Mathematical Programming: Modelling and Applications

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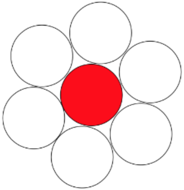
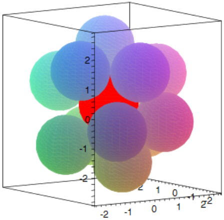
Outline

Kissing number problem

Some examples

$n = 6, K = 2$ $\qquad$ $n = 12, K = 3$





more dimensions

n	τ (lattice)	τ (nonlattice)
0	0	
1	2	
2	6	
3	12	
4	24	
5	40	
6	72	
7	126	
8	240	
9	272	(306)*
10	336	(500)*
11	438	(582)*
12	756	(840)*
13	918	(1130)*
14	1422	(1582)*
15	2340	
16	4320	
17	5346	
18	7398	
19	10668	
20	17400	
21	27720	
22	49896	

Kissing number problem: exercise 1

Use AMPL to implement a MINLP formulation of the Kissing Number Problem (as an **optimization problem**).

Hint 1: .dat files are given for small/medium instances.
Hint 2: look at Maculan, Michelon, Smith' model (1995).
Hint 3: you can use additional variables to reformulate the binary products.
Hint 4: you can use the knpcheck.run script (which is given) to check your solution (in other words, you can include it at the bottom of your .run file).
Hint 5: use Baron.

Set a threshold (for example, CPU Time limit 100 s., i.e. maxtime=100 in the options of your .run script) and test these instances:
(a) n=12 and k=3
(b) n=7 and k=2
In theory, your program should be able to accomplish (a) but not (b).
Then, download knp.mod and knp.run. Your implementation should be quite similar.

Kissing number problem: exercise 2

Use AMPL to implement the Kissing Number Problem (as a **decision problem**).

Hint: look at Kucherenko, Belotti, Liberti, Maculan' model (2007).

Set again a threshold (for example, CPU Time limit 100 s., i.e. maxtime=100 in the options of your .run script) and test again these instances:
(a) n=12 and k=3
(b) n=7 and k=2

In theory, the program should not be able to accomplish neither (a) nor (b).
Then, download knpfeas18.mod and knpfeas18.run. Your implementation should be quite similar.

Change the constraint on α, to see if something changes.

Reminder: SDP

SEMIDEFINITE PROGRAMMING (SDP) refers to optimization problems where decision variables are a symmetric matrix that is required to be semidefinite positive. The standard form is

$$\left.\begin{array}{rrcl} \min & C \bullet X & & \\ \forall i \leq m & A^i \bullet X & = & b^i \\ & X & \succeq & 0, \end{array}\right\} \quad (1)$$

where:

- C and A^i (for $i \leq m$) are $n \times n$ symmetric matrices
- X is an $n \times n$ matrix of decision variables,
- $b^i \in \mathbb{R}$ for all $i \leq m$
- for two $n \times n$ matrices $L = (\lambda_{ij})$, $M = (\mu_{ij})$ we have $L \bullet M = \sum_{i,j \leq n} \lambda_{ij}\mu_{ij}$
- $X \succeq 0$, i.e. X is positive semidefinite

Reminder: SDP

To make Eq. (1) clearer, write out the componentwise product $\bullet$ of the matrices: $C = (c_{jh})$, $A^i = (a^i_{jh})$ and $X = (x_{jh})$:

$$\begin{array}{rrcl} & \min \sum\limits_{j,h \le n} c_{jh} x_{jh} & & \\ \forall i \le m & \sum\limits_{j,h \le n} a^i_{jh} x_{jh} & = & b^i. \end{array}$$

This is just an LP subject to a semidefinite constraint $X \succeq 0$. What does $X \succeq 0$ mean?

X, square and symmetric matrix, with real values is PSD if $\forall v \in \mathbb{R}^n$, we have that the scalar $v^T X v \ge 0$.

This is the same as requiring the decision variables x_{jh} to take values such that, when they are arranged in a $n \times n$ array, the resulting matrix A has non-negative eigenvalues.

Kissing number problem: exercise 3

Use Octave to implement an SDP relaxation of the KNP (decision version, [KBLM07])

Then, download SDP_KBLM07.m (your code should not be very different).
Test some instances:
(b) n=7 and k=2
(c) n=8 and k=2
(d) n=9 and k=2
(e) …

Can you explain this behaviour, with growing n ?

Kissing number problem: exercise 4

Apparently, this first version of a SDP relaxation of KNP (i.e. SDP_KBLM07.m) is feasible independently from n. This makes it not really useful (cf. *Uselessness Theorem* in Liberti's slides)

Look for an improved SDP relaxation that deals with this drawback.
Initially, focus on **2D case** only ($k = 2$).

Hint: add a constraint on the dimension of the sdp matrix (i.e. decision variable).
Hint 2: for example, estimate the perimeter of the central sphere “consumed" by the placements.

Kissing number problem: exercise 5

Extend your code to deal with 3D case ($k = 3$).

Hint: this would be an approximation.

Kissing number problem: Upper bounds

Download delsartebnd.m, delsartelp.m and gegenbauer.m and test it in Octave in the following way:
[b,r] = delsartebnd(3, 60, 10);

Similarily, Download pfenderbnd.m and pfenderlp.m and test it in Octave in the following way:
[p,s] = pfenderbnd(3, 60, 10);

Your outcome should be, respectively:
delsartebnd: bound positivity in [0,0.000100], accepting
pfenderbnd: bound positivity in [0,0.000100], accepting

Can you explain what this means?

Kissing number problem: scripting

In the end, look in these scripts:

- how to optimize unconstrained functions within Octave using sqp() (examples in dgpnlp.m, delsartebnd.m and pfenderbnd.m).
- how to call glpk() within octave (examples in delsartelp.m and pfenderlp.m).
- how to compute Gegenbauer polynomial coefficients.