A note on
Bipartite Eulerian Planar Maps

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Abstract
Liskovets and Walsh recently counted bipartite eulerian planar maps, that is
embeddings of planar bipartite graphs with all vertices of even degrees. In this note
we present a simple one-to-one correspondence between the latter and the 4-eulerian
bijective enumeration then follows.

1 The number of rooted bipartite eulerian planar maps

A planar map is a proper embedding of a connected graph in the plane, considered up
to continuous deformations of the plane. We consider in this note the set $B$ of rooted
bipartite eulerian planar maps: (i) a root edge is chosen on the border of the infinite face
and oriented counterclockwise, (ii) the vertices are colored in black or white so that each
edge joins a black vertex to a white one, (iii) the degree of each vertex is even. Observe
that a bipartite eulerian map contains no loop nor bridge but may contain multiple edges.

Duality is an involution on the set of all planar maps in which the roles of vertices
and faces are exchanged: to construct the dual $M^*$ of a map $M$, put a vertex of $M^*$ in
each face of $M$ and draw an edge of $M^*$ across each edge of $M$. An easy classical remark
is that duality bijectively maps bipartite maps onto bicolor maps (i.e. maps in which
faces can be bicolorized), that are exactly eulerian maps. As a consequence, the set of
bipartite eulerian planar map is self dual, that is stable under duality. Hence alternative
characterizations of $B$ are as the set of rooted bipartite bicolor planar maps, or the set of
rooted eulerian dualy eulerian planar maps.

Liskovets and Walsh [4] recently obtained the following result.
Theorem 1 (Liskovets – Walsh, 2002) The number of rooted bipartite eulerian planar maps with \( n \) edges is

\[
|B_n| = \frac{4 \cdot 3^{n-1}(3n)!}{(2n+2)! \cdot n!}.
\]  

The enumerative theory of planar maps dates back to the sixties with Tutte’s seminal papers [6, 7, 8], in which he obtained a number of similar elegant formulas. However Formula (1) and the formula of Bouquet-Mêlou and Schaeffer for planar constellations [1] are the first new simple product form formulas to appear in planar maps enumeration since Walsh and Lehman’s work in the seventies [9]. Tutte used recursive decompositions to obtain functional equations for generating functions, and solved them with the quadratic method. Alternative proofs were given by physicists using matrix integrals [2]. A generating function for bipartite eulerian planar maps was obtained along similar lines [3]. Liskovets and Walsh’s formula (1) is derived from this generating function through Lagrange inversion. As opposed to this, Bousquet-Mêlou and Schaeffer’s approach for constellations is based on the conjugation of trees principle introduced by Schaeffer [5] for Tutte’s formulas.

In this note we give a bijective proof of Formula (1), based on two ingredients. The first is the edge-map or radial construction. The second is again conjugation of trees, and is a special case of the construction of Bousquet-Mêlou and Schaeffer [1]. Taken together, they allow to give a simple construction of Formula (1).

2 The edge-map construction

A map is bi-quartic if it is bipartite and all its vertices have degree four (compare to Tutte’s bi-cubic maps). The following edge-map construction (see Figure 1) is a variant of Tutte’s derivation of maps.

Theorem 2 There is a one-to-one correspondence between rooted bipartite eulerian planar maps with \( n \) edges and rooted bi-quartic planar maps with \( n \) vertices.

Proof. Let \( M \) be a rooted bipartite eulerian map, with vertices of colors black and white (filled or empty squares in Figure 1), and faces of colors green (filled down triangles) and white (empty up triangles). Define its edge-map \( M^e \) as follows.

- Place a vertex of \( M^e \) (red circle) on each edge of \( M \).
- Add an edge \((x, y)\) of \( M^e \) for each corner \( xy \) of \( M \), that is, if the two edges \( x \) and \( y \) of \( M \) are successively incident around the same vertex of \( M \).

