RANDOM PLANAR LATTICES AND INTEGRATED SUPERBROWNIAN EXCURSION

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ABSTRACT. In this paper, a surprising connection is described between a specific brand of random lattices, namely planar quadrangulations, and Aldous' Integrated SuperBrownian Excursion (ISE). As a consequence, the radius r_n of a random quadrangulation with n faces is shown to converge, up to scaling, to the width $r = R - L$ of the support of the one-dimensional ISE, or precisely:

$$n^{-1/4} r_n \xrightarrow{law} (8/9)^{1/4}\, r.$$

More generally the distribution of distances to a random vertex in a random quadrangulation is described in its scaled limit by the random measure ISE shifted to set the minimum of its support in zero.

The first combinatorial ingredient is an encoding of quadrangulations by *trees embedded in the positive half-line*, reminiscent of Cori and Vauquelin's well labelled trees. The second step relates these trees to embedded (discrete) trees in the sense of Aldous, via the *conjugation of tree principle*, an analogue for trees of Vervaat's construction of the Brownian excursion from the bridge.

From probability theory, we need a new result of independent interest: the weak convergence of the encoding of a random embedded plane tree by two contour walks $(e^{(n)}, \hat{W}^{(n)})$ to the Brownian snake description $(e, \hat{W})$ of ISE.

Our results suggest the existence of a *Continuum Random Map* describing in term of ISE the scaled limit of the dynamical triangulations considered in two-dimensional pure quantum gravity.

1. INTRODUCTION

From a distant perspective, this article uncovers a surprising, and hopefully deep, relation between two famous models: *random planar maps*, as studied in combinatorics and quantum physics, and *Brownian snakes*, as studied in probability theory and statistical physics. More precisely, our results connect some distance-related functionals of *random quadrangulations* with functionals of Aldous' *Integrated SuperBrownian Excursion* (ISE) in dimension one.

Quadrangulations. On the one hand, quadrangulations are finite plane graphs with 4-regular faces (see Figure 1 and Section 2 for precise definitions). Random quadrangulations, like random triangulations, random polyhedra, or the ϕ^4-models of physics, are instances of a general family of random lattices that has received considerable attention both in combinatorics (under the name *random planar maps*, following Tutte's terminology [35]) and in physics (under the name *Euclidean two-dimensional discretised quantum geometry*, or simply *dynamical triangulations* or *fluid lattices* [3, 10, 18]).

Many probabilistic properties of random planar maps have been studied, that are *local properties* like vertex or face degrees [7, 17], or $0-1$ laws for properties expressible in first order logic [8]. Other well documented families of properties are

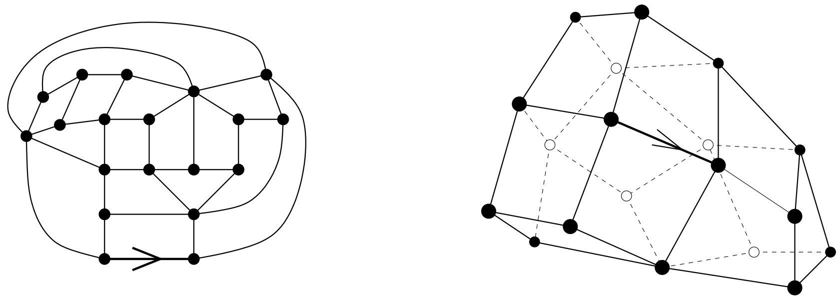


FIGURE 1. Random quadrangulations, in planar or spherical representation.

related to connectedness and constant size separators [6], also known as branchings into baby universes [21]. In this article we consider another fundamental aspect of the geometry of random maps, namely *global properties of distances.* The *profile* $(H_k^n)_{k\geq 0}$ and *radius* r_n of a random quadrangulation with n faces are defined in analogy with the classical profile and height of trees: H_k^n is the number of vertices at distance k from a basepoint, while r_n is the maximal distance reached. It had been accepted in physics since the 80's that for such discrete geometries *the internal Hausdorff dimension is 4*, but exact computations about the profile (with triangulations instead of quadrangulations) were first obtained by physicists Watabiki, Ambjørn et *al.* (see [4, 36] and ref. therein). Under strong (unproven) smoothness assumptions on the behavior of the profile, they obtain that the only possible scaling is indeed $k \sim n^{1/4}$, and they compute a transform of the profile in the scaling limit[1]. Although non rigourous (due to approximations and limits interchanges), these results are remarkably precise and beautifully derived. Independently the conjecture that $\mathbb{E}(r_n) \sim cn^{1/4}$ was proposed by Schaeffer [31].

Integrated SuperBrownian Excursion. On the other hand, ISE was introduced by Aldous as a model of random distributions of masses [1]. He considers random embedded discrete trees as obtained by the following two steps: first an abstract tree t, say a Cayley tree with n nodes, is taken from the uniform distribution and each edge of t is given length 1; then t is embedded in the regular lattice on $\mathbb{Z}^d$, with the root at the origin, and edges of the tree randomly mapped on edges of the lattice. Assigning masses to leaves of the tree t yields a random distribution of mass on $\mathbb{Z}^d$. Upon scaling the lattice to $n^{-1/4}\mathbb{Z}^d$, these random distributions of mass admit, for n going to infinity, a continuum limit $\mathcal{J}$ which is a random probability measure on $\mathbb{R}^d$ called ISE.

Derbez and Slade proved that ISE describes in dimension larger than 8 the continuum limit of a model of lattice trees [16], while Hara and Slade obtained the same limit for the incipient infinite cluster in percolation in dimension larger than 6 [19]. As opposed to these works, we shall consider ISE in dimension one and our embedded discrete trees should be thought of as folded on a line. The support of ISE is then a random interval (L, R) of $\mathbb{R}$ that contains the origin.

[1] More precisely they obtain that the singular behavior of the function $\sum_{n\geq k} H_k^{(n)} z^n$ near its dominant singularity z_0 is $ct^{3/4}\cosh(t^{1/4}k)/\sinh(t^{1/4}k)$ with c, c' constants and $t = c'(1 - z/z_0)$.

From quadrangulations to ISE. The purpose of this paper is to draw a relation between, on the one hand, random quadrangulations, and, on the other hand, Aldous' ISE: upon proper scaling, the profile of a random quadrangulations is described in the limit by ISE translated to have support $(0, R-L)$. This relation implies in particular that the radius r_n of random quadrangulations, again upon scaling, weakly converges to the width of the support of ISE in one dimension, that is the continuous random variable $r = R - L$. We shall indeed prove (Corollary 3) that

$$n^{-1/4} r_n \xrightarrow{\text{law}} (8/9)^{1/4}\, r,$$

as well as the convergence of moments. This proves the conjecture $\mathbb{E}(r_n) \sim cn^{1/4}$. In the first version of this paper, the value of the constant c was quoted as unknown. In the meantime, Delmas [14] has computed a moment generating function of R and the first and second moment of r, giving in particular

$$\mathbb{E}(r) = 6\frac{2^{3/4}\Gamma(5/4)}{\sqrt{\pi}}.$$

We shall more generally prove (Corollary 4) that the normalized cumulated profile, that is the fraction of the number of vertices at distance at most $xn^{1/4}$ of the base point, converges to the fraction of the mass of the ISE in the interval $(0, x)$, when ISE is shifted to have support $(0, R-L)$.

The path from quadrangulations to ISE consists of three main steps, the first two of combinatorial nature and the last with a more probabilistic flavor. Our first step, Theorem 1, revisits a correspondence of Cori and Vauquelin [13] between planar maps and some *well labelled trees*, that can be viewed as plane trees embedded in the positive half-line. Thanks to an alternative construction [31, Ch. 7], we show that under this correspondence the profile can be mapped to the mass distribution on the half-line. In particular, the radius r_n of a random quadrangulation is equal in law to the maximal label μ_n of a random well labelled tree.

Safe for the positivity condition, well labelled trees would be constructed exactly according to Aldous' prescription for embedded discrete trees. Well labelled trees are thus to Aldous' embedded trees what the Brownian excursion is to the Brownian bridge, and we seek an analogue of Vervaat's relation. At the discrete level a classical elegant explanation of such relations is based on Dvoretsky and Motzkin's cyclic shifts and cycle lemma (see also Otter). Our second combinatorial step, Theorem 3, consists in the adaptation of these ideas to embedded trees. More precisely, via the *conjugation of tree principle* of [31, Chap. 2], we bound the discrepancy between the mass distribution of our conditioned trees on the positive half-line and a translated mass distribution of freely embedded trees. In particular we construct a coupling between well labelled trees and freely embedded trees such that the largest label μ_n, and thus the radius r_n, is coupled to the width of the support (L_n, R_n) of random freely embedded trees:

$$|r_n - (R_n - L_n)| \leq 3.$$

We also sketch an alternative construction, directly relating freely embedded trees to quadrangulations. Since our freely embedded trees are constructed according to Aldous' prescription, one could expect to be able to conclude directly. However two obstacles still need to be bypassed at this point.

Contour walks and Brownian snakes. The first obstacle is that the construction of ISE as a continuum limit of mass distributions supported by embedded discrete trees was only outlined in Aldous' original paper. The original mathematical definition is by embedding a continuum random tree (CRT), which amounts to exchanging the embedding and the continuum limit. But Borgs et *al.* proved that indeed ISE is the limit of mass distributions supported by embedded Cayley trees [12] and their proof could certainly be adapted to other simple classes of trees and in particular to our embedded plane trees.

The second, more important, obstacle is that weak convergence of probability measures is not adequate to our purpose, since we are interested in particular in convergence of the width of the support, *which is not a continuous functional on the space of measures.* In order to circumvent this difficulty, we turn to the description of ISE in terms of superprocesses: ISE can be constructed from the Brownian snake with lifetime e, the standard Brownian excursion [1, 23].

From the discrete point of view, we consider the encoding of an embedded plane tree by a pair of contour walks (x_k, y_k), that encode respectively the height of the node visited at time k and its position on the line. Our last result, Theorem 5, is the weak convergence, upon proper scaling, of this pair of walks to the Brownian snake with lifetime e:

$$\big(e^{(n)}(s), \hat{W}^{(n)}(s)\big) \xrightarrow{\text{law}} \big(e(s), \hat{W}_s\big).$$

As $R = \sup_s \hat{W}_s$ and $L = \inf_s \hat{W}_s$ this convergence, together with some deviation bounds obtained in the proof allows us to conclude on the radius. (A similar weak convergence was independently proved by Marckert and Mokkadem [26] but without the deviation bounds we need here.)

More generally the joint convergence of the minimum and the mass distribution of discrete embedded trees implies that, upon scaling, the label distribution of well labelled trees converges to ISE translated to have the minimum of its support at the origin. The same then holds for the profile of random quadrangulations.

Dynamical triangulations and a Continuum Random Map. Although we concentrate in this article on the radius and profile of random quadrangulations, our derivation suggests a much tighter link between random quadrangulations and ISE. We conjecture that a Continuum Random Map (CRM) can be built from ISE that would describe the continuum limit of scaled random quadrangulations, in a similar way as the CRT describes the continuum limit of scaled random discrete trees. From the point of view of physics, the resulting CRM would describe in the limit the geometry of scaled dynamical triangulations as studied in discretised two-dimensional Euclidean pure quantum geometries [3, 10, 18]. We plan to discuss this connection further in future work.

Organization of the paper. Section 2 contains the definition of the combinatorial models of random lattices. Sections 3 and 4 are devoted to the first combinatorial steps. Section 5 contains the definition of probabilistic models, and the statement of the convergence result. Finally in Section 6 we give the proof of this convergence.

2. The combinatorial models of random lattices

2.1. **Planar maps and quadrangulations.** A *planar map* is a proper embedding (without edge crossings) of a connected graph in the plane. Loops and multiple

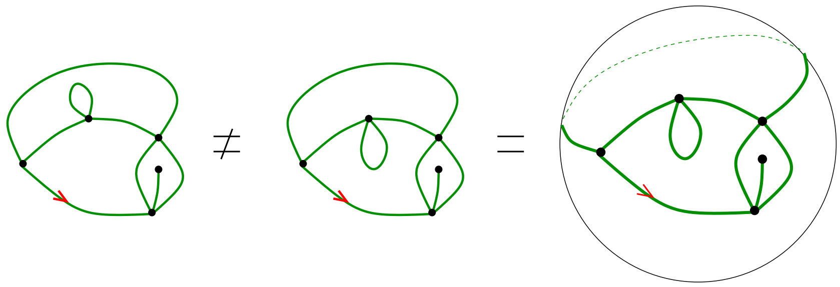


FIGURE 2. Two distinct planar maps, and a spherical representation of the second.

edges are *a priori* allowed. A planar map is *rooted* if there is a *root*, *i.e.* a distinguished edge on the border of the infinite face, which is oriented counterclockwise. The origin of the root is called the *root vertex*. Two rooted planar maps are considered identical if there exists an homeomorphism *of the plane* that sends one map onto the other (roots included).

The difference between planar graphs and planar maps is that the cyclic order of edges around vertices matters in maps, as illustrated by Figure 2. Observe that planar maps can be equivalently defined on the sphere. In particular Euler's characteristic formula applies and provides a relation between the numbers n of edges, f of faces and v of vertices of any planar map: $f + v = n + 2$.

The *degree* of a face or of a vertex of a map is its number of incidence of edges. A planar map is a *quadrangulation* if all faces have degree 4. All (planar) quadrangulations are *bipartite*: their vertices can be colored in black or white so that the root is white and any edge joins two vertices with different colors. In particular a quadrangulation contains no loop but may contain multiple edges. See Figures 1 and 3 for examples of quadrangulations.

Let $\mathcal{Q}_n$ denote the set of rooted quadrangulations with n faces. A quadrangulation with n faces has $2n$ edges (because of the degree constraint) and $n+2$ vertices (applying Euler's formula). The number of rooted quadrangulations with n faces was obtained by W.T. Tutte [35]:

$$|\mathcal{Q}_n| \;=\; \frac{2}{n+2}\,\frac{3^n}{n+1}\binom{2n}{n}. \tag{1}$$

Various alternative proofs of this result have been obtained (see *e.g.* [5, 10, 13, 31]). Our treatment will indirectly provide another proof, related to [13, 31].

2.2. **Random planar lattices.** Let L_n be a random variable with uniform distribution on $\mathcal{Q}_n$. Formally, L_n is the $\mathcal{Q}_n$-valued random variable such that for all $Q \in \mathcal{Q}_n$

$$\Pr(L_n = Q) \;=\; \frac{1}{|\mathcal{Q}_n|} \;=\; \frac{1}{\frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}}.$$

The random variable L_n is our *random planar lattice*. To explain this terminology, taken from physics, observe that locally the usual planar square lattice is a planar map whose faces and vertices all have degree 4. Our random planar lattice corresponds to a relaxation of the constraint on vertices.

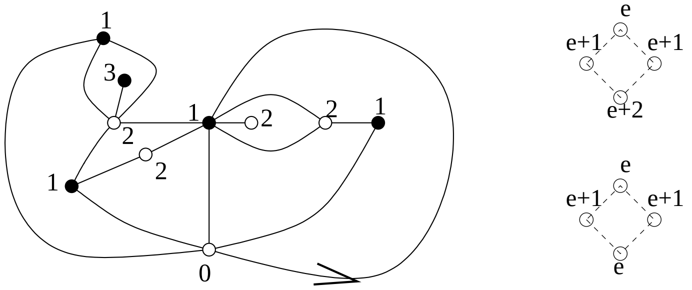


FIGURE 3. Labelling by distance from the root vertex and the two possible configurations of labels (top: a simple face; bottom: a confluent face).

Classical variants of this definition are obtained by replacing quadrangulations with n faces by triangulations with $2n$ triangles, or by (vertex-)4-regular maps with n vertices, or by all planar maps with n edges, *etc.* All these random planar lattices have been considered both in combinatorics (see [6] and references therein) and in mathematical physics (see [3] and references therein; in the physics literature, definitions are usually phrased using "symmetry weights" instead of rooted objects, but this is strictly equivalent to the combinatorial definition). Although details of local topology vary between families, most probabilistic properties are believed to be "universal", that is qualitatively analogue for all "reasonable" families. Observe also that random maps in classical families have exponentially small probability to be symmetric, so that all results hold as well in the model of uniform unrooted maps [29].

In this article we focus on quadrangulations because of their combinatorial relation, detailed in Section 3, to well labelled trees.

2.3. **The profile of a map.** The distance $d(x, y)$ between two vertices x and y of a map is the minimal number of edges on a path from x to y (in other terms all edges have abstract length 1).

The *profile* of a rooted map M is the sequence $(H_k)_{k\geq 1}$, where $H_k \equiv H_k^{[M]}$ is the number of vertices at distance k of the root vertex v_0. We shall also consider the cumulated profile $\widehat{H}_k^{[M]} = \sum_{\ell=1}^{k} H_\ell^{[M]}$. By construction the support of the profile of a rooted map is an interval *i.e.* $\{k \mid H_k > 0\} = [1, r]$ where r is the *radius* of the map (sometimes also called *eccentricity*). The radius r is closely related to the *diameter*, that is the largest distance between two vertices of a map: in particular $r \leq d \leq 2r$. The quadrangulation of Figure 3 has radius 3.

The *profile of the random planar lattice* L_n is the random variable $(H_k^{(n)})_{k\geq 1}$ that is defined by taking the profile $(H_k^{[L_n]})_{k\geq 1}$ of an instance of L_n, while $(\widehat{H}_k^{(n)})_{k\geq 1}$ denotes the *cumulated* profile of L_n. Similarly the radius of a random planar lattice is a positive integer valued random variable r_n.

3. ENCODING THE PROFILE WITH WELL LABELLED TREES

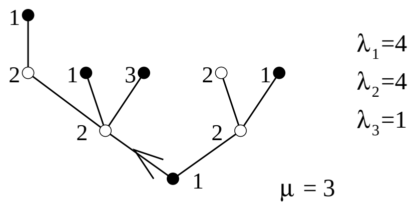


FIGURE 4. A well labelled tree with its label distribution.

3.1. **Well labelled trees and the encoding result.** A *plane tree* is a rooted planar map without cycle (and thus with only one face). Equivalently plane trees can be recursively defined as follows:

- the smallest tree is made of a single vertex,
- any other tree is a non-empty sequence of subtrees attached to a root.

In other term, each vertex has a possibly empty sequence of sons, and each vertex but the root has a father. The number of plane trees with n edges is the well known Catalan number

$$C(2n) = \frac{1}{n+1}\binom{2n}{n}.$$

A plane tree is *well labelled* if all its vertices have positive integer labels, the labels of two adjacent vertices differ at most by one, and the label of the root vertex is one. Let $\mathcal{W}_n$ denote the set of well labelled trees with n edges.

The *label distribution* of a well labelled tree T is the sequence $(\lambda_k)_{k\geq 1} \equiv (\lambda_k^{[T]})_{k\geq 1}$ where $\lambda_k^{[T]}$ is the number of vertices with label k in the tree T. The cumulated label distribution is defined by $\widehat{\lambda}_k^{[T]} = \sum_{\ell=1}^{k} \lambda_\ell^{[T]}$. By construction the support of the label distribution is an interval: there exists an integer μ such that $\{k \mid \lambda_k > 0\} = [1, \mu]$. This integer μ is the maximal label of the tree. These definitions are illustrated by Figure 4.

