Mathematical Programming: Modelling and Applications

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Mathematical formulation

- Sets:

  V, set of vertices of G

  A, set of arcs of G

- Parameters:

  μ, arc coloring of graph G

  k, a prefixed arc color

- Variables:

  x_v, binary, indicates if the vertex v is contained into the densest uniformly colored subgraph (U, B):

$$\forall v \in V \quad x_v = \begin{cases} 1 & \text{if } v \in U \\ 0 & \text{otherwise} \end{cases}$$

Mathematical model

- Objective function:

  The densest subgraph has the maximum number of arcs and the minimum number of vertices:

$$\max \left(\sum_{(u,v)\in A} x_u x_v - \sum_{v\in V} x_v \right)$$

- Constraint:

  There cannot be arcs in the subgraph having a color different from k:

$$\forall (u,v) \in A$$

$$x_u x_v \leq \min\left(\max(0, \mu(u,v) - k + 1), \max(0, k - \mu(u,v) + 1)\right)$$

Product of binary variables

Observation: we have a product of binary variables in the objective function.

Because of the product of binary variables, this formulation is nonlinear:
Mixed-Integer NonLinear problem.

$\Downarrow$

Use a suitable reformulation for removing the product in the objective function.

Mathematical model: product of binary variables

A new variable:
$y_{uv} \in [0, 1]$, represents the product between x_u and x_v.

Objective function updated:
We substitute $x_u x_v$ with y_{uv}:

$$\max \left(\sum_{(u,v) \in A} y_{uv} - \sum_{v \in V} x_v \right)$$

Mathematical model: product of binary variables

Constraint updated:
We substitute $x_u x_v$ with y_{uv}:

$$\forall (u,v) \in A$$

$$y_{uv} \leq \min\left(\max(0, \mu(u,v) - k + 1), \max(0, k - \mu(u,v) + 1)\right)$$

New Linearization Constraints:

$$\begin{array}{ll} \forall (u,v) \in A & y_{uv} \leq x_u \\ \forall (u,v) \in A & y_{uv} \leq x_v \\ \forall (u,v) \in A & y_{uv} \geq x_u + x_v - 1 \end{array}$$