An edge of \( M \) is incident to four corners, hence all vertices of \( M^e \) have degree four. Since \( M \) is bipartite bicolor, an edge of \( M \) joins vertices and separates faces of distinct colors. Thus, around each vertex of \( M^e \) the four faces have different types that form either a cycle filled square - empty triangle - empty square - filled triangle (filled red circles) or a cycle
Figure 1: A bipartite eulerian map and its (bi-quartic) edge-map, with the four colors of faces.

filled square - filled triangle - empty square - empty triangle (empty red circles). Adjacent vertices of $M^c$ have opposite cycle orientations so that the map $M^c$ is bi-quartic.

Conversely consider a bi-quartic map $E$. First $E$ is quartic, hence eulerian, and thus, bicolor. This allows to recover a 2-coloring of its faces in colors square and triangle. Upon placing vertices in square-faces of $E$ and joining them by edges through vertices of $E$, a planar map $M$ is recovered that clearly satisfies $M^c = E$.

Observe now that the degree of a vertex of $M$ is equal to the degree of the corresponding square-face of $E$, and that, similarly the degree of a face of $M$ is the degree of a corresponding triangle-face of $E$.

Since $E$ is bipartite, its dual $E^*$ is eulerian or, in other terms, the faces of $E$ have all even degree. Thus $M$ and $M^*$ are eulerian, or by duality, $M^*$ and $M$ are bipartite. Thus all colors can be recovered and $M$ is a bipartite eulerian map.

\[\square\]

3 Conjugation of trees

Formula (1) of Liskovets and Walsh is now recovered from the following theorem.

**Theorem 3 (Bousquet-Mélon – Schaeffer, 2000)** Rooted bi-quartic planar maps with $n$ vertices are in one-to-one correspondence with balanced blossom trees with $n - 1$ nodes and their number is

\[|B_n| = \frac{4}{2n+2} \cdot \frac{3^{n-1}}{2n+1} \binom{3n}{n}.\]

This result is a corollary of a more general theorem on the enumeration of planar constellations. Indeed bi-quartic maps exactly correspond to maximal 4-eulerian planar maps, so Theorem 3 follows upon taking $m = 4$, $d_1 = n$, and $d_i = 0$ for $i \geq 2$ in Theorem 2.3 and 5.2 of [1].

Since the case of bi-quartic planar maps is much simpler than the general case of constellations, we offer in Appendix a short presentation of the construction of Formula (2) using the conjugation of trees principle.
References


Appendix. A pictorial census of bi-quartic maps

The “construction” of Formula (2) is performed in five elementary steps.

1– There are

\[
\frac{1}{2n+1} \binom{3n}{n}
\]

planted ternary trees with \( n \) nodes. They have \( 2n + 2 \) leaves (root included), and \( n - 1 \) internal edges.
2- Consider all ways to attach two buds in the middle of each inner edge of a planted ternary tree. This can be done in three ways for each inner edge, yielding

$$\frac{3^n - 1}{2n + 1} \binom{3n}{n}$$

planted blossom trees with \( n \) white and \( n - 1 \) black nodes.

3- Match iteratively pairs of closest buds and leaves in counterclockwise direction around the tree. Each pair (bud, leaf) will be fused into a new edge. There are \( 2n - 1 \) buds and \( 2n + 2 \) leaves, so four leaves remain unmatched.

4- A blossom tree is balanced if its root remains unmatched.

Among the \( 2n + 2 \) possible rootings of an unplanted tree, 4 yield a balanced tree. Hence

$$\frac{4}{2n + 2} \cdot \frac{3^n - 1}{2n + 1} \binom{3n}{n}$$

is the number of balanced blossom trees with \( n \) white nodes.

5- Upon adding a black root vertex connected to the four free leaves, this closure is one-to-one between

- balanced blossom trees (\( n \) white nodes),
- and bi-quartic maps (\( 2n \) vertices).

Therefore the number of bi-quartic maps with \( 2n \) vertices is

$$\frac{4}{2n + 2} \cdot \frac{3^n - 1}{2n + 1} \binom{3n}{n}$$

This is also the number of bipartite bicolor maps with \( n \) edges.