Walks on the slit plane

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Abstract

In the first part of this paper, we enumerate exactly walks on the square lattice that start from the origin, but otherwise avoid the half-line $\mathcal{H} = \{(k,0), k \leq 0\}$. We call them *walks on the slit plane*. We count them by their length, and by the coordinates of their endpoint. The corresponding three variable generating function is algebraic of degree 8. Moreover, for any point (i,j), the length generating function for walks of this type ending at (i,j) is also algebraic, of degree 2 or 4, and involves the famous Catalan numbers.

Our method is based on the solution of a functional equation, established via a simple combinatorial argument. It actually works for more general models, in which walks take their steps in a finite subset of $\mathbb{Z}^2$ satisfying two simple conditions. The corresponding generating functions are always algebraic.

In the second part of the paper, we derive from our enumerative results a number of probabilistic corollaries. For instance, we can compute exactly the probability that an ordinary random walk starting from (i,j) hits for the first time the half-line $\mathcal{H}$ at position $(k,0)$, for any triple (i,j,k). This generalizes a question raised by R. Kenyon, which was the starting point of this paper.

Taking uniformly at random all n-step walks on the slit plane, we also compute the probability that they visit a given point $(k,0)$, and the average number of visits to this point. In other words, we quantify the transience of the walks. Finally, we derive an explicit limit law for the coordinates of their endpoint.

1 Introduction

In January 1999, Rick Kenyon posted on the "domino" mailing-list the following e-mail:

> "Take a simple random walk on $\mathbb{Z}^2$ starting on the y-axis at $(0,1)$, and stopping when you hit the nonpositive x-axis. Then the probability that you end at the origin is $1/2$.
> Since this result was obtained from a long calculation involving irrational numbers, I wonder if there is an easy proof? By way of comparison, if you start at $(1,0)$ the probability of stopping at the origin in $2-\sqrt{2}$."

This mail led Olivier Roques, a graduate student at LaBRI, to investigate the *number* of such walks of given length: he soon conjectured that exactly $4^n C_n$ walks of length $2n+1$ go from $(0,1)$ to $(0,0)$ without hitting the nonpositive x-axis before they reach their endpoint, where $C_n = \binom{2n}{n}/(n+1)$ is the nth Catalan number. This seems to confirm the statement that Catalan numbers "are probably the most frequently occurring combinatorial numbers after the binomial coefficients" [21]. Similarly, O. Roques conjectured that, if the starting point is chosen to be $(1,0)$, then the number of walks is even more remarkable, being C_{2n+1}. These conjectures directly imply Rick Kenyon's results.

In this paper, we prove O. Roques's conjectures as a special case of a more complete result. More precisely, having reversed the direction of the walks, we study the number $a_{i,j}(n)$ of walks of length n that

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start from $(0,0)$, end at (i,j), and never hit the horizontal half-axis $\mathcal{H} = \{(k,0) : k \leq 0\}$ once they have left their starting point: we call them *walks on the slit plane* (Fig. 1). We give a closed form expression for the *complete generating function*

$$S(x,y;t) = \sum_{n \geq 0} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} a_{i,j}(n) x^i y^j t^n,$$

which turns out to be algebraic of degree 8 (Theorem 1). The series $S(1,1;t)$, which counts walks by their length, has already been considered in the literature, and some asymptotic estimates for its coefficients have been obtained (see Lawler [18, Chap. 2]). However, to our knowledge, it was never realized that this walk model was exactly solvable. Note that a refinement/variation of an argument of Lawler is actually the starting point of another possible derivation of $S(x,y;t)$ (see [4]).

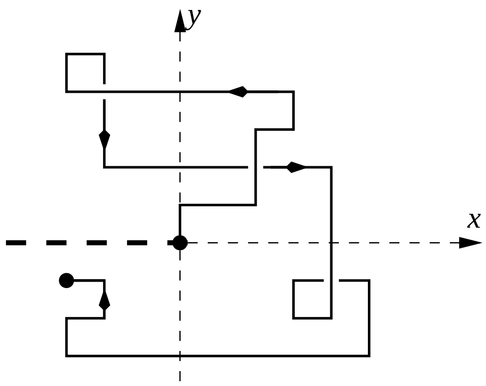

Figure 1: A walk on the slit plane.

Although O. Roques's conjectures beg for bijective proofs, our approach is far from bijective. It is based on a functional equation for $S(x,y;t)$, which is extremely simple to establish: its combinatorial counterpart is the deletion of the last step of the walk. The originality of our method lies in the solution of this equation, which might, at first, look a bit miraculous, and which we believe to be of independent interest. Indeed, after staring at the "miracle" for a while, we realized that the same approach would work just as well for a whole range of similar walk models where the walks take their steps in a finite subset of $\mathbb{Z}^2$ satisfying two simple conditions (the half-line $\mathcal{H}$ being still forbidden). These conditions are satisfied, for example, by walks with diagonal steps (Fig. 2). In all cases, the complete generating function for walks on the slit plane is algebraic and can be written explicitly. This is the central enumerative result of our paper (Section 2).

Starting from this result, we then embark on the enumeration of walks on the slit plane that end at a specified point (i,j). Again, for all the models that fit in our framework, these walks have an algebraic generating function (Section 3). In particular, for walks on the square lattice ending at $(0,1)$ or $(1,0)$, we prove the Catalan results conjectured by O. Roques. Remarkably, walks with *diagonal* steps also involve Catalan numbers. For instance, the number of walks ending at $(1,1)$ is again C_{2n+1}. Note that ordinary walks on the (unslit) square lattice ending at (i,j) do *not* have an algebraic generating function. Nor do walks that completely avoid the line $y=0$, and end at (i,j).

Section 4 is still devoted to exact enumeration, but does not really contain original techniques: mostly, we need to know, for the probabilistic applications we have in mind, the generating function for walks on the slit plane that start from a given point (k,ℓ), and this can be derived from the results previously obtained.

So much for counting. The rest of the paper uses our enumerative results to solve probabilistic questions, and focusses on the square lattice. In Section 5, we generalize R. Kenyon's problem by computing the probability that a random walk, starting from (i,j), meets for the first time the half-line $\mathcal{H}$ at abscissa $-k$. In other words, we compute exactly the *hitting distribution* of $\mathcal{H}$, starting from (i,j). Our results refine asymptotics results from [16] and [18].

In Section 6.1, walks on length n on the slit plane are taken uniformly at random, and we compute the probability p_k that they visit the point $(k, 0)$, in the limit $n \to \infty$. As can be expected, the forbidden half-line creates a long-range repulsion, and the walks are *transient*. That is, p_k is strictly less than 1. We also compute the average number of visits to the point $(k, 0)$, which is finite.

In Section 6.2, walks of length n are still taken uniformly at random, so that the position (X_n, Y_n) of their endpoint becomes a two-dimensional random variable. We prove that $(X_n/\sqrt{n}, Y_n/\sqrt{n})$ converges in distribution to an explicit law. In particular, the average abscissa $\mathbb{E}(X_n)$ grows like $c\sqrt{n}$, where $c = \Gamma(3/4)/\Gamma(1/4)$: this confirms the fact that walks are repelled away from the origin.

Notations. Given a ring $\mathbb{L}$ and k indeterminates $x_1, \ldots, x_k$, we denote by $\mathbb{L}[x_1, \ldots, x_k]$ the ring of polynomials in $x_1, \ldots, x_k$ with coefficients in $\mathbb{L}$, and by $\mathbb{L}[[x_1, \ldots, x_k]]$ the ring of formal power series in $x_1, \ldots, x_k$ with coefficients in $\mathbb{L}$; that is, formal sums

$$F(x_1, \ldots, x_k) = \sum_{n_1 \geq 0, \ldots, n_k \geq 0} a_{n_1, \ldots, n_k} x_1^{n_1} \cdots x_k^{n_k},$$

where $a_{n_1, \ldots, n_k}$ belongs to $\mathbb{L}$. If $\mathbb{L}$ is a field, we denote by $\mathbb{L}(x_1, \ldots, x_k)$ the field of rational functions in $x_1, \ldots, x_k$ with coefficients in $\mathbb{L}$. In this case, a power series $F(x_1, \ldots, x_k)$ is said to be algebraic (over the field $\mathbb{L}(x_1, \ldots, x_k)$) if there exists a non-trivial polynomial P in $k+1$ variables, with coefficients in $\mathbb{L}$, such that

$$P(x_1, \ldots, x_k, F(x_1, \ldots, x_k)) = 0.$$

A *Laurent polynomial* in the x_i is a polynomial in the x_i and the $\bar{x}_i = 1/x_i$. A *Laurent series* in the x_i is a series of the form $\sum_{n_1 \geq N_1, \ldots, n_k \geq N_k} a_{n_1, \ldots, n_k} x_1^{n_1} \cdots x_k^{n_k}$, where $N_1, \ldots, N_k \in \mathbb{Z}$.

2 The complete generating function

2.1 The ordinary square lattice

We consider walks on the square lattice made of four kinds of steps: north, east, south and west. Let $n \geq 0$, and $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. We denote by $a_{i,j}(n)$ the number of walks of length n that start from $(0, 0)$, end at (i, j), and never return to the horizontal half-axis $\mathcal{H} = \{(k, 0) : k \leq 0\}$ once they have left their starting point: we call them *walks on the slit plane*. Fig. 1 shows such a walk, with $(i, j) = (-3, -1)$ and $n = 46$. We denote by $a(n)$ the total number of walks of length n on the slit plane, regardless of their endpoint.

Let $S(x, y; t)$ be the *complete generating function* for walks on the slit plane, counted by their length and the position of their endpoint:

$$\begin{aligned} S(x, y; t) &= \sum_{n \geq 0} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} a_{i,j}(n) x^i y^j t^n, \qquad (1) \\ &= 1 + t(x + y + \bar{y}) + t^2(x^2 + 2xy + 2x\bar{y} + \bar{x}y + \bar{x}\bar{y} + y^2 + \bar{y}^2) \\ &+ (5x + x^3 + 4y + 4\bar{y} + y^3 + \bar{y}^3 + 3xy^2 + 3x\bar{y}^2 + 3x^2y + 3x^2\bar{y} + \bar{x}^2y + \bar{x}^2\bar{y} + 2\bar{x}y^2 + 2\bar{x}\bar{y}^2)t^3 + O(t^4) \end{aligned}$$

with the notations $\bar{x} = x^{-1}, \bar{y} = y^{-1}$. This is a formal power series in t with coefficients in $\mathbb{R}[x, \bar{x}, y, \bar{y}]$. We shall prove that this series is algebraic of degree 8 over the field of rational functions in x, y and t, and actually give an explicit expression for it.

Theorem 1 *The complete generating function for walks on the slit plane is*

$$S(x, y; t) = \frac{\left(1 - 2t(1 + \bar{x}) + \sqrt{1 - 4t}\right)^{1/2} \left(1 + 2t(1 - \bar{x}) + \sqrt{1 + 4t}\right)^{1/2}}{2(1 - t(x + \bar{x} + y + \bar{y}))}.$$

This series is algebraic of degree 8*. When* $x = y = 1$*, it specializes to*

$$S(1, 1; t) = \sum_{n \geq 0} a(n) t^n = \frac{(1 + \sqrt{1 + 4t})^{1/2}(1 + \sqrt{1 - 4t})^{1/2}}{2(1 - 4t)^{3/4}},$$

so that the asymptotic growth of the number of n-step walks on the slit plane is

$$a(n) \sim \frac{\sqrt{1+\sqrt{2}}}{2\Gamma(3/4)}\, 4^n n^{-1/4}.$$

In other words, the probability that a random walk on the square lattice, starting from $(0,0)$, has not met the half-line $\mathcal{H}$ after n steps is asymptotic to $cn^{-1/4}$ with $c = \sqrt{1+\sqrt{2}}/2/\Gamma(3/4)$. The decay in $n^{-1/4}$ was known [18, Eq. (2.35)], but the detailed asymptotic behaviour of this probability seems to be new.

We delay the proof of this theorem to state another result of the same type.

2.2 The diagonal square lattice

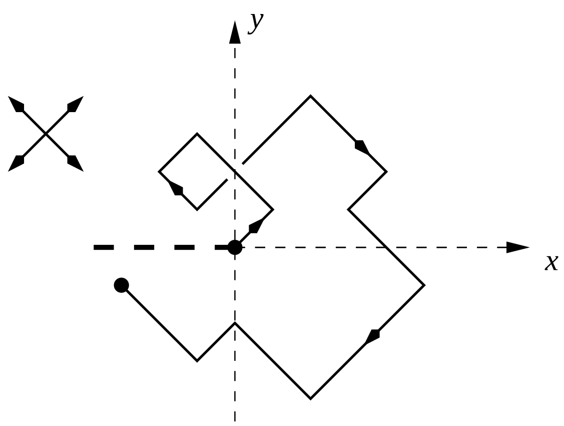

Figure 2: A walk on the slit plane with diagonal steps.

Let us consider the following variation on the previous model: the forbidden half-line $\mathcal{H}$ is unchanged, but walks now consist of *diagonal* steps $(1,1),(1,-1),(-1,1)$ and $(-1,-1)$ (Fig. 2). Let $a_{i,j}(n)$ be the number of walks of length n with diagonal steps that start from $(0,0)$, end at (i,j) (with $i+j$ even), and never return to $\mathcal{H}$. Let $S(x,y;t)$ denote the corresponding complete generating function, defined as in (1). We obtain for this series a result that is very similar to Theorem 1.

Theorem 2 *The complete generating function for walks on the slit plane with diagonal steps is*

$$S(x,y;t) = \frac{\left(1-8t^2(1+\bar{x}^2)+\sqrt{1-16t^2}\right)^{1/2}}{\sqrt{2}(1-t(x+\bar{x})(y+\bar{y}))}.$$

This series is algebraic of degree 4. When $x=y=1$, it specializes to

$$S(1,1;t) = \sum_{n\geq 0} a(n)t^n = \frac{(1+4t)^{1/4}(1+\sqrt{1-16t^2})^{1/2}}{\sqrt{2}(1-4t)^{3/4}},$$

so that the asymptotic growth of the number of walks of length n on the diagonal slit plane is

$$a(n) \sim \frac{1}{2^{1/4}\Gamma(3/4)}\, 4^n n^{-1/4}.$$

Theorems 1 and 2 look suspiciously like each other; we are going to show that they are two instances of a more general result that applies to walks with rather general steps.

