

# BRICK POLYTOPES, LATTICE QUOTIENTS, AND HOPF ALGEBRAS

VINCENT PILAUD

ABSTRACT. This paper is motivated by the interplay between the Tamari lattice, J.-L. Loday's realization of the associahedron, and J.-L. Loday and M. Ronco's Hopf algebra on binary trees. We show that these constructions extend in the world of acyclic  $k$ -triangulations, which were already considered as the vertices of V. Pilaud and F. Santos' brick polytopes. We describe combinatorially a natural surjection from the permutations to the acyclic  $k$ -triangulations. We show that the fibers of this surjection are the classes of the congruence  $\equiv^k$  on  $\mathfrak{S}_n$  defined as the transitive closure of the rewriting rule  $UacV_1b_1 \cdots V_k b_k W \equiv^k UcaV_1b_1 \cdots V_k b_k W$  for letters  $a < b_1, \dots, b_k < c$  and words  $U, V_1, \dots, V_k, W$  on  $[n]$ . We then show that the increasing flip order on  $k$ -triangulations is the lattice quotient of the weak order by this congruence. Moreover, we use this surjection to define a Hopf subalgebra of C. Malvenuto and C. Reutenauer's Hopf algebra on permutations, indexed by acyclic  $k$ -triangulations, and to describe the product and coproduct in this algebra and its dual in term of combinatorial operations on acyclic  $k$ -triangulations. Finally, we extend our results in three directions, describing a Cambrian, a tuple, and a Schröder version of these constructions.

## CONTENTS

<b>Introduction</b>	2
<b>Part 1. Acyclic twists: combinatorics, geometry, and algebra</b>	4
<b>1.1. Combinatorics of twists</b>	4
1.1.1. Pipe dreams and twists	4
1.1.2. Twists and multitriangulations	5
1.1.3. Numerology	7
1.1.4. Elementary properties of pipes and twists	8
1.1.5. Pipe insertion and deletion	9
1.1.6. $k$ -twist correspondence	10
1.1.7. $k$ -twist congruence	12
1.1.8. $k$ -twist congruence classes and WOIPs	13
1.1.9. Lattice congruences of the weak order	14
1.1.10. Increasing flip lattice	14
1.1.11. $k$ -recoil schemes	15
1.1.12. $k$ -canopy schemes	18
<b>1.2. Geometry of acyclic twists</b>	20
1.2.1. Permutahedra, brick polytopes, and zonotopes	20
1.2.2. The geometry of the surjections $\text{ins}^k$ , $\text{can}^k$ , and $\text{rec}^k$	24
<b>1.3. Algebra of acyclic twists</b>	25
1.3.1. Hopf algebras $\text{FQSym}$ and $\text{FQSym}^*$	25
1.3.2. Subalgebra of $\text{FQSym}$	26
1.3.3. Quotient algebra of $\text{FQSym}^*$	29
1.3.4. Multiplicative bases and irreducible elements	30
1.3.5. Integer point transform	32
1.3.6. $k$ -twistiform algebras	33
<b>Part 2. Three extensions</b>	36
<b>2.1. Cambrianization</b>	36
2.1.1. Combinatorics of Cambrian twists	36
2.1.2. Geometry of acyclic Cambrian twists	45
2.1.3. Algebra of acyclic Cambrian twists	45

---

Partially supported by the Spanish MICINN grant MTM2011-22792 and by the French ANR grants EGOS (12 JS02 002 01) and SC3A (15 CE40 0004 01).

<b>2.2. Tuplization</b>	48
2.2.1. Cambrian $k$ -twist $\ell$ -tuples	48
2.2.2. Geometric realization	49
2.2.3. Algebra of acyclic Cambrian twist tuples	50
2.2.4. Twin twists	51
<b>2.3. Schröderization</b>	52
2.3.1. Ordered partitions, hypertwists, and partial orientations	52
2.3.2. Algebra of acyclic hypertwists	56
<b>Acknowledgements</b>	57
<b>References</b>	57

## INTRODUCTION

The motivation of this paper comes from relevant combinatorial, geometric, and algebraic structures on permutations, binary trees and binary sequences. Classical surjections from permutations to binary trees (BST insertion) and from binary trees to binary sequences (canopy) yield:

- lattice morphisms from the weak order, via the Tamari lattice, to the boolean lattice;
- normal fan coarsenings from the permutahedron, via J.-L. Loday’s associahedron [Lod04], to the parallelepiped generated by the simple roots  $\mathbf{e}_{i+1} - \mathbf{e}_i$ ;
- Hopf algebra inclusions from C. Malvenuto and C. Reutenauer’s algebra [MR95], via J.-L. Loday and M. Ronco’s algebra [LR98], to L. Solomon’s descent algebra [Sol76].

These fascinating connections were widely extended by N. Reading in his work on “Lattice congruences, fans and Hopf algebras” [Rea05]. In particular, he proves that any lattice congruence  $\equiv$  of the weak order on the permutations of  $\mathfrak{S}_n$  (see Section 1.1.9 for proper definitions) defines a complete simplicial fan  $\mathcal{F}_\equiv$  refined by the Coxeter fan, and he characterizes in terms of simple rewriting rules the families  $(\equiv_n)_{n \in \mathbb{N}}$  of lattice congruences of the weak orders on  $(\mathfrak{S}_n)_{n \in \mathbb{N}}$  which yield Hopf subalgebras of C. Malvenuto and C. Reutenauer’s algebra on permutations. His work opens two natural questions. On the geometric side, it is not clear which of the fans  $\mathcal{F}_\equiv$  are actually normal fans of polytopes, as in the previous example of the associahedron. On the algebraic side, this construction produces a combinatorial Hopf algebra whose basis is indexed by the congruence classes of  $(\equiv_n)_{n \in \mathbb{N}}$ . However, the product and coproduct in this Hopf algebra are performed extrinsically: the algebra is embedded in C. Malvenuto and C. Reutenauer’s algebra on permutations and the computations are performed at that level. The remaining challenge is to realize the resulting Hopf algebra intrinsically by attaching a combinatorial object to each congruence class of  $(\equiv_n)_{n \in \mathbb{N}}$  and working out the rules for product and coproduct directly on these combinatorial objects. The present paper answers these two questions for a relevant family of lattice congruences of the weak order, generalizing the classical sylvester congruence [HNT05].

Our starting point is the world of acyclic multitriangulations. A  $k$ -*triangulation* of a convex  $(n+2k)$ -gon is a maximal set of diagonals such that no  $k+1$  of them are pairwise crossing. Multitriangulations were introduced by V. Capowleas and J. Pach [CP92] in the context of extremal theory for geometric graphs and studied for their rich combinatorial properties [Nak00, DKM02, Jon05, PS09]. Using classical point-line duality, V. Pilaud and M. Pocchiola interpreted  $k$ -triangulations of the  $(n+2k)$ -gon as pseudoline arrangements on  $n$ -level sorting networks [PP12], which can also be seen more combinatorially as beam arrangements in a trapezoidal shape [Pil10, Section 4.1.4]. In this paper, we call these specific arrangements  $(k, n)$ -*twists*. As observed in [Stu11, SS12], this connects multitriangulations to (specific) pipe dreams studied in [BB93, KM05]. Motivated by possible polytopal realizations of the simplicial complex of  $(k+1)$ -crossing-free sets of diagonals of a convex  $(n+2k)$ -gon, V. Pilaud and F. Santos defined in [PS12] the brick polytope of a sorting network, whose vertices correspond to certain *acyclic* pseudoline arrangements on the network. When  $k=1$ , the pseudoline arrangements on the trapezoidal network correspond to the triangulations of the  $(n+2)$ -gon. They are all acyclic and the brick polytope coincides with J.-L. Loday’s associahedron [Lod04]. The goal of this paper is to explore further properties of the acyclic  $(k, n)$ -twists for arbitrary  $k$  and  $n$ .

The first part of this paper studies the combinatorial, geometric, and algebraic structure of the set of acyclic  $(k, n)$ -twists. In Section 1.1, we present a purely combinatorial description of the natural map  $\text{ins}^k$  from the permutations of  $\mathfrak{S}_n$  to the acyclic  $(k, n)$ -twists in terms of successive insertions in a  $k$ -twist. Extending the sylvester congruence of [HNT05], we show that the fibers of this map are the classes of a congruence  $\equiv^k$  defined as the transitive closure of the rewriting rule  $UacV_1b_1V_2b_2\cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2\cdots V_kb_kW$  where  $a, b_1, \dots, b_k, c \in [n]$  are such that  $a < b_i < c$  for all  $i \in [k]$  and  $U, V_1, \dots, V_k, W$  are words on  $[n]$ . This congruence is a lattice congruence of the weak order, so that the increasing flip order on the acyclic  $(k, n)$ -twists defines a lattice, generalizing the Tamari lattice. We also define a canopy map  $\text{can}^k$  from the acyclic  $(k, n)$ -twists to the acyclic orientations of the graph  $G^k(n) = ([n], \{\{i, j\} \mid |i - j| \leq k\})$ . For  $\tau \in \mathfrak{S}_n$ , the composition  $\text{rec}^k(\tau) = \text{can}^k(\text{ins}^k(\tau))$  records the relative positions in  $\tau$  of any entries  $i$  and  $j$  such that  $|i - j| \leq k$ , thus generalizing the recoils of the permutation  $\tau$ . Note that this generalization of recoils was already considered by J.-C. Novelli, C. Reutenauer and J.-Y. Thibon in [NRT11] with a slightly different presentation. To sum up, at the combinatorial level, we obtain a commutative triangle of lattice homomorphisms from the weak order, via the increasing flip order on the acyclic  $(k, n)$ -twists, to the boolean lattice on acyclic orientations of  $G^k(n)$ . Extensions of these combinatorial properties to the increasing flip order on the vertices of the brick polytope of the subword complex  $\mathcal{SC}(c^k w_\circ(c), w_\circ)$  for a Coxeter element  $c$  of an arbitrary finite Coxeter group will be discussed in a future paper.

In Section 1.2, we survey and revisit the geometric aspects of these combinatorial maps. We recall the definitions of the classical permutahedron  $\text{Perm}^k(n)$ , of the brick polytope  $\text{Brick}^k(n)$  of the  $n$ -level trapezoidal network [PS12], and of the zonotope  $\text{Zono}^k(n)$  of the graph  $G^k(n)$ . Their vertices correspond to the permutations of  $\mathfrak{S}_n$ , to the acyclic  $(k, n)$ -twists, and to the acyclic orientations of  $G^k(n)$ , respectively. When oriented in the direction  $(n - 1, n - 3, \dots, 1 - n)$ , their 1-skeleta are the Hasse diagrams of the weak order on permutations of  $\mathfrak{S}_n$ , of the increasing flip lattice on acyclic  $(k, n)$ -twists, and of the boolean lattice on acyclic orientations of  $G^k(n)$ . The maps  $\text{ins}^k$ ,  $\text{rec}^k$  and  $\text{can}^k$  can be read as inclusions of normal cones of vertices of  $\text{Perm}^k(n)$ ,  $\text{Brick}^k(n)$  and  $\text{Zono}^k(n)$ . The reader can already glance at Figure 11 on page 21 for an illustration of the geometric situation.

In Section 1.3, we consider C. Malvenuto and C. Reutenauer's Hopf algebra on permutations [MR95], that we denote  $\text{FQSym}$ . We consider the subspace  $\text{Twist}^k$  generated by the sums of the elements of  $\text{FQSym}$  over the fibers of  $\text{ins}^k$ . Since these fibers are classes of a congruence  $\equiv^k$  which satisfy standard compatibility conditions with the shuffle and the standardization [Rea05, Hiv04, HN07, Pri13], this subspace automatically defines a Hopf subalgebra of  $\text{FQSym}$ . Our approach with  $k$ -twists provides a combinatorial interpretation for this subalgebra, and it is interesting to describe the product and the coproduct directly on acyclic  $k$ -twists. Note that our Hopf algebra  $\text{Twist}^k$  on acyclic  $k$ -twists is sandwiched in between C. Malvenuto and C. Reutenauer's Hopf algebra  $\text{FQSym}$  on permutations [MR95] and J.-C. Novelli, C. Reutenauer and J.-Y. Thibon's  $k$ -recoil Hopf algebra  $\text{Rec}^k$  on acyclic orientations of  $G^k(n)$  [NRT11]. We finally briefly study further algebraic properties of  $\text{Twist}^k$ : we define multiplicative bases and study their indecomposable elements, we connect it to integer point transforms of the normal cones of the brick polytope, and we define a natural extension of dendriform structures in the context of  $k$ -twists.

The second part of this paper is devoted to three independent extensions of these results. In Section 2.1, we show that our constructions can be parametrized by a sequence of signs, following the same direction as [Rea06, HL07, PS12, CP14]. We obtain generalizations of the Cambrian lattices, mention their connections to certain well-chosen brick polytopes, and construct a Cambrian Hopf algebra on acyclic  $k$ -twists with similar ideas as in [CP14, Part 1]. In Section 2.2, we extend the Cambrian tuple algebra of G. Chatel and V. Pilaud [CP14, Part 2], which was motivated by the Hopf algebras on diagonal rectangulations described by S. Law and N. Reading [LR12] and on twin binary trees described by S. Giraudo [Gir12]. Finally in Section 2.3, we study Hopf algebras on all faces of the brick polytope, motivated by the work of F. Chapoton [Cha00], and its extension to the Cambrian Schröder algebra in [CP14, Part 3].

## Part 1. Acyclic twists: combinatorics, geometry, and algebra

### 1.1. COMBINATORICS OF TWISTS

**1.1.1. Pipe dreams and twists.** A *pipe dream* is a filling of a triangular shape with crosses  $\times$  and elbows  $\curvearrowright$  so that all pipes entering on the left side exit on the top side. These objects were studied in the literature, under different names including “pipe dreams” [KM05], “RC-graphs” [BB93], “beam arrangements” [Pil10, PS12]. This paper is mainly concerned with the following specific family of pipe dreams.

**Definition 1.** For  $k, n \in \mathbb{N}$ , a  $(k, n)$ -*twist* (we also use just *k-twist*, or even just *twist*) is a pipe dream with  $n + 2k$  pipes such that

- no two pipes cross twice (the pipe dream is reduced),
- the pipe which enters in row  $i$  exists in column  $i$  if  $k \leq i \leq n+k$ , and in column  $n + 2k + 1 - i$  otherwise. Here and throughout the paper, rows are indexed from bottom to top and columns are indexed from left to right.

Besides the first  $k$  and last  $k$  trivial pipes, a  $(k, n)$ -twist has  $n$  relevant pipes, labeled by  $[n]$  from bottom to top, or equivalently from left to right. In other words, the  $p$ th pipe enters in row  $p + k$  and exits at column  $p + k$  of the  $(n + 2k) \times (n + 2k)$ -triangular shape. We denote by  $\mathcal{T}^k(n)$  the set of  $(k, n)$ -twists.

**Definition 2.** The *contact graph* of a  $(k, n)$ -twist  $T$  is the directed multigraph  $T^\#$  with vertex set  $[n]$  and with an arc from the SE-pipe to the WN-pipe of each elbow in  $T$  involving two relevant pipes. We say that a twist  $T$  is *acyclic* if its contact graph  $T^\#$  is (no oriented cycle), and we then denote by  $\triangleleft_T$  the transitive closure of  $T^\#$ . We denote by  $\mathcal{AT}^k(n)$  the set of acyclic  $(k, n)$ -twists.

Figure 1 illustrates examples of  $(k, 5)$ -twists and their contact graphs for  $k = 0, 1, 2, 3$ . The first two are acyclic, the last two are not. Except in Figures 2 and 4, we only represent the  $n$  relevant pipes of the  $(k, n)$ -twists and hide the other  $2k$  trivial pipes (the first  $k$  and last  $k$  pipes).

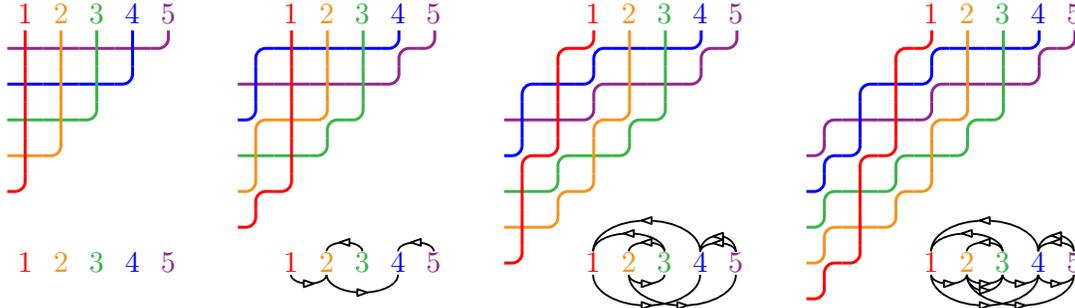


FIGURE 1.  $(k, 5)$ -twists (top) and their contact graphs (bottom) for  $k = 0, 1, 2, 3$ .

**Remark 3** (Subword complexes). Although we prefer a more visual presentation to develop an intuition which will be crucial later on, twists and their contact graphs can be defined in more algebraic terms. Namely, a  $(k, n)$ -twist could be equivalently defined as a reduced expression of the longest permutation  $w_\circ := [n, n-1, \dots, 2, 1]$  in the word  $Q := (s_1 \cdots s_{n-1})^{k+1} (s_1 \cdots s_{n-2}) \cdots (s_1 s_2) s_1$  where  $s_i$  denotes the simple transposition  $(i \ i+1)$ . In other words, it is a facet in the subword complex  $\mathcal{SC}(Q, w_\circ)$ , see [KM04]. The contact graph of the twist is then given by the root configuration of this facet [CLS14, PS15]. The twist is acyclic when this root configuration is pointed.

**Example 4** (0-twists). For any  $n \in \mathbb{N}$ , there is a unique  $(0, n)$ -twist. It is the pipe dream where all boxes on the diagonal  $\{(i, i) \mid i \in [n]\}$  are filled with elbows  $\curvearrowright$  while all other boxes are filled with crossings  $\times$ , see Figure 1 (left). Its contact graph is therefore an independent graph.

**Example 5** (1-twists, triangulations and binary trees). Following the (unpublished) work of A. Woo [Woo04], we observe that  $(1, n)$ -twists are in bijective correspondence with triangulations of a convex  $(n+2)$ -gon. This bijection is illustrated in Figure 2. To describe it explicitly, consider:

- the  $(n+2) \times (n+2)$  triangular shape where rows (resp. columns) are labeled by  $\{0, \dots, n+1\}$  from bottom to top (resp. from left to right),
- the convex  $(n+2)$ -gon where the vertices are labeled by  $\{0, \dots, n+1\}$  from left to right and where all vertices lie below the top boundary edge  $[0, n+1]$ ,
- the map which sends an elbow in row  $i$  and column  $j$  of the triangular shape to the diagonal  $[i, j]$  of the  $(n+2)$ -gon.

As illustrated in Figure 2, this map sends a  $(1, n)$ -twist  $T$  to a triangulation  $T^*$  of the  $(n+2)$ -gon:

- the  $p$ th relevant pipe of  $T$  corresponds to the  $p$ th triangle of  $T^*$  (with central vertex  $p$ ),
- the elbows of  $T$  correspond to the diagonals of  $T^*$ ,
- the crossings of  $T$  correspond to the common bisectors between triangles of  $T^*$ ,
- the contact graph of  $T$  corresponds to the dual binary tree of  $T^*$ ,
- elbow flips in  $T$  (see Figure 2 and Section 1.1.10) corresponds to diagonal flips in  $T^*$ .

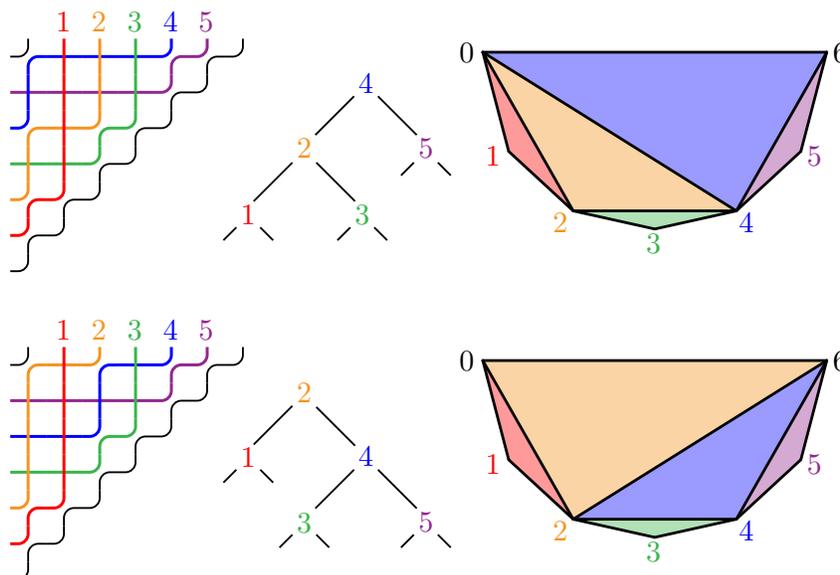


FIGURE 2. The bijection between  $(1, n)$ -twists  $T$  (left) and triangulations  $T^*$  of the  $(n+2)$ -gon (right) sends the pipes of  $T$  to the triangles of  $T^*$ , the elbows of  $T$  to the diagonals of  $T^*$ , the contact graph of  $T$  to the dual binary tree of  $T^*$  (middle), and the elbow flips in  $T$  to the diagonal flips in  $T^*$ . See also Section 2.1.

**1.1.2. Twists and multitriangulations.** Following Example 5, we now recall the bijective correspondence between  $(k, n)$ -twists and  $k$ -triangulations of a convex  $(n+2k)$ -gon. This correspondence is not strictly required for our constructions, but was at the initial motivation of our study of  $k$ -twists. The reader familiar with multitriangulations can jump directly to Section 1.1.4.

Multitriangulations appeared in the context of extremal theory for geometric graphs [CP92] and were studied for their combinatorial structure [Nak00, DKM02, Jon05]. We refer to [PS09] for a recent overview of their properties.

**Definition 6.** A  $k$ -triangulation of a convex  $(n+2k)$ -gon is a maximal set of diagonals of the  $(n+2k)$ -gon such that no  $k+1$  of them are pairwise crossing.

Figure 3 illustrates this definition with examples of  $k$ -triangulations for  $k = 0, 1, 2, 3$ . Observe from the definition that 1-triangulations are just classical triangulations. Note also that a diagonal of the  $(n+2k)$ -gon can appear in a  $(k+1)$ -crossing only if it has at least  $k$  vertices of the  $(n+2k)$ -gon

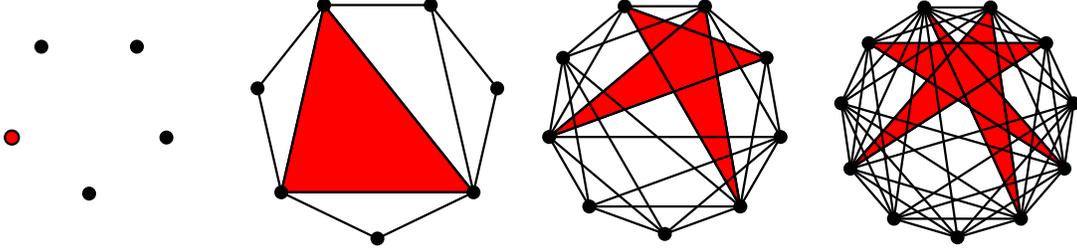


FIGURE 3.  $k$ -triangulations of the  $(5 + 2k)$ -gon with a colored  $k$ -star, for  $k = 0, 1, 2, 3$ .

on each side. We say that those diagonals are  *$k$ -relevant diagonals*, that the diagonals separating  $k - 1$  vertices of the  $(n + 2k)$ -gon from the others are  *$k$ -boundary diagonals*, and that the other ones are  *$k$ -irrelevant diagonals*. To connect  $k$ -triangulations to  $k$ -twists, we need their decompositions into  $k$ -stars, introduced in [PS09]. See Figure 3 for illustrations.