The following theorem will serve us to reduce the study of the profile of quadrangulations to the study of the label distribution of well labelled trees.

Theorem 1 (Schaeffer [31])**.** *There exists a bijection $\mathcal{T}$ between rooted quadrangulations with n faces and well labelled trees with n edges, such that the profile $(H_k^{[Q]})_{k\geq 1}$ of a quadrangulation Q is mapped onto the label distribution $(\lambda_k^{[T]})_{k\geq 1}$ of the tree $T = \mathcal{T}(Q)$.*

Theorem 1 and Tutte's formula (1) imply that the number of well labelled trees with n edges equals

$$|\mathcal{W}_n| = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}. \tag{2}$$

This result was proved already by Cori and Vauquelin [13], who introduced well labelled trees to give an encoding of all planar maps with n edges. Because of a classical bijection between the latter maps and quadrangulations with n faces, their result is equivalent to the first part of Theorem 1. Their bijection has been extended to bipartite maps by Arquès [5] and to higher genus maps by Marcus and

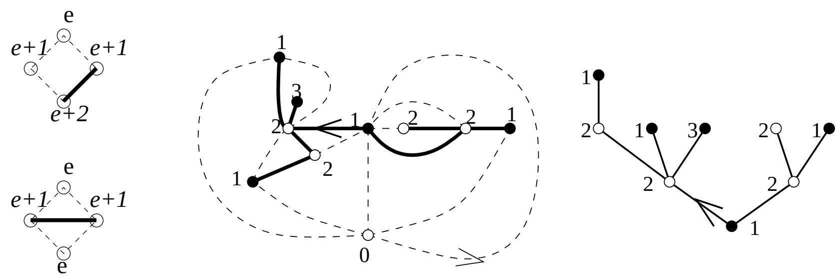


FIGURE 5. The rules of selection of edges and the example of Figure 3.

Vauquelin [25]. All these constructions were recursive and based on encodings of maps with permutations (also known as rotation systems).

However, our interest in well labelled trees lies in the relation between the profile and the label distribution, which does not appear in Cori and Vauquelin's bijection. The bijection we use here is much simpler and immediately leads to the second part of Theorem 1. This approach was extended to non separable maps by Jacquard [20] and to higher genus by Marcus and Schaeffer [24].

We postpone to Section 4 the discussion of the interesting form of Formula (2) and its relation to Catalan numbers. Instead the rest of this part is concerned with the proof of Theorem 1, which goes in three steps. First some properties of distances in quadrangulations are indicated (Section 3.2). This allows in a second step to define the encoding, as a mapping $\mathcal{T}$ from quadrangulations to well labelled trees (Section 3.3). A decoding procedure allows then to prove that $\mathcal{T}$ is faithful (Section 3.4).

3.2. **Properties of distances in a quadrangulation.** Let Q be a rooted quadrangulation and denote v_0 its root vertex. The labelling ϕ of the map Q is defined by $\phi(x) = d(x, v_0)$ for each vertex x, where $d(x,y)$ denotes the distance in Q (cf. Figure 3). Observe that the number of vertices with label k in the labelling of the map Q is precisely the number of vertices at distance k of v_0, that is $H_k^{[Q]} = |\{x \mid \phi(x) = k\}|$. This labelling satisfies the following immediate properties:

Property 1. *If x and y are joined by an edge, $|\phi(x) - \phi(y)| = 1$. Indeed the quadrangulation being bipartite, a vertex x is white if and only if $\phi(x)$ is even, black if and only if $\phi(x)$ is odd.*

Property 2. *Around a face, four vertices appear: a black x_1, a white y_1, a black x_2 and a white y_2. These vertices satisfy at least one of the two equalities $\phi(x_1) = \phi(x_2)$ or $\phi(y_1) = \phi(y_2)$ (cf. Figure 3).*

A face will be said *simple* when only one equality is satisfied and *confluent* otherwise (see Figure 3). It should be noted that one may have $x_1 = x_2$ or $y_1 = y_2$.

3.3. **Construction of the encoding $\mathcal{T}$.** Let Q be a rooted quadrangulation with its distance labelling. The map Q' is obtained by dividing all confluent faces Q into two triangular faces by an edge joining the two vertices with maximal label. Let us now define a subset $\mathcal{T}(Q)$ of edges of Q' by two selection rules:

- In each confluent face of Q, the edge that was added to form Q' is selected.

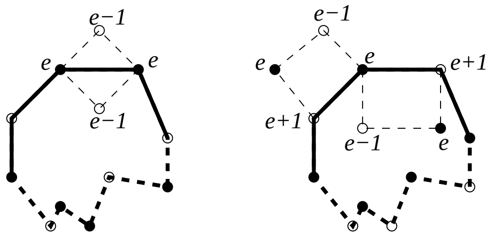


FIGURE 6. Impossibility of cycles.

- For each simple face f of Q, let v be the vertex with maximal label in f, then the edge leaving v with f on its left is selected.

Observe that since maps are defined on the sphere, the infinite face in a planar representation contributes as well to an edge. These two selection rules are illustrated by Figure 5. The root of $\mathcal{T}(Q)$ is defined to be the first edge of $\mathcal{T}(Q)$ around the endpoint of the root of Q, counterclockwise, starting from the root of Q.

The proof of Theorem 1 is now completed in two steps. First, in the rest of this section, $\mathcal{T}(Q)$, which is *a priori* only defined as a subset of edges of Q' together with their incident vertices, is shown to be in fact a well labelled tree with n edges. Second, in the next section the inverse mapping is described and used to prove that the mapping $\mathcal{T}$ is faithful.

Proposition 1. *The mapping $\mathcal{T}$ sends a quadrangulation Q with n faces on a well labelled tree with n edges.*

Proof. If the vertex x is not the root of Q, then one of its neighbors in Q, say y, has a smaller label. The edge (x, y) can be incident to: at least a confluent face; at least a simple face in which x has maximal label; or two simple faces in which x has intermediate label. In all three cases, x is incident to the selected edge of at least one face. Thus all vertices of Q but its root are also vertices of $\mathcal{T}(Q)$: in particular $\mathcal{T}(Q)$ has $n+1$ vertices. Next, the number of edges of $\mathcal{T}(Q)$ is n, because this is the number of faces of Q and two faces cannot select the same edge (as immediately follows from inspection of the selection rules). Now the planarity of Q and thus of Q' grants that each connected component of $\mathcal{T}(Q)$ is planar. Provided we can rule out cycles, this imply that $\mathcal{T}(Q)$ is a forest of trees with n edges and $n+1$ vertices, *i.e.* a single tree. This tree is then clearly well labelled.

Suppose now that there exists a cycle in $\mathcal{T}(Q)$ and let $e \geq 0$ be the value of the smallest label of a vertex of this cycle. Either all these labels are equal to e, or there is in the cycle an edge $(e, e+1)$ and an edge $(e+1, e)$. In both cases the rules of selection of edges imply that each connected component of Q' defined by the cycle contains a vertex x (resp. y) with label $e-1$, as shown by Figure 6. According to Jordan's theorem, either the shortest path from x to the root or the shortest path from y to the root has to intersect the cycle, leading to a contradiction with the definition of labels by distances. There are thus no cycles and $\mathcal{T}(Q)$ is a tree. $\square$

3.4. **The inverse $\mathcal{Q}$ of the mapping $\mathcal{T}$.** Let T be a well labelled tree with n edges. Recall that the tree T can be viewed as a planar map that has a unique

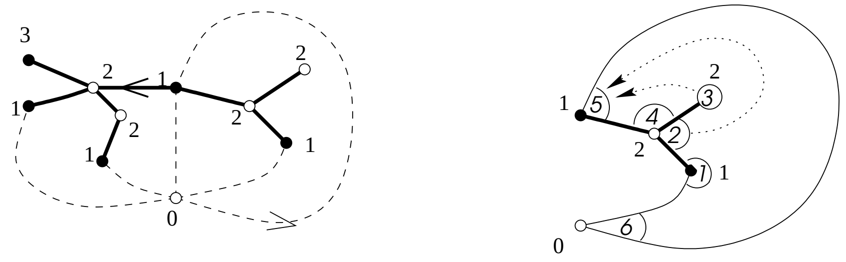


FIGURE 7. Step (1), and the chords $(i, s(i))$ in one of the faces.

face F_0. A *corner* is a sector between two consecutive edges around a vertex. A vertex of degree k defines k corners (in particular a leaf defines one corner). The total number of corners of T is $2n$. The label of a corner is by definition the label of the corresponding vertex.

The image $\mathcal{Q}(T)$ is defined in three steps.

(1) A vertex v_0 with label 0 is placed in the face F_0 and one edge is added between this vertex and each of the ℓ corners with label 1. The new root is taken to be the edge arriving from v_0 at the corner before the root of T.

After Step (1) a uniquely defined rooted map T_0 with ℓ faces has been obtained (see Figure 7, with $\ell = 5$). The next steps take place independently in each of those faces and are thus described for a generic[2] face F of T_0. Let k be the degree of F (by construction $k \geq 3$). Among the corners of F, only one is incident to v_0 and has label 0. Let the corners be numbered from 1 to k in clockwise order along the border, starting right after v_0. Let moreover e_i be the label of corner i (so that $e_1 = e_{k-1} = 1$ and $e_k = 0$). In Figure 7, the corners of a face are represented with their numbering and labels.

(2) The function successor s is defined for all corners $1, \ldots, k-1$ by

$$s(i) = \inf\{j > i \mid e_j = e_i - 1\}.$$

For each corner $i \geq 2$ such that $s(i) \neq i + 1$, a chord $(i, s(i))$ is added inside the face, in such a way that the various chords do not intersect (Property 3).

Once this construction has been carried on in each face, a planar map T' is obtained.

(3) All edges of with labels of the form (e, e) of T' are deleted. The resulting map is a quadrangulation $\mathcal{Q}(T)$ with n faces (Property 4).

The following proposition ends the proof of Theorem 1.

Proposition 2. *The mapping $\mathcal{Q}$ is the inverse bijection of the mapping $\mathcal{T}$.*

Let us first prove the two properties that validate the preceding construction.

Property 3. *The chords $(i, s(i))$ do not intersect.*

Proof. Suppose that two chords $(i, s(i))$ and $(j, s(j))$ cross each other. Upon maybe exchanging i and j one has $i < j < s(i) < s(j)$. The first two inequalities imply, together with the definition of s, that $e_j > e_{s(i)}$, while the two last inequalities imply $e_{s(i)} \geq e_j$. This is a contradiction. $\square$

[2]The infinite face is only apparently different from the others: imagine the map on the sphere.

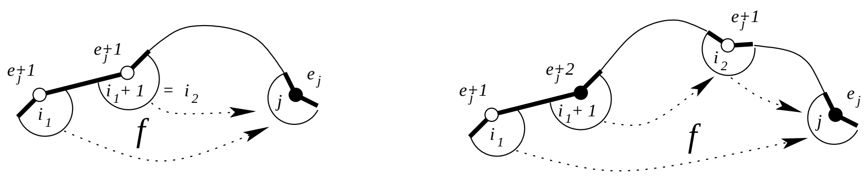


FIGURE 8. Two possible sizes for f: triangular or quadrangular.

Property 4. *The faces of T' are of one of the two types of Figure 8: either triangular with labels $e, e+1, e+1$, or quadrangular with labels $e, e+1, e+2, e+1$. The faces of $\mathcal{Q}(T)$ are all quadrangular.*

Proof. Let f be a face of T'. The face f is included in a face F of T_0 so that its corners inherit the numbering and labelling of those of F. Let j be the corner with largest number in f and $i_1 < i_2 < j$ its two neighbors in f (cf. Figure 8). Let us compute the label of the corners i_1 and i_2: the edge (i_1, j) is a chord by construction so that $j = s(i_1)$ and $e_{i_1} = e_j + 1$; moreover, as $i_1 < i_2 < j$, this imply $e_{i_2} \geq e_{i_1}$ (or j would not be $s(i_1)$) and finally $e_{i_2} = e_{i_1} = e_j + 1$.

By construction and planarity, no chord can arrive at i_1 between the unique leaving chord (i_1, j) and the edge $(i_1, i_1 + 1)$ of F. The latter edge thus borders the face f. There are then two cases, as illustrated by Figure 8:

- if $e_{i_1+1} = e_{i_1}$, then $i_2 = i_1 + 1$, and the face is triangular (left hand figure),
- otherwise $e_{i_1+1} = e_{i_1} + 1$ (recall $e_{i_1+1} \geq e_{i_1}$ since $i_1 < i_1 + 1 < j = s(i_1)$) and the chord leaving $i_1 + 1$ goes to i_2 (otherwise $s(i_1 + 1) < i_2$ with $e_{s(i_1+1)} = e_{i_1+1} - 1 = e_{i_2}$ and a chord $(s(i_1 + 1), j)$ would exclude i_2 from the face f): the face is quadrangular (right hand side in Figure 8).

Observe finally that the deletion of edges with labels of the form (e, e) will join triangular faces two by two so that $\mathcal{Q}(T)$ has only quadrangular faces. $\square$

Proof of Proposition 2. Given a well labelled tree T, faces of its image $\mathcal{Q}(T)$ are as described by Figure 8. The selection rules for $\mathcal{T}$ then shows that each face correctly selects an edge of T, so that $\mathcal{T}(\mathcal{Q}(T)) = T$. Thus $\mathcal{T}$ and $\mathcal{Q}$ are inverse bijections between well labelled trees and a subset $\tilde{\mathcal{Q}}_n$ of the set of quadrangulations. Equality of cardinalities, as granted for instance by the alternative bijection of [13], proves that $\tilde{\mathcal{Q}}_n$ is the full set of quadrangulations with n faces and concludes the proof. However, we provide below a direct proof of the equality $\mathcal{Q}(\mathcal{T}(Q)) = Q$ for any quadrangulation Q, for this provides better understanding of the bijection.

Let Q be a quadrangulation, Q' and $T = \mathcal{T}(Q)$ the map and tree as in Section 3.3, and T_0, T' as in the construction of $\mathcal{Q}(T)$. Consider first the selection rules applied around the root to construct $\mathcal{T}(Q)$. Each edge (with labels) 1-2 of $\mathcal{T}(Q)$ forms a directly oriented corner with an edge 1-0 in its face of creation, while each edge 1-1 forms two such corners (one on each side). Hence, in accordance with Step (1) of the reciprocal construction, an edge 1-0 arrives at each corner with label 1. Thus the submap T_0 of T' is also a submap of Q'. Moreover T_0 covers all vertices of T' (resp. Q'), so that edges of T' (resp. Q') not in T_0 are chords of faces of T_0.

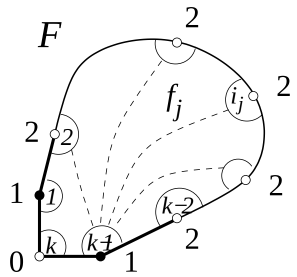


FIGURE 9. Faces of T_1 inside a face F of T_0.

Accordingly, $T' = Q'$ if, inside each face of T_0, both T' and Q' have the same chords.

The maps Q' and T' have the same vertices, and, due to Property 4, the same number of faces of degree 4 (that is, the number of edges $i - (i+1)$ in T), and the same number of faces of degree 3 (that is, the number of edges $i - i$ in T). By Euler's formula, they thus have also the same number of edges and finally, they are equal as planar maps if, inside all faces of T_0, each chord of T' is a chord of Q'.

Let us now work inside a face F of T_0 (see Figure 9). By construction of T_0 the face F has only one corner with label 0 (incident to the root) and two corners with label 1 (since such a corner is incident to an edge $(0,1)$ after Step (1)). If F has degree 3 (resp. 4), the corners around the face, in clockwise order, are labelled 0-1-1 (resp. 0-1-2-1), and there is no chord, neither in Q' nor in T'. Let us thus assume that F has degree k larger than 4, and number the k corners of F in clockwise order starting after the root corner. Let e_i be the label of the ith corner, so that $e_1 = 1$, $e_2 = 2$, $e_{k-2} = 2$, $e_{k-1} = 1$, $e_k = 0$, and $e_i \geq 2$ otherwise. This numbering corresponds to the one used to define chords of T': for each corner i with $e_i = 2$ but the last one, $s(i) = k-1$ and the chord $(i, k-1)$ appears in T'.

In order to check that these chords appear also in Q' we consider corners with label 2 in increasing order: they are numbered $2 = i_1 < i_2 < \ldots < i_p = k-2$. In Q' let f_1 be the face that contains the corners numbered k, 1 and $k-1$ (with labels 0, 1 and 1). The face f_1 contains a fourth corner, with label 2: it can be i_p (if the chord $(1, i_p)$ is in Q') or i_1 (if the chord $(i_1, k-1)$ is in Q'). In the first case the face f_1 in Q' has corners $(1, i_p, k-1, k)$ and there is a contradiction with the fact that the edge $(i_p, k-1)$ has not been selected in F by the construction $\mathcal{T}$. Hence the second case hold: the chord $(i_1, k-1)$ is in Q' and the edge $(1, i_1)$ was selected.

Now assume that the edges $(i_{j'}, k-1)$ belong to Q' for $j' < j < k-2$ and check that $(i_j, k-1)$ belongs to Q'. Consider in Q' the face f_j, included in the face F and bordering the edge $(i_j - 1, i_j)$ of the cycle F. If $e_{i_j - 1} = 2$ then $i_j - 1 = i_{j-1}$ and the face f_j is triangular (since the selection rule of confluent faces was applied by $\mathcal{T}$) and contains the chord $(i_j, k-1)$. Otherwise $e_{i_j - 1} = 3$ and the face f_j is quadrangular and of simple type (since the selection rule of simple faces was applied by $\mathcal{T}$). Therefore there is an edge from i_j to a corner with label 1, which can only be the chord $(i_j, k-1)$, for a chord $(i_j, 1)$ would cross $(i_1, k-1)$.

All chords with labels 1-2 are thus identical in T' and Q'. Let T_1 be the union of T_0 and these chords. In view of the previous discussion, the faces of T_1 are exactly the previous subdivisions into faces f_j of all faces F of T_0. Moreover each face f of T_1 has only one corner with label 1 and two with label 2, all other labels being at least 3. Shifting down all labels by one inside face f, the situation is exactly equivalent to that of the face F above (observe that the rules for the construction $\mathcal{T}$ and $\mathcal{Q}$ remain unaffected by the shift since only vertices with label greater or equal to two are considered). The identity between chords of T' and Q' of successively greater orders can thus be checked inductively.

Finally Q' contains all the edges of T', that is $Q' = T'$, and since the deletion of edges with labels of the form (e, e) in Q' (resp. T') produces Q (resp. $\mathcal{Q}(\mathcal{T}(Q))$), we obtain $\mathcal{Q}(\mathcal{T}(Q)) = Q$. □

4. Well labelled and embedded trees

4.1. **Unconstrained well labelled trees as embedded trees.** Formula (2) for the number of well labelled trees with n edges,

$$|\mathcal{W}_n| \;=\; \frac{2}{n+2}\,\frac{3^n}{n+1}\binom{2n}{n} \;=\; \frac{2}{n+2}\cdot 3^n \cdot C(2n),$$

is remarkably simple and yet not immediately clear from definition. Indeed, even though $C(2n)$ is known to be the number of plane trees, the positivity of labels makes it difficult to count labellings that make a plane tree well labelled.