2.3 A general result for walks on the slit plane

Let A be a finite subset of $\mathbb{Z}^2$. A walk with steps in A is a finite sequence $w = (w_0, w_1, \ldots, w_n)$ of vertices of $\mathbb{Z}^2$ such that $w_i - w_{i-1} \in A$ for $1 \leq i \leq n$. The number of steps, n, is the *length* of w. We say that w avoids the half-line $\mathcal{H} = \{(k,0), k \leq 0\}$ if none of the vertices $w_1, \ldots, w_n$ belong to $\mathcal{H}$. The starting point of the walk, w_0, is allowed to be on $\mathcal{H}$. For $(i,j) \in \mathbb{Z}$ and $n \geq 0$, let $a_{i,j}(n)$ denote the number of walks of length n, with steps in A, that start from $(0,0)$, end at (i,j), and avoid $\mathcal{H}$: we call them again *walks on the slit plane*. Let $S(x,y;t)$ be the associated complete generating function, given by (1). In what follows, we shall often omit the length variable t, writing, for instance, $S(x,y)$ instead of $S(x,y;t)$.

We obtain a functional equation for the series $S(x,y)$ by saying that a walk of length n is obtained by adding a step to another walk of length $n-1$. However, this procedure sometimes produces a *bridge*, that is, a nonempty walk that starts at $(0,0)$, ends on the half-line $\mathcal{H}$, but otherwise avoids $\mathcal{H}$. Hence, denoting by $B(\bar{x})$ the generating function for bridges, we have:

$$S(x,y) = 1 + tS(x,y)\left(\sum_{(i,j)\in A} x^i y^j\right) - B(\bar{x}),$$

that is,

$$K(x,y)S(x,y) = 1 - B(\bar{x}), \tag{2}$$

where

$$K(x,y) = 1 - t\sum_{(i,j)\in A} x^i y^j \tag{3}$$

is the *kernel* of Eq. (2). It turns out that Eq. (2) can be solved by elementary algebraic methods, provided the set A of allowed steps satisfies the two following conditions:

1. **Symmetry**: the set of steps is symmetric with respect to the x-axis; that is, if $(i,j) \in A$, then $(i,-j) \in A$.
2. **Small height variations**: for all $(i,j) \in A$, $|j| \leq 1$.

From now on, we shall restrict the study to sets A satisfying these conditions. Examples include the ordinary square lattice of Theorem 1, the diagonal square lattice of Theorem 2, and the (oriented) triangular lattice of Fig. 3, corresponding to $A = \{(-1,1),(-1,-1),(2,0)\}$. The latter model is equivalent, by the transformation $(i,j) \to ((i+j)/2, j)$, to the case $A = \{(0,1),(-1,-1),(1,0)\}$ recently studied by Ira Gessel via a completely different approach [14].

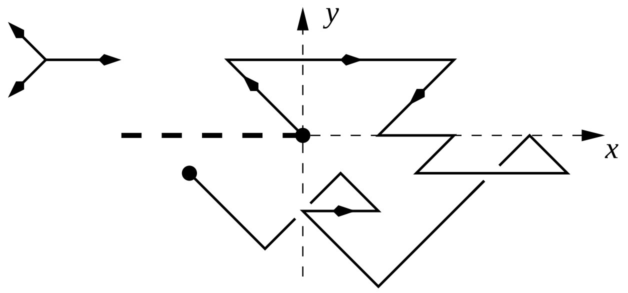

Figure 3: A walk on the slit plane with steps in $\{(-1,1),(-1,-1),(2,0)\}$.

We define two Laurent polynomials in x:

$$A_0(x) = \sum_{(i,0)\in A} x^i \quad \text{and} \quad A_1(x) = \sum_{(i,1)\in A} x^i. \tag{4}$$

We could define similarly a polynomial $A_{-1}(x)$, but the symmetry condition makes it identical to $A_1(x)$. The kernel, given by (3), can be rewritten

$$K(x,y) = 1 - tA_0(x) - t(y+\bar{y})A_1(x). \tag{5}$$

One of the main tools in the solution of this rather general model is the following lemma, which defines a canonical way of factoring polynomials in t with coefficients in $\mathbb{R}[x,\bar{x}]$.

Lemma 3 (The factorization lemma [13]) *Let $\Delta(x;t)$ be a polynomial in t with coefficients in $\mathbb{R}[x,\bar{x}]$, and assume $\Delta(x;0)=1$. There exists a unique triple $(\Delta_0(t), \Delta_+(x;t), \Delta_-(\bar{x};t)) \equiv (\Delta_0, \Delta_+(x), \Delta_-(\bar{x}))$ of formal power series in t satisfying the following conditions:*

- $\Delta(x) = \Delta_0\Delta_+(x)\Delta_-(\bar{x})$,
- *the coefficients of Δ_0 belong to $\mathbb{R}$,*
- *the coefficients of $\Delta_+(x)$ belong to $\mathbb{R}[x]$,*
- *the coefficients of $\Delta_-(\bar{x})$ belong to $\mathbb{R}[\bar{x}]$,*
- $\Delta_0(0) = \Delta_+(0;t) = \Delta_-(0;t) = \Delta_+(x;0) = \Delta_-(\bar{x};0) = 1$.

Moreover, these three series are algebraic, and $\Delta_+(x)$ (resp. $\Delta_-(\bar{x})$) is a polynomial in x (resp. $\bar{x}$).

This lemma will be proved in Section 2.4. We can now state the main theorem of this section, which gives a closed form expression for the complete generating function of walks on the slit plane.

Theorem 4 *Let A be a finite subset of $\mathbb{Z}^2$ satisfying the conditions of symmetry and small height variations. Let $\Delta(x)$ be the following polynomial in t, x and $\bar{x}$:*

$$\Delta(x) = (1 - tA_0(x) - 2tA_1(x))\,(1 - tA_0(x) + 2tA_1(x))\,. \tag{6}$$

Let $(\Delta_0, \Delta_+(x), \Delta_-(\bar{x}))$ be the factorisation of $\Delta(x)$, defined as in the lemma above. Then the complete generating function for walks on the slit plane with steps in A is

$$S(x,y;t) = \frac{\sqrt{\Delta_0\Delta_-(\bar{x})}}{K(x,y)}$$

where the kernel $K(x,y)$ is given by (5). The generating function for walks on the slit plane ending on the line $y=0$ is

$$S_0(x;t) := \sum_{n\geq 0}\sum_{i\in\mathbb{N}} a_{i,0}(n)x^i t^n = \frac{1}{\sqrt{\Delta_+(x)}}.$$

These series are algebraic.

As a by-product of this theorem, we are able to enumerate bridges. We shall use this result in Section 4.

Proposition 5 *The generating function for bridges, counted by their length (t) and abscissa of their endpoint (x), is:*

$$B(\bar{x};t) = 1 - \sqrt{\Delta_0\Delta_-(\bar{x})}.$$

In probabilistic terms, this proposition gives the joint distribution of the time T and position X where a random walk with steps in A, starting from $(0,0)$, returns for the first time to $\mathcal{H}$ (we assume that the walk takes each step of A with uniform probability $1/a$, with $a = |A|$):

$$\mathbb{E}\left(t^T x^X\right) := \sum_{n,k\geq 0} \Pr(T=n, X=-k)t^n\bar{x}^k = 1 - \sqrt{\Delta_0(t/a)\Delta_-(\bar{x};t/a)}.$$

2.4 Proofs

2.4.1 Another functional equation

Instead of working directly with Eq. (2), it is convenient to introduce the series

$$T(x,y;t) \equiv T(x,y) = \sum_{i,j,n} a_{i,j}(n) x^i y^{|j|} t^n,$$

which, due to the symmetry condition, contains as much information as $S(x,y)$, and has only nonnegative exponents in y. Again, one writes easily a functional equation for the series $T(x,y)$:

$$K(x,y)T(x,y) = 1 - B(\bar{x}) + tS_0(x)A_1(x)(y-\bar{y}), \tag{7}$$

the kernel $K(x,y)$ being the same as in Eq. (2). The role of the last term of this equation is to correct the power of y for walks whose last step starts on the line $y=0$ and ends at ordinate -1. Recall that $S_0(x;t)$ counts walks on the slit plane ending on the x-axis.

2.4.2 The kernel method

The next ingredient of the proof is the so-called kernel method. The principle is to cancel the kernel $K(x,y)$ by an appropriate choice of y, so as to obtain certain relations between the series that occur on the right-hand side of the equation. See [8, 17] for early uses of this method, and [2, 5, 9] for more recent developments.

The kernel $K(x,y)$ of our functional equation is given by (5). As a Laurent polynomial in y, it has two roots. One of them is a formal power series in t:

$$Y(x) = \frac{1 - tA_0(x) - \sqrt{\Delta(x)}}{2tA_1(x)} = A_1(x)t + A_0(x)A_1(x)t^2 + O(t^3),$$

where $\Delta(x)$ is the polynomial defined in Theorem 4. The coefficient of t^n in this series lies *a priori* in $\mathbb{Q}(x)$, but the equation

$$Y(x) = tA_0(x)Y(x) + tA_1(x)\left(1 + Y(x)^2\right)$$

implies that it is actually a Laurent polynomial in x. By the symmetry assumption, $K(x,y) = K(x,\bar{y})$, and the other root of the kernel is $1/Y(x)$.

Observe that $T(x, Y(x))$ is a well-defined series belonging to $\mathbb{Z}[x,\ \bar{x}][[t]]$: this comes from the fact that $T(x,y)$ has only nonnegative exponents in the variable y. Let us replace y by $Y(x)$ in Eq. (7): the kernel vanishes, leaving:

$$\begin{aligned} 1 - B(\bar{x}) &= tS_0(x)A_1(x)\left(\frac{1}{Y(x)} - Y(x)\right) \\ &= S_0(x)\sqrt{\Delta(x)}. \end{aligned} \tag{8}$$

Here, the facts that y only appears as $(y-\bar{y})$ in the right-hand side of (7), and that the kernel $K(x,y)$ is symmetric in y and $\bar{y}$, play a crucial role. These two properties parallel (and actually inspired) the two properties imposed on A.

2.4.3 Separation of the positive and negative parts in x

Let us assume, for the moment, the existence of the three series $\Delta_0, \Delta_+(x)$ and $\Delta_-(\bar{x})$. We replace $\Delta(x)$ by $\Delta_0\Delta_+(x)\Delta_-(\bar{x})$ in Eq. (8), and divide the resulting equation by $\sqrt{\Delta_-(\bar{x})}$. This gives:

$$\frac{1}{\sqrt{\Delta_-(\bar{x})}}\left(1 - B(\bar{x})\right) = S_0(x)\sqrt{\Delta_0\Delta_+(x)}.$$

The two sides of this equation are formal power series in t whose coefficients are Laurent polynomials in x. But we observe that the right-hand side only contains nonnegative exponents of x, while the left-hand side only contains nonpositive exponents of x: hence both sides must be independent of x, and equal to the value of the right-hand side, at, say, $x=0$. By definition of S_0 and Δ_+, this value is $\sqrt{\Delta_0}$. Hence

$$S_0(x) = \frac{1}{\sqrt{\Delta_+(x)}} \qquad \text{and} \qquad 1 - B(\bar{x}) = \sqrt{\Delta_0 \Delta_-(\bar{x})}.$$

The latter identity is exactly Proposition 5. Going back to Eq. (2), which defines the complete generating function $S(x,y)$, one finally obtains:

$$S(x,y) = \frac{\sqrt{\Delta_0 \Delta_-(\bar{x})}}{K(x,y)}.$$

This completes the proof of Theorem 4 (assuming Lemma 3 is true).

2.4.4 The factorization lemma

Let us finally prove Lemma 3. Let the smallest exponent of x occurring in $\Delta(x)$ be $-m$. Then $P(x;t) = x^m \Delta(x)$ is a polynomial in x and t such that $P(x;0) = x^m$. As a polynomial in x, P has degree, say, d, and hence admits d roots, denoted $X_1, \ldots, X_d$, which belong to the algebraic closure of $\mathbb{R}(t)$. By Newton's theorem, there exists an integer $n \geq 1$ such that all these roots can be written as Laurent series in the variable $z = t^{1/n}$ with complex coefficients (see [1, p. 89]). Assume that exactly k of these roots, say $X_1, \ldots, X_k$, are finite at $z=0$. The other $d-k$ roots contain terms of the form z^{-i}, with $i > 0$. The polynomial $P(x;t)$ can be factored as

$$P(x;t) = x^m \Delta(x) = \Delta_0 \prod_{i=1}^{k} (x - X_i) \prod_{i=k+1}^{d} \left(1 - \frac{x}{X_i}\right),$$

where Δ_0 and the X_i are algebraic functions of z. For $i > k$, the series $1/X_i$ equals 0 at $z=0$. Hence the condition $P(x;0) = x^m$ implies that $k=m$, that $\Delta_0(0) = 1$, and that the finite roots $X_1, \ldots, X_m$ equal 0 when $z=0$. Let

$$\Delta_+(x) \equiv \Delta_+(x;z) = \prod_{i=m+1}^{d} \left(1 - \frac{x}{X_i}\right) \quad \text{and} \quad \Delta_-(\bar{x}) \equiv \Delta_-(\bar{x};z) = \prod_{i=1}^{m} (1 - \bar{x} X_i).$$

Then the series $\Delta_0, \Delta_+(x)$ and $\Delta_-(\bar{x})$ satisfy all the required conditions, but two: we still need to prove that they are actually series in t (and not only in $z = t^{1/n}$) with real (rather than complex) coefficients. By taking logarithms, we obtain

$$\log \Delta(x) = \log \Delta_0 + \log \Delta_+(x) + \log \Delta_-(\bar{x}).$$

As $\Delta_+(0;z) = \Delta_+(x;0) = 1$, the series $\log \Delta_+(x)$ is a multiple of x. Similarly, the series $\log \Delta_-(\bar{x})$ is a multiple of $\bar{x}$. But $\log \Delta(x)$ can be written in a *unique* way as

$$\log \Delta = L_0(z) + x L_+(x;z) + \bar{x} L_-(\bar{x};z)$$

where L_0 is a series in z with real coefficients, L_+ is a series in z with coefficients in $\mathbb{R}[x]$ and L_- is a series in z with coefficients in $\mathbb{R}[\bar{x}]$. Moreover, L_0, L_+ and L_- are actually series in t, and not only in z. The above equation forces $\Delta_0 = \exp(L_0)$, $\Delta_+(x) = \exp(x L_+(x))$ and $\Delta_-(\bar{x}) = \exp(\bar{x} L_-(\bar{x}))$, and proves simultaneously that the series $\Delta_0, \Delta_+(x)$ and $\Delta_-(\bar{x})$ are actually series in t with real coefficients, and that they are the unique triple satisfying the required conditions. This completes the proof of Lemma 3.