**Definition 7.** A  *$k$ -star* is a star  $(2k + 1)$ -gon, given by  $2k + 1$  vertices  $s_0, \dots, s_{2k}$  cyclically ordered connected by the  $2k + 1$  edges  $[s_i, s_{i+k}]$  (indices are taken modulo  $2k + 1$ ).

The  $k$ -stars were used in [PS09] to show by double counting that the number of diagonals in a  $k$ -triangulation of the  $(n + 2k)$ -gon only depends on  $k$  and  $n$ , a property known in [Nak00, DKM02].

**Theorem 8** ([PS09, Thm. 1.4 (1 & 2)]). A  $k$ -triangulation of the  $(n + 2k)$ -gon contains precisely  $n$   $k$ -stars and  $k(2n + 2k - 1)$  diagonals,  $k(n - 1)$  of which are  $k$ -relevant. Moreover, each  $k$ -relevant (resp.  $k$ -boundary, resp.  $k$ -irrelevant) diagonal of a  $k$ -triangulation  $T$  is contained in 2 (resp. 1, resp. 0)  $k$ -stars of  $T$ .

Call *dual graph* of a  $k$ -triangulation  $T$  the graph  $T^\#$  with one vertex for each  $k$ -star of  $T$  and an edge for each  $k$ -relevant diagonal  $\delta$  of  $T$  connecting the two  $k$ -stars of  $T$  containing  $\delta$ . This graph has  $n$  nodes and  $k(n - 1)$  edges, and it is known to be a  *$k$ -arborescence*: it can be partitioned into  $k$  edge-disjoint spanning trees [Pil10, Prop. 4.20].

The  $k$ -stars were also used in [PS09] to provide a local combinatorial description of the notion of flip in  $k$ -triangulations introduced in [Nak00, DKM02, Jon05]. In the next statement, a *bisector* of a  $k$ -star  $S$  is a line passing through one of its vertices and with  $k$  vertices of  $S$  on each side.

**Theorem 9** ([Nak00, Thm. 3], [PS09, Thm. 1.4 (3)]). Consider a  $k$ -relevant diagonal  $\delta$  in a  $k$ -triangulation  $T$  of the  $(n + 2k)$ -gon. Then there exists a unique other  $k$ -triangulation  $T'$  of the  $(n + 2k)$ -gon such that  $T \setminus \{\delta\} = T' \setminus \{\delta'\}$  for some diagonal  $\delta' \in T'$ . Moreover,  $\delta'$  is the unique common bisector of the two  $k$ -stars of  $T$  containing  $\delta$ .

We are now ready to recall the correspondence between  $k$ -twists and  $k$ -triangulations. Besides the special case  $k = 1$  treated in [Woo04], this correspondence first appeared for arbitrary values of  $k$  in the work of V. Pilaud and M. Pocchiola [Pil10, Sections 3.2.3 and 4.1.4][PP12]. C. Stump [Stu11] and L. Serrano and C. Stump [SS12] rediscovered this connection and used it to get enumerative and bijective results discussed in Section 1.1.3. This correspondence is illustrated in Figures 2 and 4. The  $k$ -triangulations of Figure 3 also correspond to the  $k$ -twists of Figure 1.

**Theorem 10** ([PP12, Thm. 23]). Consider the map that sends an elbow in row  $i$  and column  $j$  of the  $(n + 2k) \times (n + 2k)$  triangular shape to the diagonal  $[i, j]$  of a convex  $(n + 2k)$ -gon. It maps a  $(k, n)$ -twist  $T$  to a  $k$ -triangulation  $T^*$  of the  $(n + 2k)$ -gon:

- the relevant pipes of  $T$  correspond to the  $k$ -stars of  $T^*$ ,
- the (relevant) elbows of  $T$  correspond to the (relevant) diagonals in  $T^*$ ,
- the crossings of  $T$  correspond to the common bisectors between  $k$ -stars of  $T^*$ ,
- the contact graph of  $T$  correspond to the dual graph of  $T^*$  (its incidence graph of  $k$ -stars),
- elbow flips in  $T$  (see Figure 4 and Section 1.1.10) correspond to diagonal flips in  $T^*$ .

Moreover, this map defines a bijective correspondence between the  $(k, n)$ -twists and the  $k$ -triangulations of the  $(n + 2k)$ -gon.

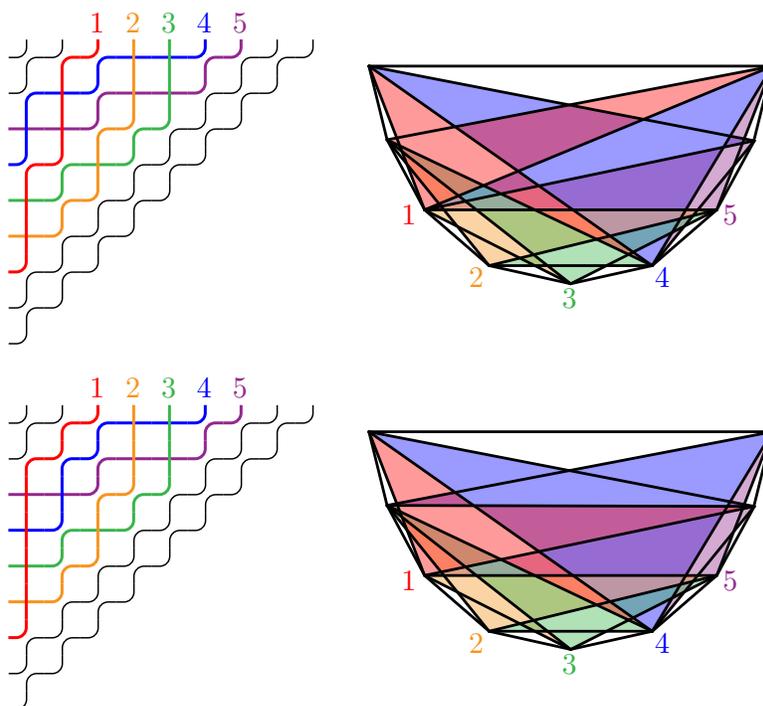


FIGURE 4. The bijection between  $(k, n)$ -twists  $T$  (left) and  $k$ -triangulations  $T^*$  of the  $(n + 2k)$ -gon (right) sends the pipes of  $T$  to the  $k$ -stars of  $T^*$ , the elbows of  $T$  to the diagonals of  $T^*$ , the contact graph of  $T$  to the dual graph of  $T^*$ , and the elbow flips in  $T$  to the diagonal flips in  $T^*$ . See also Section 2.1.

**1.1.3. Numerology.** J. Jonsson [Jon05] proved that  $k$ -triangulations are counted by an Hankel determinant of Catalan numbers, which was known to count fans of  $k$  non-crossing Dyck path by the celebrated non-intersecting path enumeration technique of I. Gessel and X. Viennot [GV85]. The correspondence between  $k$ -triangulations and  $k$ -twists in Theorem 10 thus shows that the  $k$ -twists are also counted by this Hankel determinant. L. Serrano and C. Stump revisited independently this correspondence to  $k$ -twists and gave a remarkable explicit bijection between  $k$ -triangulations and  $k$ -tuples of non-crossing Dyck paths [SS12]. We summarize these numerology results in the following statement, and Table 1 gathers the numbers of  $(k, n)$ -twists for  $k, n \leq 9$ .

**Theorem 11** ([Jon05, Thms. 14 & 15], [PP12, Thm. 23], [SS12, Thm. 1.1]). *The three combinatorial families consisting in*

- the  $k$ -twists with  $n$  relevant pipes,
- the  $k$ -triangulations of the  $(n + 2k)$ -gon,
- the fans of  $k$  non-crossing Dyck paths of length  $2n$ ,

are equinumerous and counted by the Hankel determinant  $\det(C_{n+2k-i-j})_{i,j \in [k]}$  of Catalan numbers  $C_m := \frac{1}{m+1} \binom{2m}{m}$ . See [OEIS, A000108] and [OEIS, A078920].

To our knowledge, there is no enumerative results on acyclic twists. Table 2 gathers the numbers of acyclic  $(k, n)$ -twists for  $k < n \leq 10$  (we will see later that the number of acyclic  $(k, n)$ -twists is always  $n!$  when  $k \geq n$ ). See [OEIS, A263791]. The reader is invited to compare Tables 1 and 2. We leave open the general question of finding a counting formula for  $(k, n)$ -twists:

**Question 12.** Is there a nice closed formula for the number of acyclic  $(k, n)$ -twists?

$k \setminus n$	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1	1	2	5	14	42	132	429	1430	4862
2	1	3	14	84	594	4719	40898	379236	3711916
3	1	4	30	330	4719	81796	1643356	37119160	922268360
4	1	5	55	1001	26026	884884	37119160	1844536720	105408179176
5	1	6	91	2548	111384	6852768	553361016	55804330152	6774025632340
6	1	7	140	5712	395352	41314284	6018114036	1153471856900	278563453441350
7	1	8	204	11628	1215126	204951252	51067020290	17618122000050	8012241391386375
8	1	9	285	21945	3331251	869562265	354544250775	210309203300625	171879976152056250
9	1	10	385	38962	8321170	3245256300	2085445951875	2045253720802500	2885318087540733000

TABLE 1. The number  $|\mathcal{T}^k(n)| = \det(C_{n+2k-i-j})_{i,j \in [k]}$  of  $(k, n)$ -twists for  $k, n \leq 9$ .

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	.	2	5	14	42	132	429	1430	4862	16796
2	.	.	6	22	92	420	2042	10404	54954	298648
3	.	.	.	24	114	612	3600	22680	150732	1045440
4	.	.	.	.	120	696	4512	31920	242160	1942800
5	.	.	.	.	.	720	4920	37200	305280	2680800
6	.	.	.	.	.	.	5040	39600	341280	3175200
7	.	.	.	.	.	.	.	40320	357840	3457440
8	.	.	.	.	.	.	.	.	362880	3588480
9	.	.	.	.	.	.	.	.	.	3628800

TABLE 2. The number  $|\mathcal{AT}^k(n)|$  of acyclic  $(k, n)$ -twists for  $k < n \leq 10$ . Dots indicate that the value remains constant (equal to  $n!$ ) in the column.

1.1.4. **Elementary properties of pipes and twists.** We now give some elementary properties of pipes and twists needed in the next sections. For a given pipe in a twist, we call:

- **WE-crosses** its horizontal crosses  $\vdash$ , and **SN-crosses** its vertical crosses  $\dashv$ ,
- **SE-elbows** its elbows  $\curvearrowright$  (*aka.* “pics”) and **WN-elbows** its elbows  $\curvearrowleft$  (*aka.* “valleys”),
- **internal steps** the segments between two consecutive elbows and **external steps** the first and last steps of the pipe.

The following statement gathers some elementary properties of pipes in  $k$ -twists.

**Lemma 13.** *The following properties hold for any  $k, n \in \mathbb{N}$ :*

- A  $(k, n)$ -twist has  $\binom{n}{2}$  crosses  $\vdash$  and  $kn$  elbows  $\curvearrowright$ .
- The  $p$ th pipe of a  $(k, n)$ -twist has
  - $n - 1$  crosses:  $p - 1$  WE-crosses  $\vdash$  and  $n - p$  SN-crosses  $\dashv$ ,
  - $2k + 1$  elbows:  $k$  SE-elbows  $\curvearrowright$  and  $k + 1$  WN-elbows  $\curvearrowleft$ ,
  - $2k + 2$  steps:  $k + 1$  vertical steps and  $k + 1$  horizontal steps.
- The interior of the rectangle defined by two consecutive steps of a pipe contains no elbow.

*Proof.* For (i), observe that any two pipes cross exactly once and that the number of relevant boxes in the triangular shape is  $kn + \binom{n}{2}$ . Point (ii), follows from the fact that a pipe  $p$  has a SN-cross for each pipe  $p' < p$  and a WE-cross for each pipe  $p' > p$ , and that the length of each pipe is  $n + 2k$ . (Note that the  $2k + 1$  elbows of pipe  $p$  correspond to the  $2k + 1$  edges of the dual  $k$ -star  $p^*$ .) To see (iii), consider a rectangle  $R$  defined by two consecutive steps of a pipe  $p$  and assume that  $R$  is located below  $p$ . Observe that the pipes entering on the west side of  $R$  have to exit on its east side of  $R$  (otherwise, they would exit on its north side, and would thus cross  $p$  twice). This prevents an elbow inside  $R$ . The argument is similar if the rectangle  $R$  is above  $p$ .  $\square$

The next statement provides a sufficient condition for the existence of a path between two pipes in the contact graph of a  $k$ -twist. Note that the converse statement holds when  $k = 1$ , but not in general.

**Lemma 14.** *Consider two pipes  $p, p'$  in a twist  $T$ . If there is an elbow of  $p$  located weakly south-east of an elbow of  $p'$ , then there is a path from  $p$  to  $p'$  in the contact graph  $T^\#$  of  $T$ .*

*Proof.* Up to adding one more edge at the beginning or at the end of the path from  $p$  to  $p'$ , we can assume without loss of generality that there is a WN-elbow  $e$  of  $p$  located south-east of a SE-elbow  $e'$  of  $p'$ . We claim that there exists a sequence  $e = e_0, e_1, \dots, e_\ell = e'$  of elbows such that each elbow  $e_i$  is located weakly south-east of  $e'$  and the elbows  $e_i$  and  $e_{i+1}$  are connected by a pipe  $p_{i+1}$  of  $T$ . Indeed, assume that we have constructed  $e = e_0, e_1, \dots, e_i$ , and consider the WN-pipe  $p_{i+1}$  at  $e_i$ . By Lemma 13 (iii), either the previous or the next elbow along  $p_{i+1}$  is still located weakly south-east of  $e'$ . Choose  $e_{i+1}$  accordingly (pick arbitrarily one if the two options are possible). The process ends at  $e'$  since the distance to  $e'$  decreases at each step. Finally, the sequence of pipes  $p = p_1, \dots, p_\ell = p'$  gives a path from  $p$  to  $p'$  in the contact graph  $T^\#$ .  $\square$

**Corollary 15.** (i) *If  $p < p'$  are two pipes of a twist  $T$  such that the crossing between  $p$  and  $p'$  is on an internal step of  $p'$ , then there is a path from  $p$  to  $p'$  in the contact graph  $T^\#$  of  $T$ .*  
(ii) *If  $p < p'$  are two pipes of a twist  $T$  and are incomparable in the contact graph  $T^\#$  of  $T$ , then the last vertical step of  $p$  crosses the first horizontal step of  $p'$ .*  
(iii) *If a twist  $T$  is acyclic, then no two pipes of  $T$  cross at internal steps of both.*

When  $T$  is acyclic, we denote by  $\triangleleft_T$  the transitive closure of its contact graph  $T^\#$ . Note the difference between our notations  $p < p'$  (meaning that  $p$  starts below and ends to the left of  $p'$ ) and  $p \triangleleft_T p'$  (meaning that  $p$  is smaller than  $p$  in the contact graph  $T^\#$ ).

**1.1.5. Pipe insertion and deletion.** We now define the pipe insertion and deletion, two reverse operations on  $k$ -twists: an insertion transforms a  $(k, n)$ -twist into a  $(k, n + 1)$ -twist by inserting a single pipe while a deletion transforms a  $(k, n)$ -twist into a  $(k, n + 1)$ -twist by deleting a single pipe. Insertions are always possible (see Definition 16), while only certain pipes are allowed to be deleted (see Definition 18). We start with pipe insertions.

**Definition 16.** *Consider a  $(k, n)$ -twist  $T$  with an increasing relabeling  $\lambda : [n] \rightarrow \mathbb{N}$  of its relevant pipes and an integer  $q \in \mathbb{N}$ . Let  $p \in \{0, \dots, n\}$  be such that  $\lambda(p) \leq q < \lambda(p + 1)$ , where we set the convention  $\lambda(0) = -\infty$  and  $\lambda(n + 1) = +\infty$ . The **pipe insertion** of  $q$  in the relabeled  $(k, n)$ -twist  $T$  produces the relabeled  $(k, n + 1)$ -twist  $T \hat{\wedge} q$  obtained by:*

- inserting a row and a column in the triangular shape between  $p + k - 1$  and  $p + k$ ,
- filling in with elbows  $\curvearrowright$  the new boxes  $(p + k, p), (p + k + 1, p + 1), \dots, (p + 2k, p + k)$ ,
- filling in with crosses  $\oplus$  all other new boxes  $(p + k, j)$  for  $j < p$  and  $(i, p + k)$  for  $i > p + 2k$ ,
- relabeling the  $r$ th relevant pipe of the resulting twist by  $\lambda(r)$  if  $r < p$ , by  $q$  if  $r = p$ , and by  $\lambda(r - 1)$  if  $r > p$ .

Figure 5 illustrates the insertion of 4 in the  $(k, 5)$ -twists of Figure 1 relabeled by  $[2, 3, 6, 8, 9]$ . The following statement is an immediate consequence of the definition of pipe insertion.

**Lemma 17.** *The contact graph  $(T \hat{\wedge} q)^\#$  is obtained from  $T^\#$  by connecting to some existing nodes the new node corresponding to the inserted pipe  $q$ . In particular, the node corresponding to the inserted pipe is a source of the contact graph  $(T \hat{\wedge} q)^\#$ .*

We now define the deletion, which just erases a pipe from a  $(k, n + 1)$ -twist.

**Definition 18.** *Consider a  $(k, n + 1)$ -twist  $T$  with an increasing relabeling  $\lambda : [n + 1] \rightarrow \mathbb{N}$  of its relevant pipes. Assume that the  $p$ th pipe of  $T$ , labeled by  $\lambda(p) = q$ , is a source of the contact graph  $T^\#$ . Then the **pipe deletion** of  $q$  in the relabeled  $(k, n + 1)$ -twist  $T$  produces the relabeled  $(k, n)$ -twist  $T \heartsuit q$  obtained by:*

- deleting the  $(p + k)$ th row and column of  $T$ ,
- relabeling the  $r$ th relevant pipe of the resulting twist by  $\lambda(r)$  if  $r < p$  and by  $\lambda(r + 1)$  if  $r \geq p$ .

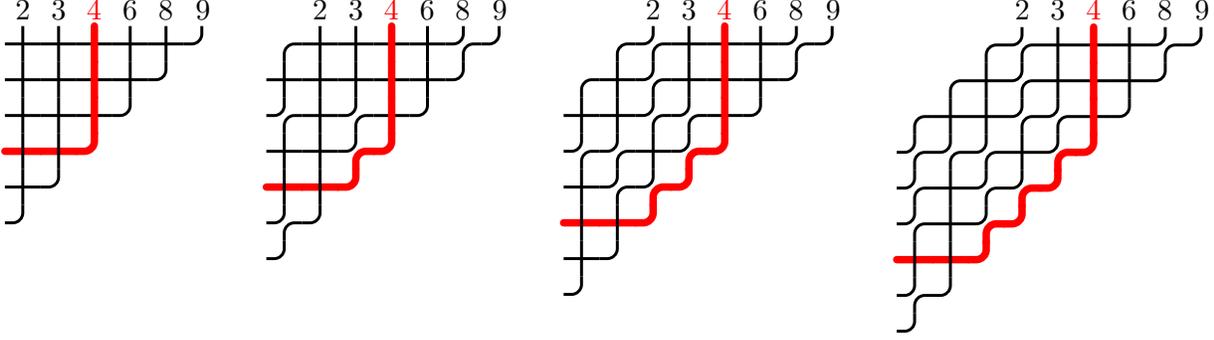


FIGURE 5. Inserting 4 in the  $(k, 5)$ -twists of Figure 1. The inserted pipe is in bold red.

The following statement is an immediate consequence of the definition of pipe deletion.

**Lemma 19.** *The contact graph  $(T \forall q)^\#$  is obtained from  $T^\#$  by deleting the node corresponding to the delete pipe  $q$ .*

Moreover, the insertion and deletion are clearly reverse to each other.

**Lemma 20.** *For any  $(k, n)$ -twist  $T$  relabeled by  $\lambda : [n] \rightarrow \mathbb{N}$  and any integer  $q \in \mathbb{N}$ , we have*

- $(T \hat{\wedge} q) \forall q = T$ , and
- $(T \forall q) \hat{\wedge} q = T$  as soon as  $q \in \lambda([n])$  labels a source of  $T$ .

**Example 21** (Insertion in 1-twists, triangulations and binary search trees). The following operations are equivalent under the bijections between 1-twists, triangulations, and binary search trees (see Example 5 and Figure 2):

- the pipe insertion of  $q$  in the (reabeled) 1-twist  $T$ ,
- the triangle insertion of  $q$  in the (reabeled) triangulation  $T^*$ ,
- the node insertion  $q$  in the (reabeled) binary search tree  $T^\#$ .

**Example 22** (Pipe insertion in  $k$ -twists and  $k$ -crossing inflation in  $k$ -triangulations). As in Example 21, a pipe insertion in a  $k$ -twist  $T$  corresponds to the inflation into a  $k$ -star of a  $k$ -crossing formed by  $k$  consecutive  $k$ -boundary edges in the  $k$ -triangulation  $T^*$ . Similarly, a pipe deletion in a  $k$ -twist  $T$  corresponds to the flattening into a  $k$ -crossing of a  $k$ -star incident to  $k$   $k$ -boundary edges. This  $k$ -crossing inflation and  $k$ -star flattening operations on  $k$ -triangulations were considered in [Nak00, Jon05, PS09], even in more generality. The detailed description of these operations is not needed here and can be found in [PS09].

**1.1.6.  $k$ -twist correspondence.** We now present a natural surjection from permutations to acyclic  $k$ -twists. It relies on an insertion operation on pipe dreams similar to the insertion in binary search trees (see Example 25 for details). It is motivated by the geometry of the normal fan of the corresponding brick polytope (see Section 1.2 and [PS12]).

We now describe this algorithm. From a permutation  $\tau := [\tau_1, \dots, \tau_n]$  (written in one-line notation), we construct a  $(k, n)$ -twist  $\text{ins}^k(\tau)$  obtain from the  $(k, 0)$ -twist by successive pipe insertions of the entries  $\tau_n, \dots, \tau_1$  of  $\tau$  read from right to left. Equivalently, starting from the empty triangular shape, we insert the pipes  $\tau_n, \dots, \tau_1$  of the twist such that each new pipe is as northwest as possible in the space left by the pipes already inserted. This procedure is illustrated in Figure 6 for the permutation 31542 and different values of  $k$ .

