The number of plane corner cuts

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In this note we give a generating function for the number $\#\binom{\mathbb{N}^2}{n}_{cut}$ of plane corner cuts with respect to their size and prove that there exists two positive constants c, c' such that, for all $n > 1$,

$$cn \log n < \#\binom{\mathbb{N}^2}{n}_{cut} < c'n \log n. \tag{1}$$

We rely on [Onn-Sturmfels] for motivations for this work and we simply recall the following definition: a (plane) corner cut of size n is a n-element subset λ of $\mathbb{N}^2$ which is cut off by a line, *i.e.* there exist w in $\mathbb{R}^2$ and w_0 in $\mathbb{R}$ such that $\lambda = \{v \in \mathbb{N}^2 \mid w \cdot v < w_0\}$. We denote by $\binom{\mathbb{N}^2}{n}_{cut}$ the set of corner cuts of size n.

1. Reduction

Fix $n \in \mathbb{N}$. Let $w \in \mathbb{R}^2_{>0}$. Let w_0 be the smallest real number such that $\#\{v \in \mathbb{N}^2 : w \cdot v \leq w_0\} \geq n$. Let $C(w) := \{v \in \mathbb{N}^2 : w \cdot v \leq w_0\}$, $L(w) := \{v \in \mathbb{N}^2 : w \cdot v < w_0\}$, and $H(w) := \{v \in \mathbb{N}^2 : w \cdot v = w_0\}$, We shall also use the notation $P(w)$ for the line defined by $w \cdot x = w_0$.

Now we define an equivalence relation on $\mathbb{R}^2_{>0}$ by saying w is equivalent to v iff $C(w) = C(v)$. Clearly, these equivalence classes are convex cones in $\mathbb{R}^2_{>0}$. Further, let $w \in \mathbb{R}^2_{>0}$, and consider the line $P(w)$. If $\#C(w) = n$, then for any v near w, there is a perturbation of $P(w)$ normal to v which cuts off the same set of points, so $C(v) = C(w)$. This shows that v and w lie in the same equivalence class. Thus it follows that in this case the equivalence class of w is an open 2-dimensional cone.

On the other hand, if $\#C(w) > n$, then $H(w)$ contains more than one point. Hence for v near w, though there will still be a perturbation of $P(w)$ normal to v which cuts off the same set of points, it will not define $C(v)$, because it will be possible to find a line normal to v which cuts off strictly fewer, but still at least n, points (in fact, exactly n). Indeed $C(v)$ will consist of the points $L(w)$ together with enough points from $H(w)$ to bring the total up to n. Thus $C(v) \neq C(w)$, so w is in a 1-dimensional equivalence class.

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Therefore, the equivalence classes consist of open two-dimensional cones separated by rays. The 2-dimensional equivalence classes are in 1-1 correspondence with the set $\binom{\mathbb{N}^2}{n}_{cut}$. But the number of 2-dimensional equivalence classes is just one more than the number of 1-dimensional equivalence classes. Thus, we could concern ourselves with counting 1-dimensional equivalence classes. These rays may be identified with the first lattice point along the ray (since the rays will always be rational). Thus, we are now concerned with counting relatively prime pairs of numbers (a, b) such that $C((a, b))$ has strictly more than n elements. In this case, we shall say that (a, b) is critical.

Let $w = (a, b)$ be a ray, $H(w) = \{(x_1, y_1), \dots, (x_k, y_k)\}$ with $k \geq 2$, $x_1 \leq \dots \leq x_k$ and $m = n - \#L(w) > 0$. The ray w separates two 2-dimensional equivalence classes. According to the previous discussion, the two corresponding corner cuts are $C_1(w) = L(w) \cup \{(x_1, y_1), \dots (x_m, y_m)\}$ and $C_2(w) = L(w) \cup \{(x_{k-m+1}, y_{k-m+1}), \dots, (x_k, y_k)\}$. Moreover the map C_1 is a 1-1 correspondence between the rays and elements of $\binom{\mathbb{N}^2}{n}_{cut} \setminus \{h_n\}$ where h_n denotes the horizontal corner cut of size n.