It is thus natural to work first without this positivity condition: define a plane tree to be an *unconstrained well labelled tree* if its vertices have integer labels, the labels of two adjacent vertices differ at most by one, and the label of the root vertex is one. Let $\mathcal{U}_n$ denote the set of unconstrained well labelled trees with n edges.

The labelling of a labelled tree can be recovered uniquely from the label of its root and the variations of labels along all edges. We shall denote $\kappa(\epsilon) \in \{-1, 0, 1\}$ the variation of labels along the edge ϵ when it is traversed away from the root. Since there is no positivity condition on the labels of unconstrained well labelled trees, all $\kappa(\epsilon)$ can be set independently and the number of labellings of a plane tree that yield an unconstrained well labelled tree is just 3^n. That is,

$$|\mathcal{U}_n| \;=\; \frac{3^n}{n+1}\binom{2n}{n} \;=\; 3^n \cdot C(2n).$$

The definition of label distribution extends to unconstrained well labelled trees. For $U \in \mathcal{U}_n$ let $(\lambda_k)_{m<k<M} \equiv (\lambda_k^{[U]})_{k\in\mathbb{Z}}$ be the number of vertices with label k in the tree U. The label distribution of U is supported by an interval $[m, M]$ with $m \le 1 \le M$. The cumulated label distribution is defined with respect to the minimum label m by $\widehat{\lambda}_k^{[U]} = \sum_{\ell=1}^{k} \lambda_{m+\ell-1}^{[U]}$. These definitions are illustrated by Figure 10.

Observe moreover that similar unconstrained labellings have been considered by D. Aldous [1] with the following interpretation (we restrict to our special one-dimensional case). The tree is folded on the lattice $\mathbb{Z}$ with the root set at position 1 and each edge mapped on an elementary vector (here $+1$, 0, or -1). The label of a node then describe its position on the line and, upon counting the number of nodes at position j, a mass distribution is obtained. More precisely, with our notations, Aldous' discrete mass distribution associated to a tree $U \in \mathcal{U}_n$ is just the empirical measure of labels

$$\mathcal{J}^{[U]} \;=\; \frac{1}{n}\sum_{k\in\mathbb{Z}} \lambda_k^{[U]}\delta_k,$$

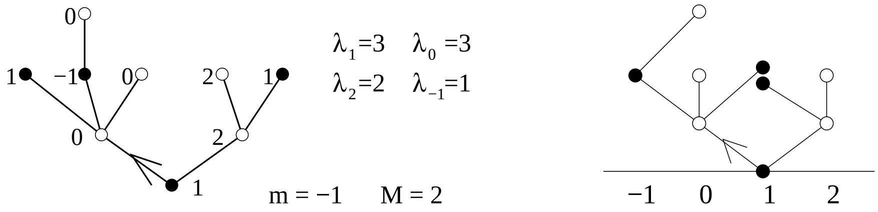


FIGURE 10. An unconstrained well labelled tree with its label distribution and a representation of the embedding on the line (the plane order structure of the tree is lost in the latter representation).

where δ_k denotes the Dirac mass at k. (Observe that for this interpretation to be completely equivalent to our model, the ordering of subtrees of a vertex must be recorded explicitely, since it does not show any more on the picture.)

In view of this interpretation and for concision's sake, let us rename *unconstrained well labelled trees* and call them instead *embedded trees.*

4.2. **Random trees and random quadrangulations.** Let W_n and U_n be random variables with uniform distribution on $\mathcal{W}_n$ and $\mathcal{U}_n$. More precisely,

$$\Pr(W_n = W) \;=\; \frac{1}{\frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}}, \qquad \text{and} \qquad \Pr(U_n = U) \;=\; \frac{1}{\frac{3^n}{n+1}\binom{2n}{n}},$$

for all $W \in \mathcal{W}_n$ and $U \in \mathcal{U}_n$.

The label distribution of the corresponding random trees are two random variables that we shall denote $(\lambda_k^{(n)})_{k\geq 1} \equiv (\lambda_k^{[W_n]})_{k\geq 1}$ for random well labelled trees, and $(\Lambda_k^{(n)})_{k\in\mathbb{Z}} \equiv (\lambda_k^{[U_n]})_{k\in\mathbb{Z}}$ for random embedded trees. For random well labelled trees we also use the notation μ_n for the maximal label, and for random embedded trees the notations m_n and M_n for the minimal and maximal label respectively. Finally cumulated profiles $\widehat{\lambda}_k^{(n)} = \sum_{\ell=1}^k \lambda_\ell^{[W_n]}$ and $\widehat{\Lambda}_k^{(n)} = \sum_{\ell=1}^k \lambda_{m_n+\ell-1}^{[U_n]}$ are defined accordingly (the minimum m_n in $\widehat{\Lambda}_k^{(n)}$ is understood for the same realisation U_n).

At this point we are given three random variables: random quadrangulations L_n, random well labelled trees W_n and random embedded trees U_n. On the one hand, according to Theorem 1, random quadrangulations "are" random well labelled trees, as illustrated by the next corollary.

Corollary 1. *The label distribution of random well labelled trees has the same distribution as the profile of quadrangulations:*

$$(\lambda_k^{(n)})_{k\geq 1} \;\stackrel{law}{=}\; (H_k^{(n)})_{k\geq 1}.$$

In particular $r_n = \mu_n$.

On the other hand, random embedded trees seem to be a simple variant of well labelled trees that has the great advantage to be defined in accordance with Aldous' prescription for discrete embedded trees. This leads us to study more precisely the relation between W_n and U_n. By definition, $\mathcal{W}_n \subset \mathcal{U}_n$, and according to Tutte's formula (2),

$$|\mathcal{W}_n| \;=\; \frac{2}{n+2}\cdot|\mathcal{U}_n|. \tag{3}$$

For combinatorialists, this relation could be reminiscent of the relation between the number of Dyck walks and the number of bilateral Dyck walks (see [33, Ch. 5]).

Equivalently, from a more probabilistic point of view, the relation reads

$$\Pr(U_n \in \mathcal{W}_n) \;=\; \frac{2}{n+2},$$

and random well labelled trees are random embedded trees conditioned to positivity. This is exactly similar to Kemperman's formula for the probability that a simple symmetric walk on $\mathbb{Z}$ starting from $k > 0$ and ending at 0 after n steps remains positive until the last step (see [28]).

4.3. **Cyclic shifts and the cycle lemma.** The idea to consider cyclic shifts originates in Dvoretsky and Motzkin's work and was used by Raney to prove Lagrange inversion formula and by Takács to prove and extend Kemperman's formula for random walks (we refer to [33, Ch. 5] and [28] for these historical references and many more). We shall prove a consequence of this idea to the study of "height distribution" of simple walks, that will be fundamental in the next section.

Let n and k be nonnegative integers and let $\mathcal{B}_{n,k}$ denote the set of walks of length $2n+k$ with n increments $+1$ and $n+k$ increments -1, that end with a negative increment -1. A walk $w \in \mathcal{B}_{n,k}$ is described either by its sequence of increments $w = (w_1,\ldots,w_{2n+k})$, $w_i \in \{+1,-1\}$, or by the partial sums $w(p) = \sum_{i=1}^{p} w_i$, $p = 0,\ldots,2n+k$. By construction, $w(0) = 0$, $w_{2n+k} = -1$, $w(2n+k) = -k$, and $w(2n+k-1) = -k+1$. Finally for $k \geq 1$, consider the subset $\mathcal{D}_{n,k}$ of $\mathcal{B}_{n,k}$ of "positive" walks defined by the condition: $w(p) > -k$ for all $0 \leq p < 2n+k$.

Two walks w and w' of $\mathcal{B}_{n,k}$ belong the same *conjugacy class* if they differ by a cyclic shift, that is, if there exists s such that

$$w = (w_1,\ldots,w_{2n+k}) \quad \text{and} \quad w' = (w_s,\ldots,w_{s+2n+k}),$$

where indices are considered modulo $2n+k$.

Define a (left-to-right) record to be a step $p \geq 1$ at which a minimum is reached for the first time: $w(q) > w(p)$ for all $q < p$. Since $w(2n+k) = -k$ there are at least k records. Let us denote $p_1 < \cdots < p_k$ the k lowest records, so that in particular $w(p_k)$ is the minimum value reached by the walk and $w(p_i) = k - i + w(p_k)$. The steps p_i, $i = 1,\ldots,k$ are called the *low records* of w.

The following immediate properties are illustrated by Figure 11.

Property 5. *A walk of $\mathcal{B}_{n,k}$ belongs to $\mathcal{D}_{n,k}$ if and only if its lowest record p_k equals $2n+k$.*

Property 6. *Cyclic shifts transport low records: Let $w = (w_1,\ldots,w_{2n+k})$ and $w' = (w_s,\ldots,w_{s+2n+k})$ and assume $\{p_1,\ldots,p_k\}$ are the low records of w. Then the low records of w' are $\{p_1+s,\ldots,p_k+s\}$ (modulo $2n+k$).*

The classical cycle lemma follows from Properties 5 and 6.

Lemma 1 (Cycle lemma). *Let C be a conjugacy class of $\mathcal{B}_{n,k}$. Then*

$$(n+k)\cdot|C \cap \mathcal{D}_{n,k}| \;=\; k\cdot|C|.$$

Proof. Let us apply a double counting argument:

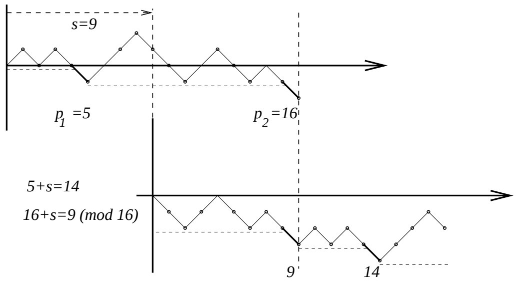


FIGURE 11. Two conjugated walks from $\mathcal{D}_{n,2}$ and $\mathcal{B}_{n,2}$ with low records indicated. Dashed lines show the zero level with respect to Dyck height.

- The left hand side counts walks in C with a (low) record at the last position ($w \in \mathcal{D}_{n,k}$) and a down step marked ($n+k$ choices).
- The right hand side counts walks in C with a down step at the last position ($w \in \mathcal{B}_{n,k}$) and a low record marked (k choices).

Now a bijection is obtained between these two sets upon sending the marked step to the last position by a cyclic shift and marking the former last step. $\square$

Given a walk w with lowest record p_k, the *height-to-min* of a step p is $\tilde{w}(p) = w(p) - w(p_k)$, which is nonnegative by definition. In order to study $\tilde{w}$, it will be convenient to consider the height of the walk relatively to the k low records: given a walk w with low records $p_1 < \cdots < p_k$, let us define the *Dyck height* at step p by

$$\bar{w}(p) = \begin{cases} w(p) - w(p_1) - 1, & \text{if } 0 \le p < p_1, \\ w(p) - w(p_i), & \text{if } p_i \le p < p_{i+1}, \text{ with } 1 \le i \le k-1 \\ w(p) - w(p_k), & \text{if } p_k \le p \le 2n+k. \end{cases}$$

The Dyck height can be understood as the height inside each of the k Dyck factors separated by low records. Let $\hat{\ell}_i(w)$ (resp. $\hat{h}_i(w)$) denote the number of down steps of w ending at Dyck height (resp. height-to-min) at most i,

$$\hat{\ell}_i(w) \;=\; |\{p \mid \bar{w}(p) \le i,\ w_p = -1\}|, \quad \text{and} \quad \hat{h}_i(w) \;=\; |\{p \mid \tilde{w}(p) \le i,\ w_p = -1\}|.$$

Then by construction, for all w and p, $\bar{w}(p) \le \tilde{w}(p) \le \bar{w}(p)+k$ and, for all i, $\hat{h}_i(w) \le \hat{\ell}_i(w) \le \hat{h}_{i+k}(w)$. The following lemma immediately follows from Property 6.

Lemma 2. *The Dyck height commutes with cyclic shift. In particular the Dyck height distribution $\hat{\ell}_i$ is invariant under cyclic shift:*

$$(\hat{\ell}_i(w))_{i\ge 0} \;=\; (\hat{\ell}_i(w'))_{i\ge 0}, \qquad \textit{for all } w \textit{ and } w' \textit{ in the same conjugacy class of } \mathcal{B}_{n,k}.$$

The height-to-min distribution thus satisfies the following weaker invariance:

$$\hat{h}_i(w) \;\le\; \hat{h}_{i+k}(w'), \qquad \textit{for all } i \ge 0 \textit{ and } w,\ w' \textit{ in the same conjugacy class of } \mathcal{B}_{n,k}.$$

From the probabilistic point of view this result can be understood as a simplified discrete version of Vervaat's relation between the Brownian excursion and the Brownian bridge and their local times relatively to the minimum.

4.4. **How to lift the positivity condition for labelled trees.** In view of Relation (3) it is tempting to use ideas of the previous section to relate well labelled and embedded trees. As a matter of fact we shall prove the following theorem.

Theorem 2. *There exists a partition of* $\mathcal{U}_n = \bigcup_{C\in\mathcal{C}_n} C$ *into disjoint* classes *each of size at most* $n+2$ *and such that in each class* $C \in \mathcal{C}_n$

- *well labelled trees are fairly represented:*
$$2\cdot|C| \;=\; (n+2)\cdot|C\cap\mathcal{W}_n|,$$
- *and for any* $W\in\mathcal{W}_n\cap C$, $U\in C$ *and* $k\geq 1$,
$$\widehat{\Lambda}_{k-2}(U) \;\leq\; \widehat{\lambda}_k(W) \;\leq\; \widehat{\Lambda}_{k+2}(U).$$

Corollary 2 (Cori-Vauquelin, 1981)**.** *The number of well labelled trees with* n *edges, (which is also the number of quadrangulations with* n *faces), is*

$$|\mathcal{W}_n| \;=\; \frac{2}{n+2}\cdot|\mathcal{U}_n| \;=\; \frac{2}{n+2}\cdot\frac{3^n}{n+1}\binom{2n}{n}.$$

The proof of Theorem 2 is presented in the next section. It relies on an encoding of plane trees in terms of another family of trees, called *blossom trees*, and on the *conjugation of trees* principle which is an analogue of the cycle lemma for blossom trees. This principle was introduced in [31] in order to give a direct combinatorial proof of Corollary 2 based on the cycle lemma. However that proof did not rely on well labelled trees and does not provide the link to the profile.

Theorem 2 admits the following probabilistic restatement.

Theorem 3. *There is a coupling* (W_n, U_n) *(*i.e. *a distribution on* $\mathcal{W}_n\times\mathcal{U}_n$ *such that the marginals are* W_n *and* U_n *as previously defined) such that the induced joint distribution* $(\lambda^{(n)},\Lambda^{(n)})$ *satisfies for all* k

$$\widehat{\Lambda}^{(n)}_{k-2} \;\leq\; \widehat{\lambda}^{(n)}_k \;\leq\; \widehat{\Lambda}^{(n)}_{k+2},$$

and in particular

$$|\mu_n - (M_n - m_n)| \leq 3.$$

Proof of Theorem 3. The distribution on $\mathcal{W}_n\times\mathcal{U}_n$ is immediately obtained from the partition $\mathcal{U}_n = \bigcup_{C\in\mathcal{C}_n} C$ as follows: for any (W,U) in $\mathcal{W}_n\times\mathcal{U}_n$, let

$$\Pr((W_n,U_n)=(W,U)) = \begin{cases} \frac{1}{2|\mathcal{U}_n|} & \text{if } U,\, W \text{ are both in } C \text{ with } |C\cap\mathcal{W}_n| = 2,\\ \frac{1}{|\mathcal{U}_n|} & \text{if } U,\, W \text{ are both in } C \text{ with } |C\cap\mathcal{W}_n| = 1,\\ 0 & \text{if } U\in C_1 \text{ and } W\in C_2 \text{ with } C_1\neq C_2.\end{cases}$$

In view of the first part of Theorem 2, the marginals are uniformly distributed. The second part of Theorem 2 gives the two inequalities. □

4.5. **Blossom trees and the conjugation of trees.** Theorem 2 is clearly analogous to Lemma 1 and 2. However we were not able to define directly conjugacy classes on embedded trees. Instead we first construct an encoding of embedded trees in terms of another family, blossom trees, and then define conjugacy classes of blossom trees and prove Theorem 2.

Following [31], let a *blossom tree* be a plane tree with the following properties:

- Vertices of degree one are of two types: arrows and flags. The root of the a blossom tree is a flag, which is said to be *special*, as opposed to the other *normal* flags.

- All inner nodes have degree 4 and each of them is adjacent to exactly one arrow.

Let $\mathcal{B}_n$ be the set of blossom trees with n inner nodes. By construction these trees have n vertices of degree 4 and thus $2n+2$ of degree one, that is n arrows and $n+2$ flags. The labelling of a blossom tree is given by the following *labelling process*:

- Start with *current label* 2 just after the root.
- Turn around the border of the tree in counterclockwise direction.
  - Each time an arrow is reached, the *current label* is increased by 1.
  - Each time a flag is reached, the *current label* is decreased by one and then written on the flag.
- Stop when the root flag is reached again (no label is written there).

This definition is illustrated by Figure 12.

Lemma 3. *Embedded trees with n edges are in one-to-one correspondence with blossom trees with n inner nodes with the same label distribution.*

Proof. In order to prove this lemma we work on a set of decorated blossom trees: in these trees, the root flag is special and any flag with label e (as given by the labelling process) can either be empty or be decorated by an embedded tree with root label e (for $e \neq 1$ the immediate generalisation of embedded trees is meant). The *combined label distribution* of a decorated blossom tree counts labels of decorations (embedded trees on flags) and of empty flags. Examples of decorated blossom trees are given in Figure 14 (in these figures, labels along edges indicate values taken by the current label during the labelling process).

The first step of the encoding of an embedded tree consists in writing it on the normal flag of the unique blossom tree with two flags and no inner node (Figure 14 top-left example). Then the encoding is performed by recursively transforming the decorated blossom tree according to the local rules of Figure 13. Each time the leftmost rule is applied to a flag decorated by an embedded tree reduced to a vertex with label e, this vertex is suppressed and the flag becomes empty, with label e (by construction, e agrees with the labelling process; the combined label distribution is left unchanged). When one of the other three rules is applied, a new inner node is created while an edge of embedded tree is suppressed. The relation between the position of the created arrow and the root labels of the embedded trees grants that the compatibility with the labelling process is preserved (observe that in the middle rule subtrees have been switched for this purpose).

As long as there are decorated flags a rule can be applied. Once there is no more decorated flag, a blossom tree is obtained. Rules are local so that rules applied in distinct subtrees commute. As a consequence the final blossom tree does not depend on the order in which rules are applied. Each rule is uniquely reversible so that the encoding is bijective. $\square$

Proof of Theorem 2. The partition $\mathcal{U}_n \equiv \mathcal{B}_n = \bigcup_{C \in \mathcal{C}_n} C$ is the partition of blossom trees in conjugacy classes: two blossom trees A and B are in the same *conjugacy class* C if B is obtained from A by first replacing the root flag of A by a normal flag and then choosing a new special flag. This operation is called a *cyclic shift* of the tree. In other terms each conjugacy class C is the set of blossom trees that can be obtained from a specific *unrooted* blossom tree (the flags of which are all

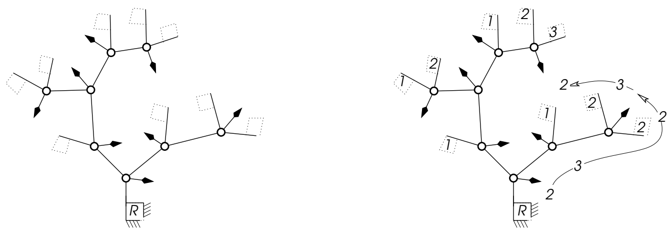


FIGURE 12. An example of blossom tree and its labelling.