**Proposition 23.** *For any  $(k, n)$ -twist  $T$ , the permutations  $\tau \in \mathfrak{S}_n$  such that  $\text{ins}^k(\tau) = T$  are precisely the linear extensions of the contact graph of  $T$ . In particular,  $\text{ins}^k$  is a surjection from the permutations of  $\mathfrak{S}_n$  to the acyclic  $(k, n)$ -twists.*

*Proof.* We prove the result by induction on  $n$ . Consider a permutation  $\tau = [\tau_1, \dots, \tau_n] \in \mathfrak{S}_n$ , and let  $\tau' = [\tau_2, \dots, \tau_n]$ . By definition, we have  $\text{ins}^k(\tau) = \text{ins}^k(\tau') \hat{\wedge} \tau_1$ . By induction hypothesis,

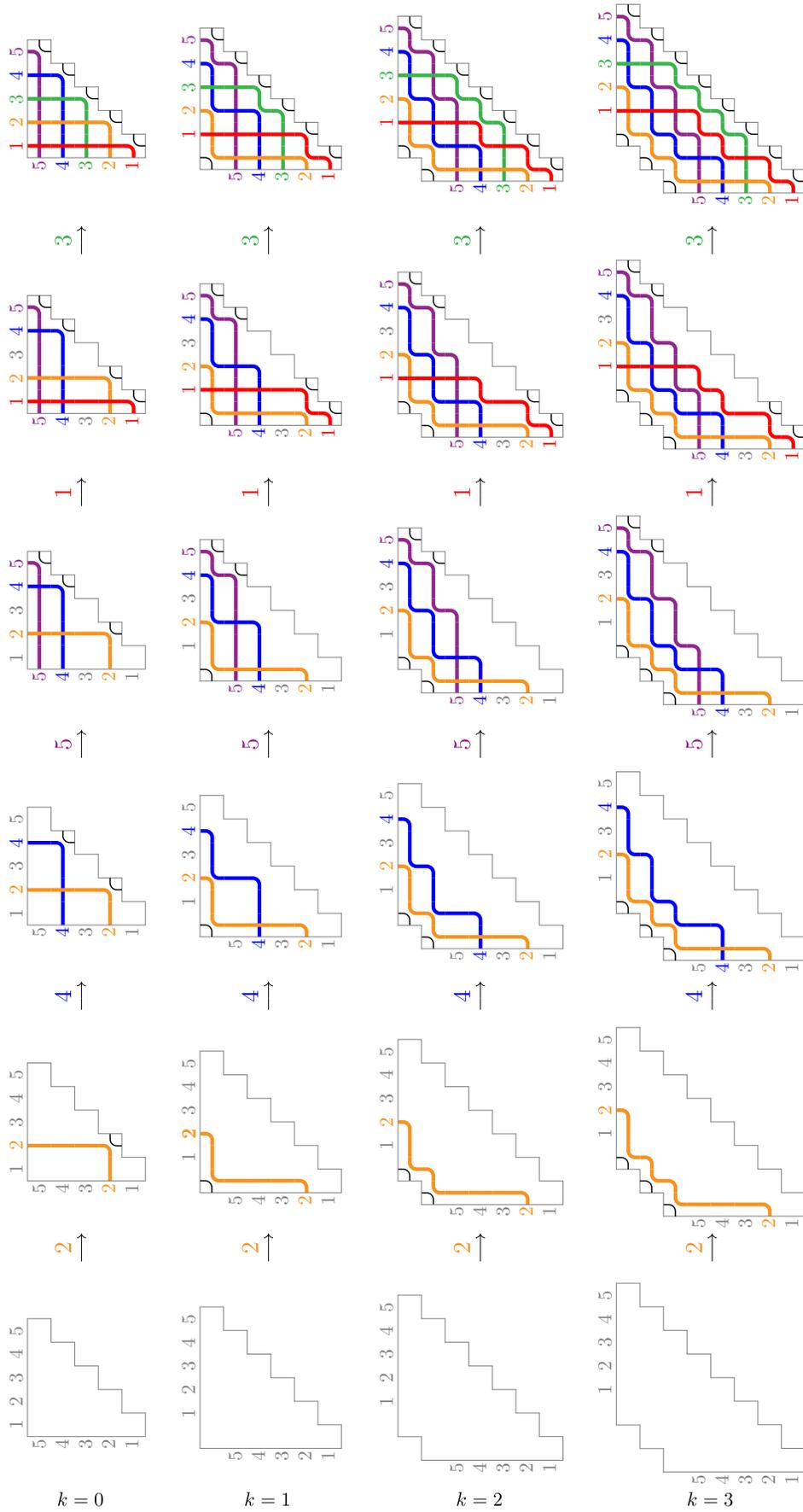


FIGURE 6. Insertion of the permutation 31542 in  $k$ -twists for  $k = 0, 1, 2, 3$ .

$\tau'$  is a linear extension of the contact graph  $\text{ins}^k(\tau')^\#$  and Lemma 17 ensures that  $\tau_1$  is a source of  $\text{ins}^k(\tau)^\# = (\text{ins}^k(\tau') \wedge \tau_1)^\#$ . It follows that  $\tau$  is a linear extension of the contact graph  $\text{ins}^k(\tau)^\#$ . Conversely, assume that  $\tau$  is a linear extension of the contact graph of a  $(k, n)$ -twist  $T$ . Since  $\tau_1$  is a source of  $T^\#$ , the twist  $T \vee \tau_1$  is well-defined, and  $\tau'$  is a linear extension of  $(T \vee \tau_1)^\#$ . By induction hypothesis, it follows that  $\text{ins}^k(\tau') = T \vee \tau_1$  and thus that  $\text{ins}^k(\tau) = \text{ins}^k(\tau') \wedge \tau_1 = (T \vee \tau_1) \wedge \tau_1 = T$  by Lemma 20.  $\square$

**Remark 24** (*k-twist correspondence*). Remembering additionally the insertion order,  $\text{ins}^k$  defines a bijection between permutations of  $\mathfrak{S}_n$  and *leveled acyclic  $(k, n)$ -twists*, i.e.  $(k, n)$ -twists whose relevant pipes have been relabeled by  $\lambda : [n] \rightarrow [n]$  so that the label of the SE-pipe is larger than the label of the WN-pipe at each elbow. We call this bijection the *k-twist correspondence*.

**Example 25** (1-twist correspondence and sylvester correspondence). According to Example 21, the contact graph of the 1-twist  $\text{ins}^1(\tau)$  is the binary search tree obtained by the successive insertions of the entries of  $\tau$  read from right to left. If we additionally remember the insertion order, we obtain the sylvester correspondence [HNT05] between permutations and leveled binary trees.

1.1.7. *k-twist congruence*. We now characterize the fibers of  $\text{ins}^k$  as classes of a congruence  $\equiv^k$  defined by a simple rewriting rule, similar to the sylvester congruence [HNT05].

**Definition 26.** The *k-twist congruence* is the equivalence relation  $\equiv^k$  on  $\mathfrak{S}_n$  defined as the transitive closure of the rewriting rule

$$UacV_1b_1V_2b_2 \cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2 \cdots V_kb_kW \quad \text{if } a < b_i < c \text{ for all } i \in [k],$$

where  $a, b_1, \dots, b_k, c$  are elements of  $[n]$  while  $U, V_1, \dots, V_k, W$  are (possibly empty) words on  $[n]$ . We say that  $b_1, \dots, b_k$  are *k-twist congruence witnesses* for the exchange of  $a$  and  $c$ .

The congruence classes of  $\equiv^k$  for  $k \in \{1, 2\}$  and  $n = 4$  are represented in Figure 7, where the underlying graph is the Hasse diagram of the (right) weak order (see Section 1.1.8).

**Proposition 27.** For any  $\tau, \tau' \in \mathfrak{S}_n$ , we have  $\tau \equiv^k \tau' \iff \text{ins}^k(\tau) = \text{ins}^k(\tau')$ . In other words, the fibers of  $\text{ins}^k$  are precisely the *k-twist congruence classes*.

*Proof.* From Proposition 23, each fiber of  $\text{ins}^k$  gathers the linear extensions of a *k-twist*. Since the set of linear extensions of a poset is connected by simple transpositions, we just need to show that  $\tau \equiv^k \tau' \iff \text{ins}^k(\tau) = \text{ins}^k(\tau')$  for any two permutations  $\tau = UacV$  and  $\tau' = UcaV$  of  $\mathfrak{S}_n$  which differ by the inversion of two consecutive values.

Let  $T = \text{ins}^k(V)$  denote the *k-twist* obtained after the insertion of  $V$ . The positions where  $a$  and  $c$  will be inserted in  $T$  are separated by the letters  $b$  in  $V$  such that  $a < b < c$ . Therefore, if there exists at least  $k$  such letters, the pipes  $a$  and  $c$  are not comparable in  $(T \wedge c) \wedge a = (T \wedge a) \wedge c$  and we have  $\text{ins}^k(\tau) = \text{ins}^k(\tau')$ . Conversely, if there are strictly less than  $k$  such letters, then  $a$  is below  $c$  in  $(T \wedge c) \wedge a$  while  $a$  is above  $c$  in  $(T \wedge a) \wedge c$ , and thus we get  $\text{ins}^k(\tau) \neq \text{ins}^k(\tau')$ .  $\square$

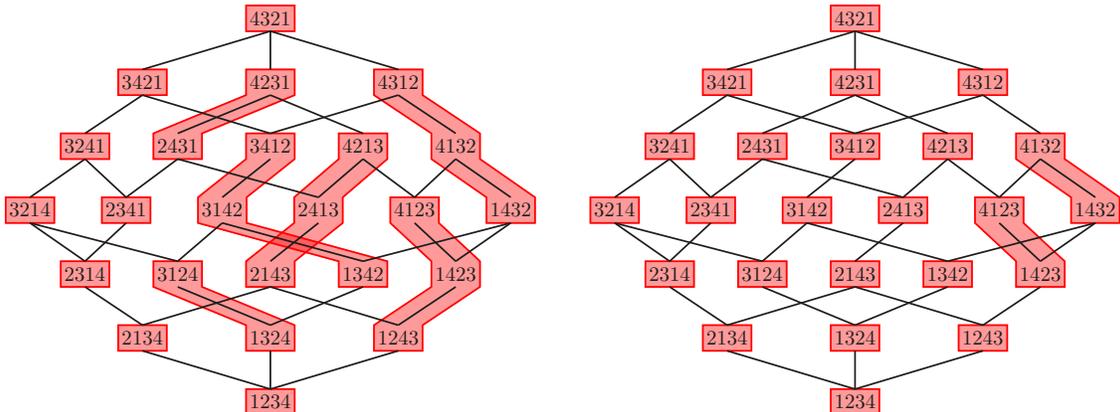


FIGURE 7. The *k-twist congruence classes* on  $\mathfrak{S}_4$  for  $k = 1$  (left) and  $k = 2$  (right).

**Example 28** (1-twist congruence and sylvester congruence). The 1-twist congruence coincides with the sylvester congruence defined in [HNT05] as the transitive closure of the rewriting rule  $UacVbW \equiv UcaVbW$  for  $a < b < c$  elements of  $[n]$  while  $U, V, W$  are (possibly empty) words on  $[n]$ .

**Remark 29** (Numerology again). The number of  $k$ -twist congruence classes of  $\mathfrak{S}_n$  is the number of acyclic  $(k, n)$ -twists. We thus obtain that

$$|\mathcal{AT}^k(n)| = n! \quad \text{if } n \leq k+1 \quad \text{and} \quad |\mathcal{AT}^k(k+2)| = (k+2)! - k!$$

since there is no possible rewriting when  $n \leq k+1$ , and the only non-trivial  $k$ -twist congruence classes on  $\mathfrak{S}_{k+2}$  are the pairs  $\{1(k+2)(\sigma_1+1) \cdots (\sigma_k+1), (k+2)1(\sigma_1+1) \cdots (\sigma_k+1)\}$  for  $\sigma \in \mathfrak{S}_k$ . The other numbers in Table 2 remain mysterious for us.

**1.1.8.  $k$ -twist congruence classes and WOIPs.** We now take a moment to study an essential property of the  $k$ -twist congruence classes with respect to the weak order on permutations. Remember that the (right) weak order on  $\mathfrak{S}_n$  is defined as the inclusion order of coinversions, where a coinversion of  $\tau \in \mathfrak{S}_n$  is a pair of values  $i, j \in \mathbb{N}$  such that  $i < j$  while  $\tau^{-1}(i) > \tau^{-1}(j)$ . For example, the Hasse diagram of the weak order on  $\mathfrak{S}_4$  is the underlying graph of Figure 7. The following definitions of weak order intervals and weak order interval posets are connected in the Proposition 31 proved by A. Bjorner and M. Wachs [BW91].

**Definition 30.** A **weak order interval (WOI)** is a subset  $[\tau, \tau']$  of  $\mathfrak{S}_n$  consisting of all permutations  $\sigma \in \mathfrak{S}_n$  such that  $\tau \leq \sigma \leq \tau'$  in weak order. A **weak order interval poset (WOIP)** is a poset  $\triangleleft$  on  $[n]$  such that for all  $a < b < c$ ,

$$a \triangleleft c \implies a \triangleleft b \text{ or } b \triangleleft c \quad \text{and} \quad a \triangleright c \implies a \triangleright b \text{ or } b \triangleright c.$$

**Proposition 31** ([BW91, Thm. 6.8]). *The WOIs are the sets of linear extensions of WOIPs:*

- (i) *the WOI  $[\tau, \tau']$  is the set of linear extensions of the WOIP obtained as the transitive closure of the relations  $i \triangleleft j$  for all non coinversions  $i, j$  of  $\tau$  and  $i \triangleright j$  for all coinversions  $i, j$  of  $\tau'$ ;*
- (ii) *the linear extensions of a WOIP  $\triangleleft$  is the WOI  $[\tau, \tau']$ , where  $\tau$  and  $\tau'$  are respectively the greedy and antigreedy linear extensions of  $\triangleleft$ .*

We use this characterization to prove the following statement, fundamental for the paper.

**Proposition 32.** *The  $k$ -twist congruence classes are intervals of the weak order.*

*Proof.* By Propositions 23 and 27, a  $k$ -twist congruence class is the insertion fiber of a  $k$ -twist  $T$ , and thus the set of linear extensions of the transitive closure  $\triangleleft_T$  of the contact graph  $T^\#$ . We therefore just need to show that  $\triangleleft_T$  satisfies the WOIP conditions of Definition 30.

Assume that  $a < b < c$  are such that  $a \triangleleft_T c$ . Decompose the triangular shape into three regions: the region  $A$  of all points located south-west of the first elbow of the  $b$ th pipe of  $T$ , the region  $B$  of all points located north-west or south-east of an elbow of the  $b$ th pipe of  $T$ , and the region  $C$  of all points located north-east of the last elbow of the  $b$ th pipe of  $T$ . Since  $a \triangleleft_T c$ , there is a path  $\pi$  from the entering point of the  $a$ th pipe of  $T$  to the exiting point of the  $c$ th pipe of  $T$  which travels along the pipes and the elbows of  $T$ . Since  $a < b < c$ , the path  $\pi$  starts in region  $A$  and ends in region  $C$ , so that it necessarily passes from region  $A$  to region  $C$ . Since the north-east corner of  $A$  is located south-west of the south-west corner of  $C$ , this forces an elbow  $e$  of  $\pi$  to lie in region  $B$ . Lemma 14 then ensures that either  $a \triangleleft_T b$  (if  $e$  is above the  $b$ th pipe), or  $b \triangleleft_T c$  (if  $e$  is below the  $b$ th pipe). The proof is similar if  $a \triangleright_T c$ .  $\square$

**Remark 33** (Interval property for arbitrary subword complexes). Surprisingly, Proposition 32 fails for arbitrary subword complexes, even in type  $A$ : it is shown in [PS15, Section 5.3] that fibers are always closed by intervals, but not necessarily intervals themselves. See *e.g.* [PS15, Figure 9].

Let us sum up the last two sections by the following enumerative statement.

**Corollary 34.** *The following combinatorial objects are in bijection:*

- *acyclic  $(k, n)$ -twists,*
- *$k$ -twist congruence classes of  $\mathfrak{S}_n$ ,*
- *permutations of  $\mathfrak{S}_n$  avoiding  $1(k+2) - (\sigma_1+1) - \dots - (\sigma_k+1)$  for all  $\sigma \in \mathfrak{S}_k$  (maximums),*
- *permutations of  $\mathfrak{S}_n$  avoiding  $(k+2)1 - (\sigma_1+1) - \dots - (\sigma_k+1)$  for all  $\sigma \in \mathfrak{S}_k$  (minimums).*

When  $k = 1$ , one easily sees that 13-2 avoiding permutations are enumerated by Catalan numbers (as are 31-2 avoiding ones). As mentioned in Question 12, it would be interesting to obtain closed formulas for the enumeration of acyclic  $(k, n)$ -twists for general  $k$ .

**Remark 35** (13-2 versus 1-3-2 avoiding permutations). It is easy to see that a permutation avoids 13-2 if and only if it avoids 1-3-2. This property fails for larger values of  $k$ . For example, the permutation 13524 avoids 14-2-3 but not 1-4-2-3. Here, we deal with permutations avoiding the pattern  $1(k+2)-(\sigma_1+1)-\dots-(\sigma_k+1)$ , where 1 and  $(k+2)$  are consecutive.

**1.1.9. Lattice congruences of the weak order.** Based on the result of the previous section, we now show that the  $k$ -twist congruence  $\equiv^k$  is a lattice congruence of the weak order. This follows as well from the work of N. Reading [Rea05], but we prefer to remind the short proof as we will use similar ideas in Part 2. We first remind the reader with the definition of these congruences and refer to [Rea04, Rea06] for further details.

**Definition 36.** An *order congruence* is an equivalence relation  $\equiv$  on a poset  $P$  such that:

- (i) Every equivalence class under  $\equiv$  is an interval of  $P$ .
- (ii) The projection  $\pi_\downarrow : P \rightarrow P$  (resp.  $\pi_\uparrow : P \rightarrow P$ ), which maps an element of  $P$  to the minimal (resp. maximal) element of its equivalence class, is order preserving.

The *quotient*  $P/\equiv$  is a poset on the equivalence classes of  $\equiv$ , where the order relation is defined by  $X \leq Y$  in  $P/\equiv$  iff there exists representatives  $x \in X$  and  $y \in Y$  such that  $x \leq y$  in  $P$ . The quotient  $P/\equiv$  is isomorphic to the subposet of  $P$  induced by  $\pi_\downarrow(P)$  (or equivalently by  $\pi_\uparrow(P)$ ).

If moreover  $P$  is a finite lattice, then  $\equiv$  is a *automatically lattice congruence*, meaning that it is compatible with meets and joins: for any  $x \equiv x'$  and  $y \equiv y'$ , we have  $x \wedge y \equiv x' \wedge y'$  and  $x \vee y \equiv x' \vee y'$ . The poset quotient  $P/\equiv$  then inherits a lattice structure where the meet  $X \wedge Y$  (resp. the join  $X \vee Y$ ) of two congruence classes  $X$  and  $Y$  is the congruence class of  $x \wedge y$  (resp. of  $x \vee y$ ) for arbitrary representatives  $x \in X$  and  $y \in Y$ .

**Theorem 37.** The  $k$ -twist congruence  $\equiv^k$  is a lattice congruence of the weak order.

*Proof.* We already observed in Proposition 32 that the  $k$ -twist congruence classes are intervals of the weak order. We therefore just need to show that  $\tau < \tau'$  implies  $\min(C) < \min(C')$  and  $\max(C) < \max(C')$  for any two permutations  $\tau, \tau' \in \mathfrak{S}_n$  from distinct  $k$ -twist congruence classes  $C, C'$ . We only prove the result for the maximums, the proof for the minimums being similar. We first observe that we can assume that  $\tau' = \tau s_i$  for some simple transposition  $s_i = (i \ i+1)$  which is not a descent of  $\tau$ . The proof then works by induction on the weak order distance between  $\tau$  and  $\max(C)$ . If  $\tau = \max(C)$ , the result is immediate as  $\max(C) = \tau < \tau' \leq \max(C')$ . Otherwise, consider a permutation  $\sigma$  in  $C$  covering  $\tau$  in weak order. We write  $\sigma = \tau s_j$ , and  $s_j$  is not a descent of  $\tau$ . We now distinguish four cases, according to the relative positions of  $i$  and  $j$ :

- (1) If  $j > i+1$ , we have  $\tau = UabVcdW$ ,  $\tau' = UbaVcdW$  and  $\sigma = UabVdcW$  for some letters  $a < b$ ,  $c < d$  and words  $U, V, W$ . Define  $\sigma' := \tau s_i s_j = \tau' s_j = \sigma s_i = UbaVdcW$ . Since  $\tau \equiv^k \sigma$ , there exist  $k$ -twist congruence witnesses  $c < w_1, \dots, w_k < d$  which all appear in  $W$ . These letters are also  $k$ -twist congruence witnesses for the equivalence  $\tau' \equiv^k \sigma'$ . Moreover,  $\sigma < \sigma'$  since  $c < d$ .
- (2) If  $j = i+1$ , we have  $\tau = UabcW$ ,  $\tau' = UbacW$  and  $\sigma = UacbW$  for some letters  $a < b < c$  and words  $U, W$ . Define  $\sigma' := \tau s_i s_{i+1} s_i = \tau' s_{i+1} s_i = \sigma s_i s_{i+1} = UcbaW$ . Since  $\tau \equiv^k \sigma$ , there exist  $k$ -twist congruence witnesses  $b < w_1, \dots, w_k < c$  which all appear in  $W$ . Since  $a < b$ , these letters are thus also  $k$ -congruence witnesses for the equivalences  $\tau' \equiv^k \tau' s_{i+1} \equiv^k \tau' s_{i+1} s_i = \sigma'$ . Moreover, since  $a < b < c$ , we have  $\sigma < \sigma'$ .
- (3) If  $j = i-1$ , we proceed similarly as in Situation (2).
- (4) If  $j < i-1$ , we proceed similarly as in Situation (1).

In all cases, we have  $\tau \equiv^k \sigma < \sigma' \equiv^k \tau'$ . Since  $\sigma$  is closer to  $\max(C)$  than  $\tau$ , we obtain that  $\max(C) < \max(C')$  by induction hypothesis.  $\square$

**1.1.10. Increasing flip lattice.** We now recall the notion of flips in pipe dreams and study the graph of increasing flips in acyclic  $k$ -twists. A detailed treatment of increasing flips in finite type subword complexes can be found in [PS13]. However, we are not aware that the restriction of this graph to acyclic pipe dreams was ever studied, besides the undirected version in connection to brick polytopes [PS12, PS15].

**Definition 38.** An **elbow flip** (or just **flip**) in a  $k$ -twist is the exchange of an elbow  $\curvearrowright$  between two relevant pipes  $p, p'$  with the unique crossing  $\oplus$  between  $p$  and  $p'$ . The flip is **increasing** if the initial elbow is located (largely) south-west of the final elbow.

**Remark 39** (Flips in  $k$ -triangulations). As already mentioned in Example 5 and Theorem 10, an elbow flip in a  $k$ -twist  $T$  corresponds to a diagonal flip in the dual  $k$ -triangulation  $T^*$ . The  $(m + 2k)$ -gon was chosen so that a flip is increasing if and only if the slope increases between the initial and the final diagonals. See Figures 2 and 4 for illustrations.

We are interested in the graph of increasing flips, restricted to the acyclic  $(k, n)$ -twists. See Figure 8 for an illustration when  $k = 2$  and  $n = 4$ . Note that although the graph of flips is regular, its restriction to acyclic twists is not anymore regular in general: the first example appears for  $k = 2$  and  $n = 5$ . It is known (see *e.g.* via subword complexes in [PS13] or via multitriangulations in [PS09]) that this graph is acyclic and it has a unique source (resp. sink) given by the  $(k, n)$ -twist where all relevant elbows are in the first columns (resp. last rows) while all crosses are on the last  $n$  columns (resp. first  $n$  rows).

We call **increasing flip order** the transitive closure of the increasing flip graph on acyclic  $k$ -twists. Be aware that it is strictly contained in the restriction to acyclic  $k$ -twists of the transitive closure of the increasing flip graph on all  $k$ -twists: namely, there are pairs of acyclic  $k$ -twists so that any path of increasing flips between them passes through a cyclic  $k$ -twist.