2.4.5 Proof of Theorem 1

We apply Theorem 4 to the ordinary square lattice, that is, to the case $A_0(x) = x + \bar{x}$ and $A_1(x) = 1$. The polynomial $\Delta(x) = (1 - t(x + \bar{x} + 2))\,(1 - t(x + \bar{x} - 2))$ has four roots X_i, $1 \leq i \leq 4$, which are quadratic functions of t and can be computed explicitly. Let $C(t)$ denote the generating function for Catalan numbers:

$$C(t) = \frac{1 - \sqrt{1-4t}}{2t} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^n.$$

Then $X_1 = C(t) - 1$ and $X_2 = 1 - C(-t)$ are the two roots that are finite at $t = 0$, and by symmetry of Δ in x and $\bar{x}$, the two other roots are

$$X_3 = (C(t) - 1)^{-1} = \frac{1 - 2t + \sqrt{1 - 4t}}{2t} \quad \text{and} \quad X_4 = (1 - C(-t))^{-1} = \frac{1 + 2t + \sqrt{1 + 4t}}{2t}. \tag{9}$$

By the previous subsection, the canonical factorization of Δ is such that

$$\begin{aligned} \Delta_+(x) = \Delta_-(x) &= (1 - xX_3^{-1})(1 - xX_4^{-1}) &(10)\\ &= (1 - x(C(t) - 1))\,(1 + x(C(-t) - 1))\,. &(11) \end{aligned}$$

Taking the coefficient of x^2 in the relation $\Delta(x) = \Delta_0 \Delta_+(x)\Delta_+(\bar{x})$ yields

$$\begin{aligned} \Delta_0 &= t^2 X_3 X_4 &(12)\\ &= (C(t)C(-t))^{-2}, &(13) \end{aligned}$$

as $(C(t) - 1) = tC(t)^2$. Now by Theorem 4, Eqs. (10) and (12),

$$S(x, y; t) = \frac{\sqrt{t^2 X_3 X_4 (1 - \bar{x}X_3^{-1})(1 - \bar{x}X_4^{-1})}}{K(x, y)} = \frac{\sqrt{(tX_3 - t\bar{x})(tX_4 - t\bar{x})}}{1 - t(x + \bar{x} + y + \bar{y})}.$$

Theorem 1 then follows from (9). We obtain the asymptotic behaviour of $a(n)$ by examining the singularities of $S(1, 1; t)$. (We refer to [12] for a description of techniques that lead from the position and nature of the singularities of a series to asymptotic estimates of its coefficients. These techniques are applied in detail in Section 6.2.)

2.4.6 Proof of Theorem 2

For the diagonal square lattice, $A_0(x) = 0$ and $A_1 = x + \bar{x}$. Again, the polynomial $\Delta(x) = 1 - 4t^2(x + \bar{x})^2$ has four roots X_i, $1 \le i \le 4$, which are quadratic functions of t and can be expressed explicitly in terms of the Catalan generating function. More precisely, $X_{1,2} = \pm 2tC(4t^2)$ and

$$X_{3,4} = \pm X_1^{-1} = \pm \frac{1 + \sqrt{1 - 16t^2}}{4t}. \tag{14}$$

We then follow the same steps as above. The formal expression of $\Delta_+(x)$ in terms of the roots X_i remains unchanged, but the actual value is of course different:

$$\begin{aligned} \Delta_+(x) = \Delta_-(x) &= (1 - xX_3^{-1})(1 - xX_4^{-1}) &(15)\\ &= 1 - 4t^2 x^2 C(4t^2)^2. &(16) \end{aligned}$$

Taking the coefficient of x^2 in the relation $\Delta(x) = \Delta_0 \Delta_+(x)\Delta_+(\bar{x})$ yields now

$$\begin{aligned} \Delta_0 &= -4t^2 X_3 X_4 &(17)\\ &= C(4t^2)^{-2}. &(18) \end{aligned}$$

Theorem 2 then follows from Theorem 4, Eqs. (15), (17) and finally (14).

2.5 Refining the enumeration

In what we have done so far, each step of a walk contributes for a weight t in the generating function. We can refine the enumeration by giving different weights to different steps, and the method works just as well. Let us, for instance, take into account the number of vertical steps in the first model (ordinary square lattice). Let us denote by $S(x, y; t, v)$ the refined generating function, in which the variable v keeps track of the vertical steps. The polynomial Δ is now

$$\Delta(x) = (1 - t(x + \bar{x} + 2v))(1 - t(x + \bar{x} - 2v)).$$

The computations then follow exactly the same lines as in Section 2.4.5. The roots of Δ that diverge when $t=0$ are

$$X_3 = \frac{1-2tv+\sqrt{\Delta_1}}{2t}, \quad X_4 = \frac{1+2tv+\sqrt{\Delta_2}}{2t},$$

with

$$\Delta_1 = (1-2t(1+v))(1+2t(1-v)) \quad \text{and} \quad \Delta_2 = (1-2t(1-v))(1+2t(1+v)). \tag{19}$$

We still have

$$\Delta_+(x) = \Delta_-(x) = (1-xX_3^{-1})(1-xX_4^{-1}) \tag{20}$$

and $\Delta_0 = t^2X_3X_4$. We thus obtain the following refinement of Theorem 1.

Theorem 6 *On the ordinary square lattice, the refined generating function $S(x,y;t,v)$ for walks on the slit plane is given by*

$$S(x,y;t,v) = \frac{\left(1-2t(v+\bar{x})+\sqrt{\Delta_1}\right)^{1/2}\left(1+2t(v-\bar{x})+\sqrt{\Delta_2}\right)^{1/2}}{2\left(1-t(x+\bar{x}+yv+\bar{y}v)\right)},$$

where Δ_1 and Δ_2 are given by (19).

Note. It is explained in the subsequent paper [4] how to solve slit plane models for *any* set of steps A. In particular, Theorem 4 is generalized to walks that have small height variations, but do not necessary fulfill the symmetry hypothesis: the expression of the complete generating function is unchanged, but the polynomial Δ is taken to be $(1-tA_0(x))^2 - 4t^2A_1(x)A_{-1}(x)$, where

$$A_{-1}(x) = \sum_{(i,-1)\in A} x^i.$$

It is also shown in [4] that some other sets of steps give transcendental generating functions.

3 Walks ending at a prescribed position

We still consider a model of walks on the slit plane where the set A of allowed steps satisfies the conditions of symmetry and small height variations that led to the expression of the complete generating function $S(x,y;t)$ given in Theorem 4. We would like to enumerate walks on the slit plane ending at a prescribed point (i,j). Let

$$S_{i,j}(t) \equiv S_{i,j} = \sum_{n\geq 0} a_{i,j}(n)t^n$$

be the corresponding generating function. This series is obtained by extracting the coefficient of x^iy^j from $S(x,y;t)$. As i and j belong to $\mathbb{Z}$, rather than $\mathbb{N}$, this is not an obvious task that could be performed by differentiation. In particular, the algebraicity of $S(x,y;t)$ does *not* automatically imply the algebraicity of $S_{i,j}(t)$. This is clearly shown by the enumeration of walks starting from $(0,0)$ in the ordinary, unslit square lattice. The complete generating function is $1/(1-t(x+\bar{x}+y+\bar{y}))$. It is rational, hence algebraic. However, for i and j in $\mathbb{Z}$, the coefficient of x^iy^j in this series is

$$\sum_{n\geq 0} \binom{n}{\frac{n+i+j}{2}}\binom{n}{\frac{n-i-j}{2}} t^n,$$

where the sum is restricted to integers n of the same parity as $i+j$. This series is transcendental: the coefficient of t^n grows like $4^n/n$, up to a multiplicative constant, revealing a logarithmic singularity in the generating function that implies its transcendence (see [11] for a discussion on the possible singularities of an algebraic series).

In contrast, we shall prove that for all models covered by Theorem 4, the series $S_{i,j}(t)$ is algebraic for all i and j. We have implemented the procedure that computes $S_{i,j}$ as a `Maple` program. This program allowed us, for instance, to compute $S_{i,j}$ for $|i|+|j|\leq 10$ for the above mentioned square, diagonal and triangular lattices.

3.1 Algebraicity: statement of the results

For the two models that we studied in detail in Section 2, the series $S_{i,j}$ admit a rational expression in terms of the following power series in t:

$$u = \frac{\sqrt{1+4t}-1}{\sqrt{1-4t}+1} = \sum_{n\geq 0}(2.4^n C_n - C_{2n+1})t^{2n+1}, \tag{21}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. Note that u is quartic over $\mathbb{Q}(t)$:

$$t = \frac{u(1-u^2)}{(1+u^2)^2}. \tag{22}$$

This equation allows us to write any rational function of t as a rational function of u, and implies that $\mathbb{Q}(u) = \mathbb{Q}(t, \sqrt{1-4t}, \sqrt{1+4t})$. We shall also use the following identities, relating the Catalan generating function $C(t)$ to the series u:

$$C(t) = \frac{1+u^2}{1-u}, \quad C(-t) = \frac{1+u^2}{1+u}, \quad tC(t)C(-t) = u, \quad \text{and} \quad C(4t^2) = \left(\frac{1+u^2}{1-u^2}\right)^2. \tag{23}$$

Theorem 7 (Ordinary square lattice: Prescribed endpoint) *For all i and j, the generating function $S_{i,j}(t) = \sum_n a_{i,j}(n)t^n$ for walks on the slit plane ending at (i,j) belongs to $\mathbb{Q}(u) = \mathbb{Q}(t, \sqrt{1-4t}, \sqrt{1+4t})$ and can be computed explicitly. It is either quadratic, or quartic over $\mathbb{Q}(t)$. In particular,*

$$S_{0,1}(t) = \frac{u}{1-u^2} = \frac{1-\sqrt{1-16t^2}}{8t} = \sum_{n\geq 0} 4^n C_n t^{2n+1},$$

and

$$S_{1,0}(t) = \frac{u(1+u^2)}{1-u^2} = \frac{2-\sqrt{1-4t}-\sqrt{1+4t}}{4t} = \sum_{n\geq 0} C_{2n+1}t^{2n+1},$$

as conjectured by O. Roques. Some other values are

$$S_{-1,1}(t) = \frac{u^2}{1-u^2} = \frac{\sqrt{1+4t}-\sqrt{1-4t}-4t}{8t} = \frac{1}{2}\sum_{n\geq 1} C_{2n}t^{2n},$$

$$S_{1,1}(t) = \frac{u^2(2-u^2)}{(1-u^2)^2} = \frac{1-24t^2+4t\sqrt{1+4t}-4t\sqrt{1-4t}-\sqrt{1-16t^2}}{32t^2} = \sum_{n\geq 1}(4^{n-1}C_n + C_{2n}/2)t^{2n}.$$

Remark. When we take into account the number of vertical steps in the enumeration, like in Section 2.5, some expressions become more intricate (for instance, the series $S_{0,1}(t,v)$ is of degree 8 while it is quadratic when $v = 1$), but certain results remain simple. In particular, one derives directly from the fact that $S_0(x;t,v) = \Delta_+(x)^{-1/2}$, where $\Delta_+(x)$ is given in (20) that

$$\begin{aligned} S_{1,0}(t,v) &= \frac{1}{2}\left(X_3^{-1}+X_4^{-1}\right) = \frac{1}{2}\left(X_1+X_2\right) \\ &= \frac{2-\sqrt{(1-2t-2tv)(1+2t-2tv)}-\sqrt{(1-2t+2tv)(1+2t+2tv)}}{4t}. \end{aligned}$$

The series X_1 satisfies $X_1 = t(1+2vX_1+X_1^2)$. Hence its coefficients can be easily computed by the Lagrange inversion formula [15, Section 1.2.4]. Moreover, $X_2(t,v) = X_1(t,-v)$. Finally, the number of walks of length $2n+1$ going from $(0,0)$ to $(1,0)$ and having $2k$ vertical steps is found to be

$$\binom{2n}{2k} 2^{2k} C_{n-k},$$

in accordance with the identity

$$\sum_{k=0}^{n} \binom{2n}{2k} 2^{2k} C_{n-k} = C_{2n+1} = a_{1,0}(2n+1).$$

This result suggests the existence of a bijection between walks ending at $(1,0)$ and *bicolored Motzkin walks* [7, Ex. 2.2], that would decrease the length by one and take the number of *vertical* steps to the number of *horizontal* steps of Motzkin walks. Such a bijection has recently been described in [3].

Conjecture. In the early days of this study, it was observed by our colleague Bétréma that the coefficients of $S_{-i,i}(t)$, for $i \geq 1$, seem to factor nicely. The program we have written to compute the series $S_{i,j}$ confirms this observation, and we conjecture that, for $i \geq 1$ and $n \geq i$,

$$a_{-i,i}(2n) = \frac{i}{2n}\binom{2i}{i}\binom{n+i}{2i}\frac{\binom{4n}{2n}}{\binom{2n+2i}{2i}}.$$

The Lagrange inversion formula, combined with a binomial identity, implies that this conjecture is equivalent to

$$S_{-i,i}(t) = \frac{(-1)^i}{(1-u^2)^{2i-1}} \sum_{k=i}^{2i-1} \binom{2i-1}{k} (-1)^k u^{2k}.$$

We have verified this conjecture up to $i = 10$.

We now state a theorem, similar to Theorem 7, for walks with diagonal steps. Recall that, for this model, the generating function for walks ending on the line $y = 0$ is especially simple: $S_0(x;t) = (1 - 4t^2x^2C(4t^2)^2)^{-1/2}$. This allows us to write a closed form expression for the series $S_{2i,0}$ and its coefficients. By some aspects, the diagonal model looks sometimes simpler than the ordinary one. By some aspects only: we shall see that computing $S_{i,j}$, for $j \neq 0$, is more difficult for the diagonal model than for the ordinary one, even though the final results look close.

