

## Spines

$T$  a tree on a signed ground set  $V = V^- \sqcup V^+$ .

**Spine** on  $T$  = directed and labeled tree  $S$  such that

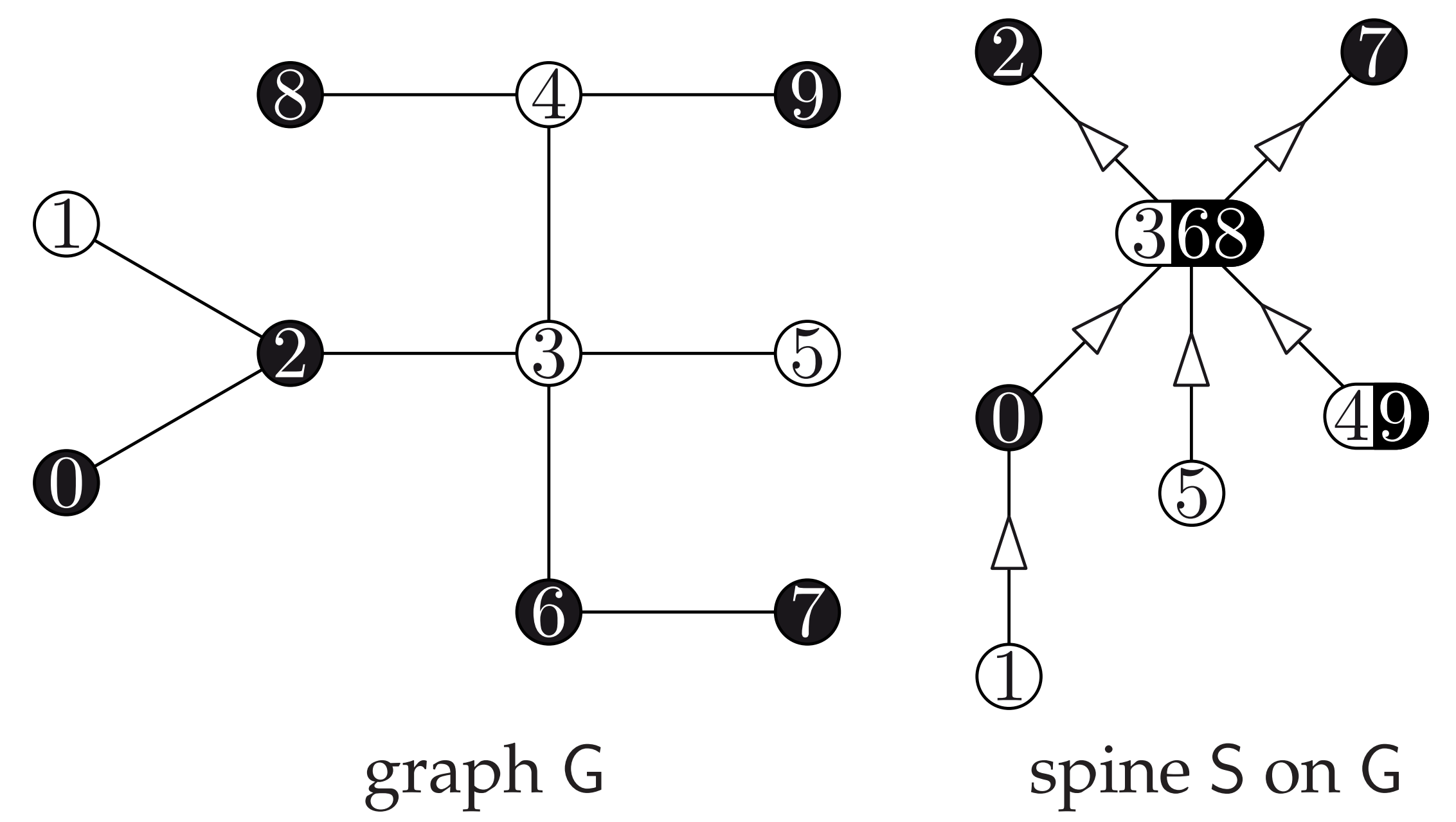
- the labels of the nodes of  $S$  form a partition of the signed ground set  $V$ ,
- at a node labeled by  $U = U^- \sqcup U^+$ , the source label sets of the incoming arcs are subsets of distinct connected components of  $T \setminus U^-$ , and the sink label sets of the outgoing arcs are subsets of distinct connected components of  $T \setminus U^+$ .

**Spine poset**  $\mathcal{S}(T)$  = poset of arc contractions on signed spines of  $T$ .

**Prop.** The spine poset  $\mathcal{S}(T)$  is a pure graded poset of rank  $|V|$ .

**Signed nested complex** = simplicial complex  $\mathcal{N}(T) = \{N(S) \mid S \in \mathcal{S}(T)\}$ ,  
where  $N(S)$  = collection of source sets of  $S$ .