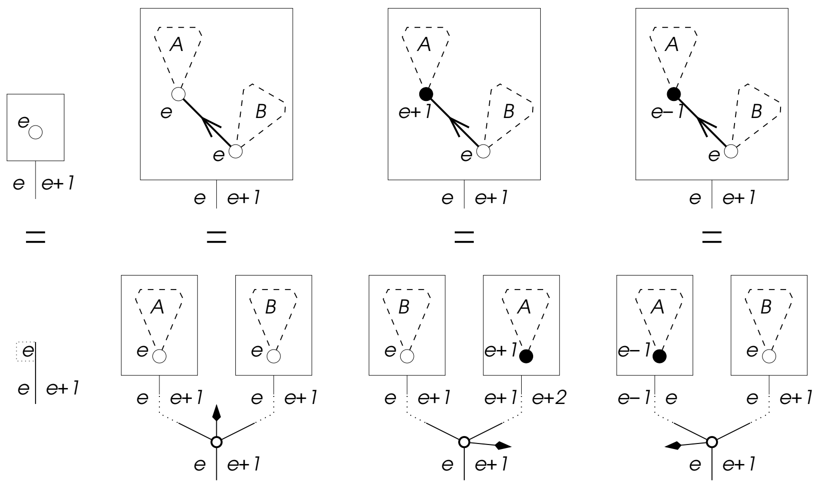


FIGURE 13. From embedded trees to blossom trees: rules.

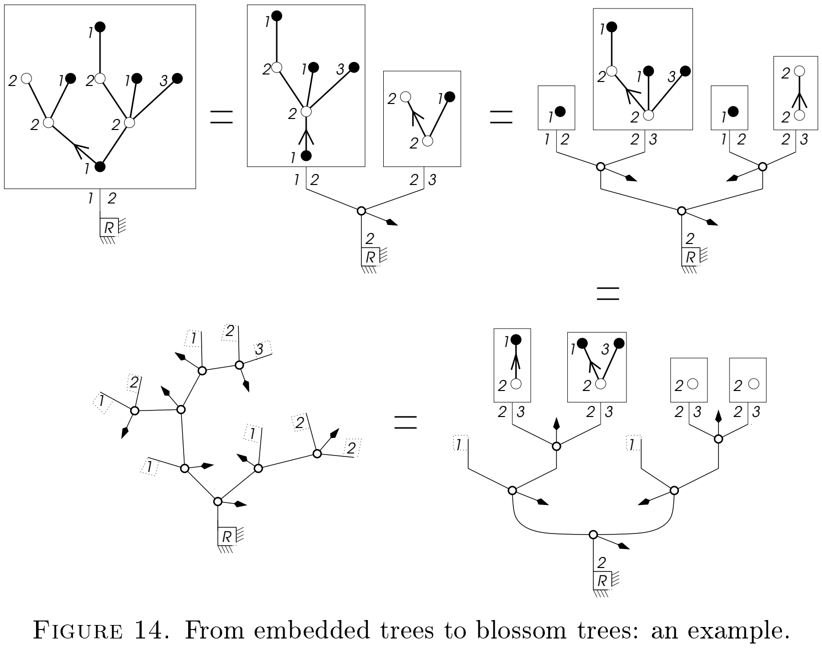


FIGURE 14. From embedded trees to blossom trees: an example.

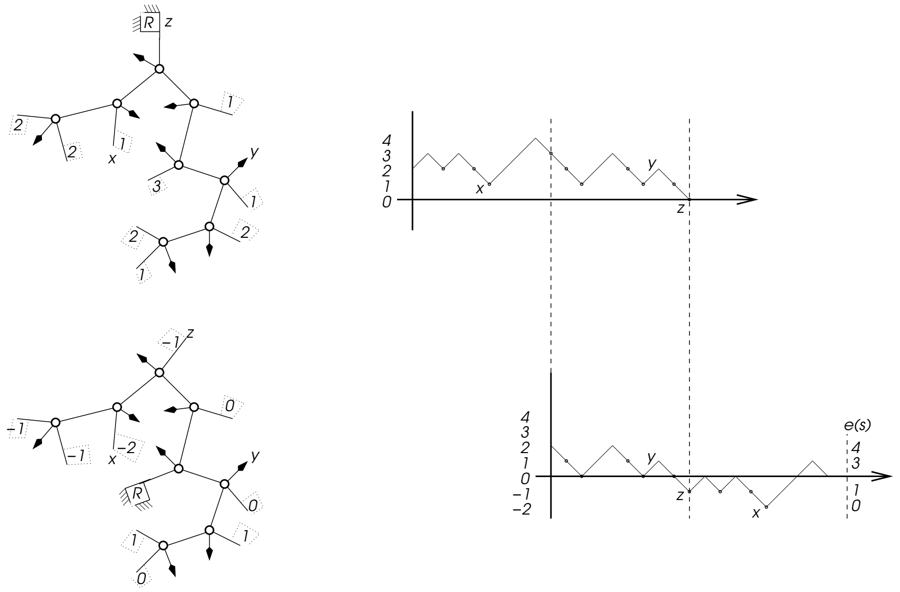


FIGURE 15. The labelling process for two conjugated blossom trees. Three steps x, y and z have been distinguished to illustrate the correspondence: up step = arrow, down step = flag.

normal) by selecting a special flag (root flag) in all the possible ways. Through the encoding, the conjugacy classes induce the partition of $\mathcal{U}_n$ mentioned in Theorem 2.

Given a blossom tree B with n arrows and $n+2$ flags, the evolution of the current label, while performing its labelling process, is a walk w_B with n increments $+1$ and $n+2$ negative increments -1, that starts from 2, and whose last step, when the process reaches again the root flag, is a negative increment. Upon decreasing all labels by two, the walk w_B is thus a walk of $\mathcal{B}_{n,2}$ as defined in Section 4.3.

Moreover each cyclic shift of the tree B induces the corresponding cyclic shift of the walk w_B. Finally a blossom tree encodes a well labelled tree if and only if all its labels are positive, that is, if and only if the walk w_B belongs to $\mathcal{D}_{n,2}$ (upon decreasing all label by two). The first statement of Theorem 2 is thus exactly the cycle lemma (Lemma 1).

Finally, let us consider label distributions. Given W a well labelled tree and U an embedded tree in the same conjugacy class of trees, the corresponding walks w_W and w_U belong to the same conjugacy class of walks. But the cumulated label distributions satisfies $\widehat{\lambda}_k^{[W]} = \hat{h}_k(w_W)$ and $\widehat{\Lambda}_k^{[U]} = \hat{h}_k(w_U)$ so that Lemma 2 gives the second statement of Theorem 2. $\square$

4.6. **A direct link between embedded trees and quadrangulations.** In the previous pages we developed an analog of the cycle lemma for trees, thus relating freely embedded trees to positively conditioned embedded trees. We believe this approach to be of interest on its own. However, one could ask for a more direct relation between Aldous' trees and quadrangulations. Observe in particular that

$$2 \cdot \frac{3^n}{n+1}\binom{2n}{n}$$

is both the number of rooted quadrangulations with a marked vertex and n faces, that is $|\mathcal{Q}_n| \cdot (n+2)$, and twice the number of embedded trees with n edges. This reflects the following deeper relation.

Theorem 4. *There is a bijection between rooted quadrangulations with a marked vertex and n faces, and couples formed of an embedded tree and a sign (+ or −). This bijection maps the profile from the marked vertex of the quadrangulation onto the shifted label distribution of the tree. (By shifted label distribution is meant the label distribution obtained by translating all labels so that the minimum label is* 1*.)*

Proof. This result is an adaptation of the construction of Section 3. Let Q be a rooted quadrangulation with a marked vertex v_0. Endow Q with the distance labelling from the marked vertex v_0 (instead of the labelling from the root vertex). Then define the tree $\mathcal{T}(Q)$ as in Section 3 except for the root, which is obtained as follows: the root face f of Q produces an edge e of $\mathcal{T}(Q)$. Depending on which of the four sides of f is the root of Q, set the orientation of e and a sign $s \in \{+,-\}$. Let $\mathcal{T}'(Q)$ be the tree obtained by rooting $\mathcal{T}(Q)$ at e and translating all labels so that the root vertex has label 1. Then the image of Q is the couple $(\mathcal{T}'(Q), s)$. In view of Section 3, this construction is one-to-one and maps the profile from the marked vertex onto the shifted label distribution. □

5. Quadrangulations, Brownian snake and ISE

5.1. **Encoding embedded trees by pairs of contour walks.** Let $\bar{\mathcal{U}}_n$ be the set of embedded trees with root label zero instead of one. These trees, that are simply obtained from trees of $\mathcal{U}_n$ by shifting all labels down by one, will be more convenient for our purpose.

Let U be an embedded tree of $\bar{\mathcal{U}}_n$ and consider the following traversal of U, where traversing an edge takes unit time:

- At time $t = 0$, the traversal arrives at the root.
- If the traversal reaches at time t a vertex v_t having k sons for the ℓth time with $\ell \leq k$, its next step is toward the ℓth son of v_t.
- If the traversal reaches at time t a vertex v_t having k sons for the $(k+1)$th time, its next step is back toward the father of v_t.

This traversal is called the *contour traversal* because, as exemplified by Figure 16, it turns around the tree. In particular every edge is traversed twice (first away from and then toward the root) and the complete traversal takes $2n$ steps. The *contour pair* of U is then defined by the height (*i.e.* distance to the root in the abstract tree), $E^{[U]}(t)$ and label $V^{[U]}(t)$ of vertex v_t traversed at time $t = 0, \ldots, 2n$. (The path E is often called the *Dyck path* associated to the tree U [33, Ch. 5], or the *contour process* in [23, Ch. I.3].)

The following proposition is immediate from the definition of contour pairs.

Proposition 3. *The contour pair construction is a one-to-one correspondence between $\bar{\mathcal{U}}_n$ (or $\mathcal{U}_n$) and the set $\mathcal{EV}_{2n}$ of pairs of walks of length $2n$ such that:*

- *the walk E is an excursion with increment ± 1 or Dyck path, that is $E(0) = E(2n) = 0$, $|E(t) - E(t+1)| = 1$ and $E(t) \geq 0$ for all $t = 0, \ldots, 2n-1$;*

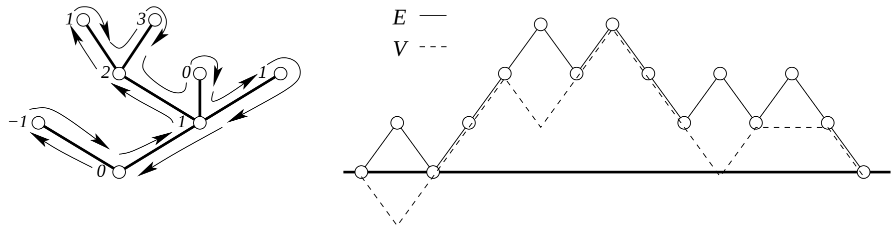


FIGURE 16. Contour traversal and contour pair (E, V) of a tree.

- *the walk V is a bridge with increment $\{-1, 0, 1\}$ or bilateral Motzkin path, that is $V(0) = V(2n) = 0$ and $(V(t) - V(t+1)) \in \{-1, 0, 1\}$ for all t;*
- *and the consistency condition hold:*
  $(\ E(t) = E(t')$ *and* $E(s) \geq E(t)$ *for all* $t < s < t'\ ) \ \Rightarrow\ V(t) = V(t')$.

The excursion E alone determines a unique unlabelled rooted plane tree, while the walk V describes one of the 3^n labelling of the tree encoded by E. Recall that for an embedded tree U, $\kappa(\epsilon) \in \{-1, 0, 1\}$ denotes the variation along edge ϵ when traversed away from the root. In particular if ϵ is traversed for the first time between time t and $t+1$ and for again between t' and $t'+1$, then

$$\kappa(\epsilon) = V(t+1) - V(t) = V(t') - V(t'+1).$$

This local condition is equivalent to the consistency condition of Proposition 3.

5.2. **Random trees as random contour pairs.** Endow now $\bar{\mathcal{U}}_n$ with the uniform distribution and let $(E^{(n)}, V^{(n)}) \equiv (E^{[U_n]}, V^{[U_n]})$ denote the contour pair of the random tree U_n. According to Proposition 3, the random contour pair $(E^{(n)}, V^{(n)})$ is uniformly distributed on $\mathcal{EV}_{2n}$ and $E^{(n)}$ is uniformly distributed on $\mathcal{E}_{2n}$, the set of Dyck walks of length $2n$. More precisely, for all $(E, V) \in \mathcal{EV}_{2n}$,

$$\Pr((E^{(n)}, V^{(n)}) = (E, V)) \ = \ \frac{1}{\frac{3^n}{n+1}\binom{2n}{n}}, \qquad \Pr(E^{(n)} = E) \ = \ \frac{1}{\frac{1}{n+1}\binom{2n}{n}}.$$

In order to state convergence results, let us now define a scaled version of these random walks: given a random tree U_n and its contour pair $(E^{(n)}, V^{(n)})$, let

$$e^{(n)} \ = \ \left(\frac{E^{(n)}(\lfloor 2ns \rfloor)}{\sqrt{2n}}\right)_{0 \leq s \leq 1} \quad \text{and} \quad \hat{W}^{(n)} \ = \ \left(\frac{V^{(n)}(\lfloor 2ns \rfloor)}{(8n/9)^{1/4}}\right)_{0 \leq s \leq 1}.$$

The random variables $e^{(n)}$ and $\hat{W}^{(n)}$ take their values in the Skorohod space $D([0, 1], \mathbb{R})$ of càdlàg real functions (right continuous with left limits).

As was proved by Kaigh [22], the scaled version $e^{(n)}$ of the contour process converges weakly to the normalised Brownian excursion e. Our aim is to state an analogous result for the random variable

$$X^{(n)} \ \equiv \ \left(e^{(n)}, \hat{W}^{(n)}\right),$$

that takes its value in the Skorohod space $D([0, 1], \mathbb{R}^2)$.

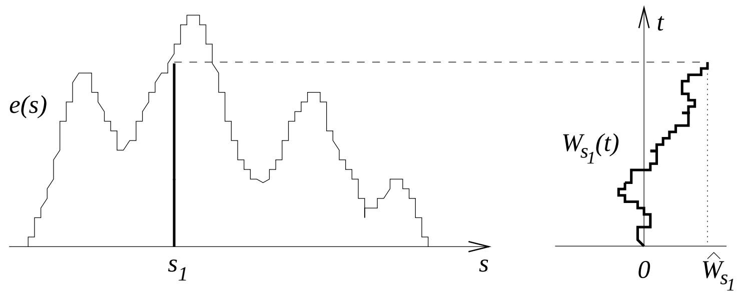


FIGURE 17. Spatial extension of the snake at time s_1.

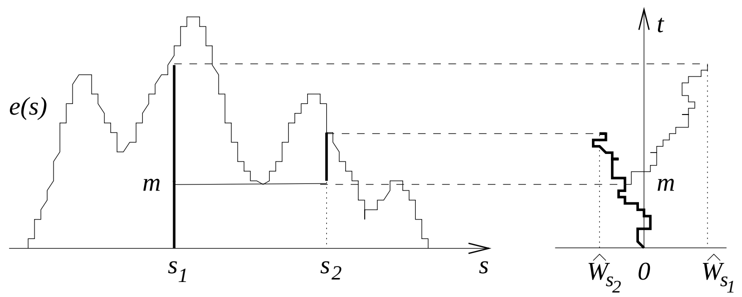


FIGURE 18. Consistency of the snake between times s_1 and s_2.

5.3. **A Brownian snake.** Let e be the normalised Brownian excursion and

$$W = (W_s(t))_{0\leq s\leq 1,\ 0\leq t\leq e(s)}$$

be the Brownian snake with lifetime e, as studied previously in [1, 12, 15, 16, 23, 32].

More precisely, given e, the conditional distribution of the process W can be described as follows:

- for all $0 \leq s \leq 1$, $t \to W_s(t)$ is a standard Brownian motion defined for $0 \leq t \leq e(s)$ (see Figure 17);
- the application $s \to W_s(.)$ is a path-valued Markov process with transition function satisfying: for $s_1 < s_2$, and for $m = \inf_{s_1\leq u\leq s_2} e(u)$, conditionally given $W_{s_1}(.)$ (see Figure 18),
  - on the one hand we have that
  $$(W_{s_1}(t))_{0\leq t\leq m} = (W_{s_2}(t))_{0\leq t\leq m},$$
  - and on the other hand $(W_{s_2}(m+t))_{0\leq t\leq e(s_2)-m}$ is a standard Brownian motion starting from $W_{s_2}(m)$, independent of $W_{s_1}(.)$.

The Brownian snake can be viewed as a branching Brownian motion, or as an embedded continuum random tree (see [1]). More precisely the excursion e can be thought of as the contour walk obtained by contour traversal of a continuum random tree, while the snake $W_s(\cdot)$ at times s describes the embedding of the branch to the root at time s.

Instead of considering the full Brownian snake $W_s(t)$ we shall concentrate, as we did in the discrete case, on its description by a contour pair (or "head of the snake" description) $X = (X_s)_{0\leq s\leq 1}$, defined by (see also Figure 19)

$$\hat{W}_s = W_s(e(s)), \quad X_s = \left(e(s), \hat{W}_s\right), \qquad \text{for } 0 \leq s \leq 1.$$

In complete analogy with the discrete case, the full Brownian snake can be reconstructed from its contour pair description since $W_s(t) = \hat{W}_{\sigma(s,t)}$ where $\sigma(s,t) = \sup\{s' \leq s \mid e(s') = t\}$. However we need only and shall content with results in terms of X (see [26] for a complete discussion of the relation between the full snake and its contour description).

5.4. **Integrated SuperBrownian Excursion.** Let $\mathcal{J}_n$ denote the empirical measure of labels of a random embedded tree:

$$\mathcal{J}_n = \frac{1}{n}\sum_k \Lambda_k^{(n)}\ \delta_k.$$

Following Aldous [1], for any simple family of trees like our embedded trees, $\mathcal{J}_n$ is expected to converge upon scaling to a random mass distribution $\mathcal{J}$ supported by a random interval $0 \in [L, R] \subset \mathbb{R}$. This random measure $\mathcal{J}$ is called Integrated SuperBrownian Excursion (ISE) by Aldous, in view of its relation to W through

$$\int g\ d\mathcal{J} = \int_0^1 g\left(\hat{W}_s\right)\, ds, \tag{4}$$

for any measurable test function g, see [23, Ch. IV.6]. In [12] the convergence of $\mathcal{J}_n$ to $\mathcal{J}$ is proved for random embedded Cayley trees. Although these trees are not exactly our random embedded *plane* trees, the proof could easily be adapted.

