.1, we obtain:

Theorem 4.2 *Let ℓ, m, n, g be non negative integers such that $\ell + m = n + 1 - 2g$. Then*

$$\mathrm{Bi}(\ell, m, n) = \frac{(n+1)_{2g}}{(n+1)2^{2g}} \binom{n+1-2g}{\ell} \sum_{g_1+g_2=g} \binom{n-1}{\ell+2g_1-1} P_{g_1}(\ell) P_{g_2}(m),$$

Eventually, summing over all ℓ and m with $\ell + m = n$, we obtain:

Theorem 4.3 *Let n, g be non negative integers. Then*

$$\mathrm{Bi}(n, g) = \frac{(n+1)_{2g}}{(n+1)2^{2g}} \sum_{g_1+g_2=g} \sum_{\substack{0 \leqslant \ell_1 \leqslant g_1 \\ 0 \leqslant \ell_2 \leqslant g_2}} c_{g_1,\ell_1} c_{g_2,\ell_2} (n+1-2g)_{\ell_1+\ell_2} \binom{2n-2g-\ell_1-\ell_2}{n-2g_1-\ell_1}$$

where the $c_{g,\ell}$ are constants given by:

$$c_{g,\ell} = \sum_{\substack{\gamma \vdash g \\ \ell(\gamma)=\ell \\ \gamma = 1^{\gamma_1} \dots g^{\gamma_g}}} \frac{1}{\prod_i \gamma_i!(2i+1)^{\gamma_i}}.$$

Corollary 4.3 *Let g be a given non negative integer, then*

$$\mathrm{Bi}(n,g) \sim_{n\to\infty} \frac{n^{3(g-\frac{1}{2})}4^n}{\sqrt{\pi}g!48^g}.$$

In particular, we recover Adrianov's result ([1], Corollary 5) that for fixed g,

$$\frac{\mathrm{Bi}(n,g)}{\mathrm{Mono}(n,g)} \sim_{n\to\infty} \left(\frac{1}{4}\right)^g.$$

We give in tables 1 and 2, the expressions of low genera for the polynomials $\mathrm{Bi}(\lambda,\mu,n)$, $P_g(x)$, $\mathrm{Bi}(\ell,m,n)$ and $\mathrm{Bi}(n,g)$ where

$$\begin{aligned} q_2(x,y) &= 13(x^2+y^2)+5xy(x+y)+86(x+y)+47xy+129, \\ q_3(x,y) &= 502(x^4+y^4)+35x^2y^2(x^2+y^2)+273xy(x^3+y^3)+70x^3y^3 \\ &\quad +9978(x^3+y^3)+1260x^2y^2(x+y)+6512xy(x^2+y^2)+71842(x^2+y^2) \\ &\quad +13410x^2y^2+54123xy(x+y)+185554xy+219918(x+y)+238480\,. \end{aligned}$$

genus	*Bi(λ,μ,n)*	$P_g(x)$
0	$\frac{n(\ell-1)!(m-1)!}{\prod_i \alpha_i!\beta_i!}$	1
1	$\frac{n(\ell-1)!(m-1)!}{3!\prod_i \alpha_i!\beta_i!}\left[\binom{\ell+1}{2}\sum_i\binom{\lambda_i-1}{2}+\binom{m+1}{2}\sum_i\binom{\mu_i-1}{2}\right]$	$\frac{x}{3}$
2	$\frac{n(\ell-1)!(m-1)!}{5!\prod_i \alpha_i!\beta_i!}\Big[\binom{\ell+3}{4}\big[36\sum_i\binom{\lambda_i-1}{4}+20\sum_{i\neq j}\binom{\lambda_i-1}{2}\binom{\lambda_j-1}{2}\big]+$ $\binom{m+3}{4}\big[36\sum_i\binom{\mu_i-1}{4}+20\sum_{i\neq j}\binom{\mu_i-1}{2}\binom{\mu_j-1}{2}\big]+$ $\frac{10}{3}\binom{\ell+1}{2}\binom{m+1}{2}\sum_i\binom{\lambda_i-1}{2}\sum_i\binom{\mu_i-1}{2}\Big]$	$\frac{x(5x+13)}{90}$

Table 1:
The number of rooted bicolored unicellular maps of genus g on n vertices with vertex degree distributions given by λ and μ.

genus	*Bi(ℓ,m,n)*	*Bi(n,g)*
0	$\frac{1}{n}\binom{n}{\ell-1}\binom{n}{m-1}$	$\frac{1}{2n+1}\binom{2n+1}{n}$ Catalan numbers
1	$\frac{\binom{n+1}{2}}{3!}\binom{n-1}{\ell-1}\binom{n-1}{m-1}$	$\frac{\binom{2n-1}{n-1}\binom{n}{3}}{(2n-1)2}$
2	$\frac{\binom{n+1}{2}q_2(\ell,m)}{6\cdot 5!}\binom{n-1}{\ell-1}\binom{n-1}{m-1}$	$\frac{\binom{2n-3}{n-2}\binom{n-1}{4}2(5n^2-7n+6)}{(2n-3)5!}$
3	$\frac{\binom{n+1}{2}q_3(\ell,m)}{36\cdot 7!}\binom{n-1}{\ell-1}\binom{n-1}{m-1}$	$\frac{\binom{2n-5}{n-2}\binom{n-2}{5}5(35n^4-182n^3+397n^2-346n+240)}{(2n-5)3(7)!}$

Table 2
The number of rooted bicolored unicellular maps
of genus g on n vertices.

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