**Proposition 40.** *The following posets are all isomorphic:*

- the increasing flip order on acyclic  $k$ -twists,
- the quotient lattice of the weak order by the  $k$ -twist congruence  $\equiv^k$ ,
- the subposet of the weak order induced by the permutations of  $\mathfrak{S}_n$  avoiding the pattern  $1(k+2) - (\sigma_1 + 1) - \cdots - (\sigma_k + 1)$  for all  $\sigma \in \mathfrak{S}_k$ ,
- the subposet of the weak order induced by the permutations of  $\mathfrak{S}_n$  avoiding the pattern  $(k+2)1 - (\sigma_1 + 1) - \cdots - (\sigma_k + 1)$  for all  $\sigma \in \mathfrak{S}_k$ .

*Proof.* Consider two distinct  $k$ -twists  $T, T'$  and their  $k$ -twist congruence classes  $C, C'$ . If there exist representatives  $\tau = UijV \in C$  and  $\tau' = UjiV \in C'$  which are adjacent in weak order, then  $T = \text{ins}^k(\tau)$  and  $T' = \text{ins}^k(\tau')$  differ by the flip of the first/last elbow between the  $i$ th and  $j$ th pipes, by definition of the map  $\text{ins}^k$ . Conversely, if  $T$  and  $T'$  differ by the flip of the first/last elbow between the  $i$ th and  $j$ th pipes, then  $i$  and  $j$  are connected in the contact graph  $T^\#$ , so that there exists a linear extension  $\tau = UijV$  of  $T^\#$  where  $i$  and  $j$  are consecutive. Let  $\tau' = UjiV$  be the permutation obtained by the switch of  $i$  and  $j$  in  $\tau$ . By definition of the map  $\text{ins}^k$ , the twist  $\text{ins}^k(\tau')$  is obtained by flipping the first elbow between the  $i$ th and  $j$ th pipes in  $\text{ins}^k(\tau)$ . The representatives  $\tau \in C$  and  $\tau' \in C'$  are thus adjacent in weak order. This proves that  $\text{ins}^k$  induces an isomorphism from the quotient lattice of the weak order by the  $k$ -twist congruence to the increasing flip order on acyclic  $k$ -twists. In turn, since the  $k$ -twist congruence is a lattice congruence, this quotient lattice is isomorphic to the subposet of the weak order induced by the minimal (resp. maximal) elements of the classes. See [Rea04, Rea06] for further details on quotient lattices.  $\square$

**Example 41** (Tamari lattice). When  $k = 1$ , the increasing flip lattice is the classical Tamari lattice [MHPS12].

**Remark 42** (Increasing flip orders for arbitrary subword complexes). Again, let us warn the reader that Proposition 40 does not hold for arbitrary type  $A$  subword complexes. First, as already mentioned in Remark 33, the fibers of the surjection map are not always intervals [PS15, Figure 9]. But in fact, there are root-independent subword complexes for which the increasing flip order on facets is not even a lattice, see *e.g.* [PS15, Example 5.12].

1.1.11.  **$k$ -recoil schemes.** To prepare the definition of the  $k$ -canopy of an acyclic  $k$ -twist, we now recall the notion of  $k$ -recoil schemes of permutations, which was already defined by J.-C. Novelli, C. Reutenauer and J.-Y. Thibon in [NRT11]. We use a description in terms of acyclic orientations of a certain graph as it is closer to the description of the vertices of the zonotope that we will use later in Section 1.2. For the convenience of the reader, we provide proof sketches in terms of acyclic orientations of the results of [NRT11] needed later in this paper.



A *recoil* in a permutation  $\tau \in \mathfrak{S}_n$  is a position  $i \in [n-1]$  such that  $\tau^{-1}(i) > \tau^{-1}(i+1)$  (in other words, it is a descent of the inverse of  $\tau$ ). The *recoil scheme* of  $\tau \in \mathfrak{S}_n$  is the sign vector  $\text{rec}(\tau) \in \{-, +\}^{n-1}$  defined by  $\text{rec}(\tau)_i = -$  if  $i$  is a recoil of  $\tau$  and  $\text{rec}(\tau)_i = +$  otherwise.

To extend this definition to general  $k$ , we consider the graph  $G^k(n)$  with vertex set  $[n]$  and edge set  $\{\{i, j\} \in [n]^2 \mid i < j \leq i+k\}$ . For example, when  $k=1$ , the graph  $G^1(n)$  is just the  $n$ -path. We denote by  $\mathcal{AO}^k(n)$  the set of acyclic orientations of  $G^k(n)$  (i.e. with no oriented cycle).

**Proposition 43** ([NRT11, Prop. 2.1]). *The number of acyclic orientations of  $G^k(n)$  is*

$$|\mathcal{AO}^k(n)| = \begin{cases} n! & \text{if } n \leq k, \\ k!(k+1)^{n-k} & \text{if } n \geq k. \end{cases}$$

*Proof.* When  $n \leq k$ , the graph  $G^k(n)$  is complete, so an acyclic orientation of  $G^k(n)$  is a permutation of  $[n]$ . The case  $n \geq k$  then follows by induction on  $n$ . Indeed, an acyclic orientation on  $G^k(n)$  corresponds to an acyclic orientation on  $G^k(n-1)$  together with a choice among the  $k+1$  possibilities of the position of  $n$  with respect to  $\{n-k, \dots, n-1\}$ .  $\square$

Table 3 gathers the numbers of acyclic orientations of  $G^k(n)$  for  $k < n \leq 10$ .

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	.	2	4	8	16	32	64	128	256	512
2	.	.	6	18	54	162	486	1458	4374	13122
3	.	.	.	24	96	384	1536	6144	24576	98304
4	.	.	.	.	120	600	3000	15000	75000	375000
5	.	.	.	.	.	720	4320	25920	155520	933120
6	.	.	.	.	.	.	5040	35280	246960	1728720
7	.	.	.	.	.	.	.	40320	322560	2580480
8	.	.	.	.	.	.	.	.	362880	3265920
9	.	.	.	.	.	.	.	.	.	3628800

TABLE 3. The numbers  $|\mathcal{AO}^k(n)| = k!(k+1)^{n-k}$  of acyclic orientations of the graph  $G^k(n)$  for  $k < n \leq 10$ . Dots indicate that the value remains constant (equal to  $n!$ ) in the column.

We use these acyclic orientations to define the  $k$ -recoil scheme of a permutation and the corresponding  $k$ -recoil congruence.

**Definition 44.** *The  $k$ -recoil scheme of a permutation  $\tau \in \mathfrak{S}_n$  is the orientation  $\text{rec}^k(\tau) \in \mathcal{AO}^k(n)$  with an edge  $i \rightarrow j$  for all  $i, j \in [n]$  such that  $|i-j| \leq k$  and  $\tau^{-1}(i) < \tau^{-1}(j)$ . We call  $k$ -recoil map the map  $\text{rec}^k : \mathfrak{S}_n \rightarrow \mathcal{AO}^k(n)$ .*

The following statement is immediate and left to the reader.

**Proposition 45.** *For  $O \in \mathcal{AO}^k(n)$ , the fiber of  $O$  by the  $k$ -recoil map is the set of linear extensions of the transitive closure of  $O$ .*

**Definition 46.** *The  $k$ -recoil congruence  $\approx^k$  on  $\mathfrak{S}_n$  is the transitive closure of the rewriting rule*

$$UijV \approx^k UjiV \quad \text{if } i+k < j,$$

where  $i, j$  are elements of  $[n]$  while  $U, V$  are (possibly empty) words on  $[n]$ .

The congruence classes of  $\approx^k$  for  $k \in \{1, 2\}$  and  $n = 4$  are represented in Figure 9, where the underlying graph is the Hasse diagram of the (right) weak order (see Section 1.1.8).

**Proposition 47** ([NRT11, Prop. 2.2]). *For any  $\tau, \tau' \in \mathfrak{S}_n$ , we have  $\tau \approx^k \tau' \iff \text{rec}^k(\tau) = \text{rec}^k(\tau')$ . In other words, the fibers of  $\text{rec}^k$  are precisely the  $k$ -recoil congruence classes.*

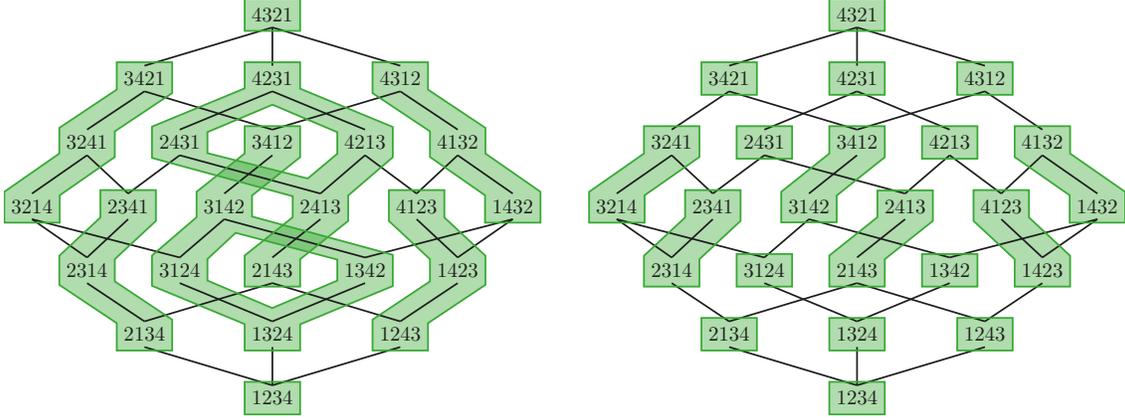


FIGURE 9. The  $k$ -recoil congruence classes on  $\mathfrak{S}_4$  for  $k = 1$  (left) and  $k = 2$  (right).

*Proof.* By Proposition 45, each fiber of  $\text{rec}^k$  gathers the linear extensions of the transitive closure of an acyclic orientation  $O$  of  $G^k(n)$ . Since the set of linear extensions of a poset is connected by simple transpositions, it suffices to show that  $\tau \approx^k \tau' \iff \text{rec}^k(\tau) = \text{rec}^k(\tau')$  for any two permutations  $\tau = UijV$  and  $\tau' = UjiV$  of  $\mathfrak{S}_n$  which differ by the inversion of two consecutive values  $i < j$ . Indeed, the orientations  $\text{rec}^k(\tau)$  and  $\text{rec}^k(\tau')$  differ if and only if  $G^k(n)$  has an edge between  $i$  and  $j$ , i.e. if and only if  $i + k \geq j$ .  $\square$

**Proposition 48** ([NRT11, Prop. 2.4]). *The  $k$ -recoil congruence classes are intervals of the weak order.*

*Proof.* Immediate from the characterization of WOIs given in Definition 30 and Proposition 31.  $\square$

**Definition 49.** A **direction flip** (or just **flip**) in an acyclic orientation is the switch of the direction of an edge of  $G^k(n)$ . The flip is **increasing** if the initial direction was increasing. Define the **increasing flip order** on  $\mathcal{AO}^k(n)$  to be the transitive closure of the increasing flip graph on  $\mathcal{AO}^k(n)$ .

**Theorem 50.** *The  $k$ -recoil congruence  $\approx^k$  is a lattice congruence of the weak order. The  $k$ -recoil map  $\text{rec}^k$  defines an isomorphism from the quotient lattice of the weak order by the  $k$ -recoil congruence  $\approx^k$  to the increasing flip lattice on the acyclic orientations of  $G^k(n)$ .*

*Proof.* Similar to that of Theorem 37 and Proposition 40.  $\square$

**1.1.12.  $k$ -canopy schemes.** We recall that the **canopy** of a binary tree  $T$  with  $n$  nodes is the sign vector  $\text{can}(T) \in \{-, +\}^{n-1}$  defined by  $\text{can}(T)_i = -$  if the node  $i$  of  $T$  is above the node  $i + 1$  of  $T$  and  $\text{can}(T)_i = +$  otherwise. This map was already used by J.-L. Loday in [LR98, Lod04], but the name “canopy” was coined by X. Viennot [Vie07]. The binary search tree insertion map and the canopy map factorize the recoil map:  $\text{can} \circ \text{ins} = \text{rec}$ . This combinatorial fact can also be understood on the geometry of the normal fans of the permutahedron, the associahedron and the cube, see Section 1.2.2.

We now define an equivalent of the canopy map for general  $k$ . To ensure that Definition 52 is valid, we need the following simple observation on comparisons of closed pipes in a  $k$ -twist.

**Lemma 51.** *If  $|i - j| \leq k$ , the  $i$ th and  $j$ th pipes in an acyclic  $k$ -twist  $T$  are comparable for  $\triangleleft_T$ .*

*Proof.* By Corollary 15 (ii), if  $i < j$  and the  $i$ th and  $j$ th pipes are incomparable, then the last vertical step of  $i$  crosses the first horizontal step of  $j$ . Since each pipe has  $k$  horizontal steps by Lemma 13 (ii), it ensures that  $j > i + k$ .  $\square$

**Definition 52.** *The  $k$ -canopy scheme of a  $(k, n)$ -twist  $T$  is the orientation  $\text{can}^k(T) \in \mathcal{AO}^k(n)$  with an edge  $i \rightarrow j$  for all  $i, j \in [n]$  such that  $|i - j| \leq k$  and  $i \triangleleft_T j$ . It indeed defines an acyclic orientation of  $G^k(n)$  by Lemma 51. We call  **$k$ -canopy** the map  $\text{can}^k : \mathcal{AT}^k(n) \rightarrow \mathcal{AO}^k(n)$ .*

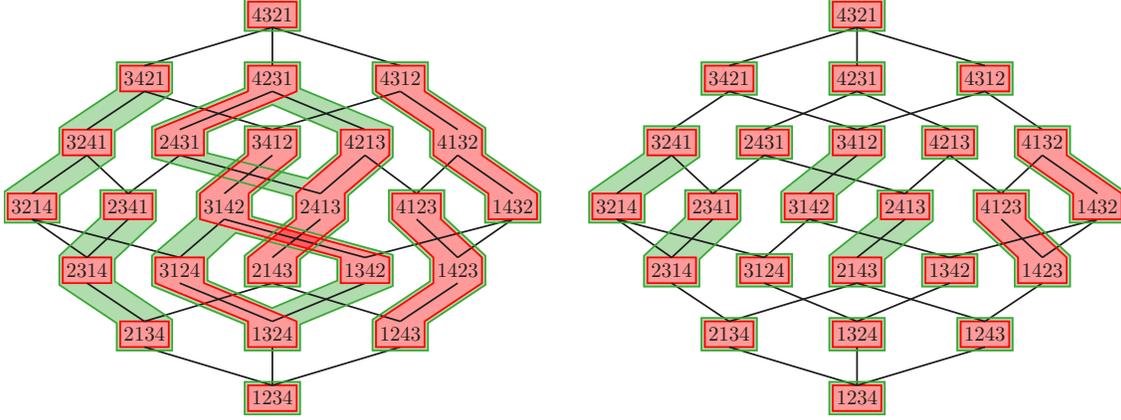


FIGURE 10. The fibers of the maps  $\text{ins}^k$  ( $k$ -twist congruence classes, red) and  $\text{rec}^k$  ( $k$ -recoils congruence classes, green) on the weak order on  $\mathfrak{S}_4$  for  $k = 1$  (left) and  $k = 2$  (right). See also Figures 7 and 9 where these fibers appear separately.

**Proposition 53.** *The maps  $\text{ins}^k$ ,  $\text{can}^k$ , and  $\text{rec}^k$  define the following commutative diagram of lattice homomorphisms:*

$$\begin{array}{ccc}
 \mathfrak{S}_n & \xrightarrow{\text{rec}^k} & \mathcal{AO}^k(n) \\
 & \searrow \text{ins}^k & \nearrow \text{can}^k \\
 & \mathcal{AT}^k(n) & 
 \end{array}$$

*Proof.* Consider a permutation  $\tau$  and let  $i, j \in [n]$  with  $|i - j| < k$  such that  $\tau^{-1}(i) < \tau^{-1}(j)$ . By definition, there is an arc  $i \rightarrow j$  in  $\text{rec}^k(\tau)$ . Moreover, the  $i$ th pipe is inserted after the  $j$ th pipe in  $\text{ins}^k(\tau)$ , so that  $i \triangleleft_{\text{ins}^k(\tau)} j$  and there is also an arc  $i \rightarrow j$  in  $\text{can}^k \circ \text{ins}^k(\tau)$ .  $\square$

The fibers of these maps for  $k \in \{1, 2\}$  and  $n = 4$  appear separately in Figures 7 and 9, and together in Figure 10. See also Section 1.2.2 for a geometric interpretation of Proposition 53.

**Remark 54** (Combinatorial inclusions). When  $k > \ell$ , the  $k$ -twist congruence  $\equiv^k$  refines the  $\ell$ -twist congruence  $\equiv^\ell$  (meaning that  $\tau \equiv^k \tau'$  implies  $\tau \equiv^\ell \tau'$ ) and the  $k$ -recoil congruence  $\approx^k$  refines the  $\ell$ -recoil congruence  $\approx^\ell$ . We can thus define surjective restriction maps  $\text{res}^{k \rightarrow \ell} : \mathcal{AT}^k(n) \rightarrow \mathcal{AT}^\ell(n)$  and  $\text{res}^{k \rightarrow \ell} : \mathcal{AO}^k(n) \rightarrow \mathcal{AO}^\ell(n)$  by:

- for an acyclic  $k$ -twist  $T$ , the  $\ell$ -twist  $\text{res}^{k \rightarrow \ell}(T)$  is obtained by insertion of any linear extension of  $T^\#$  (it is independent of the choice of this linear extension),
- for an acyclic orientation  $\theta \in \mathcal{AO}^k(n)$ , the sign vector  $\text{res}^{k \rightarrow \ell}(\theta) \in \mathcal{AO}^\ell(n)$  is obtained by restriction of  $\theta$  to the edges of  $G^\ell(n)$ .

We therefore obtain the following commutative diagram of lattice homomorphisms:

$$\begin{array}{ccc}
 \mathcal{AT}^k(n) & \xrightarrow{\text{res}^{k \rightarrow \ell}} & \mathcal{AT}^\ell(n) \\
 \downarrow \text{can}^k & \swarrow \text{ins}^k & \nearrow \text{ins}^\ell \\
 & \mathfrak{S}_n & \\
 \downarrow \text{rec}^k & \searrow \text{rec}^\ell & \\
 \mathcal{AO}^k(n) & \xrightarrow{\text{res}^{k \rightarrow \ell}} & \mathcal{AO}^\ell(n) \\
 & \downarrow \text{can}^\ell & 
 \end{array}$$

## 1.2. GEOMETRY OF ACYCLIC TWISTS

This section is devoted to the polyhedral geometry of permutations of  $\mathfrak{S}_n$ , acyclic twists of  $\mathcal{AT}^k(n)$ , and acyclic orientations of  $\mathcal{AO}^k(n)$ . It is mainly based on properties of brick polytopes of sorting networks, defined and studied by V. Pilaud and F. Santos in [PS12]. This construction has been further extended to root-independent subword complexes on any finite Coxeter group by V. Pilaud and C. Stump in [PS15]. However, the context of [PS15] does not apply here since the present paper deals with a very specific family of sorting networks which are not root-independent. We note that this family was already discussed in [PS12, Section 5].

To keep this section short, we skip all proofs of its statements as they follow directly from [PS12]. This section should be seen as a brief geometric motivation for the combinatorial and algebraic construction of this paper. The reader familiar with the geometry of the brick polytope is invited to jump directly to Section 1.3.

**1.2.1. Permutahedra, brick polytopes, and zonotopes.** We first recall the definition of three families of polytopes, which are illustrated in Figure 11. We refer to [Zie98] for background on polytopes. We denote by  $(\mathbf{e}_i)_{i \in [n]}$  the canonical basis of  $\mathbb{R}^n$  and let  $\mathbb{1} = \sum_{i \in [n]} \mathbf{e}_i$ .

PERMUTAHEDRA The permutahedron is a classical polytope [Zie98, Lecture 0] whose geometric and combinatorial properties reflect that of the symmetric group  $\mathfrak{S}_n$ .

**Definition 55.** *The permutahedron  $\text{Perm}(n)$  is the  $(n-1)$ -dimensional polytope*

$$\text{Perm}(n) := \text{conv} \{ \mathbf{x}(\tau) \mid \tau \in \mathfrak{S}_n \} = \mathbf{H}^{\neq}([n]) \cap \bigcap_{\emptyset \neq I \subsetneq [n]} \mathbf{H}^{\geq}(I) = \mathbb{1} + \sum_{1 \leq i < j \leq n} [\mathbf{e}_i, \mathbf{e}_j],$$

defined equivalently as

- the convex hull of the points  $\mathbf{x}(\tau) := [\tau^{-1}(i)]_{i \in [n]} \in \mathbb{R}^n$  for all permutations  $\tau \in \mathfrak{S}_n$ ,
- the intersection of the hyperplane  $\mathbf{H}^{\neq}([n]) := \{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2} \}$  with the half-spaces  $\mathbf{H}^{\geq}(I) := \{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in I} x_i \geq \binom{|I|+1}{2} \}$  for all proper non-empty subset  $\emptyset \neq I \subsetneq [n]$ ,
- the Minkowski sum of the point  $\mathbb{1}$  with the segments  $[\mathbf{e}_i, \mathbf{e}_j]$  for  $1 \leq i < j \leq n$ .

We consider a dilated and translated copy of the permutahedron  $\text{Perm}(n)$ , which will fit better the other two families of polytopes defined later (see Remark 62 for a precise statement). Namely, we set

$$\text{Perm}^k(n) := k \text{Perm}(n) - \frac{k(n+1)}{2} \mathbb{1}.$$

Observe that  $\text{Perm}^k(n)$  now lies in the hyperplane  $\mathbb{H} := \{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = 0 \}$ .

BRICK POLYTOPES To define the brick polytope, we essentially follow [Pil10, PS12] except that we apply again a translation in direction  $\mathbb{1}$  to obtain a polytope in the hyperplane  $\mathbb{H}$ .

**Definition 56.** *We call bricks the squares  $[i, i+1] \times [j, j+1]$  of the triangular shape. The brick area of a pipe  $\mathfrak{p}$  is the number of bricks located below  $\mathfrak{p}$  but inside the axis-parallel rectangle defined by the two endpoints of  $\mathfrak{p}$ . The brick vector of a  $k$ -twist  $\mathbf{T}$  is the vector  $\mathbf{x}(\mathbf{T}) \in \mathbb{R}^n$  whose  $i$ th coordinate is the brick area of the  $i$ th pipe of  $\mathbf{T}$  minus  $\frac{k(n+k)}{2}$ . The brick polytope  $\text{Brick}^k(n)$  is the polytope defined as the convex hull of the brick vectors of all  $(k, n)$ -twists.*

As for the permutahedron described above, we know three descriptions of the brick polytopes: its vertex description, its hyperplane description, and a Minkowski sum description. These properties are proved in [PS12] (modulo our translation in the direction  $\mathbb{1}$ ).