According to Corollary 1 and Theorem 3, the radius r_n is given by the width of the support of $\mathcal{J}_n$. However the weak convergence of $\mathcal{J}_n$ to $\mathcal{J}$, as obtained in [12] is not sufficient for our purpose since $r = R - L$, the width of the support of $\mathcal{J}$, is not a continuous functional of the measure $\mathcal{J}$.

5.5. **Convergence of snakes.** Instead of weak convergence of $\mathcal{J}_n$ to $\mathcal{J}$, we shall thus prove in Section 6 the following stronger result.

Theorem 5. *The scaled contour pair $X^{(n)}$ converges weakly to X in $D([0,1], \mathbb{R}^2)$.*

This theorem establishes weak convergence of the scaled contour (or head of the snake) description of embedded trees to the head of the snake description of the Brownian snake with lifetime e. We moreover obtain a deviation bound for the maximal extension of the snake $\hat{W}_s^{(n)}$.

Proposition 4. *There exists $y_0 > 0$ such that for all $y > y_0$ and n,*

$$\mathbb{P}\left(\sup_{0\leq s\leq 1} \hat{W}_s^{(n)}\ >\ (8/9)^{1/4} y\right)\ \leq e^{-y}.$$

Theorem 5 was independently obtained by Marckert and Mokkadem [26]. They extend the convergence result to the explicit full description of $(W_s(t))_{s,t}$ but their alternative proof does not provide the exponential bound of Proposition 4.

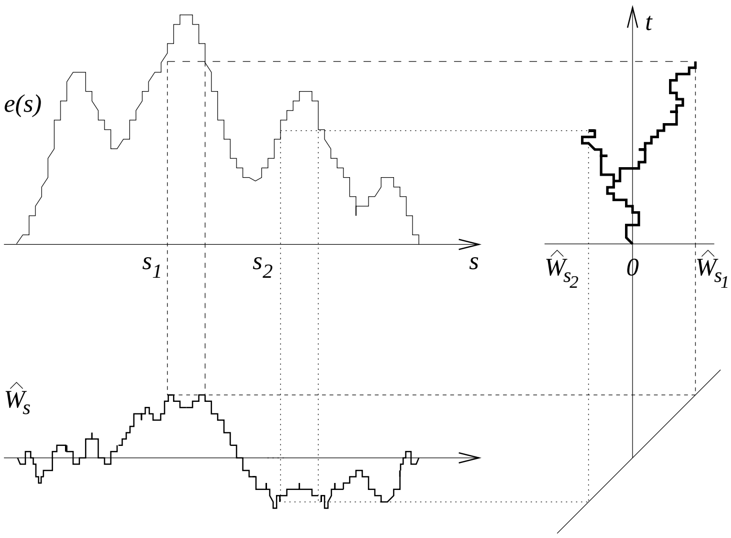


FIGURE 19. The contour description $(e, \hat{W}_s)$: the excursion e encodes the extension of the snake, the second walk describes the horizontal position of its head.

5.6. **The radius of a random quadrangulation and the width of ISE.** According to Corollary 1 and to Theorem 3, the radius r_n of the quadrangulation corresponding to U_n satisfies

$$\left| (8/9)^{1/4} \left(\sup_{0 \le s \le 1} \hat{W}_s^{(n)} - \inf_{0 \le s \le 1} \hat{W}_s^{(n)} \right) - n^{-1/4}\, r_n \right| \le 3n^{-1/4}.$$

Theorem 5 and Proposition 4 thus prove the conjecture $\mathbb{E}(r_n) = \Theta(n^{1/4})$ and lead to a much more precise characterization:

Corollary 3. *The random variable $n^{-1/4}\, r_n$ converges weakly to $(8/9)^{1/4}\, r$, in which*

$$r = \sup_{0 \le s \le 1} \hat{W}_s - \inf_{0 \le s \le 1} \hat{W}_s.$$

Furthermore, convergence of all moments holds true.

In view of Relation (4) the random variable r is also the width of ISE process $\mathcal{J}$.

5.7. **The profile and a CRM.** Actually, Theorems 1 and 3 suggest that not only the scaled radius but the full scaled profile converges (at least in distribution) to the ISE mass distribution. More precisely, define the distribution function $F(x)$ of the translated ISE by

$$W_{\min} = \inf_{0 \le s \le 1} \hat{W}_s, \qquad F(x) = \mathcal{J}\left((-\infty, W_{\min} + x]\right) = \mathcal{J}\left([W_{\min}, W_{\min} + x)]\right),$$

and the scaled distribution function of the profile of random quadrangulations by

$$F_n(x) = \frac{1}{n+1}\, \widehat{\lambda}^{(n)}_{\lfloor (8n/9)^{1/4} x \rfloor} = \frac{1}{n+1}\, \widehat{H}^{(n)}_{\lfloor (8n/9)^{1/4} x \rfloor}.$$

where $\widehat{\lambda}_k^{(n)}$ is the cumulated distribution of labels of a random well labelled tree (as defined in Section 3) and $\widehat{H}_k^{(n)}$ is the cumulated profile of a random quadrangulation (as defined in Section 2).

Then we prove the following corollary of Theorems 1, 3, 5 and Corollary 3.

Corollary 4. *The scaled profile F_n converges weakly to F in $D([0,+\infty),\mathbb{R})$.*

A natural conjecture is that there is a continuum analogue to Theorem 1 that allows to define from ISE a Continuum Random Map (CRM), such that the properties of scaled distances in random quadrangulations (distances between arbitrary pairs of points, not only with respect to a basepoint) would be described by the properties of distance in the CRM. In view of the interpretation of random quadrangulations as 2d Euclidean pure quantum geometries, this CRM might be considered as a natural candidate model of continuum 2d pure quantum geometry. We plan to discuss this connection further in a subsequent paper.

6. Random embedded trees and the Brownian snake

In this section we prove Theorem 5. Finite dimensional density functions are first calculated (Section 6.1). Section 6.2 then provides the deviation bound for the maximum of the label walk. Finally tightness is proved using the previous bound (Section 6.3). The theorem then follows from standard results on weak convergence in the space $D([0,1],\mathbb{R}^2)$ [9].

6.1. Finite dimensional density functions.

From now on in this section, p and $\tau=(\tau_0,\dots,\tau_{p+1})$ are fixed with $0=\tau_0<\tau_1<\cdots<\tau_p<\tau_{p+1}=1$, and we prove the following finite dimensional density convergence result.

Proposition 5. *The sequence of random variables*

$$X^{(n)}(\tau)=\left(\frac{E^{(n)}(\lfloor 2\tau_i n\rfloor)}{(2n)^{1/2}},\ \frac{V^{(n)}(\lfloor 2\tau_i n\rfloor)}{(8n/9)^{1/4}}\right)_{1\le i\le p}=\left(e^{(n)}(\tau_i),\ \hat{W}^{(n)}_{\tau_i}\right)_{1\le i\le p}$$

weakly converges to $X(\tau)=\left(e(\tau_i),\ \hat{W}_{\tau_i}\right)_{1\le i\le p}$, that is,

$$\lim_n \mathbb{E}\left[\Phi\left(X^{(n)}(\tau)\right)\right]=\mathbb{E}\left[\Phi\left(X(\tau)\right)\right]$$

or any bounded continuous function Φ on $\mathbb{R}^{2p}$.

For this aim it will be convenient to prove first the weak convergence of

$$Y^{(n)}(\tau)=\left(\left(e^{(n)}(\tau_i)\right)_{1\le i\le p},\left(\inf_{[\tau_i,\tau_{i+1}]}e^{(n)}\right)_{1\le i\le p-1}\right),$$

then that of

$$Z^{(n)}(\tau)=\left(Y^{(n)}(\tau),\ \left(\hat{W}^{(n)}_{\tau_i}\right)_{1\le i\le p},\left(\mu_i^{(n)}\right)_{1\le i\le p-1}\right),$$

in which $\mu_i^{(n)}$ is defined by

$$\mu_i^{(n)}\ =\ \hat{W}^{(n)}_{\tau_i'}\qquad\text{for any}\qquad \tau_i'\in\operatorname*{arginf}_{[\tau_i,\tau_{i+1}]}e^{(n)}.$$

(The consistency condition of Proposition 3 grants that $\mu_i^{(n)}$ is indeed independent of the exact choice of τ_i' in $\operatorname{arginf}_{[\tau_i,\tau_{i+1}]}e^{(n)}$.)

The weak convergence of $Y^{(n)}(\tau)$ follows from [22], as a special case, but does not fill our needs. In the next section we prove a local limit theorem for finite dimensional distributions of $e^{(n)}$ which is not a consequence of [22]. Once the local limit theorem for $Y^{(n)}(\tau)$ is proved, we recall its interpretation in terms of trees, using the key notion of *shape*. This leads us to split $\mathbb{R}^{4p-2}$ in $(p-1)!$ regions $(R_\sigma)_{\sigma\in\mathbb{S}_{p-1}}$ and to prove weak convergence of $Z^{(n)}(\tau)$ separately on each region. Finally we identify the limit and the weak convergence of $X^{(n)}(\tau)$ follows from that of $Z^{(n)}(\tau)$.

6.1.1. *A local limit theorem for* $Y^{(n)}(\tau)$. The characteristics of the walk $e^{(n)}$ we are interested in are contained in the sequence of successive heights at (resp. minima between) the τ_i. Let $(x,m) = ((x_i)_{1\le i\le p},(m_i)_{1\le i\le p-1})$ be a typical value of $Y^{(n)}(\tau)$, that is, $x_i\sqrt{2n}$ and $m_i\sqrt{2n}$ are integers, and they satisfy

$$\inf(x_i,x_{i+1}) \ge m_i, \qquad 1\le i\le p-1.$$

For the first result, we need a more handy parametrisation, by the successive up and down relative variations: let $\gamma_0 = x_1$, $\beta_p = x_p$,

$$\gamma_i = x_{i+1} - m_i, \qquad \beta_i = x_i - m_i, \qquad \text{for } i = 1,\dots,p-1, \tag{5}$$

and by convention $\beta_0 = \gamma_p = 0$.

Proposition 6. *Let K be a compact subset of*

$$\left\{(x,m)\in\mathbb{R}^p\times\mathbb{R}^{p-1} \;\middle|\; 0<m_i<\inf(x_i,x_{i+1}), 1\le i\le p-1\right\},$$

that is, of the domain of definition of coherent values of the x_i and m_i. Then, uniformly for $(x,m)\in K\cap\left((2n)^{-1/2}\mathbb{Z}\right)^{2p-1}$,

$$\mathbb{P}\left(Y^{(n)}(\tau)=(x,m)\right) \;=\; (2n)^{-2\delta}\cdot\zeta(x,m)\cdot\left(1+\mathcal{O}_K(n^{-1/2})\right),$$

where $\delta = (2p-1)/4$, and $\zeta(x,m)$ reads, in terms of the β_i and γ_i as given by relations (5):

$$\zeta(x,m) \;=\; 2^{2p}\,(2\pi)^{-p/2}\prod_{i=0}^{p}\frac{(\beta_i+\gamma_i)e^{-\frac{(\beta_i+\gamma_i)^2}{2(\tau_{i+1}-\tau_i)}}}{(\tau_{i+1}-\tau_i)^{3/2}}.$$

The notation $\mathcal{O}_K$ is used to stress the fact that the error term is uniform for fixed K.

Proof. Let, for all n and a non negative integers,

$$\begin{aligned} C(n;a) &= \frac{a+1}{n+1}\binom{n+1}{(n-a)/2} \\ &= \frac{2^{n+1}}{\sqrt{2\pi n}}\,\frac{(a+1)}{\sqrt{n}}\,e^{-\frac{a^2}{2n}}\left(1+O(\tfrac{a}{n})\right), \end{aligned} \tag{6}$$

where the error term is uniform for $a = O(n^{1/2})$.

The Catalan numbers $C(2n;0)$ are well known to give the cardinality of $\mathcal{E}_{2n}$. More generally, the reflection principle proves that $C(n;a)$ is the number of Bernoulli paths with increments ±1 that have length n, starting at 0, staying nonnegative and ending at height a. Such Bernoulli paths are often called *left factors of Dyck walks* (ending at level a). Given non negative numbers b and c, $C(n;b+c)$ is also the number of walks with increments ±1 that have length n, minimum $-b$ and

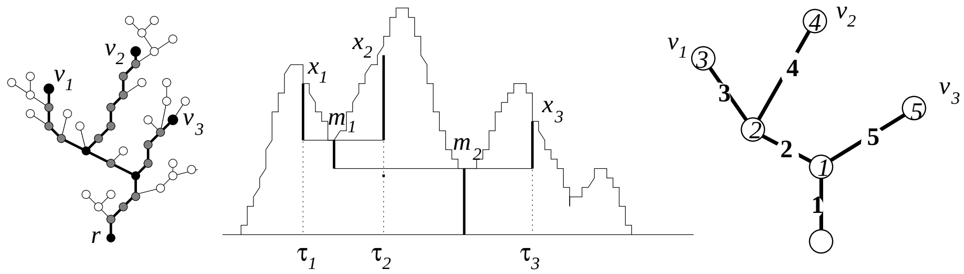


FIGURE 20. A random tree U_n, the decomposition of its normalised contour $e^{(n)}$ at times (τ_1, τ_2, τ_3) and at minima in between, and the extracted shape (prefix order shown on edges and vertices). In particular: $\mathcal{L}(\eta_4) = 7 = (x_2 - m_1)\sqrt{2n}$.

final height $c - b$, as follows from a decomposition at first and last passage at the minimum.

We thus have

$$\mathbb{P}\left(Y^{(n)}(\tau) = (x, m)\right) = \frac{\prod_{i=0}^{p} C\left(\lfloor 2n\tau_{i+1}\rfloor - \lfloor 2n\tau_i \rfloor\ ;\ (2n)^{1/2}\beta_i + (2n)^{1/2}\gamma_i\right)}{C(2n; 0)}.$$

Combined with (6), it yields Proposition 6. □

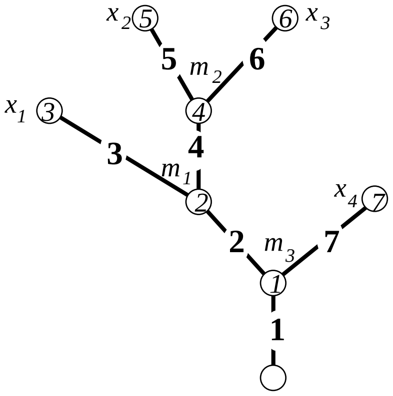
The expression

$$\zeta(x, m) = 2^{2p}(2\pi)^{-p/2} \prod_{i=0}^{p} \frac{(\beta_i + \gamma_i) e^{-\frac{(\beta_i+\gamma_i)^2}{2(\tau_{i+1}-\tau_i)}}}{(\tau_{i+1} - \tau_i)^{3/2}} \mathbf{1}_{\beta_i \geq 0} \mathbf{1}_{\gamma_i \geq 0} \mathbf{1}_{m_i \geq 0} \tag{7}$$

where the β_i and γ_i are given by relations (5), is the expected limit density of probability for the x_i and m_i: in particular it is coherent with the density of evaluations of the normalized Brownian excursion at p points, and of the $p - 1$ minima between them, as given in [32].

6.1.2. *Shapes.* Let us now define a *shape* to be a rooted plane tree T with q edges, that we call *superedges* to distinguish them from edges of embedded trees. We shall endow each superedge with a length: let $\eta_1, \ldots, \eta_q$ denote the edges of T in *prefix order* (i.e. the order induced by first visits in contour traversal) and $\mathcal{L}(\eta_i)$ the length of η_i.

Given the p normalised times $0 < \tau_1 < \cdots < \tau_p < 1$ and U an embedded tree of $\bar{\mathcal{U}}_n$, let us extract a shape T and, for each superedge η of T, a length $\mathcal{L}(\eta)$. In order to do this, let $t_i = \lfloor 2n\tau_i \rfloor$ and consider v_{t_i} the vertex visited at time t_i by the contour traversal. The p *fixed vertices* v_{t_i} and the root r span a minimal subtree of U (the union of the branches from v_{t_i} to r, see Figure 20, left hand side). Apart from the v_{t_i}, this subtree contains vertices of two types: *branchpoints* that have at least two sons in the subtree (black vertices in Figure 20), and *smooth vertices* that have exactly one son in the subtree (grey vertices). Let us define the shape T by taking the fixed vertices, the root and the branchpoints as vertex set, and the paths connecting these vertices as superedges (Figure 20, right hand side). A superedge η of T is thus by definition a set of edges of U and we let $\mathcal{L}(\eta) = |\eta|$, which is just



	η_1	η_2	η_3	η_4	η_5	η_6	η_7
m_3	1	0	0	0	0	0	0
m_1	1	1	0	0	0	0	0
x_1	1	1	1	0	0	0	0
m_2	1	1	0	1	0	0	0
x_2	1	1	0	1	1	0	0
x_3	1	1	0	1	0	1	0
x_4	1	0	0	0	0	0	1

FIGURE 21. A shape T with prefix ordering of superedges and vertices, and the matrix M_T mapping superedge lengths onto (x, m).

the length of the path η in the tree U (for instance, superedge η_4 in Figure 20 is made of 7 edges).

Observe now that the shape and superedge lengths extracted from a random tree U_n completely determine $Y^{(n)}(\tau)$. To check this assertion, let us assume that the extraction yields a shape T and superedges η_i of normalised length $\ell(\eta_i)$ with $\mathcal{L}(\eta_i) = \ell(\eta_i)\sqrt{2n}$, $i = 1, \ldots, q$. Let A_i be the set of superedges on the unique path from v_{t_i} to the root, so that $\sum_{\eta \in A_i} \ell(\eta) = e^{(n)}(\tau_i) = x_i$. The contour traversal of the shape T starts from the root $v_{t_0} = v_{t_{p+1}}$ and reaches successively each leaf v_{t_1}, $\ldots$, v_{t_p}. From v_{t_i} to $v_{t_{i+1}}$ a set B_i of superedges are first traversed toward the root, followed by a set C_i of superedges that are traversed away from the root (so that $B_i = A_i \setminus A_{i+1}$ while $C_i = A_{i+1} \setminus A_i$). Then $\sum_{\eta \in B_i} \ell(\eta) = \beta_i$ and $\sum_{\eta \in C_i} \ell(\eta) = \gamma_i$ are the lengths of these journeys, with β_i and γ_i given by Relations (5). In particular the normalised lengths $\ell(\eta)$ of superedges are all of the form

$$(x_i - m_i), \qquad (x_{i+1} - m_i) \qquad \text{or} \qquad |m_j - m_i| \text{ with } j < i.$$

In the latter case, m_j has to be a record, that is for $j < k < i$, $m_k > \sup(m_i, m_j)$. These relations are exemplified by Figure 20.

Conversely, as explained in [23, Ch. 3] or [32, Section 2], the x_i and m_i (or the β_i and γ_i as defined by Relations (5)) exactly determine the shape and superedge lengths of U_n.

From the limit density $\zeta(x, m)$ of the previous section, the probability that $m_i = m_j$ for some $i \neq j$ is seen to tend to zero as n goes to infinity. This implies that, with probability tending to one as n goes to infinity, the shape is a binary tree and $q = 2p-1$ (indeed the existence of a branchpoint of larger degree corresponds to the equality of two minima m_i and m_j). From now on we thus restrict our attention to binary shapes T.

