Exam of the course 2-38-2 of MPRI 2021
Algorithms and combinatorics of geometric graphs

Handouts and handwritten course notes allowed. Electronic devices prohibited.

Prepare three separate copies for the three parts of the exam.

You can skip the questions that block you. However, it is recommended that you try to deal with a coherent part of the topic, even incomplete, rather than dealing sporadically with questions that seem easier.

The writting and presentation of your solutions will be an important criterion of the evaluation.

1 Planar graphs and Schnyder woods: number of constrained edge orientations

Our goal is to provide a few upper bounds on the number of distinct constrained edge orientations that a plane graph can have, focusing in particular on the number of Schnyder woods of plane triangulations.

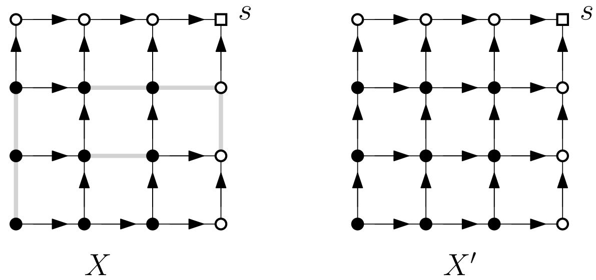

Figure 1:

In this exercise we will consider a plane graph $G = (V, E)$ provided with a planar embedding (also called planar map): we will denote by n the number of its vertices and by m the number of its edges. Given a function $\alpha : V \longrightarrow \mathbb{N}$, a *$\alpha$-orientation* X of G is an edge orientation of G such as for each vertex $v \in V$ the number of edges outgoing from v is exactly $\alpha(v)$ (observe that $\alpha(v) \leq d(v)$, where $d(v)$ denotes the degree of v in G).

1. Let us consider a subset $A \subset E$ of edges which is cycle-free and a function $\alpha : V \longrightarrow \mathbb{N}$. Let X be a given edge orientation of $E \setminus A$. Show that X can be extended to an α-orientation of G in at most one way. Show that the number of α-orientations of G is at most $2^{m-|A|}$. [*Hint: Proceed by induction on the size of A.*]

2. Using previous question, show that a plane graph G with n vertices admits at most 4^n distinct α-orientations.

We are going to illustrate how to improve the bound above.

3. Let us consider an independent set [1] of G, denoted by I_2, consisting of n_2 vertices which have degree 2 in G. Show that G has at most $(3n-6)-(n_2-1)$ edges.

4. Let us consider an independent set $I = I_1 \cup I_2$ of G, where I_2 is the subset of degree 2 vertices in I. Show that G has at most $(3n-6)-(c-1)-(|I_2|-1)$ edges, where c is the number of connected components of the sub-graph G' induced by the vertex set $V' = V \setminus I$.

5. Let us consider a plane graph G provided with an independent set $I = I_1 \cup I_2$ and a function $\alpha : V \longrightarrow \mathbb{N}$ as defined above. Show that G admits at most

$$2^{2n-4-|I_2|} \cdot \prod_{v \in I_1} \left(\frac{1}{2^{d(v)-1}} \binom{d(v)}{\alpha(v)} \right)$$

distinct α-orientations.
[*Hint: Use the results stated in previous questions.*] Pascal's formula could be useful: $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$.

Let us remind (recall lectures) that any plane triangulation T with root face [2] $f = (v_0, v_1, v_2)$ (we will denote the rooted triangulation with the pair (T, f)) can be endowed with a Schnyder wood. Such a Schnyder wood defines an orientation of the inner edges [3] of T such that every inner vertex has exactly three outgoing edges. It is possible to prove a stronger result, as stated below

Proposition 1. *Let us consider a plane triangulation T with root face $f = (v_0, v_1, v_2)$. Then there exists a bijection between the Schnyder woods of (T, f) and the α-orientations of $T \setminus \{(v_0, v_1), (v_1, v_2), (v_2, v_0)\}$ such that $\alpha(v) = 3$ for each $v \in V \setminus \{v_0, v_1, v_2\}$ and $\alpha(v) = 0$ for $v \in \{v_0, v_1, v_2\}$.*

The statement above involves a rooted triangulation, with a root face (v_0, v_1, v_2). We will show that the number of Schnyder woods actually does not depend on the choice of the root face.