**Proposition 57** ([PS12]). *The brick polytope  $\text{Brick}^k(n)$  has the following properties for any  $n, k \in \mathbb{N}$ .*

- (i) *It lies in the hyperplane  $\mathbb{H} := \{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = 0 \}$  and has dimension  $n-1$ .*
- (ii) *The vertices of  $\text{Brick}^k(n)$  are precisely the brick vectors of the acyclic  $(k, n)$ -twists.*

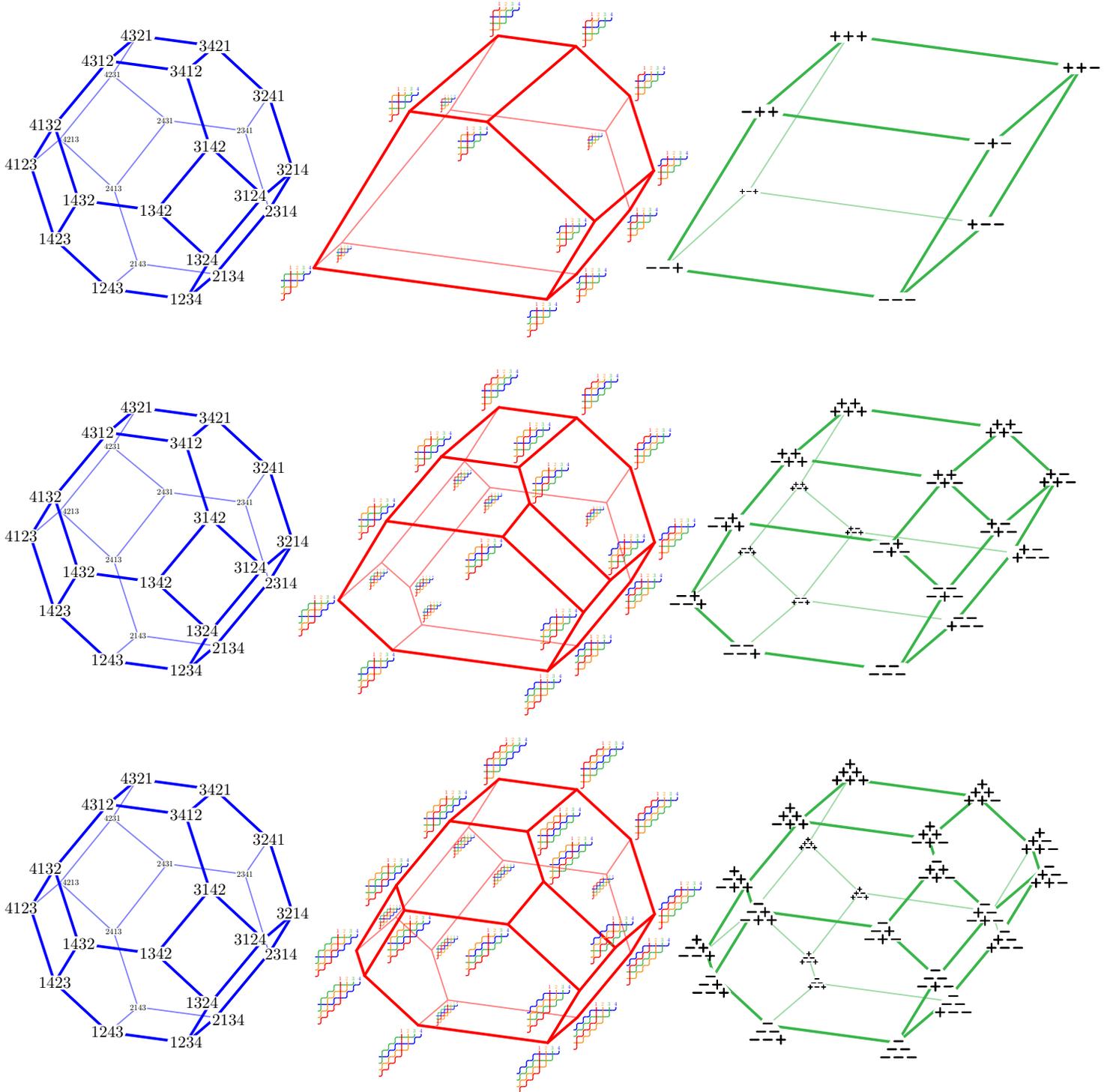


FIGURE 11. The permutahedron  $\text{Perm}^k(4)$  (left), the brick polytope  $\text{Brick}^k(4)$  (middle) and the zonotope  $\text{Zono}^k(4)$  (right) for  $k = 1$  (top),  $k = 2$  (middle) and  $k = 3$  (bottom). For readability, we represent orientations of  $G^k(n)$  by pyramids of signs.

- (iii) The normal vectors of the facets of  $\text{Brick}^k(n)$  are given by the [proper  \$k\$ -connected  \$\{0, 1\}\$ -sequences](#) of size  $n$ , i.e. the sequences of  $\{0, 1\}^n$  distinct from  $0^n$  and  $1^n$  and which do not contain a subsequence  $10^\ell 1$  for  $\ell \geq k$ . The number of facets of  $\text{Brick}^k(n)$  is therefore the coefficient of  $t^n$  in

$$\frac{t^2(2 - t^k)}{(1 - 2t + t^{k+1})(1 - t)}.$$

- (iv) For a brick  $b$ , let  $\mathbf{x}_b(\mathbb{T})$  denote the characteristic vector of the pipes of a  $(k, n)$ -twist  $\mathbb{T}$  whose brick area contain  $b$ , and define  $\text{Brick}_b^k(n)$  to be the convex hull of the vectors  $\mathbf{x}_b(\mathbb{T})$  for all  $(k, n)$ -twists  $\mathbb{T}$ . Then, up to the translation of vector  $\frac{k(n+k)}{2}\mathbf{1}$ , the brick polytope  $\text{Brick}^k(n)$  is also the Minkowski sum of the polytopes  $\text{Brick}_b^k(n)$  over all bricks  $b$ .

**Example 58** (J.-L. Loday's associahedron). When  $k = 1$ , the brick polytope  $\text{Brick}^1(n)$  coincides (up to translation) with J.-L. Loday's associahedron [Lod04]. See Figure 11 (top center).

**ZONOTOPES** Zonotopes are particularly important polytopes which are constructed equivalently as projections of cubes, or as Minkowski sums of segments. The combinatorics of a zonotope is completely determined by the matroid of the vector configuration defined by these summands. We refer to [Zie98, Lecture 7] for a presentation of these polytopes and their relations to oriented matroids. Notable examples are graphical zonotopes, defined as follows.

**Definition 59.** The [graphical zonotope](#)  $\text{Zono}(G)$  of a graph  $G$  is the Minkowski sum of the segments  $[\mathbf{e}_i, \mathbf{e}_j]$  for all edges  $\{i, j\}$  of  $G$ .

The following classical statement gives the vertex and facet descriptions of graphical zonotopes.

**Proposition 60.** The graphical zonotope  $\text{Zono}(G)$  has the following properties for any graph  $G$ .

- (i) The dimension of  $\text{Zono}(G)$  is the number of edges of a maximal cycle-free subgraph of  $G$ .
- (ii) The vertices of  $\text{Zono}(G)$  correspond to the acyclic orientations of  $G$ . The  $i$ th coordinate of the vertex  $\mathbf{x}(\mathbb{O})$  of  $\text{Zono}(G)$  corresponding to an acyclic orientation  $\mathbb{O}$  of  $G$  is the indegree of vertex  $i$  in  $\mathbb{O}$ .
- (iii) The facets of  $\text{Zono}(G)$  correspond to minimal cuts of  $G$ .

For example, the permutahedron  $\text{Perm}(n)$  is the graphical zonotope of the complete graph, its vertices correspond to permutations of  $[n]$  (acyclic tournaments), and its facets correspond to proper subsets of  $[n]$  (minimal cuts). Here, we focus on the zonotope of  $G^k(n)$  whose vertices correspond to the acyclic orientations in  $\mathcal{AO}^k(n)$  and whose facets correspond to minimal cuts of  $G^k(n)$ . As for the previous polytopes, we perturb this zonotope to fit the other two polytopes better (see Remark 62 for a precise statement), and thus define

$$\text{Zono}^k(n) := \sum_{1 \leq i < j \leq n} \lambda(i, j, k, n) \cdot [\mathbf{e}_i, \mathbf{e}_j] - \frac{(n-1)(n+3k-2)}{6} \mathbf{1},$$

where

$$\lambda(i, j, k, n) := \begin{cases} n + k - 2|i - j| & \text{if } |i - j| < k, \\ \min(i, n + 1 - j)(n + k - 1 - \min(i, n + 1 - j)) & \text{if } |i - j| = k, \\ 0 & \text{if } |i - j| > k. \end{cases}$$

Note that this perturbation is only cosmetic and preserves the combinatorics. Indeed, observe that for all  $1 \leq i < j \leq n$ , we have  $\lambda_{i,j,k,n} \neq 0$  if and only if  $|i - j| \leq k$ . Therefore, the zonotopes  $\text{Zono}^k(n)$  and  $\text{Zono}(G^k(n))$  have the same normal fan (see Section 1.2.2) and thus the same face lattice. Note that the translation ensures that  $\text{Zono}^k(n)$  lies in the hyperplane  $\mathbb{H} := \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = 0\}$ .

**Example 61** (Cube). When  $k = 1$ , the zonotope  $\text{Zono}^1(n)$  coincides (up to translation and scaling) with the parallelotope generated by the simple roots  $\mathbf{e}_{i+1} - \mathbf{e}_i$ . It has the same combinatorics as the  $n$ -dimensional cube. See Figure 11 (top right).

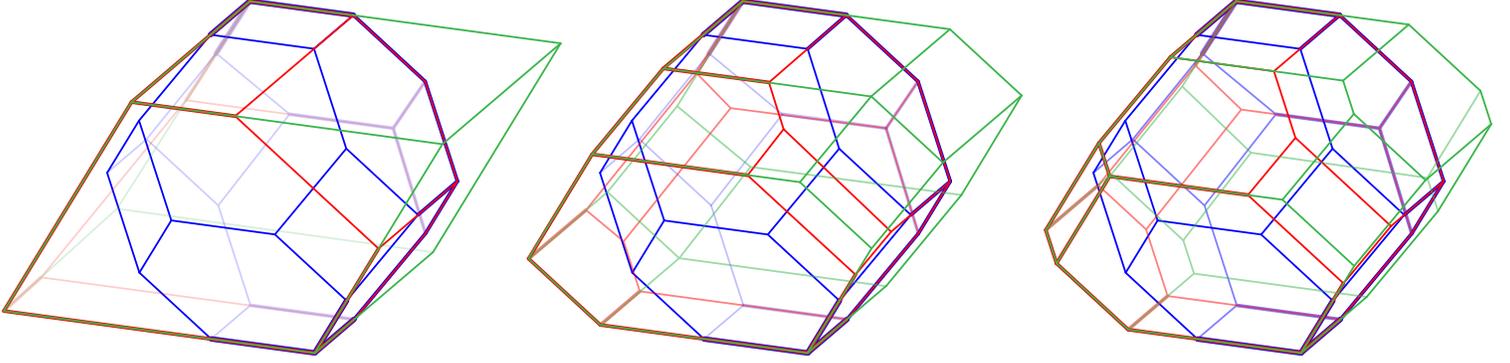


FIGURE 12. The inclusions  $\text{Perm}^k(n) \subseteq \text{Brick}^k(n) \subseteq \text{Zono}^k(n)$  for  $k = 1$  (left),  $k = 2$  (middle) and  $k = 3$  (right). The permutahedron  $\text{Perm}^k(n)$  is in blue, the brick polytope  $\text{Brick}^k(n)$  in red and the zonotope  $\text{Zono}^k(n)$  in green.

**Remark 62** (Geometric inclusions). We have chosen our normalizations (dilations and translations) so that the polytopes  $\text{Perm}^k(n)$ ,  $\text{Brick}^k(n)$  and  $\text{Zono}^k(n)$  all leave in the hyperplane  $\mathbb{H} := \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = 0\}$  and fulfill the following inclusions:

$$\begin{array}{ccc}
 \frac{1}{k} \text{Brick}^k(n) & \subseteq & \frac{1}{\ell} \text{Brick}^\ell(n) \\
 \supseteq & & \subsetneq \\
 \cap & \text{Perm}(n) & \cap \\
 \supseteq & & \subsetneq \\
 \frac{1}{k} \text{Zono}^k(n) & \subseteq & \frac{1}{\ell} \text{Zono}^\ell(n)
 \end{array}$$

These inclusions are illustrated in Figures 11, 12 and 13. Compare to Remark 54.

Observe moreover that

$$\text{Zono}^k(n) = \text{Zono}^{n-1}(n) + (k+1-n) \text{Perm}^1(n)$$

for all  $k \geq n-1$ . We therefore obtain that the rescaled polytopes  $\frac{1}{k} \text{Brick}^k(n)$  and  $\frac{1}{k} \text{Zono}^k(n)$  both converge to  $\frac{1}{k} \text{Perm}^k(n) = \text{Perm}^1(n)$  when  $k$  tend to  $\infty$  (see Figure 13).

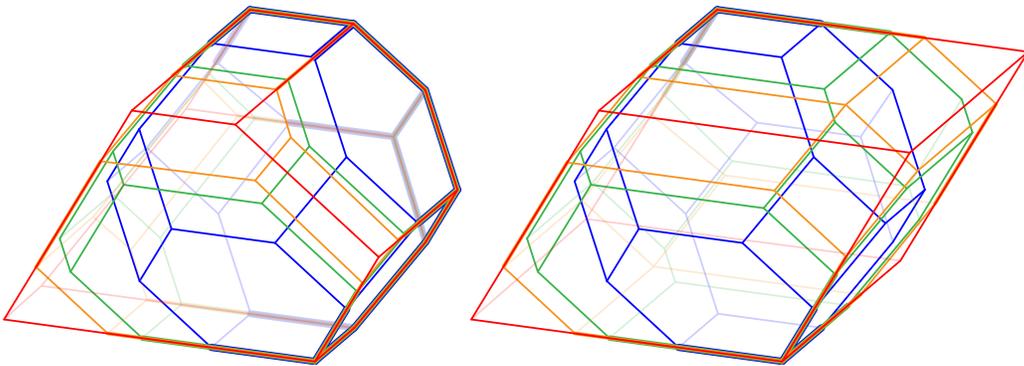


FIGURE 13. The inclusions of the brick polytopes  $\frac{1}{k} \text{Brick}^k(4)$  (left) and of the zonotopes  $\frac{1}{k} \text{Zono}^k(4)$  (right) for  $k = 1$  (red),  $k = 2$  (orange) and  $k = 3$  (green). Both tend to the classical permutahedron  $\text{Perm}^1(4)$  (blue) when  $k$  tends to  $\infty$ .

1.2.2. **The geometry of the surjections  $\text{ins}^k$ ,  $\text{can}^k$ , and  $\text{rec}^k$ .** Besides Remark 62, the main geometric connection between the three polytopes  $\text{Perm}^k(n)$ ,  $\text{Brick}^k(n)$  and  $\text{Zono}^k(n)$  is given by their normal fans. Remember that a *polyhedral fan* is a collection of polyhedral cones of  $\mathbb{R}^n$  closed under faces and which intersect pairwise along faces, see *e.g.* [Zie98, Lecture 7]. The (outer) *normal cone* of a face  $F$  of a polytope  $P$  is the cone generated by the outer normal vectors of the facets of  $P$  containing  $F$ . Finally, the (outer) *normal fan* of  $P$  is the collection of the (outer) normal cones of all its faces.

The *incidence cone*  $C(\triangleleft)$  and the *braid cone*  $C^\diamond(\triangleleft)$  of an poset  $\triangleleft$  are the polyhedral cones defined by

$$C(\triangleleft) := \text{cone} \{ \mathbf{e}_i - \mathbf{e}_j \mid \text{for all } i \triangleleft j \} \quad \text{and} \quad C^\diamond(\triangleleft) := \{ \mathbf{x} \in \mathbb{H} \mid x_i \leq x_j \text{ for all } i \triangleleft j \}.$$

These two cones lie in the space  $\mathbb{H}$  and are polar to each other. For a permutation  $\tau \in \mathfrak{S}_n$  (resp. a twist  $T \in \mathcal{AT}^k(n)$ , resp. an orientation  $O \in \mathcal{AO}^k(n)$ ), we slightly abuse notation to write  $C(\tau)$  (resp.  $C(T)$ , resp.  $C(O)$ ) for the incidence cone of the chain  $\tau_1 \triangleleft \cdots \triangleleft \tau_n$  (resp. of the transitive closure  $\triangleleft$  of the contact graph  $T^\#$ , resp. of the transitive closure of  $O$ ). We define similarly the braid cone  $C^\diamond(\tau)$  (resp.  $C^\diamond(T)$ , resp.  $C^\diamond(O)$ ). These cones (together with all their faces) form the normal fans of the polytopes of Section 1.2.1.

**Proposition 63.** *The collections of cones*

$$\{ C^\diamond(\tau) \mid \tau \in \mathfrak{S}_n \}, \quad \{ C^\diamond(T) \mid T \in \mathcal{AT}^k(n) \} \quad \text{and} \quad \{ C^\diamond(O) \mid O \in \mathcal{AO}^k(n) \},$$

*together with all their faces, are the normal fans of the permutahedron  $\text{Perm}^k(n)$ , the brick polytope  $\text{Brick}^k(n)$  and the zonotope  $\text{Zono}^k(n)$  respectively.*

Observe moreover that the normal fan of  $\text{Perm}^k(n)$  is also the collection of chambers of the Coxeter arrangement given by all hyperplanes  $\{ \mathbf{x} \in \mathbb{H} \mid x_i = x_j \}$  for all  $i, j \in [n]$ . Similarly, the normal fan of  $\text{Zono}^k(n)$  is also the collection of chambers of the graphical arrangement given by the hyperplanes  $\{ \mathbf{x} \in \mathbb{H} \mid x_i = x_j \}$  for all edges  $\{i, j\}$  in  $G^k(n)$ .

Using these normal fans, one can interpret geometrically the maps  $\text{ins}^k$ ,  $\text{can}^k$ , and  $\text{rec}^k$  as follows.

**Proposition 64.** *The insertion map  $\text{ins}^k : \mathfrak{S}_n \rightarrow \mathcal{AT}^k(n)$ , the  $k$ -canopy  $\text{can}^k : \mathcal{AT}^k(n) \rightarrow \mathcal{AO}^k(n)$  and the  $k$ -recoil map  $\text{rec}^k : \mathfrak{S}_n \rightarrow \mathcal{AO}^k(n)$  are characterized by*

$$\begin{aligned} T = \text{ins}^k(\tau) &\iff C(T) \subseteq C(\tau) \iff C^\diamond(T) \supseteq C^\diamond(\tau), \\ O = \text{can}^k(T) &\iff C(O) \subseteq C(T) \iff C^\diamond(O) \supseteq C^\diamond(T), \\ O = \text{rec}^k(\tau) &\iff C(O) \subseteq C(\tau) \iff C^\diamond(O) \supseteq C^\diamond(\tau). \end{aligned}$$

Finally, the lattices studied in Section 1.1 also appear naturally in the geometry of the polytopes  $\text{Perm}^k(n)$ ,  $\text{Brick}^k(n)$  and  $\text{Zono}^k(n)$ . Denote by  $U$  the vector

$$U := (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = \sum_{i \in [n]} (n+1-2i) \mathbf{e}_i.$$

**Proposition 65.** *When oriented in the direction  $U$ , the 1-skeleta of the permutahedron  $\text{Perm}^k(n)$  (resp. of the brick polytope  $\text{Brick}^k(n)$ , resp. of the zonotope  $\text{Zono}^k(n)$ ) is the Hasse diagram of the weak order on permutations (resp. of the increasing flip lattice on acyclic  $(k, n)$ -twists, resp. of the increasing flip lattice on acyclic orientations of  $G^k(n)$ ).*

## 1.3. ALGEBRA OF ACYCLIC TWISTS

Motivated by the Hopf algebra on binary trees constructed by J.-L. Loday and M. Ronco [LR98] as a subalgebra of the Hopf algebra on permutations of C. Malvenuto and C. Reutenauer (see also [HNT05, AS06]), we define a Hopf algebra with bases indexed by acyclic  $k$ -twists. We then give combinatorial interpretations of the product and coproduct of this algebra and its dual in terms of  $k$ -twists. We finally conclude with further algebraic properties of this algebra.

**1.3.1. Hopf algebras FQSym and FQSym\*.** We briefly recall here the definition and some elementary properties of C. Malvenuto and C. Reutenauer's Hopf algebra on permutations [MR95]. We denote this algebra by FQSym to stress out its connection to free quasi-symmetric functions. We will however not use this connection in this paper. We denote by  $\mathfrak{S} := \bigsqcup_{n \in \mathbb{N}} \mathfrak{S}_n$  the set of all permutations, of arbitrary size.

For  $n, n' \in \mathbb{N}$ , let

$$\mathfrak{S}^{(n, n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau_1 < \dots < \tau_n \text{ and } \tau_{n+1} < \dots < \tau_{n+n'}\}$$

denote the set of permutations of  $\mathfrak{S}_{n+n'}$  with at most one descent, at position  $n$ . The *shifted concatenation*  $\tau \bar{\sqcup} \tau'$ , the *shifted shuffle*  $\tau \sqcup \tau'$ , and the *convolution*  $\tau \star \tau'$  of two permutations  $\tau \in \mathfrak{S}_n$  and  $\tau' \in \mathfrak{S}_{n'}$  are classically defined by

$$\begin{aligned} \tau \bar{\sqcup} \tau' &:= [\tau_1, \dots, \tau_n, \tau'_1 + n, \dots, \tau'_{n'} + n] \in \mathfrak{S}_{n+n'}, \\ \tau \sqcup \tau' &:= \{(\tau \bar{\tau}') \circ \pi^{-1} \mid \pi \in \mathfrak{S}^{(n, n')}\} \quad \text{and} \quad \tau \star \tau' := \{\pi \circ (\tau \bar{\tau}') \mid \pi \in \mathfrak{S}^{(n, n')}\}. \end{aligned}$$

For example,

$$\begin{aligned} 12 \bar{\sqcup} 231 &= \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}, \\ 12 \star 231 &= \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}. \end{aligned}$$

**Definition 66.** We denote by FQSym the Hopf algebra with basis  $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$  and whose product and coproduct are defined by

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \bar{\sqcup} \tau'} \mathbb{F}_\sigma \quad \text{and} \quad \Delta \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}.$$

This algebra is graded by the size of the permutations.

**Proposition 67.** A product of weak order intervals in FQSym is a weak order interval: for any two weak order intervals  $[\mu, \omega]$  and  $[\mu', \omega']$ , we have

$$\left( \sum_{\mu \leq \tau \leq \omega} \mathbb{F}_\tau \right) \cdot \left( \sum_{\mu' \leq \tau' \leq \omega'} \mathbb{F}_{\tau'} \right) = \sum_{\mu \bar{\mu}' \leq \sigma \leq \bar{\omega}' \omega} \mathbb{F}_\sigma.$$

**Corollary 68.** For  $\tau \in \mathfrak{S}_n$ , define

$$\mathbb{E}^\tau = \sum_{\tau \leq \tau'} \mathbb{F}_{\tau'} \quad \text{and} \quad \mathbb{H}^\tau = \sum_{\tau' \leq \tau} \mathbb{F}_{\tau'}$$

where  $\leq$  is the weak order on  $\mathfrak{S}_n$ . Then  $(\mathbb{E}_\tau)_{\tau \in \mathfrak{S}}$  and  $(\mathbb{H}_\tau)_{\tau \in \mathfrak{S}}$  are multiplicative bases of FQSym:

$$\mathbb{E}^\tau \cdot \mathbb{E}^{\tau'} = \mathbb{E}^{\tau \setminus \tau'} \quad \text{and} \quad \mathbb{H}^\tau \cdot \mathbb{H}^{\tau'} = \mathbb{H}^{\tau / \tau'},$$

where  $\tau \setminus \tau' = \tau \bar{\tau}'$  and  $\tau / \tau' = \bar{\tau}' \tau$ . A permutation  $\tau \in \mathfrak{S}_n$  is  $\mathbb{E}$ -decomposable (resp.  $\mathbb{H}$ -decomposable) if and only if there exists  $k \in [n-1]$  such that  $\tau([k]) = [k]$  (resp. such that  $\tau([k]) = [n] \setminus [k]$ ). Moreover, FQSym is freely generated by the elements  $\mathbb{E}^\tau$  (resp.  $\mathbb{H}^\tau$ ) for the  $\mathbb{E}$ -indecomposable (resp.  $\mathbb{H}$ -indecomposable) permutations.