Associated to any binary shape T, there is a matrix M_T with size $2p-1 \times 2p-1$ and entries zero or one, that sends the $2p-1$ normalized lengths of superedges $(\ell(\eta_i))$ on the (x_i, m_i)s. Moreover, provided the lengths of superedges on the one hand, and the x_i and m_i on the other hand, are sorted according to the prefix order, the matrix M_T is lower triangular with ones on the diagonal. As a consequence, M_T is *Lebesgue measure preserving.* Indeed the kth vertex in the prefix order is reached by the kth edge and all other edges on the path to the root have already been visited.

Finally let us consider labels of U_n: the variation of label along edges extends to superedges, upon setting for any superedge η

$$\kappa(\eta) = \sum_{\epsilon \in \eta} \kappa(\epsilon).$$

Let $\tilde{M}_T$ be the double action of M_T on $\mathbb{R}^{4p-2}$. Since the matrix M_T also sends the normalized increments $(k_i)_{1\leq i\leq 2p-1}$ of labels along the $2p-1$ superedges onto the normalized labels (w_i, μ_i) of the $2p-1$ vertices of the tree, $\tilde{M}_T$ describes the one-to-one correspondence between shapes equipped with superedges' lengths and label variations on the one hand, and $Z^{(n)}(\tau)$ on the other hand. More precisely, $\tilde{M}_T$ describes the restriction of this correspondence to shape T.

6.1.3. *A local limit theorem for $Z^{(n)}(\tau)$ in a fixed region.* As already observed, the limit law of $Y^{(n)}(\tau)$ charges only the region

$$R = \left\{(x, m, w, \mu) \in (\mathbb{R}^p \times \mathbb{R}^{p-1})^2 \;\mid 0 < m_i < \inf(x_i, x_{i+1}), 1 \leq i \leq p-1\right\}.$$

For a given permutation σ on $p-1$ symbols, define

$$R_\sigma = \left\{\, (x, m, w, \mu) \in R \mid\; 0 < m_{\sigma(1)} < m_{\sigma(2)} < \cdots < m_{\sigma(p-1)} \right\}.$$

Observe that the shape is constant on R_σ and denote it T_σ. (Conversely a given shape may appear in many different regions R_σ, as the number of different shapes is the number of binary trees with $2p-1$ vertices, the Catalan number $C(2p-2;0)$ again.) Let

$$R_{\sigma,\varepsilon} = \left\{\, (x, m, w, \mu) \in R_\sigma \mid\; d\left((x, m, w, \mu), \partial R_\sigma\right) \geq \varepsilon \right\}.$$

(The distance is the usual distance in $\mathbb{R}^{4p-2}$ and ∂R_σ denote the boundary of R_σ, which is clearly a finite union of closed $(4p-3)$-dimensional cones.)

Let $\Delta(n, K)$ denote the set of possible values of $Z^{(n)}(\tau)$ that belong to a compact subset K of $\mathbb{R}^{4p-2}$, that is

$$\Delta(n, K) \;=\; K \cap \left(\left((2n)^{-1/2}\mathbb{Z}\right)^{2p-1} \times \left((8n/9)^{-1/4}\,\mathbb{Z}\right)^{2p-1} \right).$$

Furthermore, let ξ (resp. f) be defined, on $\mathbb{R}^{2p-1} \times \mathbb{R}^{2p-1}$ (resp. $(\mathbb{R}^p \times \mathbb{R}^{p-1})^2$), by

$$\begin{aligned}
\xi(\ell, k) \;&=\; (2\pi)^{-2\delta} \prod_{i=1}^{2p-1} \ell_i^{-1/2}\, \exp\left(-\frac{k_i^2}{2\,\ell_i}\right), \qquad (8)\\
\xi_T \;&=\; \xi \circ \tilde{M}_T^{-1},\\
f(x, m, w, \mu) \;&=\; \zeta(x, m) \sum_{\sigma} \xi_{T_\sigma}(x, m, w, \mu) \;\cdot\; \mathbf{1}_{R_\sigma}(x, m).
\end{aligned}$$

The function f is exactly the density of $Z(\tau)$, that is, the density of the evaluation of X at the p points τ_i and at the $p-1$ minima of e between them. This density was described in [32, Propositions 2 & 3]. The function ξ_{T_σ} is the conditional density of the labels (w, μ) given (x, m). We shall prove the following local limit law for $Z^{(n)}(\tau)$.

Lemma 4. *Let K be a compact subset of $R_{\sigma,\varepsilon}$, $\epsilon > 0$. Then, uniformly for $(x, m, w, \mu) \in \Delta(n, K)$,*

$$\begin{aligned} &\mathbb{P}\left(Z^{(n)}(\tau) = (x, m, w, \mu)\right) \\ &= (2n)^{-2\delta}(8n/9)^{-\delta} \quad f(x, m, w, \mu) \quad \left(1 + \mathcal{O}_K\left(n^{-1/2}\right)\right), \end{aligned}$$

Corollary 5. *Let K denote a compact subset of $R_{\sigma,\varepsilon}$. For any uniformly continuous function Φ with support K,*

$$\lim_n \mathbb{E}\left[\Phi\left(Z^{(n)}(\tau)\right)\right] = \mathbb{E}[\Phi(Z(\tau))].$$

Proof of Lemma 4. Our aim is to compute for $(x, m, w, \mu) \in \Delta(n, K)$ the probability

$$\begin{aligned} &\mathbb{P}\left(Z^{(n)}(\tau) = (x, m, w, \mu)\right) \\ &= \mathbb{P}\left(Y^{(n)}(\tau) = (x, m)\right) \cdot \mathbb{P}\left(Z^{(n)}(\tau) = (x, m, w, \mu) \mid Y^{(n)} = (x, m)\right). \end{aligned}$$

From Proposition 6, we already have

$$\mathbb{P}\left(Y^{(n)}(\tau) = (x, m)\right) = (2n)^{-2\delta} \quad \zeta(x, m) \quad \left(1 + O\left(n^{-1/2}\right)\right).$$

As discussed in the previous section, the normalised lengths $\ell_i = (2n)^{-1/2}\mathcal{L}_i$ of superedges, obtained from (x, m) through M_T, are of the form

$$(x_i - m_i), \qquad (x_{i+1} - m_i) \qquad \text{or} \qquad |m_j - m_i| \ \text{ with } \ j < i.$$

In particular the fact that $(x, m, \cdot, \cdot) \in K \subset R_{\sigma,\varepsilon}$ for some $\epsilon > 0$ grants that these normalized lengths are uniformly bounded away from 0. In turn the variation $\kappa^{(n)}(\eta)$ of labels along any superedge η is the sum of at least $\mathcal{O}_\varepsilon(\sqrt{n})$ i.i.d. uniform random variables on $\{1, 0, -1\}$. Therefore we can apply uniform bounds for the local limit theorem [27, pages 189–197]: if S_n denotes the sum of n i.i.d. random variables uniform on $\{+1, 0, -1\}$, we have

$$\mathbb{P}(S_n = \kappa) = \sqrt{\frac{3}{4\pi n}}\, e^{-3\kappa^2/4n} + O\left(\frac{1}{n}\right).$$

This allows us to calculate the probability that the variations of labels along superedges $\kappa^{(n)}(\eta_i)$ are equal to $\kappa_i = k_i\,(8n/9)^{1/4}$, $i = 1, \ldots, 2p-1$: uniformly for $(x, m, w, \mu) \in \Delta(n, K)$,

$$\begin{aligned} &\mathbb{P}\left(\kappa^{(n)}(\eta_i) = \kappa_i,\ 1 \le i \le 2p-1 \,\middle|\, Y^{(n)}(\tau) = (x, m)\right) \\ &= \prod_{i=1}^{2p-1}\left(\sqrt{\frac{3}{4\pi\,\mathcal{L}_i}}\, e^{-3\kappa_i^2/4\,\mathcal{L}_i} + O\left(\frac{1}{\varepsilon\sqrt{n}}\right)\right), \\ &= \left(\frac{9}{8n}\right)^{\delta} \xi(\ell, k) \quad \left(1 + \mathcal{O}_K\left(n^{-1/2}\right)\right), \end{aligned}$$

in which the sequence of normalised variations of labels along superedges, $k = (k_i)_{1 \le i \le 2p-1}$ is the inverse image of (w, μ) through M_T. In other terms,

$$(\ell, k) = \tilde{M}_T^{-1}(x, m, w, \mu),$$

and the previous relation can be written

$$\begin{aligned}&\mathbb{P}\left(Z^{(n)}(\tau)=(x,m,w,\mu)\mid Y^{(n)}=(x,m)\right)\\&\quad=\quad(8n/9)^{-\delta}\ \xi_{T_\sigma}(x,m,w,\mu)\left(1+\mathcal{O}_K\left(n^{-1/2}\right)\right),\end{aligned}$$

leading to the desired result, through Proposition 6. □

Proof of Corollary 5. In view of the measure preserving property of $\tilde{M}_T$, we have

$$\mathbb{E}(\Phi(Z(\tau)))\ =\int\Phi\circ\tilde{M}_T(\ell,k)\ f\circ\tilde{M}_T(\ell,k)\,d\ell\,dk\,.$$

Therefore, the lemma follows upon proving that

$$\lim_n\left|\mathbb{E}\left(\Phi(Z^{(n)}(\tau))\right)-\int\Phi\circ\tilde{M}_T(\ell,k)\ f\circ\tilde{M}_T(\ell,k)\,d\ell\,dk\right|=0.$$

Set

$$\phi_T=\Phi\circ\tilde{M}_T,\qquad\qquad f_T=f\circ\tilde{M}_T.$$

The function ϕ_T has a compact support $\tilde{K}=\tilde{M}_T^{-1}K$ that is included in $\tilde{M}_T^{-1}K_{\sigma,\epsilon}\subset(\epsilon,\infty)^{2p-1}\times\mathbb{R}^{2p-1}$. Since $\tilde{K}$ is compact, there exists an $\epsilon'>0$ and a compact K' such that $\tilde{K}\subset K'\subset(\epsilon,\infty)^{2p-1}\times\mathbb{R}^{2p-1}$ and $d(\partial\tilde{K},\partial K')>\epsilon'$. We shall use the fat boundary $K'\setminus\tilde{K}$, on which ϕ_T is identically zero, to deal with boundary effects.

Let finally $\tilde{\Delta}(n,K)=\tilde{M}_T^{-1}\Delta(n,K)$ be the discretized version of $\tilde{K}$ and similarly $\Delta'(n,K)=\tilde{M}_T^{-1}\Delta(n,M_TK')$ that of K'; by construction $\tilde{\Delta}(n,K)\subset\Delta'(n,K)$ are finite sets with $\mathcal{O}_K(n^{3\delta})$ elements, and ϕ_T is identically zero over $\Delta'(n,K)\backslash\tilde{\Delta}(n,K)$.

First, we have

$$\begin{aligned}\mathbb{E}\left(\Phi(Z^{(n)}(\tau))\right)&=\sum_{(\ell,k)\in\Delta'(n,K)}\phi_T(\ell,k)\,\mathbb{P}\left((\ell^{(n)},k^{(n)})=(\ell,k)\right)\\&=(2n)^{-2\delta}(8n/9)^{-\delta}\sum_{(\ell,k)\in\Delta'(n,K)}\phi_T(\ell,k)\,f_T(\ell,k)\\&\qquad+\ \|\Phi\|_\infty\cdot\mathcal{O}_K(n^{-1/2}),\end{aligned}$$

the second equality due to the local limit convergence (Lemma 4). Next, difference between the discrete summation and the integral is bounded in terms of the modulus of continuity $\omega(\phi_T\cdot f_T,K',\cdot)$ of $\phi_T\cdot f_T$ on $\tilde{K}$ (recall that the modulus of continuity of a function g on a compact K is $\omega(g,K,\epsilon)=\sup_{0<d(x,y)<\epsilon}|g(x)-g(y)|$, and that if g is uniformly continuous on K, it satisfies $\omega(g,K,\epsilon)=o(1)$ as ϵ tends to zero). This yields

$$\begin{aligned}&\left|(2n)^{-2\delta}(8n/9)^{-\delta}\sum_{(\ell,k)\in\Delta'(n,K)}\phi_T(\ell,k)\,f_T(\ell,k)-\int\phi_T(\ell,k)\,f_T(\ell,k)\,d\ell\,dk\right|\\&\qquad\le\ \mathrm{measure}(K')\ \cdot\ \omega(\phi_T\cdot f_T,K',n^{-1/4}).\end{aligned}$$

Observe that in this summation the compact K' has been approximated by a union of boxes of diameter $O(n^{-1/4})$ and boundary effect should be considered. However ϕ_T is identically zero on a region $K'\setminus K$ with $d(\delta K,\delta K')>\varepsilon'$, that contains all boxes intersecting the boundary for n large enough. The boundary effect is thus null. □

6.1.4. *Weak convergence of* $Z^{(n)}(\tau)$. According to the Porte-Manteau Theorem [9, Ch. 1], we need

$$\lim_n \mathbb{E}\ \left[\Phi\left(Z^{(n)}(\tau)\right)\right] = \int \Phi\ f = \mathbb{E}[\Phi\,(Z(\tau))]$$

to hold for any bounded uniformly continuous Φ. Now consider

$$K_m \ = \ \overline{B}(0,\rho_m)\bigcap\left(\bigcup_\sigma R_{\sigma,\varepsilon_m}\times\mathbb{R}^{2p-1}\right),$$

in which $\overline{B}(0,\rho_m)$ is the closed ball, in $\mathbb{R}^{4p-2}$, with radius ρ_m and let simultaneously ρ_m increase to $+\infty$ and ε_m decrease to 0. We obtain a increasing sequence of compacts K_m, each of these compacts having $(p-1)!$ connected components. As the limit of this sequence has a Lebesgue–negligible complement in $\mathbb{R}^{4p-2}$, we can choose the sequences $(\rho_m)_{m>1}$ and $(\varepsilon_m)_{m>1}$ in such a way that

$$\mathbb{P}\,(Z(\tau)\in K_m)\geq 1-\frac{1}{m}.$$

There exist uniformly continuous functions $\Psi_m:\ \ \mathbb{R}^{4p-2}\longrightarrow[0,1]$, such that

$$\Psi_m|_{K_m}\equiv 1, \qquad \Psi_m|_{K^c_{m+1}}\equiv 0.$$

By construction,

$$\left|\mathbb{E}\Big(\Phi(Z(\tau))\Big)\ -\ \mathbb{E}\Big(\Psi_m\cdot\Phi\,(Z(\tau))\Big)\right| \ \leq\ \frac{\|\Phi\|_\infty}{m}.,$$

Moreover the product $\Psi_m\cdot\Phi$ is now a finite sum of functions satisfying the assumptions of Corollary 5: this yields

$$\lim_n\ \mathbb{E}\left(\Psi_m\cdot\Phi\,(Z^{(n)}(\tau))\right)\ \ =\ \ \mathbb{E}\ \left(\Psi_m\cdot\Phi\,(Z(\tau))\right).$$

Next observe that, by definition of Ψ_m,

$$\mathbb{P}\left(Z^{(n)}(\tau)\in K_m\right)\ \geq\ \mathbb{E}\left(\Psi_{m-1}(Z^{(n)}(\tau))\right),$$

and

$$\mathbb{E}\big(\Psi_{m-1}(Z(\tau))\big)\ \geq\ \mathbb{P}\big(Z(\tau)\in K_{m-1}\big).$$

Therefore, applying Corollary 5 to Ψ_{m-1},

$$\liminf_n\ \mathbb{P}\left(Z^{(n)}(\tau)\in K_m\right)\ \ \geq\ \ \mathbb{P}\,(Z(\tau)\in K_{m-1})\ \geq\ 1-\frac{1}{m-1}.$$

Moreover

$$\begin{aligned}\mathbb{E}\left((1-\Psi_m)\cdot\Phi\,(Z^{(n)}(\tau))\right)\ &\leq\ \mathbb{P}\left(\Psi_m(Z^{(n)}(\tau))<1\right)\cdot\|\Phi\|_\infty,\\ &\leq\ \mathbb{P}\left(Z^{(n)}(\tau)\notin K_m\right)\cdot\|\Phi\|_\infty.\end{aligned}$$

Thus, taking limit for n going to infinity and applying the previous lower bound,

$$\limsup_n \mathbb{E}\left((1-\Psi_m)\cdot\Phi\,(Z^{(n)}(\tau))\right)\ \ \leq\ \ \frac{\|\Phi\|_\infty}{m-1}.$$

Finally, the decomposition

$$\mathbb{E}\left(\Phi(Z^{(n)}(\tau))\right)\ \ =\ \ \mathbb{E}\left(\Psi_m\cdot\Phi\,(Z^{(n)}(\tau))\right)\ +\ \mathbb{E}\left((1-\Psi_m)\cdot\Phi\,(Z^{(n)}(\tau))\right),$$

yields

$$\limsup_n \left| \mathbb{E}\left(\Phi(Z^{(n)}(\tau)) \right) - \mathbb{E}\big(\Phi(Z(\tau))\big) \right| \le \frac{2\|\Phi\|_\infty}{m-1}.$$

Letting m go to infinity gives the weak convergence of $Z^{(n)}(\tau)$ to $Z(\tau)$ as claimed. The convergence of $X^{(n)}(\tau)$ is a by-product.

6.2. **A deviation bound for the largest label.** In this section, a rough but exponential deviation bound for the value of the largest label in a forest of k embedded trees with n edges is obtained. For $k = 1$, Proposition 4 is exactly obtained.

Let $\mathcal{EV}_{k,2n}$ denote the set of k-uples of element of $\mathcal{EV}$ of total length $2n$: an element $[(E_1, V_1), \ldots, (E_k, V_k)]$ of $\mathcal{EV}_{k,2n}$ codes for a forest of k embedded trees (each (E_i, V_i) codes for a tree, according to Proposition 3). Equivalently, one may concatenate the k pairs and view any element of $\mathcal{EV}_{k,2n}$ as a pair $(E, V) = (E_1 \cdots E_k, V_1 \cdots V_k) \in \mathcal{EV}_{2n}$ together with a set of concatenation times $0 = t_0 \le t_1 \le \cdots \le t_k = 2n$, subject to the conditions $E(t_i) = V(t_i) = 0$ for all $i = 1, \ldots, k$. In this identification, E_i has length $2n_i = t_i - t_{i-1}$.

Let $(E^{k,n}, V^{k,n})$ denote a random forest of $\mathcal{EV}_{k,2n}$ under the uniform distribution. Let us insist that the uniform distribution is indeed meant here with equality between the t_is allowed. This is more convenient in particular for Section 6.3.

Proposition 7. *There exists $y_0 > 0$ such that, for all n, k and $y \ge y_0$,*

$$\mathbb{P}\left(\sup_{0 \le t \le 2n} V^{k,n}(t) \; > \; y n^{1/4} \right) \; < \; e^{-y}. \tag{9}$$

This is the key fact in the proof of tightness of the sequence $X^{(n)}$, given in the last subsection, and it also leads to the convergence of moments in Corollary 3 through the following weaker formulation (with $k = 1$): for $y > y_0$, and for all n,

$$\mathbb{P}\left(\sup_{0 \le s \le 1} \hat{W}_s^{(n)} \; > \; (8/9)^{1/4} y \right) \; \le e^{-y}, \tag{10}$$

which is exactly Proposition 4.