6. Let us consider two neighboring $f = (v_0, v_1, v_2)$ and $f' = (v'_0, v_1, v_2)$ (sharing the edge (v_1, v_2)) of a plane triangulation T. Show that it is possible to transform in a bijective way each Schnyder wood of (T, f) into a Schnyder wood of (T, f').

7. Let us consider two arbitrary faces $f = (v_0, v_1, v_2)$ and $f' = (v'_0, v'_1, v'_2)$ of a plane triangulation T. Show that there is a bjection between the Schnyder woods of T having f as root face and the Schnyder woods of T as f' as root face.

8. Show that a plane triangulation T with n vertices admits at most $2^{2n-4}(\frac{5}{8})^{\frac{n}{4}}$ distinct Schnyder woods.
[*Hint: recall the result stated in question 5 and the bound $2^{1-d}\binom{d}{3} \leq \frac{5}{8}$, which holds for any integer value $d \geq 3$. Remark: planar graphs do admit a 4-coloration of the vertices (by the* Four colors *theorem).*]

[1] A subset of vertices $I \subset V$ is *independent* if the vertices of I are pairwise non adjacent.
[2] In our drawings the root face does coincide sometimes with the (infinite) outer face: observe that the choice of the root face is arbitrary.
[3] The *inner* edges of (T, f) are the edges not lying on the root face: all the edges but $(v_0, v_1), (v_1, v_2), (v_2, v_0)$.

2 Polytopes and triangulations (VP) : Minimal number of partial triangulations

Recall that a partial triangulation of a planar point set P is a set of triangles whose interiors are disjoint, which cover the convex hull $\mathrm{conv}(P)$ of P, and whose vertices all belong to P (but not all points of P need to be vertices of triangles). The goal of the problem is to show[4] that any set of $n+2$ points in general position in the plane has at least as many partial triangulations as a set of $n+2$ points in convex position.[5]

1. We denote by T_n the number of triangulations of a set of $n+2$ points in convex position in the plane.

(a) Give a formula for T_n and an idea of proof in less than 3 lines (details should be omitted).

(b) Prove that $T_{n_1} \cdot T_{n_2} \cdots T_{n_k} \le T_{n_1+\cdots+n_k}$ for any $n_1, \dots, n_k \in \mathbb{N}$.
[*Hint: The formula of Point (i) is not needed to prove Point (ii).*]

2. Let P be a set of $n+2$ points in general position in the plane. We say that an interior point p of P is *almost exterior* if there exists an edge qr of $\mathrm{conv}(P)$ such that any triangle qrs with $s \in P$ contains p. We say that P is *almost convex* if all its interior points are almost exterior. See Figure 2.

(a) Give two examples of almost convex sets studied in class.

(b) Show that an almost convex set with $n+2$ points has T_n partial triangulations.[6]

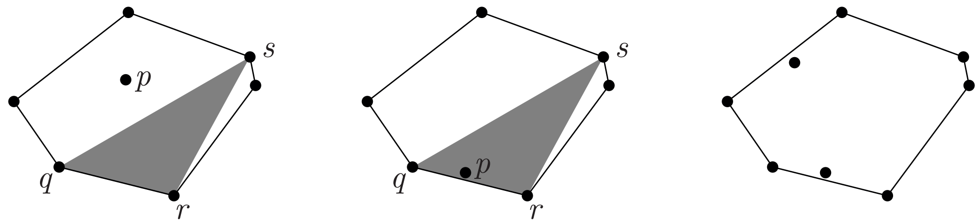

Figure 2: An interior point p which is not almost exterior (left), an interior point p almost exterior (middle), and an almost convex set with two interior points (right).

3. We now consider an arbitrary set P of $n+2$ points in general position in the plane. We choose an arbitrary vertex p of $\mathrm{conv}(P)$ and we denote by $p_0, \dots, p_n$ the other points of P in counterclockwise order around p in such a way that p_0 and p_n are the neighbors of p on the boundary of $\mathrm{conv}(P)$. See Figure 3 (left). We call *polyline* any polygonal line connecting p_0 to p_n in counterclockwise order, or said differently any sequence of points $\pi = p_0, p_{i_1}, \dots, p_{i_k}, p_n$ with $0 < i_1 < \cdots < i_k < n$. We call *signature* of the polyline $\pi = p_0, p_{i_1}, \dots, p_{i_k}, p_n$ the word $\sigma(\pi) = \sigma(\pi)_1 \dots \sigma(\pi)_{n-1}$ with $n-1$ letters on the alphabet $\{u, d\}$ defined as follows. For any $1 \le j \le n-1$, we denote by ℓ and r the indices such that $i_\ell < j < i_r$ and $r - \ell$ is minimal, and we define

- if p_j is a point of π, then $\sigma(\pi)_j = u$ if the segments $[p, p_j]$ and $[p_{i_\ell}, p_{i_r}]$ cross, and $\sigma(\pi)_j = d$ otherwise,

[4] Following the recent proof of A. Kupavskii, A. Volostnov and Y. Yarovikov in arxiv:2104.05855.