Theorem 8 (Diagonal square lattice: Prescribed endpoint) *For all i and j, the generating function $S_{i,j}(t) = \sum_n a_{i,j}(n)t^n$ for walks on the slit plane ending at (i,j) belongs to $\mathbb{Q}(u) = \mathbb{Q}(t, \sqrt{1-4t}, \sqrt{1+4t})$ and can be computed explicitly. It is either quadratic, or quartic over $\mathbb{Q}(t)$. In particular,*

$$S_{1,1}(t) = \frac{u(1+u^2)}{1-u^2} = \frac{2-\sqrt{1-4t}-\sqrt{1+4t}}{4t} = \sum_{n\geq 0} C_{2n+1}t^{2n+1},$$

$$S_{-1,1}(t) = u = \frac{\sqrt{1+4t}-1}{\sqrt{1-4t}+1} = \sum_{n\geq 0}(2.4^nC_n - C_{2n+1})t^{2n+1},$$

$$S_{0,2}(t) = \frac{2u^2}{1-u^2} = \frac{\sqrt{1+4t}-\sqrt{1-4t}-4t}{4t} = \sum_{n\geq 1} C_{2n}t^{2n},$$

and for $i \geq 1$,

$$S_{2i,0}(t) = \binom{2i}{i} t^{2i} C(4t^2)^{2i} = \sum_{n\geq i} \frac{i}{n}\binom{2i}{i}\binom{2n}{n-i} 4^{n-i}t^{2n}.$$

Remark. Performing a counterclockwise rotation of 45 degrees on Fig. 2 shows that counting walks on the slit plane with diagonal steps is equivalent to counting walks on the *ordinary* square lattice that avoid the diagonal half-line $\{(k,k), k \leq 0\}$. Theorems 7 and 8 imply that, among the walks of length $2n+1$ that go from $(0,0)$ to $(1,0)$ on the ordinary square lattice, exactly as many avoid the horizontal half-line $\{(k,0), k \leq 0\}$ as the diagonal half-line $\{(k,k), k \leq 0\}$ (and this number is C_{2n+1}). It would be interesting to find a combinatorial explanation for this fact.

Once again, Theorems 7 and 8 are instances of a more general result that applies to walks with symmetry and small height variations. We define the polynomials $A_0(x), A_1(x)$ and $\Delta(x;t)$ as in Section 2.3 (see (4) and (6)). The canonical factorisation of $\Delta(x)$ is still denoted $(\Delta_0(t), \Delta_+(x;t), \Delta_-(\bar{x},t))$.

Theorem 9 (General model: Prescribed endpoint) *Let $\mathbb{Q}_A$ be the finite extension of $\mathbb{Q}(t)$ generated by:*

– *the series $\sqrt{\Delta_0}$,*
– *the coefficients of the Laurent polynomials $\Delta_+(x)$ and $\Delta_-(\bar{x})$ (these coefficients are series in t),*
– *the roots α_i of the Laurent polynomial $A_1(x)$ (these roots are complex numbers),*
– *the series $\sqrt{\Delta_+(\alpha_i)}$.*

Then for all i and j, the generating function $S_{i,j}(t) = \sum_n a_{i,j}(n)t^n$ for walks on the slit plane ending at (i,j) belongs to $\mathbb{Q}_A$.

3.2 Computing $S_{i,j}$ in the general case

We start, naturally, from the complete generating function $S(x,y;t) = \sum_{i,j\in\mathbb{Z}} S_{i,j}(t)x^iy^j$ derived in Section 2. Our first step is fairly simple, and consists in extracting from $S(x,y;t)$ the coefficient of y^j. By symmetry, we can assume $j \geq 0$.

Lemma 10 *For $j \geq 0$, the generating function for walks on the slit plane ending on the line $y = j$ is algebraic, and admits the following expression:*

$$S_j(x;t) := \sum_{n\geq 0}\sum_{i\in\mathbb{Z}} a_{i,j}(n)x^it^n = \frac{1}{(2tA_1(x))^j}\left(\frac{f_j^+(x;t)}{\sqrt{\Delta_+(x)}} - f_j^-(x;t)\sqrt{\Delta_0\Delta_-(\bar{x})}\right), \tag{24}$$

where $f_j^+(x;t)$ and $f_j^-(x;t)$ are the following polynomials in $x, \bar{x}$ and t:

$$f_j^+(x;t) = \sum_{k=0}^{\lfloor j/2\rfloor}\binom{j}{2k}\Delta(x)^k(1-tA_0(x))^{j-2k}, \tag{25}$$

$$f_j^-(x;t) = \sum_{k=0}^{\lfloor (j-1)/2\rfloor}\binom{j}{2k+1}\Delta(x)^k(1-tA_0(x))^{j-2k-1}. \tag{26}$$

Proof. We start from the expression of $S(x,y;t)$ given in Theorem 4. Note that $1/K(x,y)$ is the generating function for unrestricted walks on the plane. In Section 2.4.2, we have seen that the two roots of $K(x,y)$, seen as a polynomial in y, are the series $Y(x)$ and $1/Y(x)$, where

$$Y(x) = \frac{1 - tA_0(x) - \sqrt{\Delta(x)}}{2tA_1(x)}.$$

Let us convert the rational function $1/K(x,y)$ into partial fractions of y:

$$\frac{1}{K(x,y)} = \frac{1}{\sqrt{\Delta(x)}}\left(\frac{\bar{y}Y(x)}{1-\bar{y}Y(x)} + \frac{1}{1-yY(x)}\right).$$

As $\Delta(x) = 1 + O(t)$ and $Y(x) = O(t)$, this identity splits $1/K(x,y)$ as the sum of two power series in t with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$, one with only negative powers of y, and the other with only nonnegative powers of y. Hence, by Theorem 4, the coefficient of y^j in $S(x,y;t)$ is, for $j \geq 0$,

$$S_j(x) = [y^j]S(x,y;t) = \sqrt{\Delta_0\Delta_-(\bar{x})}\,[y^j]\frac{1}{K(x,y)} = \frac{\sqrt{\Delta_0\Delta_-(\bar{x})}}{\sqrt{\Delta(x)}}Y(x)^j = \frac{Y(x)^j}{\sqrt{\Delta_+(x)}}. \tag{27}$$

The binomial formula, applied to $Y(x)^j$, combined with the fact that $\Delta(x) = \Delta_0\Delta_+(x)\Delta_-(\bar{x})$, gives the announced expression of $S_j(x;t)$. This expression is certainly much bigger than (27), but also more convenient for the purpose we have in mind[1].

■

[1] Note that the generating function for unconstrained walks that take their steps in A, which is $Y(x)^j/\sqrt{\Delta(x)}$, has not, in general, algebraic coefficients, even though it does not look very different from $S_j(x)$.

Our next step will be to separate in $S_j(x;t)$ the terms with a negative power of x from the other terms. In general, given a series $S(x;t)$ belonging to $\mathbb{Q}[x,\bar{x}][[t]]$, we define the positive and negative parts of S by:

$$S(x;t) = \sum_{n\geq 0} t^n \sum_{i\in\mathbb{Z}} a_i(n)x^i \Longrightarrow S^+(x;t) = \sum_{n\geq 0} t^n \sum_{i\geq 0} a_i(n)x^i \quad \text{and} \quad S^-(x;t) = \sum_{n\geq 0} t^n \sum_{i<0} a_i(n)x^i.$$

Let us examine Expression (24) of $S_j(x;t)$. It is an easy task to extract the positive part of the portion of the expression that lies between the brackets: as $f_j^+(x;t)$ and $f_j^-(x;t)$ are Laurent polynomials, expanding $1/\sqrt{\Delta_+(x)}$ in x allows us to remove from $f_j^+(x;t)/\sqrt{\Delta_+(x)}$ the (finitely many) terms with a negative exponent of x, and similarly, expanding $\sqrt{\Delta_-(\bar{x})}$ in $\bar{x}$ allows us to select in $f_j^-(x;t)\sqrt{\Delta_-(\bar{x})}$ the (finitely many) terms with a nonnegative exponent of x. Hence the positive part of the expression between brackets is $f_j^+(x;t)/\sqrt{\Delta_+(x)}$, plus some corrections: these corrections are polynomials in x and $\bar{x}$, whose coefficients involve t, $\sqrt{\Delta_0}$, and the coefficients of $\Delta_+(x)$ and $\Delta_-(\bar{x})$, seen as polynomials in x and $\bar{x}$.

To illustrate this procedure, let us consider the case $j=1$ of the ordinary square lattice:

$$S_1(x;t) = \frac{1}{2t}\left(\frac{1-t(x+\bar{x})}{\sqrt{\Delta_+(x)}} - \sqrt{\Delta_0\Delta_+(\bar{x})}\right),$$

where $\Delta_+(x)$ and Δ_0 are given by (11) and (13). This readily gives

$$S_1^+(x;t) = \sum_{i\geq 0, n\geq 0} a_{i,1}(n)x^i t^n = \frac{1}{2t}\left(\frac{1-t(x+\bar{x})}{\sqrt{\Delta_+(x)}} + t\bar{x} - \sqrt{\Delta_0}\right), \tag{28}$$

$$S_1^-(x;t) = \sum_{i<0, n\geq 0} a_{i,1}(n)x^i t^n = \frac{1}{2t}\left(\sqrt{\Delta_0} - t\bar{x} - \sqrt{\Delta_0\Delta_+(\bar{x})}\right). \tag{29}$$

In general, however, problems will arise from nontrivial polynomials $A_1(x)$. For instance, $A_1(x) = x+\bar{x}$ for the diagonal square lattice model, and we have:

$$S_1(x;t) = \frac{1}{2t(x+\bar{x})}\left(\frac{1}{\sqrt{\Delta_+(x)}} - \sqrt{\Delta_0\Delta_+(\bar{x})}\right) \tag{30}$$

where $\Delta_+(x)$ and Δ_0 are now given by (16) and (18). Hence we need to extract the positive part of certain series of the form $U(x;t)/A(x)$, where $A(x)$ is a polynomial. The following proposition solves this problem.

Proposition 11 *Let $A(x)$ be a non-zero polynomial in x with complex coefficients. Assume $A(x)$ has exactly d distinct roots, $\alpha_1,\dots,\alpha_d$, of multiplicities $m_1,\dots,m_d$. Let $W(x;t)$ be a series of $\mathbb{C}[x,\bar{x}][[t]]$, and define $U(x;t) = A(x)W(x;t)$. Clearly, $U(x;t)$ also belongs to $\mathbb{C}[x,\bar{x}][[t]]$. There exist polynomials $P_{i,k}(x)$ with complex coefficients such that*

$$W^+(x;t) = \frac{1}{A(x)}\left(U^+(x;t) - \sum_{i=1}^{d}\sum_{k=0}^{m_i-1} P_{i,k}(x)U^{+(k)}(\alpha_i;t)\right)$$

where $U^{+(k)}$ denotes the kth derivative of U^+ with respect to x. Moreover, if $A(x)$ and $W(x;t)$ have their coefficients in $\mathbb{Q}$ rather than $\mathbb{C}$, then the polynomials $P_{i,k}(x)$ have their coefficients in $\mathbb{Q}(\alpha_1,\dots,\alpha_d)$.

We shall prove this proposition in the following subsection. Meanwhile, let us derive from it our main result.

Proposition 12 *Let A be a finite subset of $\mathbb{Z}^2$ satisfying the conditions of symmetry and small height variations. Let $j\geq 0$. There exists a Laurent polynomial in x, with coefficients in the field $\mathbb{Q}_A$ defined in Theorem 9, denoted $g_j(x;t)$, such that*

$$S_j^+(x;t) = \sum_{n\geq 0}\sum_{i\geq 0} a_{i,j}(n)x^i t^n = \frac{1}{(2tA_1(x))^j}\left(\frac{f_j^+(x;t)}{\sqrt{\Delta_+(x)}} - g_j(x;t)\right),$$

and

$$S_j^-(x;t) \quad = \quad \sum_{n\geq 0}\sum_{i<0} a_{i,j}(n)x^i t^n \quad = \quad \frac{1}{(2tA_1(x))^j}\left(g_j(x;t) - f_j^-(x;t)\sqrt{\Delta_0\Delta_-(\bar{x})}\right),$$

where $f_j^+(x;t)$ and $f_j^-(x;t)$ are the polynomials in $x, \bar{x}$ and t given by (25) *and* (26).

Proof. Let m be the smallest integer (positive or not) such that $A(x) := x^m A_1(x)$ is a polynomial in x. Then $A(0) \neq 0$. By Lemma 10,

$$S_j(x;t) = \frac{1}{(2tA(x))^j}\left(\frac{x^{mj} f_j^+(x;t)}{\sqrt{\Delta_+(x)}} - x^{mj} f_j^-(x;t)\sqrt{\Delta_0\Delta_-(\bar{x})}\right).$$

The positive part of the expression between brackets is of the form

$$\frac{x^{mj} f_j^+(x;t)}{\sqrt{\Delta_+(x)}} - h_j(x;t),$$

where $h_j(x;t)$ is a Laurent polynomial in x with coefficients in the extension of $\mathbb{Q}(t)$ generated by $\sqrt{\Delta_0}$ and the coefficients of $\Delta_+(x)$ and $\Delta_-(\bar{x})$. We then apply Proposition 11 to obtain the expression of $S_j^+(x;t)$. The value of $S_j^-(x;t)$ follows from the fact that $S_j = S_j^- + S_j^+$.

■

The above proposition expresses $S_j^+(x;t)$ as the product of two terms, which are Laurent series in x with coefficients in $\mathbb{Q}_A$. Hence the coefficient of x^i in $S_j^+(x;t)$ also belongs to this field, and this proves Theorem 9 for $i \geq 0$. A similar argument, starting from $S_j^-(x;t)$, gives the announced result for $i < 0$.

■

3.3 The extraction procedure in practice

3.3.1 Proof of Proposition 11

Lemma 13 *Let α be a nonzero complex number, and m a positive integer. We define an operator $T_{\alpha,m}$ on series of $\mathbb{C}[x,\bar{x}][[t]]$ by:*

$$T_{\alpha,m}(U(x;t)) = \frac{1}{(x-\alpha)^m}\left(U(x;t) - \sum_{k=0}^{m-1} U^{(k)}(\alpha;t)\frac{(x-\alpha)^k}{k!}\right).$$

If $U(x;t)$ belongs to $\mathbb{C}[x,\bar{x}][[t]]$ (resp. $\mathbb{C}[x][[t]]$, resp. $\bar{x}\mathbb{C}[\bar{x}][[t]]$), then so does the series $T_{\alpha,m}(U(x;t))$.

Proof. The operator $T_{\alpha,m}$ acts coefficient-wise on power series of t, so that we can concentrate on its effect on $\mathbb{C}[x,\bar{x}]$. By linearity, it suffices to prove the statement when $U(x) = x^n$, for $n \in \mathbb{Z}$. In this case, let

$$R(x) := U(x) - \sum_{k=0}^{m-1} U^{(k)}(\alpha)\frac{(x-\alpha)^k}{k!}.$$

Taylor's formula implies that $R^{(\ell)}(\alpha) = 0$ for all $0 \leq \ell < m$. When $R(x)$ is a polynomial (*i.e.* when $n \geq 0$), this immediately implies its divisibility by $(x-\alpha)^m$, and we conclude that $T_{\alpha,m}(U(x)) = R(x)/(x-\alpha)^m$ is a polynomial in x as well.