We will also consider the dual Hopf algebra of FQSym, defined as follows.

**Definition 69.** We denote by FQSym\* the Hopf algebra with basis  $(\mathbb{G}_\tau)_{\tau \in \mathfrak{S}}$  and whose product and coproduct are defined by

$$\mathbb{G}_\tau \cdot \mathbb{G}_{\tau'} = \sum_{\sigma \in \tau \star \tau'} \mathbb{G}_\sigma \quad \text{and} \quad \Delta \mathbb{G}_\sigma = \sum_{\sigma \in \tau \bar{\sqcup} \tau'} \mathbb{G}_\tau \otimes \mathbb{G}_{\tau'}.$$

This algebra is graded by the size of the permutations.

**1.3.2. Subalgebra of FQSym.** We denote by  $\text{Twist}^k$  the vector subspace of FQSym generated by the elements

$$\mathbb{P}_T := \sum_{\substack{\tau \in \mathfrak{S} \\ \text{ins}^k(\tau) = T}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(T^\#)} \mathbb{F}_\tau,$$

for all acyclic  $k$ -twists  $T$ . For example, for the  $(k, 5)$ -twists of Figure 6, we have

$$\begin{array}{l} \mathbb{P}_{\begin{array}{c} \color{red}{1} \color{green}{2} \color{blue}{3} \color{purple}{4} \color{orange}{5} \\ \color{red}{1} \color{green}{2} \color{blue}{3} \color{purple}{4} \color{orange}{5} \end{array}} = \sum_{\tau \in \mathfrak{S}_5} \mathbb{F}_\tau \\ \mathbb{P}_{\begin{array}{c} \color{red}{1} \color{green}{2} \color{blue}{3} \color{purple}{4} \color{orange}{5} \\ \color{red}{1} \color{green}{2} \color{blue}{3} \color{purple}{4} \color{orange}{5} \end{array}} = \mathbb{F}_{13542} + \mathbb{F}_{15342} \\ \quad + \mathbb{F}_{31542} + \mathbb{F}_{51342} \\ \quad + \mathbb{F}_{35142} + \mathbb{F}_{53142} \\ \quad + \mathbb{F}_{35412} + \mathbb{F}_{53412} \\ \mathbb{P}_{\begin{array}{c} \color{red}{1} \color{green}{2} \color{blue}{3} \color{purple}{4} \color{orange}{5} \\ \color{red}{1} \color{green}{2} \color{blue}{3} \color{purple}{4} \color{orange}{5} \end{array}} = \mathbb{F}_{31542} \\ \quad + \mathbb{F}_{35142} \\ \mathbb{P}_{\begin{array}{c} \color{red}{1} \color{green}{2} \color{blue}{3} \color{purple}{4} \color{orange}{5} \\ \color{red}{1} \color{green}{2} \color{blue}{3} \color{purple}{4} \color{orange}{5} \end{array}} = \mathbb{F}_{31542}. \end{array}$$

**Theorem 70.**  $\text{Twist}^k$  is a Hopf subalgebra of FQSym.

*Proof.* This statement is a particular case of the results of [Rea05]. Alternatively, we could also invoke the formalism of [Hiv04, HN07, Pri13] and just observe that the  $k$ -twist congruence  $\equiv^k$  is compatible with the standardization and the restriction to intervals. However, we prefer to repeat the straightforward argument for the reader who sees it for the first time.

We need to show that  $\text{Twist}^k$  is stable by product and coproduct. For the product, consider two  $k$ -twists  $T \in \mathcal{AT}^k(n)$  and  $T' \in \mathcal{AT}^k(n')$  and two permutations  $\sigma = UacV_1b_1 \cdots V_kb_kW$  and  $\tilde{\sigma} = UcaV_1b_1 \cdots V_kb_kW$  with  $a < b_1, \dots, b_k < c$ . Assume that  $\mathbb{F}_\sigma$  appears in the product  $\mathbb{P}_T \cdot \mathbb{P}_{T'}$ . Thus, there exist  $\tau \in \mathcal{L}(T)$  and  $\tau' \in \mathcal{L}(T')$  such that  $\sigma$  belongs to  $\tau \sqcup \tau'$ . Then

- if  $a \leq n < c$ , then  $\tilde{\sigma}$  also belongs to  $\tau \sqcup \tau'$ , so that  $\mathbb{F}_{\tilde{\sigma}}$  also appears in the product  $\mathbb{P}_T \cdot \mathbb{P}_{T'}$ .
- if  $c \leq n$ , then  $\tau = XacY_1b_1 \cdots Y_kb_kZ$  is  $\equiv^k$ -congruent to  $\tilde{\tau} = XcaY_1b_1 \cdots Y_kb_kZ$ . Since  $\tilde{\sigma}$  belongs to  $\tilde{\tau} \sqcup \tau'$ , the element  $\mathbb{F}_{\tilde{\sigma}}$  also appears in the product  $\mathbb{P}_T \cdot \mathbb{P}_{T'}$ .
- If  $n < a$ , the argument is similar, exchanging  $ac$  to  $ca$  in  $\tau'$ .

We conclude that if  $\sigma \equiv^k \tilde{\sigma}$ , then  $\mathbb{F}_\sigma$  appears in the product  $\mathbb{P}_T \cdot \mathbb{P}_{T'}$  if and only if  $\mathbb{F}_{\tilde{\sigma}}$  does. Thus, the product of two  $k$ -twist congruence classes decomposes into a sum of  $k$ -twist congruence classes.

For the coproduct, consider a  $k$ -twist  $S$  and three permutations  $\tau, \tau' = UacV_1b_1 \cdots V_kb_kW$  and  $\tilde{\tau}' = UcaV_1b_1 \cdots V_kb_kW$  with  $a < b_1, \dots, b_k < c$ . Assume that the tensor  $\tau \otimes \tau'$  appears in the coproduct  $\Delta \mathbb{P}_S$ . Thus, there exist  $\sigma \in \mathcal{L}(S) \cap (\tau \star \tau')$ . By definition of the convolution, we can thus write  $\sigma = \hat{\tau} \hat{U} \hat{a} \hat{c} \hat{V}_1 \hat{b}_1 \cdots \hat{V}_k \hat{b}_k \hat{W}$  for some  $\hat{a} < \hat{b}_1, \dots, \hat{b}_k < \hat{c}$ . Then the permutation  $\tilde{\sigma} := \hat{\tau} \hat{U} \hat{c} \hat{a} \hat{V}_1 \hat{b}_1 \cdots \hat{V}_k \hat{b}_k \hat{W}$  belongs to  $\mathcal{L}(S) \cap (\tau \star \tilde{\tau}')$ , so that the tensor  $\tau \otimes \tilde{\tau}'$  also appears in the coproduct  $\Delta \mathbb{P}_S$ . By symmetry, we conclude that if  $\tau \equiv^k \tilde{\tau}$  and  $\tau' \equiv^k \tilde{\tau}'$ , the tensor  $\mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$  appears in the coproduct  $\Delta \mathbb{P}_S$  if and only if the tensor  $\mathbb{F}_{\tilde{\tau}} \otimes \mathbb{F}_{\tilde{\tau}'}$  does. In other words, the coproduct of a  $k$ -twist congruence class decomposes as a sum of tensor products of  $k$ -twist congruence classes.  $\square$

**Example 71** (Polynomial ring). The bijection which sends the unique  $(0, n)$ -twist to  $X^n/n!$  defines an isomorphism from the 0-twist algebra  $\text{Twist}^0$  to the polynomial ring  $\mathbb{K}[X]$ .

**Example 72** (J.-L. Loday and M. Ronco's algebra). The bijection given in Example 5 (see also Figure 2) defines an isomorphism from the 1-twist algebra  $\text{Twist}^1$  to M. Ronco and J.-L. Loday's Hopf algebra PBT on planar binary trees [LR98, HNT05].

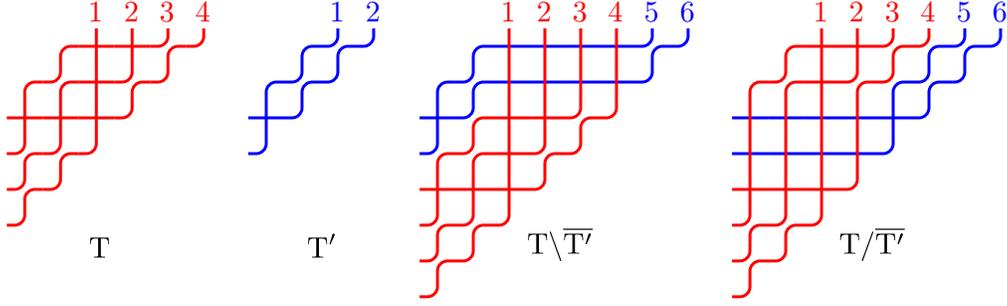
We now aim at understanding the product and the coproduct in  $\text{Twist}^k$  directly on  $k$ -twists. Although they are not always as satisfactory, our descriptions naturally extend classical results on the binary tree Hopf algebra PBT described in [LR98, AS06, HNT05].

**PRODUCT** To describe the product in  $\text{Twist}^k$ , we need the following notation, which is illustrated in Figure 14. For a  $(k, n)$ -twist  $T$  and a  $(k, n')$ -twist  $T'$ , we denote by  $T \setminus T'$  the  $(k, n + n')$ -twist obtained by inserting  $T$  in the first rows and columns of  $T'$  and by  $T/T'$  the  $(k, n + n')$ -twist obtained by inserting  $T'$  in the last rows and columns of  $T$ .

**Proposition 73.** For any acyclic  $k$ -twists  $T \in \mathcal{AT}^k(n)$  and  $T' \in \mathcal{AT}^k(n')$ , the product  $\mathbb{P}_T \cdot \mathbb{P}_{T'}$  is given by

$$\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_S \mathbb{P}_S,$$

where  $S$  runs over the interval between  $T \setminus T'$  and  $T/T'$  in the  $(k, n + n')$ -twist lattice.

FIGURE 14. Two twists  $T, T'$  (left) and the two twists  $T \setminus \overline{T'}$  and  $T/\overline{T'}$  (right).

*Proof.* Consider two acyclic  $k$ -twists  $T, T'$ . By Proposition 32, their fibers under  $\text{ins}^k$  are intervals of the weak order, which we denote by  $[\mu, \omega]$  and  $[\mu', \omega']$  respectively. By Proposition 67, the product  $\mathbb{P}_T \cdot \mathbb{P}_{T'}$  is therefore the weak order interval  $[\mu\bar{\mu}', \bar{\omega}'\omega]$ . Theorem 70 ensures that this interval is partitioned into various fibers of  $\text{ins}^k$ . In particular, the fiber of  $T \setminus \overline{T'}$  contains  $\mu\bar{\mu}'$  while the fiber of  $T/\overline{T'}$  contains  $\bar{\omega}'\omega$ . Theorem 37 finally ensures that  $[\mu\bar{\mu}', \bar{\omega}'\omega]$  is precisely the union of the fibers of the increasing flip interval  $[T \setminus \overline{T'}, T/\overline{T'}]$ .  $\square$

Figure 15 illustrates the product between two 2-twists in  $\text{Twist}^2$ .

$$\begin{aligned}
 \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \end{array}} \cdot \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \end{array}} &= (\mathbb{F}_{1423} + \mathbb{F}_{4123}) \cdot \mathbb{F}_{21} \\
 &= \begin{bmatrix} \mathbb{F}_{142365} \\ + \mathbb{F}_{412365} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{142635} \\ + \mathbb{F}_{146235} \\ + \mathbb{F}_{412635} \\ + \mathbb{F}_{416235} \\ + \mathbb{F}_{461235} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164235} \\ + \mathbb{F}_{614235} \\ + \mathbb{F}_{641235} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{142653} \\ + \mathbb{F}_{146253} \\ + \mathbb{F}_{412653} \\ + \mathbb{F}_{416253} \\ + \mathbb{F}_{461253} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164253} \\ + \mathbb{F}_{614253} \\ + \mathbb{F}_{641253} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{146523} \\ + \mathbb{F}_{416523} \\ + \mathbb{F}_{461523} \\ + \mathbb{F}_{465123} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164523} \\ + \mathbb{F}_{614523} \\ + \mathbb{F}_{641523} \\ + \mathbb{F}_{645123} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{165423} \\ + \mathbb{F}_{615423} \\ + \mathbb{F}_{651423} \\ + \mathbb{F}_{654123} \end{bmatrix} \\
 &= \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \\ \diagup \\ 5 \\ \diagdown \\ 6 \end{array}} + \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \\ \diagup \\ 5 \\ \diagdown \\ 6 \end{array}} + \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \\ \diagup \\ 5 \\ \diagdown \\ 6 \end{array}} + \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \\ \diagup \\ 5 \\ \diagdown \\ 6 \end{array}} + \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \\ \diagup \\ 5 \\ \diagdown \\ 6 \end{array}} + \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \\ \diagup \\ 5 \\ \diagdown \\ 6 \end{array}} + \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \\ \diagup \\ 5 \\ \diagdown \\ 6 \end{array}} + \mathbb{P}_{\begin{array}{c} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \\ \diagdown \\ 4 \\ \diagup \\ 5 \\ \diagdown \\ 6 \end{array}}
 \end{aligned}$$

FIGURE 15. An example of product in the 2-twist algebra  $\text{Twist}^2$ .

**COPRODUCT** Our description of the coproduct in  $\text{Twist}^k$  is unfortunately not as simple as the coproduct in PBT. It is very closed to the description of the direct computation using the coproduct of  $\text{FQSym}$ . We need the following definition. A *cut* in a  $k$ -twist  $S$  is a set  $\gamma$  of edges of the contact graph  $S^\#$  such that any path in  $S^\#$  from a leaf to the root contains precisely one edge of  $\gamma$ . We then denote by  $A^\#(S, \gamma)$  (resp.  $B^\#(S, \gamma)$ ) the restriction of the contact graph  $S^\#$  to the nodes above (resp. below)  $\gamma$ . Moreover,  $A^\#(S, \gamma)$  is the contact graph of the  $k$ -twist  $A(S, \gamma)$  obtained from  $S$  by deleting all pipes below  $\gamma$  in  $S^\#$ . Nevertheless, note that  $B^\#(S, \gamma)$  is not *a priori* the contact graph of a  $k$ -twist.

**Proposition 74.** *For any acyclic  $k$ -twist  $S \in \mathcal{AT}^k(m)$ , the coproduct  $\Delta \mathbb{P}_S$  is given by*

$$\Delta \mathbb{P}_S = \sum_{\gamma} \left( \sum_{\tau} \mathbb{P}_{\text{ins}^k(\tau)} \right) \otimes \mathbb{P}_{A(S, \gamma)},$$

where  $\gamma$  runs over all cuts of  $S$  and  $\tau$  runs over a set of representatives of the  $k$ -twist congruence classes of the linear extensions of  $B^\#(S, \gamma)$ .



1.3.3. **Quotient algebra of  $\text{FQSym}^*$ .** The following statement is automatic from Theorem 70.

**Theorem 76.** *The graded dual  $\text{Twist}^{k*}$  of the  $k$ -twist algebra is isomorphic to the image of  $\text{FQSym}^*$  under the canonical projection*

$$\pi : \mathbb{C}\langle A \rangle \longrightarrow \mathbb{C}\langle A \rangle / \equiv^k,$$

where  $\equiv^k$  denotes the  $k$ -twist congruence. The dual basis  $\mathbb{Q}_{\mathbb{T}}$  of  $\mathbb{P}_{\mathbb{T}}$  is expressed as  $\mathbb{Q}_{\mathbb{T}} = \pi(\mathbb{G}_{\tau})$ , where  $\tau$  is any permutation such that  $\text{ins}^k(\tau) = \mathbb{T}$ .

Similarly as in the previous section, we try to describe combinatorially the product and coproduct of  $\mathbb{Q}$ -basis elements of  $\text{Twist}^{k*}$  in terms of operations on Cambrian trees.

**PRODUCT** Once more, our description of the product in the dual twist algebra is not as simple as to coproduct in PBT, and is very closed to the description of the direct computation using the coproduct of  $\text{FQSym}$ . We use the following notation. For  $X = \{x_1 < \dots < x_n\} \in \binom{[n+n']}{n}$ ,  $\tau \in \mathfrak{S}_n$ , and  $\mathbb{T}' \in \mathcal{AT}^k(n')$ , we denote by  $\mathbb{T}' \hat{\wedge}(\tau \cdot X)$  the result of iteratively inserting  $x_{\tau_n}, \dots, x_{\tau_1}$  in the  $k$ -twist  $\mathbb{T}'$  relabeled increasingly by  $[n+n'] \setminus X$ .

**Proposition 77.** *For any acyclic  $k$ -twists  $\mathbb{T} \in \mathcal{AT}^k(n)$  and  $\mathbb{T}' \in \mathcal{AT}^k(n')$ , the product  $\mathbb{Q}_{\mathbb{T}} \cdot \mathbb{Q}_{\mathbb{T}'}$  is given by*

$$\mathbb{Q}_{\mathbb{T}} \cdot \mathbb{Q}_{\mathbb{T}'} = \sum_X \mathbb{Q}_{\mathbb{T}' \hat{\wedge}(\tau \cdot X)}$$

where  $X$  runs over  $\binom{[n+n']}{n}$  and  $\tau$  is an arbitrary permutation such that  $\text{ins}^k(\tau) = \mathbb{T}$ .

*Proof.* Consider  $\tau \in \mathfrak{S}_n$  and  $\tau' \in \mathfrak{S}_{n'}$  such that  $\text{ins}^k(\tau) = \mathbb{T}$  and  $\text{ins}^k(\tau') = \mathbb{T}'$ , and a permutation  $\sigma$  in the convolution  $\tau \star \tau'$ . Let  $X$  denote the first  $n$  values in  $\sigma$ . Since the relative order of the last  $n'$  entries in  $\sigma$  is that of the entries of  $\tau'$ , the insertion of the last  $n'$  values creates a copy of  $\mathbb{T}'$ . The remaining entries are then inserted in this copy of  $\mathbb{T}'$  at the positions given by  $X$  according to the order given by  $\tau$ . The result immediately follows.  $\square$

Figure 17 illustrates the product of two 2-twists in  $\text{Twist}^{2*}$ .

$$\begin{aligned} \mathbb{Q}_{12} \cdot \mathbb{Q}_{21} &= \mathbb{G}_{12} \cdot \mathbb{G}_{21} \\ &= \mathbb{G}_{1243} + \mathbb{G}_{1342} + \mathbb{G}_{1432} + \mathbb{G}_{2341} + \mathbb{G}_{2431} + \mathbb{G}_{3421} \\ &= \mathbb{Q}_{1234} + \mathbb{Q}_{1234} + \mathbb{Q}_{1234} + \mathbb{Q}_{1234} + \mathbb{Q}_{1234} + \mathbb{Q}_{1234} \end{aligned}$$

FIGURE 17. An example of product in the dual 2-twist algebra  $\text{Twist}^{2*}$ .

**COPRODUCT** Our description of the coproduct is more satisfactory. It is a special case of a coproduct on arbitrary pipe dreams studied by N. Bergeron and C. Ceballos [BC14]. We need the following notations, illustrated in Figure 18. For an acyclic  $(k, m)$ -twist  $\mathbb{S}$  and a position  $p \in \{0, \dots, m\}$ , we define two  $k$ -twists  $L(\mathbb{S}, p) \in \mathcal{AT}^k(p)$  and  $R(\mathbb{S}, p) \in \mathcal{AT}^k(m-p)$  as follows. The twist  $L(\mathbb{S}, p)$  is obtained by erasing the last  $m-p$  pipes in  $\mathbb{S}$  and glide the elbows of the remaining pipes as northwest as possible. More precisely, each elbow  $e$  of one of the first  $p$  pipes is translated one step north (resp. west) for each of the last  $m-p$  pipes passing north (resp. west) of  $e$ . The definition is similar for  $R(\mathbb{S}, p)$ , except that we erase the first  $p$  pipes instead of the last  $m-p$  pipes.

**Proposition 78.** *For any acyclic  $k$ -twist  $\mathbb{S} \in \mathcal{AT}^k(m)$ , the coproduct  $\Delta \mathbb{Q}_{\mathbb{S}}$  is given by*

$$\Delta \mathbb{Q}_{\mathbb{S}} = \sum_{p \in \{0, \dots, m\}} \mathbb{Q}_{L(\mathbb{S}, p)} \otimes \mathbb{Q}_{R(\mathbb{S}, p)}.$$

*Proof.* Consider  $\sigma \in \mathfrak{S}_m$  such that  $\text{ins}^k(\sigma) = \mathbb{S}$ , let  $p \in \{0, \dots, m\}$ , and let  $\tau \in \mathfrak{S}_p, \tau' \in \mathfrak{S}_{m-p}$  be the two permutations such that  $\sigma \in \tau \sqcup \tau'$ . By definition,  $\tau$  (resp.  $\tau'$ ) is given by the relative order of the first  $p$  (resp. last  $m-p$ ) values of  $\sigma$ . It is immediate to see that the insertion process then gives  $\text{ins}^k(\tau) = L(\mathbb{S}, p)$  and  $\text{ins}^k(\tau') = R(\mathbb{S}, p)$ . The result follows.  $\square$

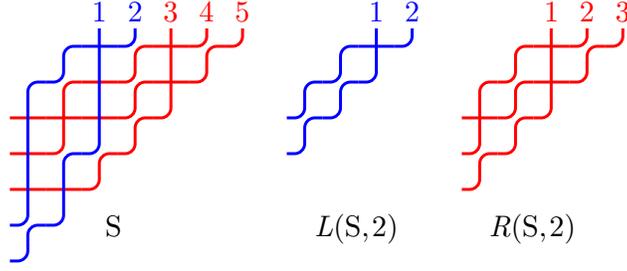


FIGURE 18. A twist  $S$  (left) and the two twists  $L(S,p)$  (middle) and  $R(S,p)$  (right).