[5] In fact, the result still holds if we consider
- only regular triangulations of the points sets,
- all polyhedral subdivisions of the points sets.

We thus obtain that the f-vector of the secondary polytope of a set of $n+2$ points in general position in the plane is (componentwise) larger or equal to the f-vector of the associahedron of dimension $n-1$. This result was recently proved by A. Fernandez and F. Santos in arxiv:2110.00544.

[6] One can even show that a set of $n+2$ points in general position in the plane has exactly T_n partial triangulations if and only if it is almost convex. See arxiv:2104.05855.

- if p_j is not a point of π, then $\sigma(\pi)_j = d$ if the segments $[p, p_j]$ and $p_{i_\ell}, p_{i_r}]$ cross, and $\sigma(\pi)_j = u$ otherwise.

Intuitively, if we place nails at the points of P and an elastic strip connecting p_0 to p_n and passing between p and p_j if and only if $\sigma(\pi)_j = d$, then we obtain π by stretching the elastic strip. See Figure 3 (middle and right).

(a) What is the number of different polylines connecting p_0 to p_n?
(b) Describe the signatures of the polylines p_0, p_n and $p_0, p_1, \dots, p_{n-1}, p_n$.
(c) Describe the polylines of signature u^{n-1} and d^{n-1}.
(d) Show that σ defines a bijection between the polylines connecting p_0 to p_n and the words with $n-1$ letters on the alphabet $\{u, d\}$.
[*Hint: To show that any signature σ corresponds to a polyline, we could give an intuitive argument based on the elastic strip, but we will prefer a proof by induction considering the point p_k which minimizes the angle $\widehat{pp_0p_k}$ and distinguishing two cases according to σ_k.*]

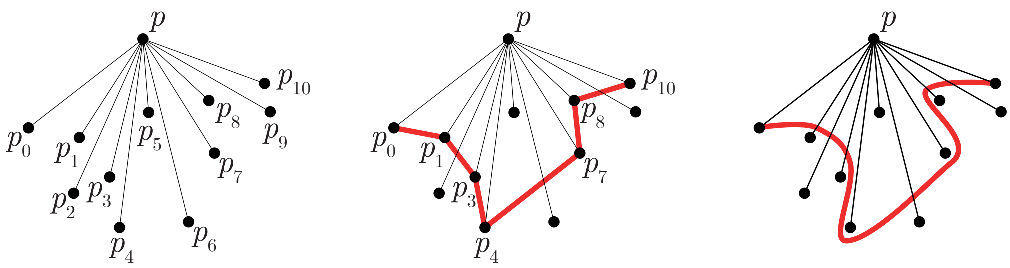

Figure 3: A point set P ordered counterclockwise around p (left), a polyline of signature $ddduududd$ (middle), et its interpretation by an elastic strip (right).

4. We now consider a partial triangulation T of P. We call *polyline* of T the polygonal line formed by the point of P adjacent to p in T. We call *signature* of T the signature of the polyline of T. See Figure 4 (left). We fix a signature $\sigma \in \{u, d\}^{n-1}$. Let $n_1 < \cdots < n_{k-1}$ denote the positions $0 < n_j < n$ such that $\sigma_{n_j} = u$, and let $n_0 = 0$ and $n_k = n$ by convention.