Now if $U(x) = \bar{x}^n$ with $n > 0$, then $P_1(x) = x^n R(x)$ is a polynomial, with degree at most $n + m - 1$. Using Leibnitz formula for successive derivatives, we also have $P_1^{(\ell)}(\alpha) = 0$ for $0 \leq \ell < m$ and this implies that $P_1(x) = (x-\alpha)^m P_2(x)$ with $P_2(x)$ of degree at most $n-1$. Finally $T_{\alpha,m}(U(x)) = \bar{x}^n P_2(x)$ belongs to $\bar{x}\mathbb{C}[\bar{x}]$.

■

As a straightforward corollary of this lemma, we obtain a proof of Proposition 11 in the case where the polynomial $A(x)$ has a single root.

Corollary 14 *Let α be a complex number, and m a positive integer. Let $W(x;t)$ be a series of $\mathbb{C}[x,\bar{x}][[t]]$, and define $U(x;t) = (x-\alpha)^m W(x;t)$. Clearly, $U(x;t)$ also belongs to $\mathbb{C}[x,\bar{x}][[t]]$. Moreover,*

$$W^+(x;t) = T_{\alpha,m}(U^+(x;t)).$$

(If $\alpha = 0$, the operator $T_{\alpha,m}$ can be defined on series, which, like $U^+(x;t)$, have their coefficients in $\mathbb{C}[x]$.)

Proof. The result is obvious if $\alpha = 0$. Otherwise, we derive from the fact that $U(x;t) = (x-\alpha)^m W(x;t)$ that, for $0 \le k < m$, $U^{(k)}(\alpha;t) = 0$. Consequently,

$$\begin{aligned} W(x;t) &= \frac{U(x;t)}{(x-\alpha)^m} = \frac{1}{(x-\alpha)^m}\left(U(x;t) - \sum_{k=0}^{m-1} U^{(k)}(\alpha;t)\frac{(x-\alpha)^k}{k!}\right) \\ &= T_{\alpha,m}(U(x;t)) \\ &= T_{\alpha,m}(U^-(x;t)) + T_{\alpha,m}(U^+(x;t)). \end{aligned}$$

The statement now follows from Lemma 13.

■

Proposition 11 follows by induction on the number d of distinct roots of $A(x)$.

3.3.2 Proof of Theorem 7

Let us apply Theorem 9 to the ordinary square lattice. As already observed, $A_1(x) = 1$, so that we are not bothered by roots of $A_1(x)$. Moreover, the symmetry of the model with respect to the line $x = 0$ implies that $\Delta_+(x) = \Delta_-(x)$. Therefore the field $\mathbb{Q}_A$ is generated by $\sqrt{\Delta_0}$ and the coefficients of $\Delta_+(x)$. Eqs. (11), (13) and (23) give

$$\Delta_+(x) = \left(1 - xu\,\frac{1+u}{1-u}\right)\left(1 - xu\,\frac{1-u}{1+u}\right) \quad \text{and} \quad \sqrt{\Delta_0} = t/u,$$

so that $\mathbb{Q}_A = \mathbb{Q}(u)$. Theorem 9 states that $S_{i,j}(t)$ belongs to $\mathbb{Q}(u)$. In particular, expanding in x (or $\bar{x}$) the series $S_0(x;t) = 1/\sqrt{\Delta_+(x)}$, $S_1^+(x;t)$ and $S_1^-(x;t)$ (given by (28) and (29)), provides the announced expressions of $S_{1,0}$, $S_{0,1}$, $S_{1,1}$ and $S_{-1,1}$.

As any element of $\mathbb{Q}(u)$, the series $S_{i,j}$ is either rational, or quadratic, or quartic. Let us rule out rationality. Take a walk w going from the point $(1,0)$ to (i,j), and avoiding the forbidden half-line. Let m be the length of w. By adding w at the end of any walk ending at $(1,0)$, we obtain that for all $n \ge m$,

$$a_{1,0}(n-m) \le a_{i,j}(n).$$

Similarly, $a_{i,j}(n) \le a_{1,0}(n+m)$. But $a_{1,0}(n) = C_n \sim c\,4^n n^{-3/2}$ (for n odd). Hence there exist two positive constants c_1 and c_2 such that, assuming $n = i+j \bmod 2$,

$$c_1 4^n n^{-3/2} \le a_{i,j}(n) \le c_2 4^n n^{-3/2}.$$

This cannot be the asymptotic behaviour of the coefficients of a rational series.

■

3.3.3 Proof of Theorem 8

For the diagonal square lattice, the canonical factorization of $\Delta(x)$ is given by (16) and (18). Again, $\Delta_+(x) = \Delta_-(x)$. These values imply that the extension of $\mathbb{Q}(t)$ generated by $\sqrt{\Delta_0}$ and the coefficients of $\Delta_+(x)$ is simply $\mathbb{Q}(t, C(4t^2)) = \mathbb{Q}(t, \sqrt{1-16t^2})$, and has degree 2 over $\mathbb{Q}(t)$. However, $A_1(x) = x + \bar{x}$ has two roots $\pm i$, so that the field $\mathbb{Q}_A$ also contains $\sqrt{\Delta_+(\pm i)}$ (and i). Using (22) and (23), the series $\Delta_+(x)$ and Δ_0 can be written in terms of u:

$$\Delta_+(x) = \Delta_-(x) = 1 - \frac{4u^2x^2}{(1-u^2)^2} \quad \text{and} \quad \sqrt{\Delta_0} = \frac{(1-u^2)^2}{(1+u^2)^2}.$$

In particular, $\sqrt{\Delta_+(\pm i)} = (1+u^2)/(1-u^2)$. This series is of degree 4, so that finally $\mathbb{Q}_A = \mathbb{Q}(i,u)$. Theorem 9 implies that $S_{i,j}$ belongs to $\mathbb{Q}(i,u)$. But a series with *real* coefficients belonging to $\mathbb{Q}(i,u)$ also belongs to $\mathbb{Q}(u)$. This proves the first part of Theorem 8. Rationality is ruled out as for the ordinary square lattice.

Let us now derive the announced expressions of the series $S_{i,j}$. The value of $S_{2i,0}$ follows from the fact that $S_0(x;t) = 1/\sqrt{\Delta_+(x)}$. The other three series require to apply explicitly the extraction procedure described above, and actually provide a good illustration of it. The series $S_1(x;t)$ is given by (30). Observe that, for this model, $S_1(x;t)$ has only odd powers of x, and in particular, no constant term. Let us divide it by x, in order to have even series,

$$\frac{S_1^+(x;t)}{x} = \left(\frac{S_1(x;t)}{x}\right)^+ = \left(\frac{U(x;t)}{2t(1+x^2)}\right)^+$$

where

$$U(x;t) = \frac{1}{\sqrt{\Delta_+(x)}} - \sqrt{\Delta_0\Delta_+(\bar{x})}.$$

This remark allows us to work with the variable $X = x^2$, and to apply only once Corollary 14. The positive part of $U(x;t)$ is $U^+(x;t) = \Delta_+(x)^{-1/2} - \Delta_0^{1/2}$; with the notations of Corollary 14, $\alpha = -1$, $m = 1$ and we obtain

$$\frac{S_1^+(x;t)}{x} = \frac{1}{2t(1+x^2)}\left(\frac{1}{\sqrt{\Delta_+(x)}} - \frac{1}{\sqrt{\Delta_+(i)}}\right),$$

that is

$$S_1^+(x;t) = \frac{x}{2t(1+x^2)}\left(\frac{1}{\sqrt{\Delta_+(x)}} - \frac{1-u^2}{1+u^2}\right).$$

By difference, this gives:

$$S_1^-(x;t) = \frac{\bar{x}}{2t(1+\bar{x}^2)}\left(\frac{1-u^2}{1+u^2} - \sqrt{\Delta_0\Delta_+(\bar{x})}\right).$$

The values of $S_{1,1}(t)$ and $S_{-1,1}(t)$ follow, by extracting the coefficient of x (or $\bar{x}$) in these series.

Finally, we need to evaluate the positive part of $S_2(x;t)$ in order to compute $S_{0,2}(t)$. By Lemma 10,

$$S_2(x;t) = \frac{U(x;t)}{2t^2(1+x^2)^2},$$

where

$$U(x;t) = \frac{x^2 - 2t^2(1+x^2)^2}{\sqrt{\Delta_+(x)}} - x^2\sqrt{\Delta_0\Delta_+(\bar{x})}.$$

Our first step is the extraction of the positive part of U:

$$U^+(x;t) = \frac{x^2 - 2t^2(1+x^2)^2}{\sqrt{\Delta_+(x)}} - x^2\sqrt{\Delta_0} + \frac{2u^2}{(1-u^2)^2}.$$

We now apply Corollary 14 (again, for the sake of efficiency, to the variable $X = x^2$) with $\alpha = -1$ and $m = 2$. We need:

$$U^+(i;t) = \frac{-1}{\sqrt{\Delta_+(i)}} + \sqrt{\Delta_0} + \frac{2u^2}{(1-u^2)^2}$$

and
$$\frac{\partial U^+}{\partial X}(i;t) = \frac{1}{\sqrt{\Delta_+(i)}} - \frac{2u^2}{(1-u^2)^2\Delta_+(i)^{3/2}} - \sqrt{\Delta_0}.$$

Corollary 14 gives:
$$\begin{aligned} S_2^+(x;t) &= \frac{1}{2t^2(1+x^2)^2}\left(U^+(x;t) - U^+(i;t) - (x^2+1)\frac{\partial U^+}{\partial X}(i;t)\right) \\ &= \frac{1}{2t^2(1+x^2)^2}\left(\frac{x^2-2t^2(1+x^2)^2}{\sqrt{\Delta_+(x)}} + \frac{2u^2(1-u^2)}{(1+u^2)^3} - x^2\frac{(1-u^2)(1+u^4)}{(1+u^2)^3}\right). \end{aligned}$$

Taking the constant term in x gives the announced expression of $S_{0,2}$.

4 Other starting points

From the results of Sections 2 and 3, some elementary operations on walks allow us to solve a number of related enumerative questions. In this section we consider walks avoiding the half-line $\mathcal{H}$ and starting from points other than the origin. The results we obtain will be useful in Sections 5 and 6, where probabilistic results are derived.

Even though the techniques we use here work for all models satisfying the conditions of symmetry and small height variations, the results are easier to state if we assume, in addition, that the set A of steps is symmetric with respect to reversal of the walks: $(i,j) \in A \Leftrightarrow (-i,-j) \in A$. This assumption is satisfied by the ordinary and diagonal square lattices. Under this assumption, $A_0(x) = A_0(\bar{x})$, and similarly $A_1(x)$ and $\Delta(x)$ are symmetric in x and $\bar{x}$. Consequently, the canonical factorization of Δ satisfies $\Delta_+(x) = \Delta_-(x)$.

We generalize the notations used so far by denoting $a_{i,j}^{[k,\ell]}(n)$ the number of n-step walks that start from (k,ℓ), end at (i,j), and have no contact with $\mathcal{H}$ except at their starting point (if $k \le 0$, $\ell = 0$). Similarly, we denote by $S^{[k,\ell]}(x,y;t)$ the corresponding generating function:
$$S^{[k,\ell]}(x,y;t) = \sum_{n\ge 0}\sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}^{[k,\ell]}(n)x^iy^jt^n.$$

When $\ell = 0$, we use for the sake of briefness the notations $a_{i,j}^{[k]}(n)$ and $S^{[k]}(x,y;t)$. As above, the length variable t is sometimes omitted. Section 4.1 is devoted to the case $\ell = 0$, $k \le 0$, and Section 4.2 to the other cases.

4.1 Starting on the forbidden half-line

Proposition 15 *With the notations of Section 2, the generating function for walks that start on $\mathcal{H}$, but never return to this half-line is:*
$$S^-(z;x,y;t) := \sum_{k\ge 0} S^{[-k]}(x,y;t)z^k = \frac{\sqrt{\Delta_0\Delta_+(\bar{x})\Delta_+(z)}}{(1-z\bar{x})K(x,y)} = \frac{\sqrt{\Delta_+(z)}}{1-z\bar{x}}S(x,y;t).$$

Proof. First, observe that $S^-(0;x,y;t)$ counts walks starting at the origin, and hence, coincides with the complete generating function $S(x,y;t)$ for walks of the slit plane. Now take a walk starting from $(-k,0)$, with $k>0$, and move it one step to the right: we obtain a walk of the same type starting from $(-k+1,0)$. Conversely, a walk starting from $(-k+1,0)$, translated one step to the left, provides either a walk of the right type starting from $(-k,0)$, or the concatenation uvw of a (reversed) bridge u going from $(0,0)$ to $(-k,0)$, a (possibly empty) sequence v of bridges ending at $(0,0)$, and a usual walk w on the slit plane (recall the definition of bridges, at the beginning of Section 2.3). In terms of generating functions:
$$z\bar{x}S^-(z;x,y;t) = S^-(z;x,y;t) - S^-(0;x,y;t) + \frac{B(z)-B(0)}{1-B(0)}S(x,y;t).$$

We conclude thanks to Theorem 4 and Proposition 5.

∎

Let us call *pseudo-bridge* a walk that starts from an arbitrary point (i,j), ends on $\mathcal{H}$ but otherwise avoids $\mathcal{H}$. These walks generalize the bridges counted by $B(z)$, which start from $(0,0)$. By the reversal assumption, the generating function for these pseudo-bridges is the coefficient of x^iy^j in the series $S^-(z;x,y;t)$ given in Proposition 15.