Figure 19 illustrates the coproduct of a 2-twist in  $\text{Twist}^{2*}$ .

$$\begin{aligned}
\Delta \mathbb{Q} \begin{array}{c} 1\ 2\ 3\ 4\ 5 \\ \text{diagram} \end{array} &= \Delta \mathbb{G}_{31542} \\
&= 1 \otimes \mathbb{G}_{31542} + \mathbb{G}_1 \otimes \mathbb{G}_{2431} + \mathbb{G}_{12} \otimes \mathbb{G}_{132} + \mathbb{G}_{312} \otimes \mathbb{G}_{21} + \mathbb{G}_{3142} \otimes \mathbb{G}_1 + \mathbb{G}_{31542} \otimes 1 \\
&= 1 \otimes \mathbb{Q} \begin{array}{c} 1\ 2\ 3\ 4\ 5 \\ \text{diagram} \end{array} + \mathbb{Q} \begin{array}{c} \text{diagram} \end{array} \otimes \mathbb{Q} \begin{array}{c} 1\ 2\ 3\ 4\ 5 \\ \text{diagram} \end{array} + \mathbb{Q} \begin{array}{c} 1\ 2 \\ \text{diagram} \end{array} \otimes \mathbb{Q} \begin{array}{c} 1\ 2\ 3\ 4\ 5 \\ \text{diagram} \end{array} \\
&\quad + \mathbb{Q} \begin{array}{c} 1\ 2\ 3\ 4\ 5 \\ \text{diagram} \end{array} \otimes \mathbb{Q} \begin{array}{c} 1\ 2 \\ \text{diagram} \end{array} + \mathbb{Q} \begin{array}{c} 1\ 2\ 3\ 4\ 5 \\ \text{diagram} \end{array} \otimes \mathbb{Q} \begin{array}{c} 1\ 2\ 3 \\ \text{diagram} \end{array} + \mathbb{Q} \begin{array}{c} 1\ 2\ 3\ 4\ 5 \\ \text{diagram} \end{array} \otimes 1
\end{aligned}$$

FIGURE 19. An example of coproduct in the dual 2-twist algebra  $\text{Twist}^{2*}$ .

**1.3.4. Multiplicative bases and irreducible elements.** In this section, we define multiplicative bases of  $\text{Twist}^k$  and study the indecomposable elements of  $\text{Twist}^k$  for these bases. For an acyclic  $(k,n)$ -twist  $T$ , we define

$$\mathbb{E}^T := \sum_{T \leq T'} \mathbb{P}_{T'} \quad \text{and} \quad \mathbb{H}^T := \sum_{T' \leq T} \mathbb{P}_{T'},$$

where  $\leq$  denotes the increasing flip lattice on acyclic  $(k,n)$ -twists. As the elements  $\mathbb{E}^T$  and  $\mathbb{H}^T$  have symmetric properties, we focus our analysis on  $\mathbb{E}^T$ . The reader is invited to translate the statements and proofs below to  $\mathbb{H}^T$ . We first observe that these elements can also be seen as elements of the multiplicative bases  $(\mathbb{E}^\tau)_{\tau \in \mathfrak{S}}$  and  $(\mathbb{H}^\tau)_{\tau \in \mathfrak{S}}$  of  $\text{FQSym}$ .

**Lemma 79.** *For any acyclic  $k$ -twist  $T$ , we have  $\mathbb{E}^T = \mathbb{E}^\mu$  and  $\mathbb{H}^T = \mathbb{H}^\omega$ , where  $\mu$  and  $\omega$  respectively denote the weak order minimal and maximal permutations in the fiber of  $T$  under  $\text{ins}^k$ .*

*Proof.* We directly obtain from the definition that

$$\mathbb{E}^T = \sum_{T \leq T'} \mathbb{P}_{T'} = \sum_{T \leq T'} \sum_{\substack{\tau' \in \mathfrak{S}_n \\ \text{ins}^k(\tau') = T'}} \mathbb{F}_{\tau'} = \sum_{\substack{\tau' \in \mathfrak{S}_n \\ T \leq \text{ins}^k(\tau')}} \mathbb{F}_{\tau'} = \sum_{\substack{\tau' \in \mathfrak{S}_n \\ \mu \leq \tau'}} \mathbb{F}_{\tau'} = \mathbb{E}^\mu. \quad \square$$

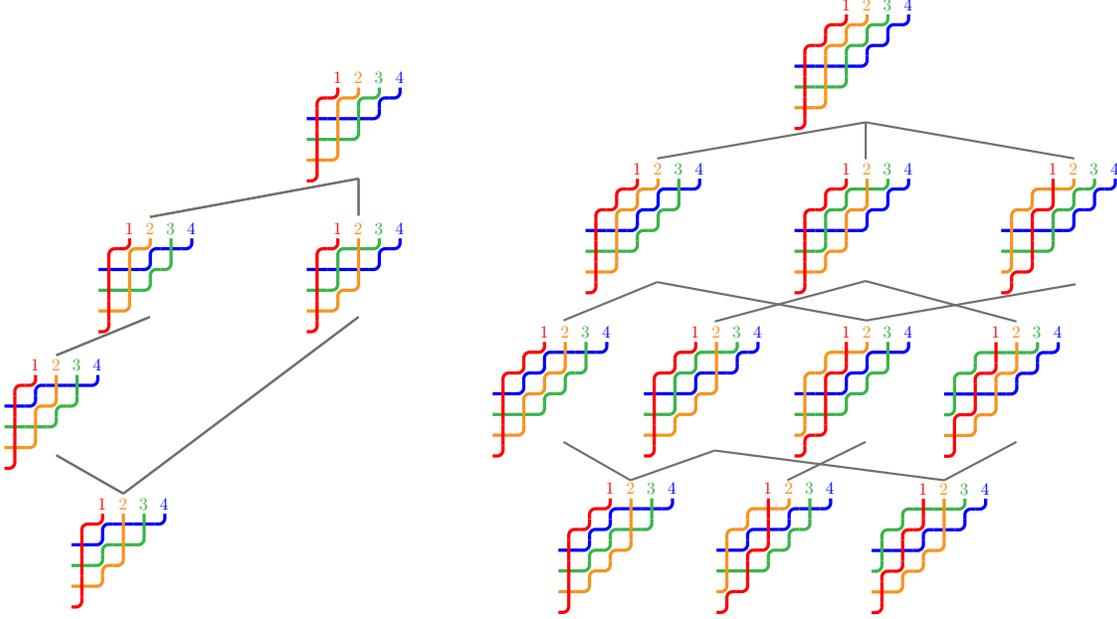
To describe the product of two elements of the  $\mathbb{E}$ - or  $\mathbb{H}$ -basis, remember that the twist  $T \setminus T'$  (resp.  $T/T'$ ) is obtained by inserting  $T$  in the first rows and columns of  $T'$  (resp.  $T'$  in the last rows and columns of  $T$ ). Examples are given in Figure 14.

**Proposition 80.**  $(\mathbb{E}^T)_{T \in \mathcal{AT}^k}$  and  $(\mathbb{H}^T)_{T \in \mathcal{AT}^k}$  are multiplicative bases of  $\text{Twist}^k$ :

$$\mathbb{E}^T \cdot \mathbb{E}^{T'} = \mathbb{E}^{T \setminus T'} \quad \text{and} \quad \mathbb{H}^T \cdot \mathbb{H}^{T'} = \mathbb{H}^{T/T'}.$$

*Proof.* Let  $\mu$  and  $\mu'$  respectively denote the minimal elements of the fibers of  $T$  and  $T'$  under  $\text{ins}^k$ . Using Lemma 79 and the fact that  $\text{ins}^k(\mu \setminus \mu') = T \setminus T'$  and  $\mu \setminus \mu'$  is minimal in its  $k$ -twist congruence class, we write

$$\mathbb{E}^T \cdot \mathbb{E}^{T'} = \mathbb{E}^\mu \cdot \mathbb{E}^{\mu'} = \mathbb{E}^{\mu \setminus \mu'} = \mathbb{E}^{T \setminus T'}. \quad \square$$

FIGURE 20. The  $\mathbb{E}$ -indecomposable acyclic  $(k, 4)$ -twists for  $k = 1, 2$ .

We now consider multiplicative decomposability. We call *cut* of an acyclic oriented graph any partition  $(X \parallel Y)$  of its vertices such that all edges between  $X$  and  $Y$  are oriented from  $X$  to  $Y$ .

**Proposition 81.** *The following properties are equivalent for an acyclic  $k$ -twist  $S$ :*

- (i)  $\mathbb{E}^S$  can be decomposed into a product  $\mathbb{E}^S = \mathbb{E}^T \cdot \mathbb{E}^{T'}$  for non-empty acyclic  $k$ -twists  $T, T'$ ;
- (ii)  $([k] \parallel [n] \setminus [k])$  is a cut of  $S^\#$  for some  $k \in [n-1]$ ;
- (iii) at least one linear extension  $\tau$  of  $S^\#$  is  $\mathbb{E}$ -decomposable, i.e.  $\tau([k]) = [k]$  for some  $k \in [n]$ ;
- (iv) the weak order minimal linear extension of  $S^\#$  is  $\mathbb{E}$ -decomposable.

The  $k$ -twist  $S$  is then called  **$\mathbb{E}$ -decomposable**. Otherwise, it is called  **$\mathbb{E}$ -indecomposable**, and we denote by  $\mathcal{IAT}^k(n)$  the set of  $\mathbb{E}$ -indecomposable acyclic  $(k, n)$ -twists.

*Proof.* The equivalence (i)  $\iff$  (ii) is an immediate consequence of the description of the product  $\mathbb{E}^T \cdot \mathbb{E}^{T'} = \mathbb{E}^{T \setminus T'}$  in Proposition 80. The implication (ii)  $\implies$  (iii) follows from the fact that for any cut  $(X \parallel Y)$  of a directed acyclic graph  $G$ , there exists a linear extension of  $G$  which starts with  $X$  and finishes with  $Y$ . The implication (iii)  $\implies$  (iv) follows from the fact that the  $\mathbb{E}$ -indecomposable permutations form an up ideal of the weak order. Finally, if  $\tau$  is a decomposable linear extension of  $S$ , then the insertion algorithm on  $\tau$  first creates a twist labeled by  $[n] \setminus [k]$  and then inserts the pipes labeled by  $[k]$ . Any arc between  $[k]$  and  $[n] \setminus [k]$  in  $S = \text{ins}^k(\tau)$  will thus be directed from  $[k]$  to  $[n] \setminus [k]$ .  $\square$

**Example 82** (Right-tilting  $k$ -twists). Say that a  $k$ -twist is **right-tilting** when it has no elbow in its first column. When  $k = 1$ , the  $\mathbb{E}$ -indecomposable 1-twists are precisely the right-tilting 1-twists. Therefore, the number of  $\mathbb{E}$ -indecomposable  $(1, n)$ -twists is the Catalan number  $C_{n-1}$ , and the  $\mathbb{E}$ -indecomposable  $(1, n)$ -twists form a principal ideal of the increasing flip lattice. When  $k \geq 2$ , right-tilting  $k$ -twists are  $\mathbb{E}$ -indecomposable, but are not the only ones. The  $\mathbb{E}$ -indecomposable  $(k, n)$ -twists form an upper ideal of the increasing flip lattice, but this ideal is not principal. Figure 20 illustrates the  $\mathbb{E}$ -indecomposable acyclic  $(k, 4)$ -twists for  $k = 1, 2$ .

**Proposition 83.** *The  $k$ -twist algebra is freely generated by the elements  $\mathbb{E}^T$  such that  $T$  is  $\mathbb{E}$ -indecomposable.*

*Proof.* Let  $T$  be an acyclic  $k$ -twist and let  $\mu$  be the weak order minimal permutation such that  $\text{ins}^k(\mu) = T$ . Decompose  $\mu = \mu_1 \setminus \dots \setminus \mu_p$  into  $\mathbb{E}$ -indecomposable permutations  $\mu_1, \dots, \mu_p$ .

For  $i \in [p]$ , define  $T_i := \text{ins}^k(\mu_i)$ . Since  $\mu_i$  avoids the patterns  $(k+2)1-(\sigma_1+1)-\dots-(\sigma_k+1)$  for all  $\sigma \in \mathfrak{S}_k$  (because  $\mu$  avoids these patterns), it is the weak order minimal permutation in the fiber of  $T_i$ . Since  $\mu_i$  is  $\mathbb{E}$ -indecomposable, we get by Proposition 81 (iv) that  $T_i$  is  $\mathbb{E}$ -indecomposable. Using Lemma 79, we thus obtained a decomposition  $\mathbb{E}^T = \mathbb{E}^\mu = \mathbb{E}^{\mu_1} \dots \mathbb{E}^{\mu_p} = \mathbb{E}^{T_1} \dots \mathbb{E}^{T_p}$  of  $T$  into  $\mathbb{E}$ -indecomposable  $k$ -twists  $T_1, \dots, T_p$ .

Now, there is no relation between the elements  $\mathbb{E}^\tau$  of  $\text{FQSym}$  corresponding to the  $\mathbb{E}$ -indecomposable permutations. Hence, by Lemma 79 and Proposition 80, there is no relation between the elements  $\mathbb{E}^T$  of  $\text{Twist}^k$  corresponding to the  $\mathbb{E}$ -indecomposable  $k$ -twists.  $\square$

**Corollary 84.** *The generating functions of the numbers of  $\mathbb{E}$ -indecomposable acyclic  $(k, n)$ -twists and of the numbers of all acyclic  $(k, n)$ -twists are related by*

$$\frac{1}{1 - \sum_{n \in \mathbb{N}} |\mathcal{IAT}^k(n)| t^n} = \sum_{n \in \mathbb{N}} |\mathcal{AT}^k(n)| t^n.$$

Table 4 gathers the numbers of  $\mathbb{E}$ -indecomposable acyclic  $(k, n)$ -twists for  $k < n \leq 10$ . One recognizes on the diagonal the number of  $\mathbb{E}$ -indecomposable permutations of  $\mathfrak{S}_n$ , see [OEIS, A003319].

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	.	1	2	5	14	42	132	429	1430	4862
2	.	.	3	11	47	219	1085	5619	30099	165555
3	.	.	.	13	65	365	2229	14465	98461	696337
4	.	.	.	.	71	437	2967	21773	170047	1395733
5	.	.	.	.	.	461	3327	26213	222103	2000125
6	.	.	.	.	.	.	3447	28373	253183	2419645
7	.	.	.	.	.	.	.	29093	268303	2668045
8	.	.	.	.	.	.	.	.	273343	2789005
9	.	.	.	.	.	.	.	.	.	2829325

TABLE 4. The number of  $\mathbb{E}$ -indecomposable acyclic  $(k, n)$ -twists for  $k < n \leq 10$ . Dots indicate that the value remains constant (equal to the number of  $\mathbb{E}$ -indecomposable permutations) in the column.

Again, the  $\mathbb{E}$ -indecomposable 1-twists are precisely the right-tilting 1-twists, and are therefore counted by the Catalan number  $C_{n-1}$ . Analogous results for  $k \geq 2$  remains to be found.

**Question 85.** Is there a simple characterization and a simple enumeration formula for the  $\mathbb{E}$ -indecomposable acyclic  $(k, n)$ -twists?

**1.3.5. Integer point transform.** In this section, we observe that the product in the  $k$ -twist Hopf algebra  $\text{Twist}^k$  can be interpreted in terms of the integer point transforms of the normal cones of the brick polytope  $\text{Brick}^k(n)$ . To make this statement precise, we introduce some notations.

**Definition 86.** *The integer point transform  $\mathbb{Z}_S$  of a subset  $S$  of  $\mathbb{R}^n$  is the multivariate generating function of the integer points inside  $S$ :*

$$\mathbb{Z}_S(t_1, \dots, t_n) = \sum_{(i_1, \dots, i_n) \in \mathbb{Z}^n \cap S} t_1^{i_1} \dots t_n^{i_n}.$$

For a poset  $\triangleleft$ , we denote by  $\mathbb{Z}_{\triangleleft}$  the integer point transform of the cone

$$C^\diamond(\triangleleft) := \left\{ \mathbf{x} \in \mathbb{R}_+^n \mid \begin{array}{l} x_i \leq x_j \text{ for all } i \triangleleft j \text{ with } i < j \\ x_i < x_j \text{ for all } i \triangleleft j \text{ with } i > j \end{array} \right\}.$$

Note that this cone differs in two ways from the cone  $C^\diamond(\triangleleft)$  defined in Section 1.2.2: first it leaves in  $\mathbb{R}_+^n$  and not in  $\mathbb{H}$ , second it excludes the facets of  $C^\diamond(\triangleleft)$  corresponding to the decreasing relations of  $\triangleleft$  (i.e. the relations  $i \triangleleft j$  with  $i > j$ ).

Following the notations of Section 1.2.2, we denote by  $\mathbb{Z}_\tau$  the integer point transform of the chain  $\tau_1 \triangleleft \dots \triangleleft \tau_n$  for a permutation  $\tau \in \mathfrak{S}_n$ . The following statements are classical.

**Proposition 87.** (i) For any permutation  $\tau \in \mathfrak{S}_n$ , the integer point transform  $\mathbb{Z}_\tau$  is given by

$$\mathbb{Z}_\tau(t_1, \dots, t_n) = \frac{\prod_{\substack{i \in [n-1] \\ \tau_i > \tau_{i+1}}} t_{\tau_i} \cdots t_{\tau_n}}{\prod_{i \in [n]} (1 - t_{\tau_i} \cdots t_{\tau_n})}.$$

(ii) The integer point transform of an arbitrary poset  $\triangleleft$  is given by

$$\mathbb{Z}_{\triangleleft} = \sum_{\tau \in \mathcal{L}(\triangleleft)} \mathbb{Z}_\tau,$$

where the sum runs over the set  $\mathcal{L}(\triangleleft)$  of linear extensions of  $\triangleleft$ .

(iii) The product of the integer point transforms  $\mathbb{Z}_\tau$  and  $\mathbb{Z}_{\tau'}$  of two permutations  $\tau \in \mathfrak{S}_n$  and  $\tau' \in \mathfrak{S}_{n'}$  is given by the shifted shuffle

$$\mathbb{Z}_\tau(t_1, \dots, t_n) \cdot \mathbb{Z}_{\tau'}(t_{n+1}, \dots, t_{n+n'}) = \sum_{\sigma \in \tau \boxplus \tau'} \mathbb{Z}_\sigma(t_1, \dots, t_{n+n'}).$$

In other words, the linear map from  $\text{FQSym}$  to the rational functions defined by  $\Psi : \mathbb{F}_\tau \mapsto \mathbb{Z}_\tau$  is an algebra morphism.

*Proof.* For Point (i), we just observe that the cone  $\{\mathbf{x} \in \mathbb{R}_+^n \mid x_{\tau_i} \leq x_{\tau_{i+1}} \text{ for all } i \in [n-1]\}$  is generated by the vectors  $\mathbf{e}_{\tau_i} + \cdots + \mathbf{e}_{\tau_n}$ , for  $i \in [n]$ , which form a (unimodular) basis of the lattice  $\mathbb{Z}^n$ . A straightforward inductive argument shows that the integer point transform of the cone  $\{\mathbf{x} \in \mathbb{R}_+^n \mid x_{\tau_i} \leq x_{\tau_{i+1}} \text{ for all } i \in [n-1]\}$  is thus given by  $\prod_{i \in [n]} (1 - t_{\tau_i} \cdots t_{\tau_n})^{-1}$ . The numerator of  $\mathbb{Z}_\tau$  is then given by the facets which are excluded from the cone  $\mathbf{C}^\diamond(\tau)$ .

Point (ii) follows from the fact that the cone  $\mathbf{C}^\diamond(\triangleleft)$  is partitioned by the cones  $\mathbf{C}^\diamond(\tau)$  for the linear extensions  $\tau$  of  $\triangleleft$ .

Finally, the product  $\mathbb{Z}_\tau(t_1, \dots, t_n) \cdot \mathbb{Z}_{\tau'}(t_{n+1}, \dots, t_{n+n'})$  is the integer point transform of the poset formed by the two disjoint chains  $\tau$  and  $\bar{\tau}'$ , whose linear extensions are precisely the permutations which appear in the shifted shuffle of  $\tau$  and  $\tau'$ . This shows Point (iii).  $\square$

For an acyclic  $k$ -twist  $\mathbb{T}$ , we denote by  $\mathbb{Z}_\mathbb{T}$  the integer point transform of the transitive closure of the contact graph  $\mathbb{T}^\#$ . It follows from Proposition 87 that the product of the integer point transforms of two acyclic  $k$ -twists behaves as the product in the  $k$ -twist algebra  $\text{Twist}^k$ .

**Corollary 88.** For any two acyclic  $k$ -twists  $\mathbb{T}, \mathbb{T}'$ , we have

$$\mathbb{Z}_\mathbb{T}(t_1, \dots, t_n) \cdot \mathbb{Z}_{\mathbb{T}'}(t_{n+1}, \dots, t_{n+n'}) = \sum_{\mathbb{T} \setminus \mathbb{T}' \leq \mathbb{S} \leq \mathbb{T}/\mathbb{T}'} \mathbb{Z}_\mathbb{S}(t_1, \dots, t_{n+n'}).$$

*Proof.* Hidding the variables  $(t_1, \dots, t_{n+n'})$  for concision, we have

$$\mathbb{Z}_\mathbb{T} \cdot \mathbb{Z}_{\mathbb{T}'} = \Psi(\mathbb{P}_\mathbb{T}) \cdot \Psi(\mathbb{P}_{\mathbb{T}'}) = \Psi(\mathbb{P}_\mathbb{T} \cdot \mathbb{P}_{\mathbb{T}'}) = \Psi\left(\sum_{\mathbb{S}} \mathbb{P}_\mathbb{S}\right) = \sum_{\mathbb{S}} \Psi(\mathbb{P}_\mathbb{S}) = \sum_{\mathbb{S}} \mathbb{Z}_\mathbb{S},$$

where the sums run over the  $k$ -twists  $\mathbb{S}$  of the increasing flip lattice interval  $[\mathbb{T} \setminus \mathbb{T}', \mathbb{T}/\mathbb{T}']$ .  $\square$

**1.3.6.  $k$ -twistiform algebras.** In this section, we extend the notion of dendriform algebras to  $k$ -twistiform algebras. Dendriform algebras were introduced by J.-L. Loday in [Lod01, Chap. 5]. In a dendriform algebra, the product  $\cdot$  is decomposed into two partial products  $\prec$  and  $\succ$  satisfying:

$$x \prec (y \cdot z) = (x \prec y) \prec z, \quad x \succ (y \prec z) = (x \succ y) \prec z, \quad x \succ (y \succ z) = (x \cdot y) \succ z.$$

In our context, we will still decompose the product of  $\text{FQSym}$  (and of  $\text{Twist}^k$ ) into partial products, but we will use  $2^k$  partial products satisfying  $3^k$  relations. In this paper, we just give the definition and observe that the algebras  $\text{FQSym}$  and  $\text{Twist}^k$  are naturally endowed with a  $k$ -twistiform structure, as they motivated the definition. A detailed study of combinatorial and algebraic properties of  $k$ -twistiform algebras and operads is in progress in a joint work with F. Hivert [HP15].

We need to fix some natural notations on words. We denote by  $|W|$  the length of a word  $W$ . For a subset  $P$  of positions in  $W$  and a subset  $L$  of letters of  $W$ , we denote by  $W_P$  the subword of  $W$  consisting only of the letters at positions in  $P$  and by  $W^L$  the subword of  $W$  consisting only of the letters which belong to  $L$ .