Exm.  $V^- = \{1, 3, 4, 5\}$   $V^+ = \{0, 2, 6, 7, 8, 9\}$



## Spine fan

Ambiant space  $\mathbb{H} = \{\mathbf{x} \in \mathbb{R}^V \mid \sum_{v \in V} x_v = \binom{|V|+1}{2}\}$ .

**Cone**  $C(S)$  of a spine  $S = \{\mathbf{x} \in \mathbb{H} \mid x_u \leq x_v \text{ for all } u \rightarrow v \text{ in } S\}$ .

**Theo.** The collection of cones  $\mathcal{F}(T) = \{C(S) \mid S \in \mathcal{S}(T)\}$  defines a complete simplicial fan on  $\mathbb{H}$ , called the **spine fan** of  $T$ .

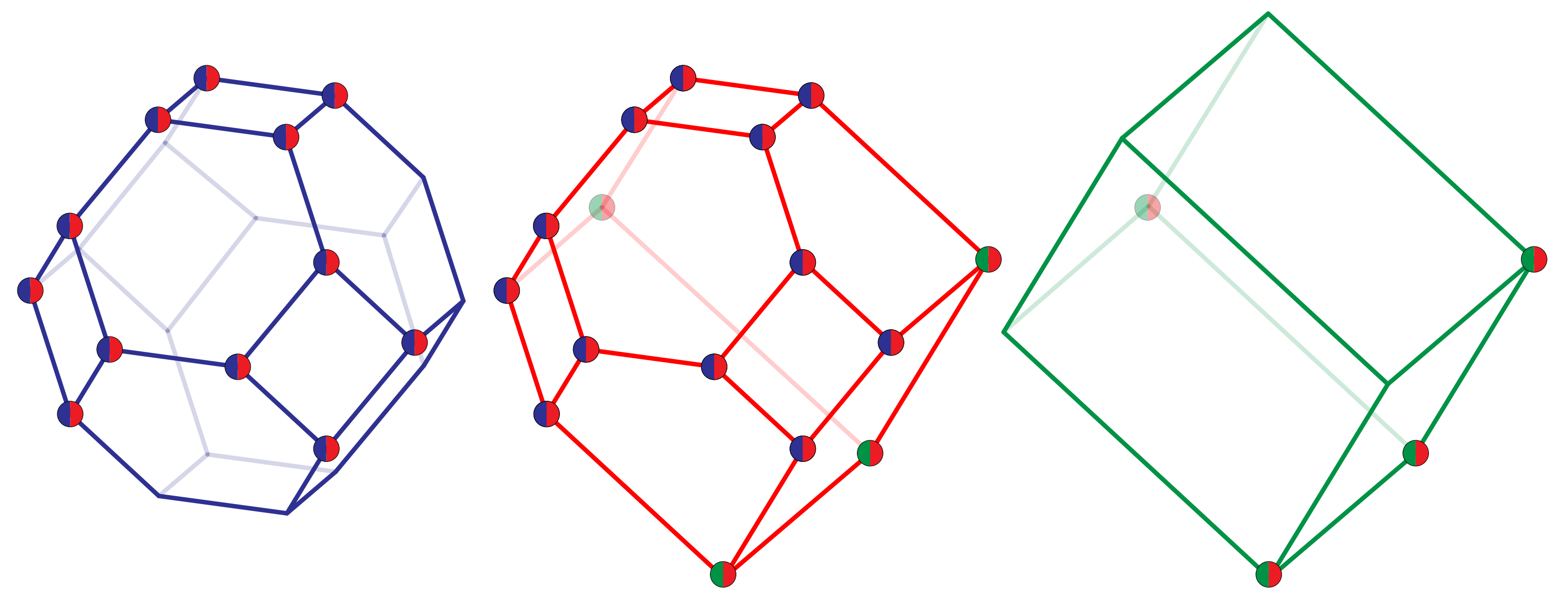
The spine fan  $\mathcal{F}(T)$  coarsens the braid fan on  $\mathbb{H}$ . It defines a map  $\kappa$  from linear orders on  $V$  to maximal spines on  $T$ .

**Prop.** The fibers of  $\kappa$  are the classes of **T-congruence** defined by  $XuwY \equiv_T XvuY$  iff there is  $w \in V$  in between  $u$  and  $v$  in  $T$  and such that  $w \in X \cap V^+$  or  $w \in Y \cap V^-$ .