(a) Prove that if P is in convex position, then it admits exactly $\prod_{j \in [k]} T_{n_j - n_{j-1}}$ partial triangulations of signature σ.
(b) Assume now that P is not in convex position. Let π be the polyline of P with signature σ. Let $m_1 < \cdots < m_{\ell-1}$ denote the indices $0 < m_j < n$ such that $\sigma_{m_j} = u$ and p_{m_j} is a point of π, and let $m_0 = 0$ and $m_\ell = n$ by convention. In other words, the polyline π is convex (with respect to the point p) at the points $p_{m_1}, \dots, p_{m_{\ell-1}}$ and concave (with respect to the point p) between these points. Let S be the subdivision of $\mathrm{conv}(P)$ whose edges are
  - all the boundary edges of $\mathrm{conv}(P)$,
  - all edges connecting p to a points of π,
  - for any $1 \le j \le \ell$, the edges of the convex hull of

$$\{p_{m_{j-1}}\} \cup \{p_i \mid m_{j-1} < i < m_\ell \text{ and } \sigma_i = d\} \cup \{p_{m_j}\}.$$

See Figure 4 (right). Prove that the number of partial triangulations of S with signature σ is at least $\prod_{j \in [\ell]} T_{m_j - m_{j-1}}$.
(c) Using Question 1 (ii), prove that

$$\prod_{j \in [k]} T_{n_j - n_{j-1}} \le \prod_{j \in [\ell]} T_{m_j - m_{j-1}}.$$

(d) Deduce that any set P of $n+2$ points in general position in the plane has at least as many partial triangulations as a set of $n+2$ points in convex position in the plane.

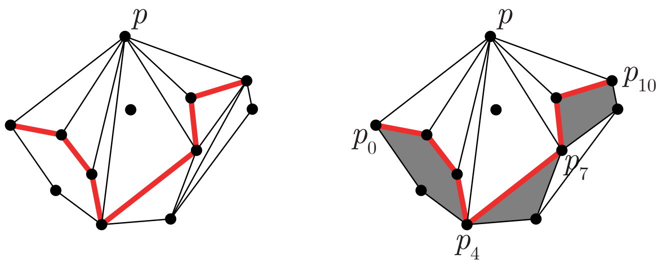

Figure 4: The polyline of a partial triangulation T of P (left) and the subdivision S of the polyline π, where the convex hulls of the $\{p_{m_{j-1}}\} \cup \{p_i \mid m_{j-1} < i < m_\ell \text{ and } \sigma_i = d\} \cup \{p_{m_j}\}$ are highlighted (right).

3 Graphs on surfaces (ECdV): Shortest non-contractible cycles

Let $\mathscr{S}$ be a compact, orientable surface without boundary of genus g. Let G be a graph cellularly embedded on $\mathscr{S}$, with complexity n. Here, G is unweighted; the length of a path (closed or not) in G is simply the number of edges of G that are used (with multiplicity, if the path uses some edges several times). *Questions 1, 2, and 3 are independent.*

1. Show that a shortest non-contractible closed path in G uses each vertex of G at most once.

2. Let s be an arbitrary vertex of G.

(a) Justify that one can, in $O(gn)$ time, compute a shortest system of loops based at s in the cross-metric surface $(\mathscr{S}, G^*)$ where G^* is the dual graph of G.

Reminder: This system of loops is made of $4g$ loops, all obtained by the concatenation of a shortest path from s, a path crossing a single edge of G^*, and a shortest path to s.

(b) Show that each non-contractible closed path in G intersects at least one loop of the system of loops.

(c) Deduce that one can, in $O(gn)$ time, compute a set of vertices S of G such that:

i. each closed path in G that is non-contractible goes through a vertex of S;

ii. for each integer i, there are at most $4g$ vertices in S whose distance to s is exactly i.

3. (a) Let T be the set of edges of a spanning tree of G. For each edge $e \in G \setminus T$, let $\sigma(e)$ be the closed path that is the concatenation of e and of the path in T connecting the endpoints of e. Show that one can, in $O(n)$ time, determine the edges e of $G \setminus T$ such that $\sigma(e)$ is non-contractible.

(b) Let $k \geq 1$ be an integer. Let V be a set of vertices of G such that the distance between any pair of distinct points in V is at least $k+1$.

Show that one can, in $O(n)$ time, compute a shortest non-contractible closed path among those passing through a vertex of V, if there exists a such a path of length at most k, or determine that each non-contractible closed path passing through a vertex of V has length at least $k+1$.

Hint: Choose T such that it contains, for each $v \in V$, a shortest path between v and any vertex at distance at most $\lfloor k/2 \rfloor$ of v.

4. Using Questions 2 and 3, show that one can, for each k, determine in $O(gnk)$ time a shortest non-contractible closed path in G, if there is such a path of length at most k.

5. Deduce an algorithm that computes a shortest non-contractible closed path in G in time $O(gnk)$, where k is the (initially unknown) length of such a path.