Corollary 16 *For $(i,j)\in\mathbb{Z}^2\setminus\mathcal{H}$, the generating function for pseudo-bridges that start from (i,j) is*

$$B_{i,j}(z;t) := \sum_{n\geq 0}\sum_{k\geq 0} a_{i,j}^{[-k]}(n)z^kt^n = \sqrt{\Delta_+(z)}S^+_{i,j}(z;t),$$

where $S^+_{i,j}(x;t)$ is the following section of the complete generating function $S(x,y;t)$:

$$S^+_{i,j}(x;t) = \sum_{k\geq 0} x^k S_{i+k,j}(t).$$

4.2 Starting outside the forbidden half-line

Proposition 17 *With the notations of Corollary* 16*, the generating function for walks that start from $(k,\ell)\in\mathbb{Z}^2\setminus\mathcal{H}$ and avoid $\mathcal{H}$ is:*

$$S^{[k,\ell]}(x,y;t) = \frac{x^ky^\ell - B_{k,\ell}(\bar{x};t)}{K(x,y)}.$$

In the case $\ell=0$, $k>0$, an alternative expression is

$$S^{[k]}(x,y;t) \;=\; \frac{S(x,y;t)}{\sqrt{\Delta_0}}\sum_{i=0}^{k-1} S_{i,0}(t)\,x^{k-i}.$$

Proof. The step-by-step construction of Section 2.3 yields

$$S^{[k,\ell]}(x,y;t) = x^ky^\ell + t\,S^{[k,\ell]}(x,y;t)\cdot(A_0(x)+(y+\bar{y})A_1(x)) - B_{k,\ell}(\bar{x};t),$$

since the forbidden steps are those completing a pseudo-bridge. The first result follows.

The alternative expression in the case $\ell=0$ is an immediate consequence of the following equalities:

$$\frac{x^k - B_{k,0}(\bar{x})}{K(x,y)} \;=\; \frac{(x^k - B_{k,0}(\bar{x}))S(x,y)}{\sqrt{\Delta_0\Delta_-(\bar{x})}} \;=\; \frac{x^kS_0(\bar{x})S(x,y)}{\sqrt{\Delta_0}} - \frac{S^+_{k,0}(\bar{x})S(x,y)}{\sqrt{\Delta_0}},$$

obtained by combining the expressions of $S(x,y)$ and $S_0(x)$ (Theorem 4), $B_{i,j}(x)$ (Corollary 16) and the assumption $\Delta_+(x)=\Delta_-(x)$.

■

Let us call *loop* a walk that starts and ends at the same point of the positive x-axis and avoids $\mathcal{H}$. The length generating function $L_k(t)$ for loops starting at $(k,0)$ is the coefficient of x^ky^0 is the series $S^{[k]}$. The last expression of Proposition 17 gives the following result.

Corollary 18 *The generating function for loops starting and ending at $(k,0)$ with $k>0$ is*

$$L_k(t) \;=\; \frac{1}{\sqrt{\Delta_0(t)}}\sum_{i=0}^{k-1} S_{i,0}(t)^2.$$

Our last proposition will be used in Section 6 to obtain the average number of visits of a (long) walk on the slit plane to the point $(k,0)$.

Corollary 19 *For $k>0$, the generating function for walks on the slit plane (starting from $(0,0)$) that visit the point $(k,0)$ is*

$$V_k(x,y;t,v) = \frac{vS_{k,0}(t)S^{[k]}(x,y;t)}{L_k(t)^2\,(1-v(1-1/L_k(t)))}.$$

This series counts walks by their length (variable t), number of visits to $(k,0)$ (variable v), and position of their endpoint (variables x,y).

Proof. Let us say that a non-empty loop is *primitive* if it visits exactly twice its starting (and ending) point. Let $P_k(t)$ be the length generating function for primitive loops starting at $(k,0)$. Clearly, $L_k = (1-P_k)^{-1}$. Equivalently, $P_k = 1 - 1/L_k$.

Similarly, a walk on the slit plane ending at $(k,0)$ is called *primitive* if it visits only once its endpoint. The generating function for primitive walks ending at $(k,0)$ is $S_{k,0}/L_k$. Finally, $S^{[k]}/L_k$ counts walks starting from $(k,0)$ that never return to their starting point.

A walk on the slitplane visiting $(k,0)$ can be seen in a unique way as the concatenation $up_1p_2\cdots p_\ell w$ of a primitive walk u going from the origin to $(k,0)$, a sequence of ℓ primitive loops starting at $(k,0)$, and a primitive walk w starting from $(k,0)$. Such a walk visits exactly $\ell+1$ times the point $(k,0)$. The result follows.

■

5 The hitting distribution of a half-line

We focus in this section on the ordinary square lattice. The results obtained in Sections 3 and 4 allow us to solve a number of probabilistic questions "à la Kenyon". Let (i,j) be a point of $\mathbb{Z}^2$. A random walk starting from (i,j) hits the half-line $\mathcal{H} = \{(k,0), k \le 0\}$ with probability 1. The probability that the *first* hitting point is $(0,0)$ is

$$p_{i,j} = \sum_{n\ge 0} \frac{a_{i,j}(n)}{4^n} = S_{i,j}(1/4). \tag{31}$$

More precisely, $a_{i,j}(n)/4^n$ is the probability that this event occurs after n steps. Theorem 7 states that $S_{i,j}(t)$ belongs to $\mathbb{Q}(u)$, where the series u is given by (21). As $u = \sqrt{2}-1$ when $t = 1/4$, this theorem implies that $p_{i,j} \in \mathbb{Q}[\sqrt{2}]$ and in particular, that $p_{0,1} = 1/2$ and $p_{1,0} = 2-\sqrt{2}$, as stated in R. Kenyon's e-mail. We have written a program to compute $p_{i,j}$: for $|i|+|j| \le 10$, the probability $p_{i,j}$ is irrational, unless $(i,j) = (0,1)$ or $(0,-1)$.

Lemma 10 and Proposition 12 tell us how to compute the series

$$S_j^+(x;t) = \sum_{i\ge 0} S_{i,j}(t)x^i \quad \text{and} \quad S_j^-(x;t) = \sum_{i<0} S_{i,j}(t)x^i.$$

Setting $t = 1/4$ in these expressions provides explicit values for generating functions of the form

$$\sum_{i\ge 0} p_{i,j}x^i \quad \text{and} \quad \sum_{i<0} p_{i,j}x^i.$$

For instance, $S_0(x) = \Delta_+(x)^{-1/2}$, where $\Delta_+(x)$ is given by (11), and consequently,

$$\sum_{i\ge 0} p_{i,0}x^i = \frac{1}{\sqrt{(1-x)\left(1-x(\sqrt{2}-1)^2\right)}}.$$

More generally, given $k \ge 0$, one can ask about the probability $p_{i,j}^{[k]}$ that the first hitting point of a random walk starting from (i,j) is $(-k,0)$. For k fixed, the function f defined by $f(i,j) = p_{i,j}^{[k]}$ is the unique bounded function on $\mathbb{Z}^2$, harmonic on $\mathbb{Z}^2 \setminus \mathcal{H}$, such that $f(-\ell,0) = \delta_{k,\ell}$ for $\ell \ge 0$ (see [18, Theorem 1.4.8]). The *hitting distribution* of the half-line $\mathcal{H}$, starting from (i,j), is condensed in the following series,

$$\sum_{k\ge 0} p_{i,j}^{[k]} z^k = B_{i,j}(z;1/4),$$

where we have used the notations of Corollary 16. This corollary, combined with Proposition 12, allows us to compute this series explicitly, for a fixed value of (i,j).

Theorem 20 *Let $(i,j) \in \mathbb{Z}^2 \setminus \mathcal{H}$. The hitting distribution of the half-line $\mathcal{H}$, starting from (i,j), is of the following form:*

$$\sum_{k\geq 0} p_{i,j}^{[k]} z^k = f(z) - g(z)\sqrt{(1-z)\left(1-z(\sqrt{2}-1)^2\right)}$$

where $f(z)$ and $g(z)$ are Laurent polynomials in z, with coefficients in $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{2}]$ respectively, that satisfy $f(1)=1$, $g(1)\neq 0$. Consequently, as $k\to\infty$, the probability that the first hitting point is $(-k,0)$ is

$$p_{i,j}^{[k]} \sim g(1)\sqrt{\frac{\sqrt{2}-1}{2\pi}}\; k^{-3/2}$$

and the probability that the first hitting point is to the left of $(-k,0)$ is

$$\sum_{\ell>k} p_{i,j}^{[\ell]} \sim g(1)\sqrt{\frac{2(\sqrt{2}-1)}{\pi}}\; k^{-1/2}.$$

The Laurent polynomials $f(z)$ and $g(z)$ can be computed explicitly. For instance, $f(z)=g(z)=\bar{z}$ if $(i,j)=(1,0)$, so that in this case $g(1)=1$.

Proof. Corollary 16 expresses $B_{i,j}(z;t)$ in terms of the section $S_{i,j}^{+}(z;t)$. The latter series only differs from the section $S_j^+(z;t)/z^i$ (the generic form of which is given by Proposition 12) by a finite number of series $S_{k,j}$; in other words, according to Theorem 7, by a Laurent polynomial $h(z;u)$ in z with coefficients in $\mathbb{Q}(u)$. Hence the combination of Corollary 16 and Proposition 12 imply that $B_{i,j}(z;t)$ is of the following form:

$$B_{i,j}(z;t) = f(z;t) - g(z;u)\sqrt{\Delta_+(z)}, \tag{32}$$

where f is a Laurent polynomial in z and t with rational coefficients, and g is a Laurent polynomial in z with coefficients in $\mathbb{Q}(u)$. In particular, for $(i,j)=(1,0)$, we have $S_{1,0}^+(z;t) = (S_0(z;t)-1)/z$ and

$$B_{1,0}(z;t) = \frac{1-\sqrt{\Delta_+(z)}}{z}.$$

Setting $t=1/4$ (and $u=\sqrt{2}-1$) in (32) provides the announced form for $B_{i,j}(z;1/4) = \sum_k p_{i,j}^{[k]} z^k$.

The recurrence of random walks on $\mathbb{Z}^2$ implies that the half-line is visited almost surely, so that

$$\sum_{k\geq 0} p_{i,j}^{[k]} = f(1) = 1.$$

The asymptotic behaviour of $p_{i,j}^{[k]}$ follows from the nature of the singularities of the series $B_{i,j}(z;1/4)$. The dominant singularity is at $z=1$, and is a square root singularity, provided that $g(1)\neq 0$. This is, at least, the case when $(i,j)=(1,0)$. In general, if $z=1$ was a root of $g(z)$, say, of multiplicity m, then $p_{i,j}^{[k]}$ would decay like $k^{-m-3/2}$, that is, much faster than $p_{1,0}^{[k]}$. As in Section 3.3.2, considering a (fixed) walk going from (i,j) to $(1,0)$ on the slit plane proves that this is impossible.

Similarly, the study of the behaviour of $(1-B_{i,j}(z;1/4))/(1-z)$ around its dominant singularity provides the asymptotic behaviour of the probability that the hitting abscissa is smaller than $-k$.

■

Remarks

1. This result has to be compared with Lemma 6 in [16], where it is proved (with our notations), that for $0\leq i\leq k/2$,

$$\sum_{\ell\geq k} p_{-i,1}^{[\ell]} \leq c\;((i+1)k)^{-1/2}, \tag{33}$$

for a constant c independent of i and k. This statement contains some uniformity in i which is absent from our result. It also implies that, for all i and j, there exists a constant $c_{i,j}$ such that for all $k \geq 0$,

$$\sum_{\ell \geq k} p_{i,j}^{[\ell]} \leq c_{i,j}\ k^{-1/2},$$

but this, in turn, is weaker than Theorem 20. Related results are described in [18, Section 2.4]. For instance, the bound (33) follows directly from Eq. (2.40) in [18]. Analogously, the probability that a planar brownian motion starting from $(1,0)$ hits the half-line for the first time at abscissa smaller than $-k$ also decays like $k^{-1/2}$ (see e.g. [20, with $\alpha = 2$]).
2. The probability $p_{i,j}^{[k]}$ admits another probabilistic interpretation. Let $(W_n)_{n\geq 0}$, be a random walk on the square lattice starting from $W_0 = (-k, 0)$. Let $T = \min\{n > 0 : W_n \in \mathcal{H}\}$, and let V denote the number of visits of the walk to the point (i,j) before T. Then the expectation of V is

$$\begin{aligned} \mathbb{E}(V) &= \sum_{n>0} \Pr(n < T, W_n = (i,j)) \\ &= \sum_{n>0} \frac{1}{4^n} a_{i,j}^{[-k]}(n) \\ &= p_{i,j}^{[k]}. \end{aligned}$$

As $p_{i,j}^{[k]}$ has been seen to be a probability, this average number of visits is always less than 1.

6 Properties of long walks on the slit plane

Again, we focus on the ordinary square lattice.

6.1 Transience and Green function

It is well-known that random walks on the square lattice are *recurrent*: any given point (i,j) of the lattice is visited with probability 1, and is actually visited infinitely many times. In more enumerative terms, the proportion of walks of length n visiting (i,j) tends to 1 as n goes to infinity, and the average number of visits of n-step walks to this point tends to infinity.

This is no longer the case for walks on the slit plane. The forbidden half-line creates a long-range repulsion of the walks, which become transient. This result is not surprising, and can probably be proved by various methods; but the enumerative results we have obtained, and more especially Corollary 19, can be used to obtain *exact quantitative information*. For instance, we can compute, for any point $(k,0)$ with $k > 0$, the probability that it is visited by an n-step walk. This probability is

$$\frac{[t^n]V_k(1,1;t,1)}{a(n)} = \frac{1}{a(n)}[t^n]\left(\frac{S_{k,0}(t)S^{[k]}(1,1;t)}{L_k(t)}\right). \tag{34}$$

In this expression, $a(n)$ denotes the total number of n-step walks on the slit plane, and the notation $[t^n]$ means "the coefficient of t^n". We shall prove that this probability converges as n goes to infinity to a limit that is strictly less than 1. By differentiating V_k with respect to v, we can also compute the average number of visits to $(k,0)$ of n-step walks, which is

$$\frac{[t^n]V_k'(1,1;t,1)}{a(n)} = \frac{1}{a(n)}[t^n]\left(S_{k,0}(t)S^{[k]}(1,1;t)\right). \tag{35}$$

Again, this quantity will be shown to converge to a *finite* limit as n goes to infinity. By analogy with ordinary random walks, we call it the value at $(k,0)$ of the *Green function* of our model.
Recall that for any i,j, the series $S_{i,j}(1/4)$ is always finite (and at most 1, by (31)).