The proof is based on a *branch decomposition*, that is discussed in the next paragraph. Then, after two preliminary results on parameters of the middle branch of a random tree (Paragraphs 6.2.2 and 6.2.3), Proposition 7 is proved by induction (Paragraphs 6.2.4, 6.2.5 and 6.2.6). At the price of more technical details in these latter paragraphs, the bound could be improved to $e^{-c_\epsilon y^{4/3-\epsilon}}$ for any fixed $\epsilon > 0$.

6.2.1. *The branch decomposition at time t.* Let $(E, V) \in \mathcal{EV}_{k,2n}$ with concatenation times $0 \le t_1 \le \ldots \le t_k \le 2n$ and let $t \in (0, 2n)$. Suppose moreover that $t_{p-1} < t < t_p$, that is t occurs during the contour traversal of (E_p, V_p), the pth component of the forest encoded by (E, V) and let U be the tree encoded by (E_p, V_p).

To any vertex v of U is associated the set of edges in the unique simple path from v to the root of U, denoted $\mathrm{br}(v)$ (for the *branch* of v). Recall that v_t denotes the vertex visited at time t of the contour traversal of U. Observe that the height $E(t)$ is $|\mathrm{br}(v_t)|$, the length of the branch from the root r to v_t, while the label $V(t)$ of v_t is given by

$$V(t) = \sum_{\epsilon \in \mathrm{br}(v_t)} \kappa(\epsilon). \tag{11}$$

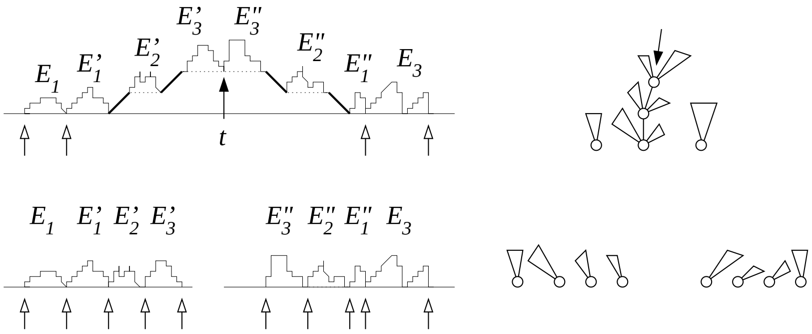


FIGURE 22. Decomposition of a forest at time t. Arrows under the walks indicate concatenation times.

Let $\ell = E(t) = |\mathrm{br}(v_t)|$, and call ϵ_i the edge of $\mathrm{br}(v_t)$ between heights $i-1$ and i, for $i = 1, \ldots, \ell$. The maximal label on the branch $\mathrm{br}(v_t)$ is

$$\bar{H}(t) \;=\; \sup_{0 \le j \le \ell} \sum_{i=1}^{j} \kappa(\varepsilon_i).$$

The branch of v_t induces a decomposition of U into two forests of trees, the *branch decomposition*, which we now phrase in terms of (E, V).

- From time t_{p-1} to t, the edges $\epsilon_1, \ldots, \epsilon_\ell$ are successively traversed away from the root on the branch $\mathrm{br}(v_t)$, at times $t'_1 < \cdots < t'_\ell < t$. Let (E'_i, V'_i) be the part of the contour walks (E, V) between t'_i and t'_{i+1} (with the convention that $t'_0 = t_{p-1}$ and $t'_{\ell+1} = t$).
- From time t to t_p, the edges $\epsilon_\ell, \ldots, \epsilon_1$ are successively traversed back toward the root on the branch $\mathrm{br}(v_t)$, at times $t''_{\ell+1} < \cdots < t''_1$. Let (E''_i, V''_i) be the part of the contour walks (E, V) between t''_{i+1} and t''_i (with the convention that $t''_{\ell+1} = t$ and $t''_0 = t_p$).

The contour walks (E'_i, V'_i) (resp. (E''_i, V''_i)) encodes for the left (resp. right) subtree attached at the ith vertex of the branch $\mathrm{br}(v_t)$. One can see t'_i as the time of the last upcrossing of heights $(i-1, i)$ before time t, and t''_i as the time of the first downcrossing of heights $(i, i-1)$ after time t.

Upon shifting the walks E'_i and E''_i down by i so that they start from zero, and also shifting the walks V'_i and V''_i by $V(t'_i)$ (resp. $V(t''_{i+1})$): the two forests

$$\begin{aligned}
\mathrm{left}(t) &= [(E_1, V_1), \ldots, (E_{p-1}, V_{p-1}), (E'_0, V'_0), \ldots, (E'_\ell, V'_\ell)] \\
\mathrm{right}(t) &= [(E''_\ell, V''_\ell), \ldots, (E''_0, V''_0), (E_{p+1}, V_{p+1}), \ldots, (E_k, V_k)]
\end{aligned}$$

belong respectively to $\mathcal{EV}_{k',n'}$ and $\mathcal{EV}_{k'',n''}$ with $k' = p + \ell$, $n' = t - \ell$, $k'' = k - p + 1 + \ell$, and $n'' = 2n - t - \ell$.

Let us apply this branch decomposition to a random forest $(E^{k,n}, V^{k,n})$. In view of expression (11), conditionally on $E^{(k,n)}(t) = \ell$,

$$V^{(k,n)}(t) \overset{\text{law}}{=} S_\ell,$$

where S_k denotes the sum of k i.i.d. random variables uniform on $\{-1,0,1\}$. Moreover, again conditionally on $E^{(k,n)}(t)=\ell$,

$$\mathrm{left}^{(k,n)}(t) \overset{\text{law}}{=} (E^{k',n'},V^{k',n'}), \qquad \mathrm{right}^{(k,n)}(t) \overset{\text{law}}{=} (E^{k'',n''},V^{k'',n''}).$$

6.2.2. *The middle branch length in a random forest.* The first step of the proof is a bound for the tail probability of $E^{k,n}(n)$, the length of the middle branch ($t=n$).

Lemma 5. *For $4\ell^2 > 27\cdot n$,*

$$\Pr(E^{k,n}(n)>\ell) \quad < \quad A\cdot\frac{\ell}{n^{1/2}}\exp\left(-\frac{4\ell^2}{27n}\right).$$

Proof. With the notations of Subsection 6.2.1, for $1\le p\le k$, let $t_p^{k,n}$ denote the pth concatenation time of $E^{k,n}$. Then

$$\Pr\left(E^{k,n}(n)=\ell \,\middle|\, t_{p+1}^{k,n}-t_p^{k,n}=2m,\ 0\le n-t_p^{k,n}=a\le 2m\right)=\Pr\left(E^{(m)}(a)=\ell\right),$$

so the proof reduces to bound $\Pr\left(E^{(m)}(a)>\ell\right)$ uniformly on pairs (a,m) such that $0\le a\le 2m\le 2n$. We have

$$\begin{aligned}\Pr\left(E^{(m)}(a)=\ell\right) &= \frac{C(a;\ell)\,C(2m-a;\ell)}{C(2m;0)}\\ &= \frac{C(\alpha;\ell)\,C(\beta;\ell)}{C(2m;0)},\end{aligned}$$

with an obvious change of variables. From [11, Ch. I.3, Th 2], for $k>n/2+h$ and $1\le h\le n/6$

$$\sqrt{\frac{\pi n}{2}}\binom{n}{k} \;<\; 2^n\exp\left(-\frac{2h^2}{n}+\frac{2h}{n}+\frac{4h^3}{n^2}\right).$$

With $n=\alpha+1$, $k=(\alpha+\ell)/2+1$, $1\le h=\ell/6\le\alpha/6\le n/6$, this bound yields:

$$\begin{aligned}C(\alpha;\ell) &= \frac{\ell+1}{\alpha+1}\binom{\alpha+1}{(\alpha+\ell)/2+1}\\ &< \sqrt{\frac{8}{\pi}}\cdot\frac{\ell}{n^{3/2}}\,2^n\exp\left(-\frac{\ell^2}{18n}+\frac{\ell}{3n}+\frac{\ell^3}{54n^2}\right),\\ &< e^{1/3}\sqrt{\frac{8}{\pi}}\cdot\frac{\ell}{n^{3/2}}\,2^n\exp\left(-\frac{\ell^2}{27n}\right)\\ &< 2e^{10/27}\sqrt{\frac{8}{\pi}}\cdot\frac{\ell}{\alpha^{3/2}}\,2^\alpha\exp\left(-\frac{\ell^2}{27\alpha}\right),\end{aligned}$$

Thus,

$$\begin{aligned}\Pr\left(E^{(m)}(a)=\ell\right) &< B_1\cdot\frac{\ell^2m^{3/2}}{(\alpha\beta)^{3/2}}\exp\left(-\frac{\ell^2}{27\alpha}-\frac{\ell^2}{27\beta}\right),\\ &< B_2\cdot\frac{\ell^2}{m^{3/2}}\exp\left(-\frac{4\ell^2}{27m}\right),\end{aligned}$$

the latter inequality since the maximum of the function $(xy)^{-3/2}\exp\left(-x^{-1}-y^{-1}\right)$, subject to $x+y=27m/\ell^2<1$, is obtained for $x=y$. The last inequality entails that, for $4\ell^2 > 27m$,

$$\begin{aligned}\Pr\left(E^{(m)}(a) > \ell\right) &< B_3 \cdot \frac{\ell}{m^{1/2}} \exp\left(-\frac{4\ell^2}{27m}\right) \\ &< B_3 \cdot \frac{\ell}{n^{1/2}} \exp\left(-\frac{4\ell^2}{27n}\right).\end{aligned}$$

□

6.2.3. *The largest label on the middle branch.*

Lemma 6. *Let $\bar{H}^{k,n}$ be the largest label on the branch between the root and the vertex reached at $t = n$ by the contour traversal of $(E^{k,n}, V^{k,n})$. Then there exists c_0 such that for all k, n and h,*

$$\mathbb{P}(\bar{H}^{k,n} > h) \le c_0\ e^{-\frac{1}{9}(hn^{-1/4})^{4/3}}. \tag{12}$$

Proof. As already discussed the conditional probability that the largest label is h knowing that the branch has length ℓ is exactly the probability that a random walk with steps $\{1, 0, -1\}$ of length ℓ has maximal value h. Using the reflection principle, Azuma's inequality [2, Th. 2.1, p. 85] then reads

$$\mathbb{P}(\bar{H}^{k,n} > h \mid E^{k,n}(n) = \ell) \quad \le \quad 2\ e^{-h^2/2\ell}.$$

Next, as previously calculated, for $4\ell^2 \ge 27n$,

$$\mathbb{P}(E^{k,n}(n) > \ell) \ < \ A \cdot (\ell/n^{1/2})\ e^{-4\ell^2/27n}$$

Finally, assuming $h > (27/4)^{3/4} n^{1/4}$, so that the previous inequality holds,

$$\begin{aligned}\mathbb{P}(\bar{H}^{k,n} > h) \quad &\le \quad \mathbb{P}(\bar{H}^{k,n} > h \mid E^{k,n}(n) \le h^{2/3}n^{1/3}) \ + \ \mathbb{P}(E^{k,n}(n) > h^{2/3}n^{1/3}) \\ &\le \quad 2e^{-h^{4/3}/2n^{1/3}} \ + \ A \cdot (h^{2/3}n^{-1/6})\, e^{-4h^{4/3}/27n^{1/3}}.\end{aligned}$$

The lemma follows for $h > (27/4)^{3/4} n^{1/4}$, taking c_0 large enough, and also for $h \le (27/4)^{3/4} n^{1/4}$, taking $c_0 \ge e^{3/4}$. □

6.2.4. *Conditional induction in the case $E(n) > 0$.*

Lemma 7. *Assume the bound (9) holds true for some y_0 for $V^{k,m}$ with $m < n$. Then, for all $\ell > 0$ and $k \ge p \ge 0$, the probability*

$$\begin{aligned}&p_{k,n}(y; h, \ell, p) \ = \\ &\qquad \mathbb{P}\Big(\sup_{0 \le t \le n} V^{k,n}(t) \ > \ yn^{1/4} \ \Big|\ \bar{H}^{k,n} = h,\, E^{k,n}(n) = \ell,\, t^{k,n}_{p-1} < n < t^{k,n}_{p}\Big),\end{aligned}$$

satisfies

$$p_{k,n}(y; h, \ell, p) \le 4e^{-2^{1/4}(y - hn^{-1/4})}, \qquad \textit{provided } h \le n^{1/4}(y - y_0/2^{1/4}). \tag{13}$$

Proof. Assume $(E^{k,n}, V^{k,n}) = (E_1 \cdots E_k, V_1 \cdots V_k)$ is such that $E^{k,n}(n) = \ell > 0$ and $t^{k,n}_{p-1} < n < t^{k,n}_p$, so that $t = n$ occurs inside (E_p, V_p). Apply the branch decomposition (see Section 6.2.1) at $t = n$ to $(E^{k,n}, V^{k,n})$ and let

$$\begin{aligned}(\bar{E}^{k,n}, \bar{V}^{k,n}) \quad &= \quad \mathrm{left}^{k,n}(n) \ = \ [(E_1, V_1), \ldots, (E_{p-1}, V_{p-1}), (E'_0, V'_0), \ldots, (E'_\ell, V'_\ell)], \\ (\bar{\bar{E}}^{k,n}, \bar{\bar{V}}^{k,n}) \quad &= \quad \mathrm{right}^{k,n}(n) \ = \ [(E''_\ell, V''_\ell), \ldots, (E''_0, V''_0), (E_{p+1}, V_{p+1}), \ldots, (E_k, V_k)].\end{aligned}$$

Upon taking $k' = p + \ell$, $k'' = k - p + \ell + 1$ and $n' = (n - \ell)/2$,

$$(\bar{E}^{k,n}, \bar{V}^{k,n}) \overset{\text{law}}{=} (E^{k',n'}, V^{k',n'}) \quad \text{and} \quad (\bar{\bar{E}}^{k,n}, \bar{\bar{V}}^{k,n}) \overset{\text{law}}{=} (E^{k'',n'}, V^{k'',n'})$$

Observe now that, in the previous decomposition,

$$\sup_{0\le t\le 2n} V(t) \;\le\; \sup\Big(\bar{H} + \sup_{0\le t\le 2n'} \bar{V}(t)\,,\; \bar{H} + \sup_{0\le t\le 2n'} \bar{\bar{V}}(t)\Big),$$

so that

$$\begin{aligned} &p_{k,n}(y;h,\ell,p) \le \\ &\qquad \mathbb{P}\Big(\sup_{0\le t\le 2n'} \bar{V}^{k,n}(t) > yn^{1/4} - h \;\Big|\; \bar{H}^{k,n} = h,\, E^{k,n}(n) = \ell,\, t^{k,n}_{p-1} < n < t^{k,n}_p\Big) \\ &\quad + \;\mathbb{P}\Big(\sup_{0\le t\le 2n'} \bar{\bar{V}}^{k,n}(t) > yn^{1/4} - h \;\Big|\; \bar{H}^{k,n} = h,\, E^{k,n}(n) = \ell,\, t^{k,n}_{p-1} < n < t^{k,n}_p\Big). \end{aligned}$$

Hence, in view of the preceding identities in law,

$$\begin{aligned} p_{k,n}(y;h,\ell,p) \;&\le\; \mathbb{P}\Big(\sup_{0\le t\le 2n'} V^{k',n'}(t) > yn^{1/4} - h\Big) \\ &\quad + \mathbb{P}\Big(\sup_{0\le t\le 2n'} V^{k'',n'}(t) > yn^{1/4} - h\Big). \end{aligned}$$

Observe that $2n' \le n$. Hence

$$\begin{aligned} p_{k,n}(y;h,\ell,p) \;&\le\; \mathbb{P}\Big(\sup_{0\le t\le 2n'} V^{k',n'}(t) > 2^{1/4}(y - hn^{-1/4})n'^{1/4}\Big) \\ &\quad + \mathbb{P}\Big(\sup_{0\le t\le 2n'} V^{k'',n'}(t) > 2^{1/4}(y - hn^{-1/4})n'^{1/4}\Big). \end{aligned}$$

The induction hypothesis now implies, for $2^{1/4}(y - hn^{-1/4}) \ge y_0$, that is, for all $h \le n^{1/4}(y - y_0/2^{1/4})$,

$$p_{k,n}(y;h,\ell,p) \le 2e^{-2^{1/4}(y-hn^{-1/4})},$$

which is exactly the lemma, up to a factor 2 added for later convenience. □

6.2.5. *Conditional induction in the case $E(n) = 0$.*

Lemma 8. *Assume the bound (9) holds true for some y_0 for $V^{k,m}$ with $m < n$. Then, provided $y \ge y_0/2^{1/4}$,*

$$\mathbb{P}\Big(\sup_{0\le t\le 2n} V^{k,n}(t) \;>\; yn^{1/4} \;\Big|\; E^{k,n}(n) = 0\Big) \;\le\; 4e^{-2^{1/4}y}.$$

The bound (13) thus remain valid in the case $\ell = 0$.

Proof. In this case the decomposition at $t = n$ is even simpler. Let $p = \sup\{i \mid t_i < n\}$, and $q = \inf\{i \mid t_i > n\}$, and consider a decomposition in four parts, cutting at times t_p, n and t_q. The two contour walks for $0 \le t \le t_p$ and $t_q \le t \le 2n$ are uniform on $\mathcal{EV}_{p,t_p}$ and $\mathcal{EV}_{k-q,2n-t_q}$. The other two contour walks are uniform on $\mathcal{EV}_{n-t_p}$ and $\mathcal{EV}_{t_q-n}$.

The result then follows using the induction hypothesis on the 4 parts. □

6.2.6. *Complete induction and proof of Proposition 7.*

Observe that the bounds of Lemmas 7 and 8 do not depend on ℓ or p, so that, assuming (9) holds for some y_0 for $V^{k,m}$ with $m<n$,

$$\mathbb{P}\Big(\sup_{0\leq t\leq 2n} V^{k,n}(t) \;>\; yn^{1/4} \;\Big|\; \bar{H}^{k,n}=h\Big) \;\leq\; 4e^{-2^{1/4}(y-hn^{-1/4})},$$

provided $h\leq n^{1/4}(y-y_0/2^{1/4})=h_0$. Using this bound,

$$\begin{aligned} f_{k,n}(y) &= \mathbb{P}\Big(\sup_{0\leq t\leq 2n} V^{k,n}(t) \;>\; yn^{1/4}\Big) \\ &\leq 4e^{-2^{1/4}y}\sum_{h=0}^{h_0} e^{2^{1/4}hn^{-1/4}}\mathbb{P}(\bar{H}^{k,n}=h) \;+\; \mathbb{P}\left(\bar{H}^{k,n}>h_0\right), \\ &\leq 4e^{-2^{1/4}y}\left(1+\sum_{h=1}^{\infty}(e^{2^{1/4}hn^{-1/4}}-e^{2^{1/4}(h-1)n^{-1/4}})\mathbb{P}(\bar{H}^{k,n}\geq h)\right) \\ &\quad +\mathbb{P}(\bar{H}^{k,n}>n^{1/4}(y-y_0/2^{1/4})). \end{aligned}$$

In view of Lemma 6, the summation is bounded by a convergent integral that evaluates to a constant c_1. Lemma 6 allows also to dispose of the second term:

$$f_{k,n}(y) \;\leq\; 4e^{-2^{1/4}y}(1+c_1)+c_0e^{-\frac{1}{9}(y-y_0/2^{1/4})^{4/3}}$$

Now observe that

$$\begin{aligned} e^{-\frac{1}{9}(y-y_0/2^{1/4})^{4/3}} &= e^{-y}\; e^{y_0/2^{1/4}}\; e^{(y-y_0/2^{1/4})-\frac{1}{9}(y-y_0/2^{1/4})^{4/3}} \\ &\leq e^{-y}\; e^{y_0/2^{1/4}}\; e^{(y_0-y_0/2^{1/4})-\frac{1}{9}(y_0-y_0/2^{1/4})^{4/3}} \\ &= e^{-y}\; e^{y_0-\frac{1}{9}(1-1/2^{1/4})^{4/3}y_0^{4/3}} \end{aligned}$$

for $y\geq y_0$, as soon as $x\to x-\frac{1}{9}x^{4/3}$ is decreasing on the interval $\left[y_0-y_0/2^{1/4},+\infty\right[$, that is, for $y_0\geq 1933$. The bound can thus be rewritten as

$$f_{k,n}(y) \;\leq\; e^{-y}\left(4e^{-(2^{1/4}-1)y_0}(1+c_1)+c_0e^{y_0-\frac{1}{9}(1-1/2^{1/4})^{4/3}y_0^{4/3}}\right)$$

which is smaller than e^{-y} for y_0 large enough, so that induction can be carried on (Recall that c_0 and c_1 do not depend on n). The case $n=1$ holds true for $y_0\geq 1$, and the proof of Proposition 7 is complete.