Remark. Let $\lambda = C_1(w)$, then $(x_m, y_m) \in \max(\lambda)$ and $(x_{m+1}, y_{m+1}) \in \min(\mathbb{N}^2 \setminus \lambda)$, $x_m < x_{m+1}$ and $\frac{y_{m+1} - y_m}{x_{m+1} - x_m} = U_\lambda$ as defined p.8 in [Onn-Sturmfels]. Conversely for $\lambda = C_2(w)$, we have $(x_{k-m}, y_{k-m}) \in \mathbb{N}^2 \setminus \lambda$, $(x_{k-m+1}, y_{k-m+1}) \in \lambda$ and $\frac{y_{k-m} - y_{k-m+1}}{x_{k-m} - x_{k-m+1}} = L_\lambda$.

2. Generating function

Let λ be a corner cut that is not horizontal and let $(a, b) = C_1^{-1}(\lambda)$. Then we may define k, m, x_1 and y_k as before. As $H(w)$ contains all non negative lattice points of $P(w)$, we have $x_1 < b$, $y_k < a$. Moreover by construction $1 \leq m < k$. Now the map $\lambda \mapsto (a, b, k, m, x_1, y_k)$ is injective. Indeed given a relatively prime pair (a, b) and integers k, m, x_1 and y_k subject to the previous conditions, let $x_{i+1} = x_1 + ib$ and $y_{k-i-1} = y_k - ia$ for $0 < i < k$. Then the corner cut is well defined and $n = N(a, b, k, m, x_1, y_k)$ is given by (see Figure 1)

$$\begin{aligned} N(a, b, k, m, i, j) &= (k-1)\left(\frac{(a+1)(b+1)}{2} - 1\right) + \binom{k-1}{2} ab + ij \\ &\quad + i(k-1)a + j(k-1)b + T(a, b, j) + T(b, a, i) + m, \end{aligned}$$

where $T(a, b, j) = \sum_{r=1}^{j} (\lfloor rb/a \rfloor + 1)$.

Thanks to this bijection we can express the generating function of corner

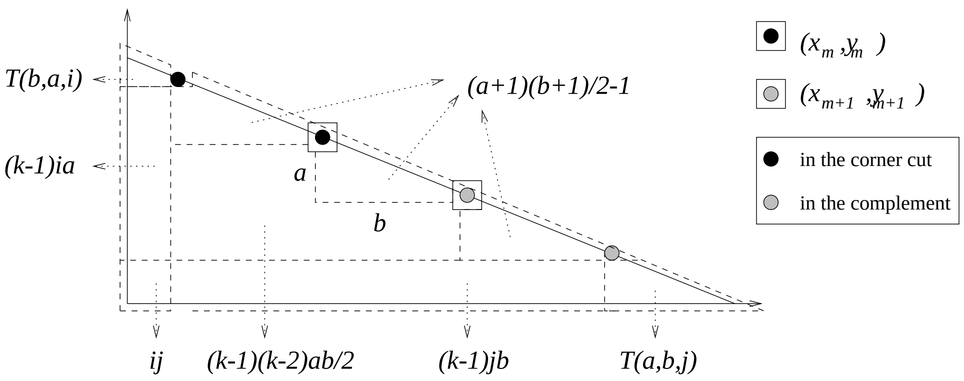

Figure 1: Decomposition of a corner cut

cuts with respect to their size as

$$\sum_n \# \binom{\mathbb{N}^2}{n}_{cut} z^n = \frac{z}{1-z} + \sum_{\gcd(a,b)=1} \sum_{\substack{0 \leq j < a \\ 0 \leq i < b}} \sum_{1 \leq m < k} z^{N(a,b,k,m,i,j)}.$$

Using computer algebra (Maple) we obtain:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\binom{\mathbb{N}^2}{n}_{cut}$	1	2	3	4	6	7	8	10	12	13	16	16	18	20	23

n	16	17	18	19	20	21	22	23	24	25	26	27
$\binom{\mathbb{N}^2}{n}_{cut}$	24	26	26	30	32	35	34	38	38	42	44	46

n	28	29	30	31	32	33	34	35	36	37	38	39
$\binom{\mathbb{N}^2}{n}_{cut}$	47	54	52	54	52	56	60	66	67	68	66	72

3. Upper bound

To establish the upper bound in (1) we fix n and consider a critical pair (a,b). Then from the expression of $N(a,b,\ldots)$ it is clear that $ab/2$ is smaller that n.

Thus, we can restrict our attention to the number of relatively prime pairs (a,b) such that $ab \leq 2n$. But this number is smaller than the total number of pairs (a,b) with $ab \leq 2n$ which, in turn, is smaller than $2n \log n + 4n$.