Proposition 21 *As n tends to infinity, the proportion of walks of length n visiting $(k,0)$ tends to*

$$p_k = S_{k,0}(1/4)\ \frac{\sum_{\ell=0}^{k-1} S_{\ell,0}(1/4)}{\sum_{\ell=0}^{k-1} S_{\ell,0}(1/4)^2} < 1,$$

while the average number of visits to this point goes to

$$v_k = 4(\sqrt{2}-1)\, S_{k,0}(1/4) \sum_{\ell=0}^{k-1} S_{\ell,0}(1/4).$$

These expressions, together with

$$S_0(x;1/4) = \sum_{\ell \geq 0} S_{\ell,0}(1/4) x^\ell = \frac{1}{\sqrt{(1-x)\left(1-x(\sqrt{2}-1)^2\right)}},$$

allow us to compute p_k and v_k for any $k>0$. For instance, walks are more likely to visit $(1,0)$ than $(2,0)$:

$$p_1 = 2-\sqrt{2} \approx 0.586, \quad p_2 = \frac{5}{34}(19-11\sqrt{2}) \approx .506$$

but spend more time at $(2,0)$ than $(1,0)$:

$$v_1 = 4(3\sqrt{2}-4) \approx 0.97, \quad v_2 = 10(22\sqrt{2}-31) \approx 1.13.$$

Proof. The series $S^{[k]}(1,1;t)$ and $L_k(t)$ are given respectively by Proposition 17 and Corollary 18. The series $\Delta_0(t)$ is given by (13), which implies $1/\sqrt{\Delta_0(1/4)} = 4(\sqrt{2}-1)$. The announced expressions of p_k and v_k then follow from (34) and (35) by analysis of the singularities of the series into consideration.

Finally, one derives from the expression of $S_0(x;1/4)$ that $S_{k,0}(1/4)$ is a strictly decreasing function of k. Consequently, the numerator of p_k is strictly smaller than its denominator, and $p_k < 1$. ■

6.2 Limit law for the coordinates of the endpoint

When all walks of length n on the slit plane are taken equally likely, the coordinates of their endpoints become random variables X_n and Y_n. It is well-known that for an ordinary random walk, these coordinates, normalized by $\sqrt{n}$, converge to a two-dimensional centered normal law. This suggests to try the same normalization for walks on the slit plane. By expanding in t the complete generating function of Theorem 1, the probabilities

$$\Pr((X_n, Y_n) = (i,j)) \ = \ \frac{a_{i,j}(n)}{a(n)}$$

can be explicitly computed for small values of n and any (i,j). The plots of the marginals, $\sqrt{n}\Pr(X_n = i)$ against $i/\sqrt{n}$, and $\sqrt{n}\Pr(Y_n = j)$ against $j/\sqrt{n}$, shown on Fig. 4, suggest that the normalized random variables $X_n/\sqrt{n}$ and $Y_n/\sqrt{n}$ also converge in distribution. These plots actually even suggest the existence of a *local* limit law. We have proved this for the ordinate $Y_n/\sqrt{n}$, but not for the abscissa, and we shall simply prove here the convergence in distribution of the normalized endpoint. Unsurprisingly, the limit law we obtain corresponds to a two-dimensional Brownian conditioned (with care) not to hit a half-line [19], and is related to the solution of the associated diffusion equation (see [6, Eq. (29)]).

Theorem 22 *The sequence of joint random variables $(X_n/\sqrt{n}, Y_n/\sqrt{n})$ converges in distribution towards a pair (X,Y) of density*

$$f(x,y) = \frac{\sqrt{2}}{\Gamma(1/4)}\ e^{-(x^2+y^2)} \sqrt{x+\sqrt{x^2+y^2}}$$

with respect to the Lebesgue measure on $\mathbb{R}^2$. This density is shown in Fig. 5.

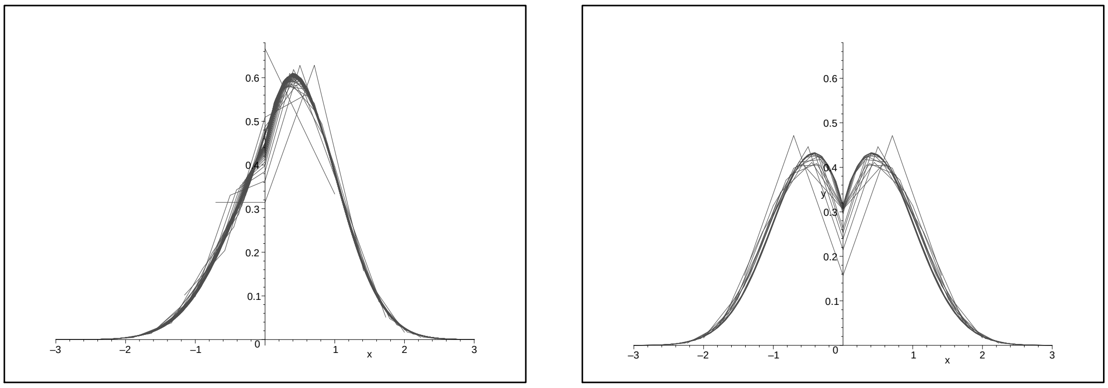

Figure 4: The convergence of $\sqrt{n}\,\Pr(X_n = i)$, plotted against $i/\sqrt{n}$ (left), and $\sqrt{n}\,\Pr(Y_n = j)$ against $j/\sqrt{n}$ (right), for $n = 2, 3, \ldots, 10, 20, 30, \ldots, 100$.

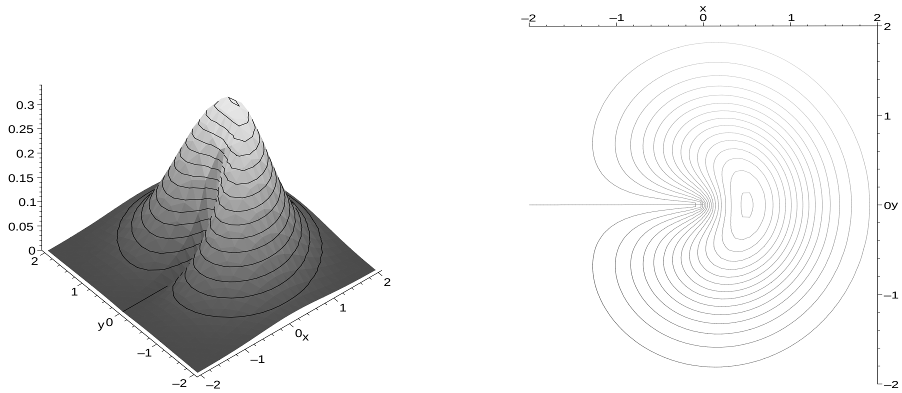

Figure 5: The density $f(x, y)$ of (X, Y).

The pair (X, Y) actually admits a simpler description in polar coordinates. Let $R = |X + iY|$ and $\Theta = \arg(X + iY) \in [-\pi, \pi[$. Then (R, Θ) has density

$$g(\rho, \theta) = \frac{2}{\Gamma(1/4)}\, \rho^{3/2} e^{-\rho^2} \cos(\theta/2)$$

with respect to the Lebesgue measure on $\mathbb{R}^+ \times [-\pi, \pi]$. Using this expression of the density, we can easily compute the moments of (X, Y). From Theorem 22, we shall deduce in particular the following asymptotic results.

Corollary 23 *As n goes to infinity, the endpoint (X_n, Y_n) of a random n-step walk on the slit plane satisfies:*

$$\mathbb{E}(X_n) \sim \frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{n}, \qquad \mathbb{E}(Y_n) = 0, \qquad \mathbb{E}\left(\sqrt{X_n^2 + Y_n^2}\right) \sim 3\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{n},$$

$$\mathbb{E}\left(X_n^2\right) \sim \frac{7}{12} n \qquad \textit{and} \qquad \mathbb{E}\left(Y_n^2\right) \sim \frac{2}{3} n.$$

Remark. When the square lattice is replaced by the diagonal lattice, elementary steps undergo a dilatation of a factor $\sqrt{2}$. One can actually prove that Theorem 22 holds for the diagonal case, upon normalizing the variables X_n and Y_n by $\sqrt{2n}$ instead of $\sqrt{n}$.

Our proof of Theorem 22 is (as the rest of this paper) based on the exact enumerative results of Section 2 and provides an alternative to more classical probabilistic proofs. Let Φ_n be the characteristic function of

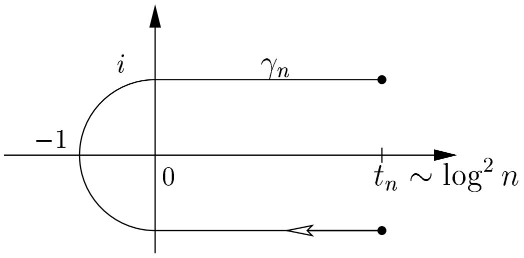

Figure 6: The contours γ_n and γ_∞.

$(X_n/\sqrt{n}, Y_n/\sqrt{n})$. We shall see that the *analysis of singularities* developed by Flajolet and Odlyzko [12] implies automatically the pointwise convergence of Φ_n and gives an expression for its limit. This limit will then be identified as the characteristic function of (X, Y).

Proposition 24 (Convergence of the characteristic functions) *Let σ and τ be real numbers. Then*

$$\lim_{n\to+\infty} \Phi_n(\sigma,\tau) = \Phi(\sigma,\tau),$$

where

$$\Phi(\sigma,\tau) = \frac{2}{i\Gamma(1/4)} \int_{\gamma_\infty} \frac{(2\sqrt{-t}+i\sigma)^{1/2} e^{-t}}{\sigma^2+\tau^2-4t} dt,$$

with γ_∞ the Hankel contour around $[0,+\infty[$ shown on the right side of Fig. 6.

Proof. The characteristic function $\Phi_n(\sigma,\tau)$ can be expressed in terms of the complete generating function $S(x,y;t)$:

$$\Phi_n(\sigma,\tau) = \mathbb{E}\left(e^{i(\sigma X_n+\tau Y_n)/\sqrt{n}}\right) = \frac{[t^n] S(e^{i\sigma/\sqrt{n}}, e^{i\tau/\sqrt{n}}; t)}{a(n)}$$

where $a(n)$ is the total number of n-step walks on the slit plane. Recall that, from Theorem 1, this number grows like $4^n n^{-1/4}$, up to an explicit multiplicative constant. As we want to prove that $\Phi_n(\sigma,\tau)$ converges, this means that the coefficient of t^n in $S(e^{i\sigma/\sqrt{n}}, e^{i\tau/\sqrt{n}}; t)$ also has to grow like $4^n n^{-1/4}$.

We shall estimate this coefficient thanks to the analysis of singularities of [12]. For the sake of completeness, we give all the details of the calculation, but this is a rather direct application of [12]. We begin with Cauchy's formula, and force the factor 4^n to appear by setting $t = z/4$:

$$2i\pi \frac{a(n)}{4^n} \Phi_n(\sigma,\tau) = \int_{\mathcal{C}} S(e^{i\sigma/\sqrt{n}}, e^{i\tau/\sqrt{n}}; z/4) \frac{dz}{z^{n+1}}, \tag{36}$$

where $\mathcal{C}$ is any simple contour positively encircling the origin, inside the domain of analycity of $S(e^{i\sigma/\sqrt{n}}, e^{i\tau/\sqrt{n}}; z/4)$. We expect this integral to behave like $n^{-1/4}$. By Theorem 1,

$$S(x,y;z/4) = \frac{\left(2 - z(1+e^{-i\sigma/\sqrt{n}}) + 2\sqrt{1-z}\right)^{1/2} \left(2 + z(1-e^{-i\sigma/\sqrt{n}}) + 2\sqrt{1+z}\right)^{1/2}}{4 - 2z\left(\cos(\sigma/\sqrt{n}) + \cos(\tau/\sqrt{n})\right)} \tag{37}$$

where we denote $x = e^{i\sigma/\sqrt{n}}$ and $y = e^{i\tau/\sqrt{n}}$. We choose the principal determination of the square root on $\mathbb{C} \setminus \mathbb{R}^-$, given by

$$\sqrt{\rho e^{i\theta}} = \sqrt{\rho} e^{i\theta/2} \quad \text{for } \rho > 0 \text{ and } \theta \in]-\pi,\pi[.$$

The singularities of the series $S(x,y;z/4)$ may, at first sight, have three sources. More precisely,

- the inner radicals restrict the domain of analycity inside $D = \mathbb{C} \setminus (]-\infty,-1] \cup [1,+\infty[)$;
- the pole $z_n = 2/(\cos(\sigma/\sqrt{n}) + \cos(\tau/\sqrt{n}))$ belongs, for n large enough, to $[1,+\infty[$ and does not interfere;

- the outer radicals do not give further singularities inside, say, $|z| < 2$: indeed,

$$\begin{aligned} 2 - z(1 + e^{-i\sigma/\sqrt{n}}) + 2\sqrt{1-z} &= 2\cos\theta e^{-i\theta}\left(\sqrt{1-z}+1\right)\left(\sqrt{1-z}+i\tan\theta\right), \\ 2 + z(1 - e^{-i\sigma/\sqrt{n}}) + 2\sqrt{1+z} &= 2\cos\theta e^{-i\theta}\left(\sqrt{1+z}+1\right)\left(1+i\tan\theta\sqrt{1+z}\right), \end{aligned}$$

with $\theta = \frac{\sigma}{2\sqrt{n}}$, and these factorizations imply that the arguments of these expressions belong to $]-\pi, \pi[$ as soon as $|z| < 2$ and $\theta < \pi/6$.

Consequently, for σ and τ fixed, and n large enough, the function $S(x, y; z/4)$ is analytic in $D \cap \{|z| < 2\}$. In this domain, the modulus of the numerator of (37) is bounded by 10. The sum of cosines that occurs at the denominator is larger than 1 for n large enough, so that finally,

$$|S(x, y; z/4)| \leq \frac{5}{|z_n - z|}. \tag{38}$$

We choose a contour $\mathcal{C}$ which depends on n and consists of four parts $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ (see Fig. 7):

- $\mathcal{C}_1$ and $\mathcal{C}_3$ are two symmetric arcs of radius $r_n = 1 + \log^2 n/n$, centered at the origin,
- $\mathcal{C}_2$ a Hankel contour around 1, at distance $1/n$ of the real axis, which meets $\mathcal{C}_1$ and $\mathcal{C}_3$:

$$\mathcal{C}_2 = \left\{1 + \frac{t-i}{n},\ t \in [0, t_n]\right\} \cup \left\{1 - \frac{e^{i\theta}}{n},\ \theta \in [-\pi/2, \pi/2]\right\} \cup \left\{1 + \frac{t+i}{n},\ t \in [0, t_n]\right\}$$

  where $(1 + t_n/n)^2 + 1/n^2 = r_n^2$, so that $t_n \leq \log^2 n$ and $t_n = \log^2 n + O(1/n)$;
- $\mathcal{C}_4 = -\mathcal{C}_2$ is the symmetric Hankel contour around -1.

