This homework is due November 18, via e-mail at arnaud.de-mesmay@univ-eiffel.fr. Also do not hesitate to ask me if you have questions. The language of your homework can be either English or French. The exercises are independent and can be treated in any order.

Exercise 1:

For $G = (V, E)$ a simple graph (without loops nor multiple edges), the **complement** of $G$ is the graph with the same vertex set $V$, and where two vertices $u$ and $v$ are connected if and only if they are not connected in $G$.

1. Let $G$ be a simple planar graph with 11 vertices. Prove that the complement of $G$ is not planar.

2. Let $G$ be a simple graph embeddable on an orientable surface of genus $g$ with $n$ vertices. For which values of $n$ (depending on $g$) can we prove that the complement of $G$ is not embeddable on an orientable surface of genus $g$? Bonus points depending on how good your bound is.

Exercise 2:

A convenient way to represent a graph on a non-orientable surface is to draw it on top of its canonical polygonal scheme $a_1a_1 \ldots a_ga_g$. For example, here is a cellular embedding of $K_5$ on a non-orientable surface of genus two.

![Cellular embedding of K5](image)

1. Provide an explicit cellular embedding of the graph pictured below on a non-orientable surface of genus 3.

![Graph](image)

2. Let $G$ be a simple graph with $v$ vertices, $e$ edges cellularly embedded on a non-orientable surface of genus $g$. Prove that $g \leq e - v + 1$. 
3. Let $G$ be a simple graph with $v$ vertices and $e$ edges, and let $g_1$ be the smallest genus of a non-orientable surface on which $G$ embeds. Prove that for any $g$ such that $g_1 \leq g \leq e - v + 1$, $G$ can be cellularly embedded on a non-orientable surface of genus $g$.

4. In particular, $G$ can always be cellularly embedded on a non-orientable surface of genus $e - v + 1$. Provide a linear-time algorithm to compute such an embedding.

**Exercise 3:**

1. Let $G$ be a graph embedded on an orientable surface of genus $g$, not necessarily cellularly. Prove that $v - e + f \geq 2 - 2g$, where $v$, $e$ and $f$ denote respectively the number of vertices, edges and faces of the graph embedding.

2. Let $G$ be a simple graph cellularly embedded on an orientable surface of genus $g$, with the properties that (1) all the faces have degree three (i.e., are incident to three edges), and (2) each cycle of length 3 in the graph bounds a face. The set of such (triangular) faces is denoted by $T$. Use the previous question to show that in any embedding of $G$, the number of faces is $|T|$. Deduce that the embedding of $G$ on an orientable surface of genus $g$ is unique up to homeomorphism.

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1In the graph-theoretical sense: a walk in the graph without repeated edges or faces