## Some properties

**Prop.** The signed tree associahedron  $\text{Asso}(T)$  is sandwiched between the permutahedron  $\text{Perm}(V)$  and the parallelepiped  $\text{Para}(T)$ :

$$\sum_{u \neq v \in V} [e_u, e_v] = \text{Perm}(T) \subset \text{Asso}(T) \subset \text{Para}(T) = \sum_{uv \in T} \pi_{uv} [e_u, e_v]$$



Common vertices of

- $\text{Asso}(T)$  and  $\text{Para}(T) \equiv$  orientations of  $T$  which are spines on  $T$ ,
  - $\text{Asso}(T)$  and  $\text{Perm}(T) \equiv$  linear orders on  $V$  which are spines on  $T$ ,
- $\Rightarrow$  no common vertex of the three polytopes except if  $T$  = signed path.

**Prop.**  $\text{Asso}(T)$  and  $\text{Asso}(T')$  isometric  $\iff T$  and  $T'$  isomorphic or anti-isomorphic up to the signs of their leaves, i.e. there is a bijection  $\theta : V \rightarrow V'$  st.  $\forall u, v \in V$

- $u-v$  edge in  $T \iff \theta(u)-\theta(v)$  edge in  $T'$ ,
- if  $u$  is not a leaf of  $T$ , the signs of  $u$  and  $\theta(u)$  coincide (resp. differ).

## Signed tree associahedron

**Theo.** The spine fan  $\mathcal{F}(T)$  is the normal fan of the signed tree associahedron  $\text{Asso}(T)$  with

- a vertex  $\mathbf{a}(S) \in \mathbb{R}^V$  for each maximal  $S \in \mathcal{S}(T)$ , with coordinates

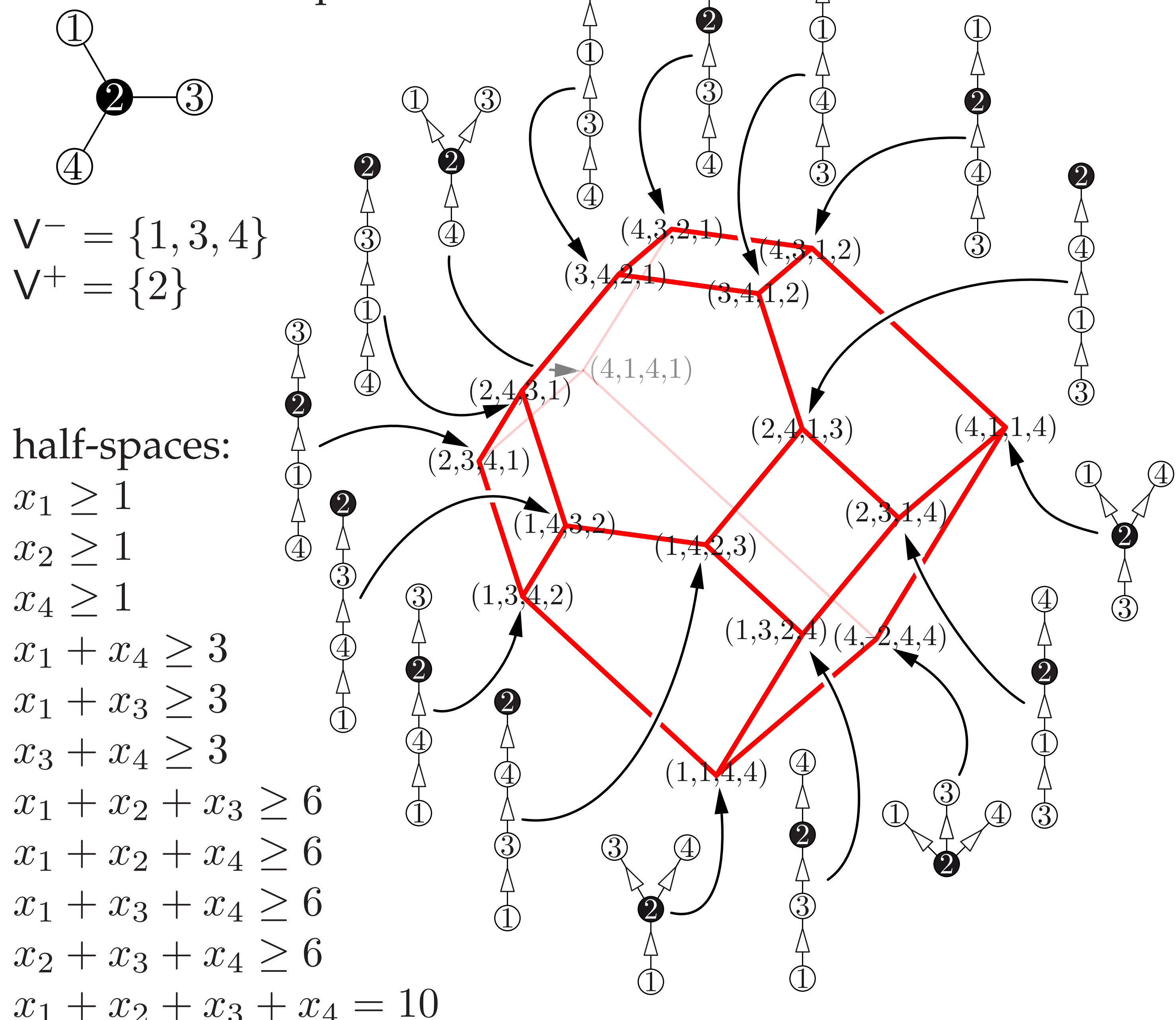
$$\mathbf{a}(S)_v = \begin{cases} |\{\pi \in \Pi(S) \mid v \in \pi \text{ and } r_v \notin \pi\}| & \text{if } v \in V^- \\ |V| + 1 - |\{\pi \in \Pi(S) \mid v \in \pi \text{ and } r_v \notin \pi\}| & \text{if } v \in V^+ \end{cases}$$