6.3. **Tightness.** Tightness for the bidimensional path follows from the tightness of the two projections. The tightness of the first projection was proved by Kaigh [22]. Thus we only have to prove the following proposition (cf. [9, Ch. 3]).

Proposition 8. *For all $\varepsilon>0$ and $\delta>0$ there exists m such that for n large enough*

$$\mathbb{P}\left(\sup_{0\leq s\leq 1}\left|\hat{W}_s^{(n)}-\hat{W}_{\lfloor ms\rfloor/m}^{(n)}\right| \;\geq\; \delta\right)\leq\varepsilon.$$

We proceed by bounding, for m large enough and all i with $1\leq i\leq m$,

$$\begin{aligned} p_{m,i}(n) &= \mathbb{P}\left(\sup_{(i-1)/m\leq s\leq i/m}\left|\hat{W}_s^{(n)}-\hat{W}_{(i-1)/m}^{(n)}\right| \;\geq\; \delta\right) \\ &= \mathbb{P}\left(\sup_{t_1\leq t\leq t_2}\left|V^{(n)}(t)-V^{(n)}(t_1)\right| \;\geq\; \delta' n^{1/4}\right), \end{aligned}$$

where $t_1=\lfloor 2(i-1)n/m\rfloor$, $t_2=\lfloor 2in/m\rfloor$ and $\delta'=(8/9)^{1/4}\delta$.

The proposition is an immediate consequence of the following lemma, upon taking m large enough and summing on i.

Lemma 9. *For all $\varepsilon > 0$ and $\delta > 0$ there exist m_0 such that for all $m > m_0$ and $1 \le i \le m$,*

$$\exists n_0,\ \forall n > n_0, \qquad p_{m,i}(n) \ \le\ \frac{\varepsilon}{2m} + C_\delta(m),$$

where $C_\delta(m)$ is exponentially decreasing in a positive power of m.

Proof. For simplicity of notations let us assume $t_2 - t_1 = 2n/m$. (The general case is identical but obscured by a collection of $\lfloor n/m \rfloor$.)

Let us consider the shape T associated with the times t_1 and t_2, as defined in Section 6.1.2. The branches $\mathrm{br}(v_{t_1})$ from v_{t_1} to the root and $\mathrm{br}(v_{t_2})$ from v_{t_2} to the root meet at the unique branchpoint v of the shape T. Let $B = \mathrm{br}(v_{t_1}) \setminus \mathrm{br}(v_{t_2})$ be the branch between v_{t_1} and v and $C = \mathrm{br}(v_{t_2}) \setminus \mathrm{br}(v_{t_1})$ the branch between v_{t_2} and v. Between t_1 and t_2, a total of k edges ϵ_i, $i = 1, \ldots, k$ of U are successively traversed on the branch from v_{t_1} to v_{t_2} by the contour walk. The k' first are along the branch B and traversed toward the root, while the k'' next are along the branch C and traversed away from the root. By construction,

$$\begin{aligned} k' &= E^{(n)}(t_1) - \inf_{[t_1,t_2]} E^{(n)} = \sqrt{2n}\left[e^{(n)}\left(\frac{i-1}{m}\right) - \inf_{[\frac{i-1}{m},\frac{i}{m}]} e^{(n)}\right], \\ k'' &= E^{(n)}(t_2) - \inf_{[t_1,t_2]} E^{(n)} = \sqrt{2n}\left[e^{(n)}\left(\frac{i}{m}\right) - \inf_{[\frac{i-1}{m},\frac{i}{m}]} e^{(n)}\right], \end{aligned}$$

and the total length $\Delta_i^{(n)} = k = k' + k''$ of the branch from v_{t_1} to v_{t_2} is

$$\Delta_i^{(n)} \ =\ E^{(n)}(t_1) + E^{(n)}(t_2) - 2 \inf_{[t_1,t_2]} E^{(n)} \ =\ \sqrt{2n}\,\Delta_{m,i}\, e^{(n)},$$

with the notation

$$\Delta_{m,i} f = \left| f\left(\frac{i-1}{m}\right) + f\left(\frac{i}{m}\right) - 2 \inf_{[\frac{i-1}{m},\frac{i}{m}]} f \right|.$$

The walk $(E^{(n)}(t))_{t_1 \le t \le t_2}$ is decomposed along the branch from v_{t_1} to v_{t_2} into a sequence

$$E_0, \epsilon_1, E_1, \epsilon_2, \ldots, E_{k-1}, \epsilon_k, E_k,$$

where the $\Delta_i^{(n)} = k$ passages on the branch separate $\Delta_i^{(n)}$ subtrees (each coded by a E_i) of total size (number of edges) $N_i^{(n)} = (2n/m - \Delta_i^{(n)})/2$.

In this decomposition, conditionally given $\Delta_i^{(n)} = k$ (and $N_i^{(n)} = n' = n/m - k/2$), the forest satisfies

$$[(E_0 \cdots E_k, V_0 \cdots V_k)] \overset{\text{law}}{=} (E^{k+1,n'}, V^{k+1,n'}),$$

upon resetting all walks to start at zero. Under the same conditions, the variation of labels on edges of B and C are i.i.d. random variables $\zeta(\epsilon)$, uniform on $\{-1, 0, +1\}$: the random variable

$$\bar{H}_i^{(n)} \ =\ \sup_{0 \le \ell \le \Delta_i^{(n)}} \sum_{j=0}^{\ell} \zeta(\epsilon_j)$$

has tail distribution bounded by Azuma's inequality.

Our strategy is to bound $\Delta_i^{(n)}$ and, conditionally given $\Delta_i^{(n)}$, to bound separately the variations on the forest and on the branch:

- First fix $\alpha > 0$ and let a_1 denote the tail probability for $\Delta_i^{(n)}$:

$$a_1 \;=\; \mathbb{P}\left(\Delta_i^{(n)} \geq m^{-\alpha}\sqrt{2n}\right) \;=\; \mathbb{P}\left(\Delta_{m,i}e^{(n)} \geq m^{-\alpha}\right).$$

- On the complementary event for $\Delta_i^{(n)}$ we consider the variation in a forest,

$$\begin{aligned} a_2 \;&=\; \mathbb{P}\left(\sup_{0\leq t\leq 2N_i^{(n)}} \left|V^{\Delta_i^{(n)}+1,N_i^{(n)}}(t)\right| \;\geq\; \frac{\delta'}{4}\,n^{1/4} \;\text{ and }\; \Delta_{m,i}e^{(n)} \leq m^{-\alpha}\right), \\ &\leq\; 2\mathbb{P}\left(\sup_{0\leq t\leq 2N_i^{(n)}} V^{\Delta_i^{(n)}+1,N_i^{(n)}}(t) \;\geq\; \frac{\delta'}{4}\,n^{1/4} \;\text{ and }\; \Delta_{m,i}e^{(n)} \leq m^{-\alpha}\right), \end{aligned}$$

- and the variations on the branch from t_1 to t_2,

$$\begin{aligned} a_3 \;&=\; \mathbb{P}\left(\sup_{0\leq \ell\leq \Delta_i^{(n)}} \left|\sum_{j=0}^{\ell}\zeta(\epsilon_j)\right| \;\geq\; \frac{\delta'}{4}\,n^{1/4} \;\text{ and }\; \Delta_i^{(n)} \leq m^{-\alpha}n^{1/4}\right), \\ &\leq\; 2\mathbb{P}\left(\bar{H}_i^{(n)} \;\geq\; \frac{\delta'}{4}\,n^{1/4} \;\text{ and }\; \Delta_i^{(n)} \leq m^{-\alpha}n^{1/4}\right). \end{aligned}$$

In view of the decomposition we have

$$p_{m,i}(n) \;\leq\; a_1 + a_2 + a_3.$$

We now bound separately each term a_1, a_2, and a_3. Let us start with a_2 and write

$$a_2 \;=\; \sum_{k=0}^{m^{-\alpha}\sqrt{2n}} \mathbb{P}\left(\sup_{0\leq t\leq 2n'} V^{k+1,n'}(t) \;\geq\; \frac{\delta'}{4}\,n^{1/4}\right)\mathbb{P}\left(\Delta_i^{(n)} = k\right),$$

where $n' = n/m - k/2$. In order to apply the tail estimate of the previous subsection (Proposition 7), we need

$$y \;:=\; \frac{\delta'}{4}(n/n')^{1/4} \;\geq\; y_0.$$

Since $n' \leq n/m$ this condition is satisfied as soon as

$$\frac{\delta'}{4}\,m^{1/4} \;\geq y_0, \qquad \text{that is} \qquad m \geq (4y_0/\delta')^4.$$

Then,

$$\mathbb{P}\left(\sup_{0\leq t\leq 2n'} V^{k+1,n'}(t) \;\geq\; \frac{\delta'}{4}\,n^{1/4}\right) \;\leq\; \exp\left(-\delta' m^{1/4}/4\right).$$

The latter bound being independent of k we obtain

$$a_2 \;\leq\; 2\exp\left(-\delta' m^{1/4}/4\right).$$

Let us now turn to a_3. Conditionally given that $\Delta_i^{(n)} = k$, the maximum on the branch $\bar{H}_i^{(n)}$ is distributed as the maximum of a random walk with k steps that are independent increments uniform on $\{-1,0,1\}$. Thus, applying Azuma's inequality and the reflection principle, we have, for $k \leq m^{-\alpha}\sqrt{2n}$,

$$\mathbb{P}\left(\bar{H}_i^{(n)} \geq \frac{\delta'}{4} n^{1/4} \;\middle|\; \Delta_i^{(n)} = k\right) \leq 2 \exp\left(-\frac{\delta'^2}{32\sqrt{2}} m^{\alpha}\right).$$

Again the latter bound is independent of k so that

$$a_3 \leq \exp\left(-\frac{\delta'^2}{32\sqrt{2}} m^{\alpha}\right).$$

Finally let us deal with a_1. For n large enough, the convergence of $e^{(n)}$ to the normalised excursion e entails

$$\left|a_1 - \mathbb{P}\left(\Delta_{m,i} e \geq m^{-\alpha}\right)\right| \leq \frac{\varepsilon}{2m}.$$

Let us thus consider, for $1 \leq i \leq m$, the probability $\pi_{m,i} = \mathbb{P}\left(\Delta_{m,i}\, e \geq m^{-\alpha}\right)$, and restrict the choice of α to $0 < \alpha < 1/2$. The finite dimensional distribution of the value of e at two points and the minimum between them was considered in Section 6.1 (take $p = 2$ in the continuum limit of Proposition 5 or see [32]). For $i \neq 1, m$ this entails,

$$\pi_{m,i} = \frac{m^{3/2}}{\pi(\tau_1\tau_2')^{3/2}} \int_{\substack{x>0,y>0,\\ \beta>m^{-\alpha},\ y+x>\beta>|x-y|}} xy\beta\ e^{-y^2/2\tau_1\ -x^2/2\tau_2'} e^{-m\beta^2/2} dx dy d\beta,$$

in which

$$\tau_1 = \frac{i-1}{m}, \qquad \tau_2 = \frac{i}{m}, \qquad \text{and} \quad \tau_2' = 1 - \tau_2.$$

Thus

$$\begin{aligned}
\pi_{m,i} &\leq \frac{m^{3/2}}{\pi(\tau_1\tau_2')^{3/2}} \int_{\substack{x>0,y>0,\\ \beta>m^{-\alpha}}} xy\beta\ e^{-y^2/2\tau_1\ -x^2/2\tau_2'} e^{-m\beta^2/2} dx dy d\beta \\
&\leq \frac{m^{1/2}}{\pi(\tau_1\tau_2')^{1/2}}\ e^{-m^{1-2\alpha}/2} \\
&\leq \frac{m^{3/2}}{\pi}\ e^{-m^{1-2\alpha}/2}.
\end{aligned}$$

For the remaining two cases $i = 1$ and $i = m$, we have with $\tau_1 = \frac{m-1}{m}$,

$$\pi_{m,1} = \pi_{m,m} = \sqrt{\frac{2m^3}{\pi\tau_1^3}} \int y^2 e^{-m\,y^2/2} e^{-y^2/2\tau_1} 1_{y>m^{-\alpha}} dy,$$

Now $x \to x e^{-x^2/a}$ is bounded by $\sqrt{\frac{a}{2e}}$, so that

$$\begin{aligned}
\pi_{m,1} &\leq \sqrt{\frac{2m}{\pi\tau_1^2 e}} \int u e^{-u^2/2}\ 1_{u>m^{0.5-\alpha}} du, \\
&\leq m^{1/2}\ e^{-m^{1-2\alpha}/2}.
\end{aligned}$$

Thus for all i, $\pi_{m,i}$ is bounded by an exponentially decreasing function of m.

The proof of the lemma is then concluded by summing the contribution of a_1, a_2 and a_3: for all m there exists n_0 such that for all $n > n_0$,

$$\sum_{i=1}^{m} p_{m,i}(n) \leq \frac{\varepsilon}{2} + C_\delta(m),$$

in which $C_\delta(m)$ is exponentially small in a power of m. □

Taking m and then n large enough the tightness is proved, and together with Lemma 5 this concludes the proof of Theorem 5.

6.4. **Convergence of the profile.** In view of the Skorohod representation theorem [30, II.86.1], we may assume the joint existence, on some probabilistic triple $(\Omega, A, \mathbb{P})$, of a sequence of copies of $X^{(n)}$, and of a copy of X (and we keep the same notation $X^{(n)}$ and X, as for the original), such that, for almost any $\omega \in \Omega$, $\left(X_t^{(n)}(\omega)\right)_{0\leq t\leq 1}$ converges to $(X_t(\omega))_{0\leq t\leq 1}$ in the Skorohod topology of $D([0,1], \mathbb{R}^2)$. In this section we build copies of F_n and F, such that, almost surely, F_n converges to F.

First, the set

$$\Omega_1 = \{\omega \mid t \longrightarrow X_t(\omega) \text{ is continuous}\}$$

has probability 1, so that uniform convergence of $\left(X_t^{(n)}(\omega)\right)_{0\leq t\leq 1}$ to $(X_t(\omega))_{0\leq t\leq 1}$ holds almost surely. Set

$$\begin{aligned}
W_{\min}^{(n)} &= \inf_{0\leq s\leq 1} W_s^{(n)}(\omega), \\
\delta_n(\omega) &= \sup_{0\leq s\leq 1} \left| W_s^{(n)}(\omega) - \hat{W}_s(\omega) \right|, \\
\Psi_n(x,\omega) &= \int_0^1 \mathbf{1}_{\hat{W}_s^{(n)}(\omega)\leq x} \, ds, \\
\tilde{F}_n(x) &= \Psi_n\left(W_{\min}^{(n)} + x, \omega\right), \\
\Psi(x,\omega) &= \int_0^1 \mathbf{1}_{\hat{W}_s(\omega)\leq x} \, ds \\
&= \mathcal{J}\left((-\infty, x]\right).
\end{aligned}$$

It follows from general results on superprocessus that Φ, the distribution function of the random measure ISE, is almost surely continuous [34]. Now from

$$\Psi(x - \delta_n) \leq \Psi_n(x) \leq \Psi(x + \delta_n)$$

and the almost sure continuity of Ψ, it follows that the set

$$\Omega_3 = \{\omega \mid \forall x,\ \lim_n \Psi_n(x,\omega) = \Psi(x,\omega) \text{ and } x \to \Psi(x,\omega) \text{ is continuous}\}$$

has probability 1. In Ω_3, as we deal with increasing functions, *uniform* convergence of Ψ_n to Ψ holds true. Hence, on the set

$$\Omega_4 = \{\omega \in \Omega_3 \mid \quad \lim_n W_{\min}^{(n)}(\omega) = W_{\min}(\omega)\},$$

i.e. almost surely, uniform convergence of $\tilde{F}_n$ to F holds true.

On the other hand, we explain below why $\tilde{F}_n$ is close to some copy of F_n, that we shall denote F_n too. As in section 5.2, from $X^{(n)}$ one recovers a random uniform contour pair $(E^{(n)}, V^{(n)})$, and a random uniform embedded tree $U_n \in \bar{\mathcal{U}}_n$. Now we can choose at random a well labelled tree W_n in the conjugacy class of U_n in such a way that Theorem 3 holds for (W_n, U_n). Theorem 3 entails that $\widehat{\lambda}^{(n)}$ and $\widehat{\Lambda}^{(n)}$ have the same asymptotic behavior, and F_n is just $\widehat{\lambda}^{(n)}$ suitably normalised. So we have now to establish the relation between $\widehat{\Lambda}^{(n)}$ and $\tilde{F}_n(x)$. First, set

$$\widehat{f}_y^{(n)} = 2n\ \tilde{F}_n\left(y(8n/9)^{-1/4}, \omega\right).$$

As we have

$$m_n = (8n/9)^{1/4}\ W_{\min}^{(n)},$$

$\widehat{f}_y^{(n)}$ is the number of visits of the contour traversal of U_n to a node whose label is not larger than $m_n + y$, but the number of visits, by the contour traversal, of a given node, is exactly 1 plus the number of children of this node, so that

$$\widehat{\Lambda}_y^{(n)} + \widehat{\Lambda}_{y-1}^{(n)} - 1 \leq \widehat{f}_y^{(n)} \leq \widehat{\Lambda}_y^{(n)} + \widehat{\Lambda}_{y+1}^{(n)}.$$

Hence, due to Theorem 3,

$$2\widehat{\lambda}_{y-3}^{(n)} - 1 \leq \widehat{f}_y^{(n)} \leq 2\widehat{\lambda}_{y+3}^{(n)},$$

or, equivalently,

$$\frac{n}{n+1}\ \tilde{F}_n\left(x - cn^{-1/4}, \omega\right) \leq F_n\left(x, \omega\right) \leq \frac{n}{n+1}\ \tilde{F}_n\left(x + cn^{-1/4}, \omega\right) + \frac{1}{2n+2},$$

c being a constant. That is, on Ω_4, uniform convergence of F_n to F holds true.

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