4. Partial lower bound

We first establish here the existence of a constant c'' such that

$$\# \binom{\mathbb{N}^2}{n}_{cut} + \# \binom{\mathbb{N}^2}{n-1}_{cut} > c'' nlogn.$$

For any pair (a,b) and $n>0$, let $H((a,b))=\{(x_1,y_1),\ldots,(x_k,y_k)\}$. Then $C((a,b))$ is included in the triangle with vertices (x_1-b,y_1+a), (x_1-b,y_k-a), (x_k+b,y_k-a). By Pick's theorem this triangle contains $\frac{1}{2}(k^2ab+k(a+b+1)+2)$ lattice points, which is less than $2abk^2$. Thus $n<2abk^2$.

In particular if $n>8ab$, $H((a,b))$ contains at least two lattice points, so if (a,b) is not critical for n then it is for $n-1$. Thus, it suffices to show that the number $P(n)$ of relatively prime pairs (a,b) with $ab<n/8$ is on the order of $n\log n$.

For simplicity of notation, denote $\lfloor n/8\rfloor$ by s. Let Q be the set of square-free numbers less than or equal to s, and pick $q\in Q$. Consider m a multiple of q less than or equal to s. Now m has a prime factorization $m=ab$ where a is the largest divisor of m all of whose prime divisors are also divisors of q, and $b=m/a$. There are $(\lfloor s/q\rfloor)$ such multiples of q, which gives us at least $(s/q)-1$ relatively prime pairs (a,b) with $ab\le s$. Considering the pairs obtained for different values of q, we see that every relatively prime pair whose product is less than or equal to s occurs for exactly one value of q. Thus

$$P(n)\ge\sum_{q\in Q}(\frac{s}{q}-1)$$

where the sum is taken over square-free $q\le s$. Clearly the -1 terms contribute at most a linear factor and can be ignored. Now consider that

$$\sum_{q\in Q}\frac{1}{q}\cdot\sum_{j=1}^{\infty}\frac{1}{j^2}>\sum_{i=1}^{s}\frac{1}{i}$$

since every number less than or equal to s can be written as the product of a square-free number less than or equal to s and a square. But the expression on the right-hand side is very close to $\log s$, and $\sum_{j=1}^{\infty}\frac{1}{j^2}=\pi^2/6$. Thus, $\sum_{q\in Q}\frac{1}{q}$ is at least a constant times $\log n$, from which it follows that $P(n)$ is at least a constant times $n\log n$, as desired.

5. Lower bound

To obtain our lower bound, we now prove

$$\#\binom{\mathbb{N}^2}{n}_{cut}\ge\frac{1}{2}\#\binom{\mathbb{N}^2}{n-1}_{cut}. \tag{2}$$

First let us show that in a 2-dimensional equivalence class there is at most one value w for which $\#H(w)>1$. Indeed, if there were two of them, v and w, then consider $P(v)$ and $P(w)$. These two lines intersect at a point

A of $\mathbb{R}^2_{>0}$. Let also $V_x = (v_x, 0)$ and $V_y = (0, v_y)$ denote the intersection of $P(v)$ with the axes of coordinates, and define as well W_x and W_y. We may suppose $v_x < w_x$, so that the segment AV_x contains $H(v)$ whereas the segment AW_y contains H_w. But, either AV_y is longer than AV_x, or AW_x than AW_y. The first case contradicts $H(v) \subset AV_x$, whereas the second contradicts $H(w) \subset AW_y$.

Then let λ be a corner cut of size n and $\mathcal{C}$ the corresponding 2-dimensional equivalence class. According to the previous discussion, either H is constant on $\mathcal{C}$ or $\mathcal{C}$ is made of two convex 2-dimensional cones on which H is constant, separated by a 1-dimensional cone w with $H(w) > 1$.

Now as a point A can be removed from λ to obtain a corner cut of size $n-1$ if and only if there exists w in $\mathcal{C}$ such that $H(w) = \{A\}$, we see that at most two points can be removed.

Finally, for any corner cut of size $n-1$ we pick a lattice point that can be added to form a corner cut of size n. As the latter is obtained at most twice, we have proved Equation (2). In turn, using the result of the previous section, it yields the lower bound in (1).

Reference

[Onn-Sturmfels] S. Onn and B. Sturmfels, Cutting corners, Advances in Applied Mathematics, this issue.