where  $r_v$  = unique incoming (outgoing) arc at  $v \in V^-$  ( $v \in V^+$ ),  
 $\Pi(S) = \{(undirected) \text{ paths in } S\}$ ,

- a facet for each  $B \in \bigcup_{S \in \mathcal{S}(T)} N(S)$  defined by the half-space

$$\left\{ \mathbf{x} \in \mathbb{R}^V \mid \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}.$$

Exm. For the tripod



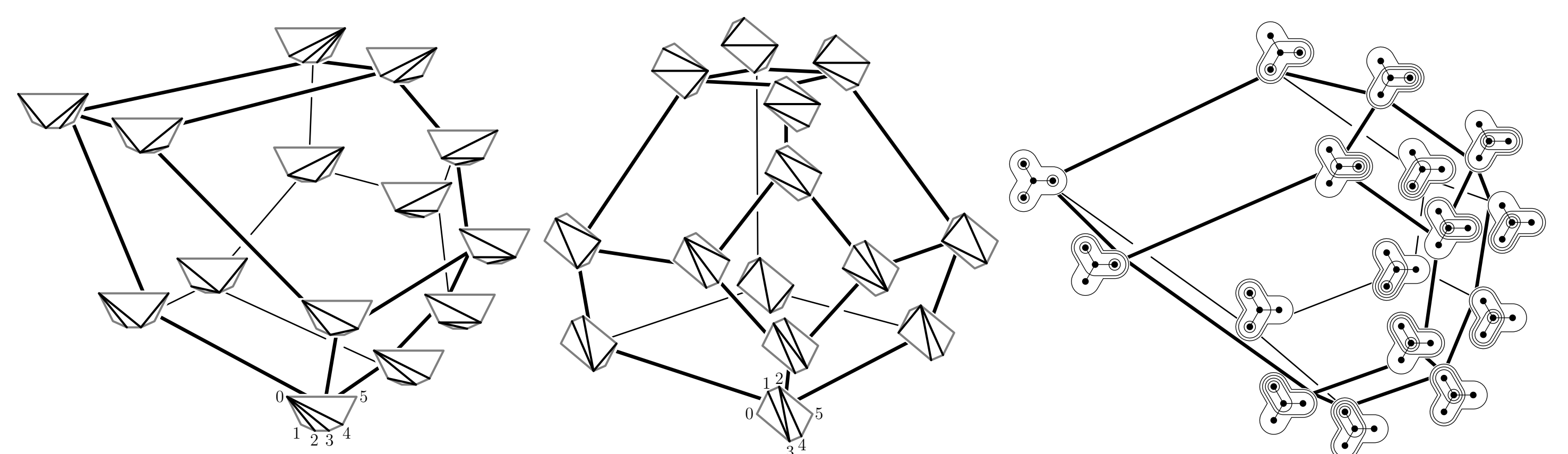
## Examples

For a signed path  $P$ ,  $\text{Asso}(P)$  is the classical associahedron

faces  $\longleftrightarrow$  dissections  $\longleftrightarrow$  Schröder trees,  
vertices  $\longleftrightarrow$  triangulations  $\longleftrightarrow$  binary trees.

Loday, Realization of the Stasheff polytope, 2004

Hohlweg & Lange, Realizations of the associahedron and cyclohedron, 2007



For an unsigned tree  $T$ ,  $\text{Asso}(T)$  is the  $T$ -associahedron

facets  $\longleftrightarrow$  tubes = connected induced subgraphs of  $T$ ,  
faces  $\longleftrightarrow$  tubings = collections of tubes which are pairwise nested, or disjoint and non-adjacent.

Carr & Devadoss, Coxeter complexes and graph associahedra, 2006