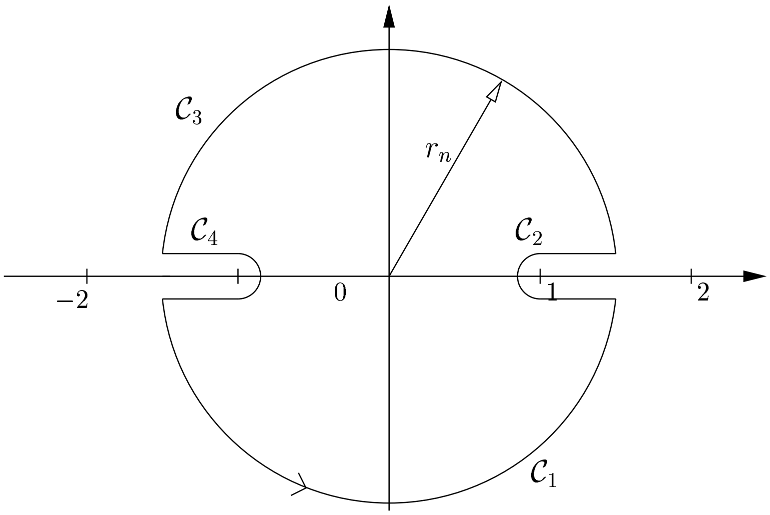

Figure 7: The contour $\mathcal{C}$.

The integral (36) on $\mathcal{C}$ is the sum of the contributions of the contours $\mathcal{C}_i$. We shall see that the dominant contribution is that of $\mathcal{C}_2$, because of the vicinity of the pole z_n. Let us consider first the contours $\mathcal{C}_1$ and $\mathcal{C}_3$. On these contours, $|z - z_n| > 1/n$ so that by (38), $|S(x, y; z/4)| < 5n$. Therefore the modulus of the integral on these arcs is bounded by $10\pi\, n\, r_n^{-n} = O(n^{1-\log n}) = o(1/n)$.

Consider next the contour $\mathcal{C}_4$. There, $|z - z_n| > 1$, so that by (38), $|S(x, y; z/4)| < 5$ and the integral is small because the contour itself is small: the integral on $\mathcal{C}_4$ is bounded by $5(2t_n/n + \pi/n)(1 - 1/n)^{-n-1} = O(\log^2 n/n)$. There remains the integral on $\mathcal{C}_2$. As z varies along $\mathcal{C}_2$, the variable t defined by $z = 1 + t/n$

varies along the contour γ_n shown on the left side of Fig. 6. As n goes to infinity, this contour converges to γ_∞. Let $t \in \gamma_\infty$. Then $t \in \gamma_n$ for n large enough, and, as n goes to infinity, the following approximations hold with error terms independant of t:

$$\begin{aligned}
\left(2 - z(1+\bar{x}) + 2\sqrt{1-z}\right)^{1/2} &= n^{-1/4}(2\sqrt{-t}+i\sigma)^{1/2}\left(1+O\left(\log^2 n/\sqrt{n}\right)\right)\\
\left(2 + z(1-\bar{x}) + 2\sqrt{1+z}\right)^{1/2} &= \sqrt{2}\sqrt{1+\sqrt{2}}\left(1+O(1/\sqrt{n})\right)\\
4 - z(x+\bar{x}+y+\bar{y}) &= n^{-1}(\sigma^2+\tau^2-4t)\left(1+O\left(\log^2 n/n\right)\right)\\
z^{-n-1} &= e^{-t}\left(1+O(\log^4 n/n)\right).
\end{aligned}$$

Observe that in the first approximation, $\Re(2\sqrt{-t}+i\sigma) > 0$ for $t \in \gamma_\infty$, so that the square root causes no difficulties. Hence, uniformly in $z \in \mathcal{C}_2$, we have

$$S(x,y;z/4)z^{-n-1} = n^{3/4}\sqrt{2}\sqrt{1+\sqrt{2}}\,\frac{(2\sqrt{-t}+i\sigma)^{1/2}e^{-t}}{\sigma^2+\tau^2-4t}\left(1+O(\log^2 n/\sqrt{n})\right)$$

with $z = 1+t/n$. For $t \in \gamma_n$, $|\sigma^2+\tau^2-4t| \geq 4$ and $|\exp(-t)| = |\exp(-\Re(t))| \leq e$. Moreover, $|t| \leq \log^2 n$, so that $|i\sigma + 2\sqrt{-t}| = O(\log n)$. Hence the previous identity implies that

$$S(x,y;z/4)z^{-n-1} = n^{3/4}\sqrt{2}\sqrt{1+\sqrt{2}}\,\frac{(2\sqrt{-t}+i\sigma)^{1/2}e^{-t}}{\sigma^2+\tau^2-4t} + O(n^{1/4}\log^3 n).$$

Let us now integrate this over $\mathcal{C}_2$:

$$\begin{aligned}
\int_{\mathcal{C}_2} S(e^{i\sigma/\sqrt{n}}, e^{i\tau/\sqrt{n}}; z/4)\frac{dz}{z^{n+1}} &= \frac{1}{n}\int_{\gamma_n} S(e^{i\sigma/\sqrt{n}}, e^{i\tau/\sqrt{n}}; (1+t/n)/4)\frac{dt}{(1+t/n)^{n+1}}\\
&= n^{-1/4}\sqrt{2}\sqrt{1+\sqrt{2}}\int_{\gamma_n}\frac{(2\sqrt{-t}+i\sigma)^{1/2}e^{-t}}{\sigma^2+\tau^2-4t}dt + O(n^{-3/4}\log^5 n).
\end{aligned}$$

As n goes to infinity,

$$\int_{\gamma_n}\frac{(2\sqrt{-t}+i\sigma)^{1/2}e^{-t}}{\sigma^2+\tau^2-4t}dt \longrightarrow \int_{\gamma_\infty}\frac{(2\sqrt{-t}+i\sigma)^{1/2}e^{-t}}{\sigma^2+\tau^2-4t}dt.$$

Hence finally,

$$\int_{\mathcal{C}_2} S(e^{i\sigma/\sqrt{n}}, e^{i\tau/\sqrt{n}}; z/4)\frac{dz}{z^{n+1}} = n^{-1/4}\sqrt{2}\sqrt{1+\sqrt{2}}\int_{\gamma_\infty}\frac{(2\sqrt{-t}+i\sigma)^{1/2}e^{-t}}{\sigma^2+\tau^2-4t}dt\ \left(1+o(1)\right).$$

Thus $\mathcal{C}_2$ is really the part of the contour that yields the dominant contribution to the integral of Eq. (36). We now inject in (36) the following ingredients:
– the four estimates of the integrals on the contours $\mathcal{C}_i$,
– the asymptotic behaviour of $a(n)$, which follows Theorem 1,
– the complement formula, according to which $\Gamma(1/4)\Gamma(3/4) = \sqrt{2}\pi$.
Proposition 24 follows.

■

Proof of Theorem 22. There remains to check that the limit function Φ, given in Proposition 24, coincides with the characteristic function Ψ of the distribution defined in Theorem 22. We use the expression of the density in polar coordinates to express this characteristic function:

$$\Psi(\sigma,\tau) = \mathbb{E}\left(e^{i\sigma X + i\tau Y}\right) = \frac{2}{\Gamma(1/4)}\int_{-\pi}^{\pi} d\theta \int_0^\infty d\rho\ e^{i\rho(\sigma\cos\theta+\tau\sin\theta)}\ \rho^{3/2}e^{-\rho^2}\cos(\theta/2).$$

One possible approach is to expand $\Psi(\sigma,\tau)$ and $\Phi(\sigma,\tau)$ in series of σ and τ, and to check that the coefficients coincide (these coefficients are, essentially, the moments of the pair (X,Y)). This natural approach works, but requires a few more calculations than the method we present below.

We choose to work with polar coordinates, not only for the density of (X,Y), but also for the variables σ and τ, which we take to be $r\cos\phi$ and $r\sin\phi$ respectively. The characteristic function Ψ becomes

$$\Psi(\sigma,\tau)=\Psi(r\cos\phi,r\sin\phi)=\frac{2}{\Gamma(1/4)}\int_{-\pi}^{\pi}d\theta\int_0^\infty d\rho\; e^{ir\rho\cos(\theta-\phi)}\;\rho^{3/2}e^{-\rho^2}\cos(\theta/2).$$

Let us expand the integrand in r:

$$\Psi(\sigma,\tau)=\frac{2}{\Gamma(1/4)}\int_{-\pi}^{\pi}d\theta\int_0^\infty d\rho\sum_{n\geq 0}\frac{(ir)^n}{n!}\rho^n\rho^{3/2}e^{-\rho^2}\cos(\theta-\phi)^n\cos(\theta/2).$$

This (triple) sum is absolutely convergent so that we can exchange the sum and the integrals, and then separate the integrals on ρ and θ. Using the definition of the Gamma function,

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt=2\int_0^\infty\rho^{2s-1}e^{-\rho^2}d\rho,$$

we then evaluate the integral over ρ, and obtain:

$$\Psi(r\cos\phi,r\sin\phi)=\frac{1}{\Gamma(1/4)}\sum_{n\geq 0}\frac{(ir)^n}{n!}\Gamma\left(\frac{n}{2}+\frac{5}{4}\right)\int_{-\pi}^{\pi}d\theta\cos(\theta-\phi)^n\cos(\theta/2).$$

Our aim is to transform this expression into the expression of Proposition 24. In particular, we need to introduce the Hankel contour γ_∞, which is known to occur in the Hankel representation of the *inverse* of the Gamma function:

$$\frac{1}{\Gamma(s)}=\frac{1}{2\pi i}\int_{\gamma_\infty}(-t)^{-s}e^{-t}dt. \tag{39}$$

The trouble is that the Gamma function appears only as a numerator, and not as a denominator, in the expression of Ψ. We shall remedy this thanks to the duplication formula:

$$2^{2s-1}\Gamma(s)\Gamma(s+1/2)=\sqrt{\pi}\,\Gamma(2s).$$

Applied to $s=n/2+3/4$, it allows us to rewrite $\Psi(\sigma,\tau)$ as

$$\Psi(\sigma,\tau)=\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(1/4)}\sum_{n\geq 0}\frac{(ir)^n}{2^n n!}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{4}\right)}\int_{-\pi}^{\pi}d\theta\cos(\theta-\phi)^n\cos(\theta/2)$$

and to introduce, at last, the contour γ_∞:

$$\Psi(\sigma,\tau)=\frac{1}{2i\sqrt{2\pi}\Gamma(1/4)}\sum_{n\geq 0}\left(\frac{ir}{2}\right)^n\frac{\Gamma\left(n+\frac{3}{2}\right)}{n!}\int_{\gamma_\infty}dt(-t)^{-\frac{n}{2}-\frac{3}{4}}\,e^{-t}\int_{-\pi}^{\pi}d\theta\cos(\theta-\phi)^n\cos(\theta/2).$$

But the summation over n is now subject to an explicit resummation. Indeed, for $|z|<1$,

$$\sum_{n\geq 0}z^n\frac{\Gamma\left(n+\frac{3}{2}\right)}{n!}=\frac{\sqrt{\pi}}{2(1-z)^{3/2}}. \tag{40}$$

Upon exchanging the sum and the integrals in the above expression of $\Psi(\sigma,\tau)$ we obtain:

$$\Psi(\sigma,\tau)=\frac{1}{2i\sqrt{2\pi}\Gamma(1/4)}\int_{\gamma_\infty}dt(-t)^{-\frac{3}{4}}\,e^{-t}\int_{-\pi}^{\pi}d\theta\cos(\theta/2)\sum_{n\geq 0}\left(\frac{ir\cos(\theta-\phi)}{2\sqrt{-t}}\right)^n\frac{\Gamma\left(n+\frac{3}{2}\right)}{n!}.$$

As $|t|\geq 1$, this exchange of summations is valid if $r<2$ by virtue of the absolute convergence of the power series (40). For larger values of r, it suffices to replace the Hankel contour γ_∞ by $2r^2\gamma_\infty$ in (39) and the above lines to obtain an absolutely convergent series. Finally,

$$\Psi(\sigma,\tau)=\frac{1}{4i\sqrt{2}\Gamma(1/4)}\int_{\gamma_\infty}dt(-t)^{-\frac{3}{4}}\,e^{-t}\int_{-\pi}^{\pi}d\theta\frac{\cos(\theta/2)}{(1-ir(-t)^{-1/2}\cos(\theta-\phi)/2)^{3/2}}. \tag{41}$$

The integration on θ can be performed easily, because the integrand has an explicit primitive:

$$\frac{\cos(\theta/2)}{(1 - z\cos(\theta - \phi))^{3/2}} = \frac{2}{1 - z^2}\frac{\partial}{\partial\theta}\left(\frac{\sin(\theta/2) + z\sin(\theta/2 - \phi)}{(1 - z\cos(\theta - \phi))^{1/2}}\right).$$

Using this primitive, we obtain

$$\int_{-\pi}^{\pi} d\theta \frac{\cos(\theta/2)}{(1 - z\cos(\theta - \phi))^{3/2}} = \frac{4(1 + z\cos\phi)^{1/2}}{1 - z^2}.$$

Applying the case $z = ir(-t)^{-1/2}/2$ of this identity to Eq. (41) gives, after a few reductions,

$$\Psi(\sigma, \tau) = \frac{2}{i\Gamma(1/4)} \int_{\gamma_\infty} dt\, e^{-t} \frac{(2\sqrt{-t} + ir\cos\phi)^{1/2}}{r^2 - 4t} = \Phi(r\cos\phi, r\sin\phi).$$

■

Proof of Corollary 23. It is sufficient to prove the convergence of moments of $(X_n/\sqrt{n}, Y_n/\sqrt{n})$ to moments of (X, Y), which can be computed from Theorem 22.

Integral moments are expressible in terms of generating functions: for any nonnegative k, ℓ,

$$\mathbb{E}(X_n^k Y_n^\ell) \quad = \quad \frac{1}{a(n)}[t^n]\frac{(x\partial)^k (y\partial)^\ell S(x, y; t)}{\partial x^k \partial y^\ell}\Big|_{x=1, y=1}.$$

In view of the explicit expression of $S(x, y; t)$, singularity analysis applies directly [12]. This involves nothing but a simplified version of the proof of Proposition 24.

Next, the convergence of all integral moments implies that of all moments and in particular of the expectation of the end-to-end distance [10, Chap. 8.1, p. 245, Ex. (e)].

■

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