**Definition 89.** A  *$k$ -twistiform algebra* is a vector space  $\text{Alg}$  endowed with a collection  $\mathfrak{B} := \{\prec, \bowtie, \succ\}^k$  of  $3^k$  bilinear operations which satisfy the following  $k3^{k-1} + 3^k$  relations:

**Split relations:** For any  $\mathfrak{b}, \mathfrak{b}' \in \{\prec, \bowtie, \succ\}^k$  with  $|\mathfrak{b}| + |\mathfrak{b}'| = k - 1$ , the operation  $\mathfrak{b}\bowtie\mathfrak{b}' \in \mathfrak{B}$  decomposes into the operations  $\mathfrak{b}\prec\mathfrak{b} \in \mathfrak{B}$  and  $\mathfrak{b}\succ\mathfrak{b} \in \mathfrak{B}$ :

$$\times \mathfrak{b}\bowtie\mathfrak{b}' y = \times \mathfrak{b}\prec\mathfrak{b}' y + \times \mathfrak{b}\succ\mathfrak{b}' y \quad \text{for all } x, y \in \text{Alg}.$$

**Associativity relations:** For any  $W \in \{x, y, z\}^k$ , the operations  $\mathfrak{b}_W, \mathfrak{b}'_W, \mathfrak{b}''_W, \mathfrak{b}'''_W \in \mathfrak{B}$  defined by

$$(\mathfrak{b}_W)_p := \begin{cases} \prec & \text{if } W_p = x \\ \succ & \text{if } W_p \in \{y, z\} \end{cases} \quad (\mathfrak{b}'_W)_p := \begin{cases} \prec & \text{if } |W^{\{y,z\}}| \geq p \text{ and } (W^{\{y,z\}})_p = y \\ \succ & \text{if } |W^{\{y,z\}}| \geq p \text{ and } (W^{\{y,z\}})_p = z \\ \bowtie & \text{otherwise} \end{cases}$$

$$(\mathfrak{b}''_W)_p := \begin{cases} \prec & \text{if } |W^{\{x,y\}}| \geq p \text{ and } (W^{\{x,y\}})_p = x \\ \succ & \text{if } |W^{\{x,y\}}| \geq p \text{ and } (W^{\{x,y\}})_p = y \\ \bowtie & \text{otherwise} \end{cases} \quad (\mathfrak{b}'''_W)_p := \begin{cases} \prec & \text{if } W_p \in \{x, y\} \\ \succ & \text{if } W_p = z \end{cases}$$

satisfy the associativity relation

$$\times \mathfrak{b}_W (y \mathfrak{b}'_W z) = (\times \mathfrak{b}''_W y) \mathfrak{b}'''_W z \quad \text{for all } x, y, z \in \text{Alg}.$$

**Example 90** (1- and 2-twistiform algebras). 1-twistiform algebras are precisely dendriform algebras, *i.e.* vector spaces endowed with three operations  $\prec, \bowtie, \succ$  which fulfill the 4 relations:

$$\begin{aligned} \times \bowtie y &= \times \prec y + \times \succ y, \\ \times \prec (y \bowtie z) &= (\times \prec y) \prec z, \quad \times \succ (y \prec z) = (\times \succ y) \prec z, \quad \times \succ (y \succ z) = (\times \bowtie y) \succ z. \end{aligned}$$

2-twistiform algebras are vector spaces endowed with 9 operations  $\prec\prec, \prec\bowtie, \prec\succ, \bowtie\prec, \bowtie\bowtie, \bowtie\succ, \succ\prec, \succ\bowtie, \succ\succ$  which satisfy the following 15 relations:

$$\begin{aligned} \times \bowtie\prec y &= \times \prec\prec y + \times \succ\prec y, & \times \bowtie\bowtie y &= \times \prec\bowtie y + \times \succ\bowtie y, & \times \bowtie\succ y &= \times \prec\succ y + \times \succ\succ y, \\ \times \prec\bowtie y &= \times \prec\prec y + \times \prec\succ y, & \times \bowtie\bowtie y &= \times \bowtie\prec y + \times \bowtie\succ y, & \times \succ\bowtie y &= \times \succ\prec y + \times \succ\succ y, \\ \times \prec\prec (y \bowtie\bowtie z) &= (\times \prec\prec y) \prec\prec z, & \times \prec\prec (y \prec\bowtie z) &= (\times \prec\prec y) \prec\prec z, & \times \prec\prec (y \succ\bowtie z) &= (\times \prec\bowtie y) \prec\prec z, \\ \times \prec\succ (y \prec\bowtie z) &= (\times \prec\succ y) \prec\prec z, & \times \prec\succ (y \prec\prec z) &= (\times \prec\succ y) \prec\prec z, & \times \prec\succ (y \prec\succ z) &= (\times \prec\bowtie y) \prec\prec z, \\ \times \succ\prec (y \succ\bowtie z) &= (\times \succ\prec y) \succ\prec z, & \times \succ\prec (y \succ\prec z) &= (\times \succ\prec y) \succ\prec z, & \times \succ\prec (y \succ\succ z) &= (\times \bowtie\bowtie y) \succ\prec z. \end{aligned}$$

**Remark 91.** Adding up all associativity relations, one obtains that

$$\times \bowtie^k (y \bowtie^k z) = (\times \bowtie^k y) \bowtie^k z \quad \text{for all } x, y, z \in \text{Alg},$$

so that the  $k$ -twistiform algebra  $(\text{Alg}, \{\prec, \bowtie, \succ\}^k)$  defines in particular a structure of associative algebra  $(\text{Alg}, \bowtie^k)$ . Reciprocally, we say that an associative algebra  $(\text{Alg}, \cdot)$  admits a  *$k$ -twistiform structure* if it is possible to split the product  $\cdot$  into  $3^k$  operations  $\mathfrak{B} := \{\prec, \bowtie, \succ\}^k$  defining a  $k$ -twistiform algebra on  $\text{Alg}$ .

We now show that C. Malvenuto and C. Reutenauer's Hopf algebra on permutations  $\text{FQSym}$  can be endowed with a structure of  $k$ -twistiform algebra. For an operation  $\mathfrak{b} \in \mathfrak{B}$  and two words  $X := x\underline{X}$  and  $Y := y\underline{Y}$ , we define

$$X \mathfrak{b} Y = \begin{cases} X \sqcup Y & \text{if } \mathfrak{b} = \emptyset, \\ x(\underline{X} \mathfrak{b} Y) & \text{if } \mathfrak{b} = \prec\mathfrak{b}, \\ x(\underline{X} \mathfrak{b} Y) \cup y(X \mathfrak{b} \underline{Y}) & \text{if } \mathfrak{b} = \bowtie\mathfrak{b}, \\ y(X \mathfrak{b} \underline{Y}) & \text{if } \mathfrak{b} = \succ\mathfrak{b} \end{cases}$$

with the initial conditions  $X \succ\mathfrak{b} \emptyset = \emptyset \prec\mathfrak{b} Y = 0$ .

In other words, we consider the shuffle of  $X$  and  $Y$ , except that the  $i$ th letter of  $X \mathfrak{b} Y$  is forced to belong to  $X$  (resp. to  $Y$ ) if the  $i$ th letter of  $\mathfrak{b}$  is  $\prec$  (resp. is  $\succ$ ). For example, when  $k = 1$ , the three operators are given by

$$X \prec Y = x(\underline{X} \sqcup Y), \quad X \asymp Y = X \sqcup Y, \quad X \succ Y = y(X \sqcup \underline{Y}),$$

with the initial conditions  $X \succ \emptyset = \emptyset \prec Y = 0$ .

Now for an operation  $\mathfrak{b} \in \mathfrak{B}$  and two permutations  $\tau \in \mathfrak{S}_n$  and  $\tau' \in \mathfrak{S}_{n'}$ , we define  $\tau \mathfrak{b} \tau' = \tau \mathfrak{b} \bar{\tau}'$ , where  $\bar{\tau}'$  is the permutation  $\tau'$  shifted by the length  $n$  of  $\tau$ . Equivalently,  $\tau \mathfrak{b} \tau'$  is the set of permutations  $\sigma \in \tau \sqcup \tau'$  such that for all  $i \in [k]$ , we have  $\sigma_i \leq n$  if  $\mathfrak{b}_i = \prec$  while  $\sigma_i > n$  if  $\mathfrak{b}_i = \succ$ . Finally, we define the operations  $\mathfrak{B}$  on the Hopf algebra  $\text{FQSym}$  itself by

$$\mathbb{F}_\tau \mathfrak{b} \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \mathfrak{b} \tau'} \mathbb{F}_\sigma.$$

**Proposition 92.** *The Hopf algebra  $\text{FQSym}$ , endowed with the operations  $\mathfrak{B}$  described above, defines a  $k$ -twistiform algebra. The product of  $\text{FQSym}$  is then given by  $\cdot = \asymp^k$ .*

*Proof.* We have to show that the operations defined above on  $\text{FQSym}$  indeed satisfy the  $k3^{k-1} + 3^k$  relations of Definition 89.

**Split relations:** Let  $\mathfrak{b}, \mathfrak{b}' \in \{\prec, \asymp, \succ\}^*$  with  $|\mathfrak{b}| + |\mathfrak{b}'| = k - 1$ . It is immediate from the definitions that  $X \mathfrak{b} \asymp \mathfrak{b}' Y = (X \mathfrak{b} \prec \mathfrak{b}' Y) \cup (X \mathfrak{b} \succ \mathfrak{b}' Y)$  for any two words  $X, Y$ .

**Associativity relations:** Let  $W \in \mathfrak{B}$ . It follows from the definition of the operations  $\mathfrak{b}_W$ ,  $\mathfrak{b}'_W$ ,  $\mathfrak{b}''_W$  and  $\mathfrak{b}'''_W$  that for any words  $X, Y, Z$

$$X \mathfrak{b}_W (Y \mathfrak{b}'_W Z) = (X \mathfrak{b}''_W Y) \mathfrak{b}'''_W Z$$

is the set of all words in  $X \sqcup Y \sqcup Z$  whose  $p$ th letter is in the word  $X$  if  $W_p = x$ , in the word  $Y$  if  $W_p = y$  and in the word  $Z$  if  $W_p = z$ .

These equalities of sets then translate to the desired linear relations on the corresponding operations in  $\text{FQSym}$ .  $\square$

We say that the operations  $\mathfrak{B}$  define the *forward  $k$ -twistiform structure* on  $\text{FQSym}$ . There is also a *backward  $k$ -twistiform structure* on  $\text{FQSym}$  which considers the last  $k$  letters rather than the first  $k$  ones. Namely, for each operation  $\mathfrak{b} \in \mathfrak{B}$ , define an operation  $\mathfrak{b}^\bullet$  by  $V \mathfrak{b}^\bullet W = (V^\bullet \mathfrak{b} W^\bullet)^\bullet$  where  $W^\bullet = w_n \cdots w_1$  denotes the mirror of a word  $W = w_1 \cdots w_n$ . Clearly, the operations  $\mathfrak{b}^\bullet$  for  $\mathfrak{b} \in \mathfrak{B}$  still fulfill the relations of Definition 89. We have chosen to define the forward  $k$ -twistiform structure as it leads to a simpler presentation, but we need this backward  $k$ -twistiform structure in the next statement to be coherent with the insertion in  $k$ -twists (whose direction was chosen consistently with J.-L. Loday and M. Ronco's conventions).

**Proposition 93.** *The subalgebra  $\text{Twist}^k$  of  $\text{FQSym}$  is stable by the operations  $\mathfrak{b}^\bullet$  for  $\mathfrak{b} \in \mathfrak{B}$  and therefore inherits a  $k$ -twistiform structure.*

*Proof.* Let  $T$  be an acyclic  $k$ -twist. We claim that the last  $k$  entries are the same in all linear extensions of  $T^\#$ . Indeed, pick a linear extension  $\tau$  of  $T^\#$  and let  $T' := \emptyset \hat{\wedge} \tau_n \hat{\wedge} \cdots \hat{\wedge} \tau_{n-k+1}$  denote the  $k$ -twist obtain after the insertion of the last  $k$  values of  $\tau$ . All pipes of  $T'$  are then comparable by Lemma 51 and will be comparable to all other pipes in  $T = T' \hat{\wedge} \tau_{n-k} \hat{\wedge} \cdots \hat{\wedge} \tau_1$ . Thus the last  $k$  values of  $\tau$  form a chain at the end of the contact graph  $T^\#$ .

It then follows that the operation  $\mathfrak{b}^\bullet$  stabilizes  $\text{Twist}^k$  for any  $\mathfrak{b} \in \mathfrak{B}$ . Indeed, for any two acyclic twists  $T \in \mathcal{AT}^k(n)$  and  $T' \in \mathcal{AT}^k(n')$ , we have

$$\mathbb{P}_T \mathfrak{b}^\bullet \mathbb{P}_{T'} = \sum_S \mathbb{P}_S$$

where the sum runs over all acyclic twists  $S \in \mathcal{AT}^k(n + n')$  such that  $T \setminus T' \leq S \leq T/T'$  and  $\sigma_{n+n'+1-i} \leq n$  if  $\mathfrak{b}_i = \prec$  and  $\sigma_{n+n'+1-i} > n$  if  $\mathfrak{b}_i = \succ$  for any linear extension  $\sigma$  of  $S^\#$ .  $\square$

**Remark 94.** One can also define similarly  $k$ -cotwistiform coalgebras, and such a structure on both  $\text{FQSym}$  and  $\text{Twist}^k$ . Details will be given in [HP15].

## Part 2. Three extensions

In the second part of this paper, we present three independent extensions of the combinatorial, geometric, and algebraic constructions of Part 1:

**Cambrianization:** Following the same direction as [Rea06, HL07, PS12, CP14], we show that our constructions can be parametrized by a sequence of signs (*i.e.* a type  $A$  Coxeter element). We obtain generalizations of the Cambrian lattices, mention their connections to certain well-chosen brick polytopes, and construct a Cambrian twist Hopf algebra with similar ideas as in [CP14, Part 1].

**Tuplization:** Motivated by the Hopf algebras on diagonal rectangulations described by S. Law and N. Reading [LR12] and on twin binary trees described by S. Giraudo [Gir12], G. Chatel and V. Pilaud [CP14, Part 2] defined Cambrian tuple algebras. We briefly extend this construction to Cambrian twist tuples.

**Schröderization:** In [Cha00], F. Chapoton described Hopf algebras on all faces of the permutahedra, associahedra and cubes. This was extended to the Cambrian algebras in [CP14, Part 3]. We show that these algebras and the corresponding combinatorics can be translated as well to all faces of the brick polytopes and zonotopes.

These extensions can moreover be combined: we could present in a single structure all these generalizations. For the sake of clarity, we have preferred to present these extensions separately, and we leave it to the reader to combine them.

### 2.1. CAMBRIANIZATION

Our first extension concerns Cambrian twists, which are natural generalizations of twists parametrized by the choice of a signature  $\varepsilon \in \pm^n$ . When  $k = 1$ , Cambrian 1-twists correspond to Cambrian trees [CP14], whose

- combinatorial properties are encoded in N. Reading’s Cambrian lattices [Rea06],
- geometric properties are encoded in C. Hohlweg and C. Lange’s realizations of the associahedron [HL07], and
- algebraic properties are encoded in G. Chatel and V. Pilaud’s Cambrian algebras [CP14].

We show how these three structures extend to arbitrary  $k$ .

**2.1.1. Combinatorics of Cambrian twists.** Given a signature  $\varepsilon := \varepsilon_1 \cdots \varepsilon_n \in \pm^n$ , we denote by  $|\varepsilon|_-$  and  $|\varepsilon|_+$  the number of  $-$  and  $+$  in  $\varepsilon$ , respectively. For any integer  $k \in \mathbb{N}$  and signature  $\varepsilon \in \pm^n$ , we define a *shape*  $\text{Sh}_\varepsilon^k$  formed by four monotone lattices paths:

- (i) *enter path*: from  $(|\varepsilon|_+, 0)$  to  $(0, |\varepsilon|_-)$  with  $p$ th step north if  $\varepsilon_p = -$  and west if  $\varepsilon_p = +$ ,
- (ii) *exit path*: from  $(|\varepsilon|_+ + k, n + k)$  to  $(n + k, |\varepsilon|_- + k)$  with  $p$ th step east if  $\varepsilon_p = -$  and south if  $\varepsilon_p = +$ ,
- (iii) *accordion paths*: the path  $(NE)^{|\varepsilon|_+ + k}$  from  $(0, |\varepsilon|_-)$  to  $(|\varepsilon|_+ + k, n + k)$  and the path  $(EN)^{|\varepsilon|_- + k}$  from  $(|\varepsilon|_+, 0)$  to  $(n + k, |\varepsilon|_- + k)$ .

We call *pth entry* (resp. *pth exit*) of  $\text{Sh}_\varepsilon^k$  the  $p$ th step of its enter path (resp. exit path). Figure 21 illustrates these families of shapes for different integers  $k$  and signatures  $\varepsilon$ .

**Definition 95.** A *Cambrian  $(k, \varepsilon)$ -twist* is a filling of the shape  $\text{Sh}_\varepsilon^k$  with crosses  $\oplus$  and elbows  $\lrcorner$  such that

- no two pipes cross twice (the pipe dream is reduced),
- the pipe which enters at the  $p$ th entry exits at the  $p$ th exit, and is called the  $p$ th pipe.

**Example 96** (Constant signatures). The Cambrian  $(k, -^n)$ -twists are precisely the  $(k, n)$ -twists. Similarly, the Cambrian  $(k, +^n)$ -twists are obtained from the  $(k, n)$ -twists by reflection with respect to the diagonal  $x = y$ .

Contact graphs of Cambrian twists are defined precisely as for classical twists in Definition 2 of Section 1.1.1.

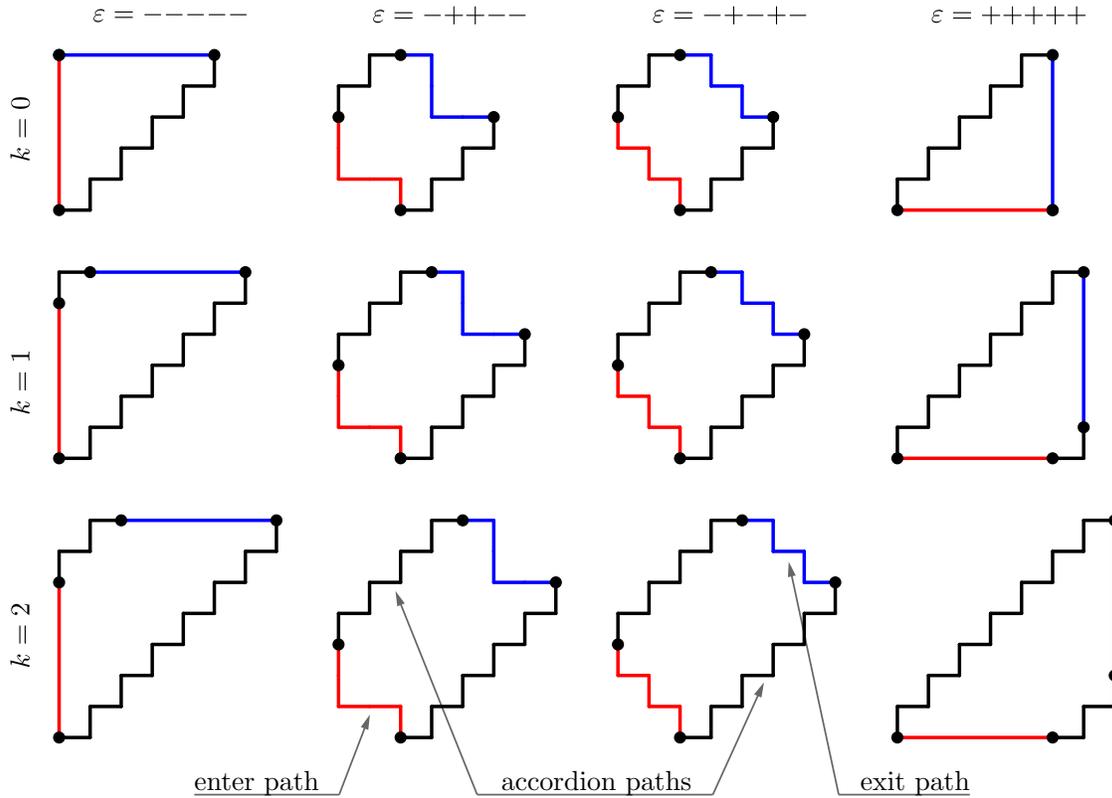


FIGURE 21. The shapes  $\text{Sh}_\varepsilon^k$  for  $k \in \{0, 1, 2\}$  and different signatures  $\varepsilon$ . The enter path is in red, the exit path is in blue, and the accordion paths are in black.

**Definition 97.** The **contact graph** of a Cambrian  $(k, \varepsilon)$ -twist  $T$  is the multigraph  $T^\#$  with vertex set  $[n]$  and with an arc from the SE-pipe to the WN-pipe of each elbow in  $T$ . A Cambrian twist  $T$  is **acyclic** if its contact graph  $T^\#$  is (no oriented cycle). We denote by  $\mathcal{AT}^k(\varepsilon)$  the set of acyclic Cambrian  $(k, \varepsilon)$ -twists.

Figure 22 illustrates examples of Cambrian  $(k, \varepsilon)$ -twists and their contact graphs for  $k = 0, 1, 2, 3$  and the signature  $\varepsilon = -+-$ . The first two are acyclic, the last two are not.

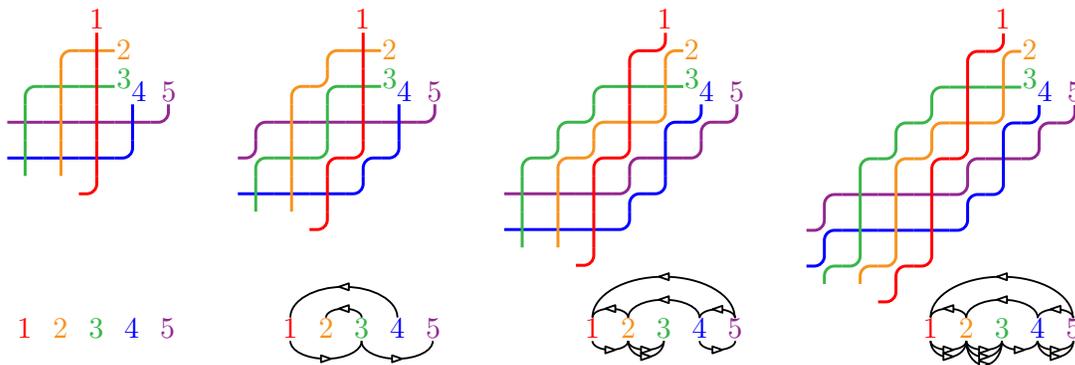


FIGURE 22. Cambrian  $(k, \varepsilon)$ -twists and their contact graphs for  $k = 0, 1, 2, 3$  and  $\varepsilon = -+-$ .

**Remark 98** (Subword complexes). Following Remark 3, a Cambrian  $(k, \varepsilon)$ -twist could be equivalently defined as a reduced expression of the longest permutation  $w_o := [n, n - 1, \dots, 2, 1]$  in the word  $Q_\varepsilon := c_\varepsilon^k w_o(c_\varepsilon)$ , where  $c_\varepsilon$  is a reduced expression for the type  $A$  Coxeter element given by the signature  $\varepsilon$  (meaning that  $s_i$  appears before  $s_{i+1}$  in  $c_\varepsilon$  if  $\varepsilon_i = -$ , and after  $s_{i+1}$  if  $\varepsilon_i = +$ ).

**Example 99** (Cambrian 1-twists, triangulations and Cambrian trees). Following Example 5, Cambrian  $(1, \varepsilon)$ -twists are in bijective correspondence with triangulations of a convex  $(n + 2)$ -gon. This bijection, illustrated in Figure 23, is similar to that described in Example 5, except that the polygon is replaced by the  $(n + 2)$ -gon  $P^\varepsilon$  whose vertices are labeled by  $\{0, \dots, n + 1\}$  from left to right and where vertex  $i$  is located above the diagonal  $[0, n + 1]$  if  $\varepsilon_i = +$  and below it if  $\varepsilon_i = -$ . See [PP12, PS12]. The contact graph of a Cambrian  $(1, \varepsilon)$ -twist  $T$  coincides with the dual tree of  $T^*$  where each node is labeled by the middle vertex of the corresponding triangle, and each arc is oriented from the triangle below to the triangle above the corresponding diagonal. These trees were called “spines” in [LP13], “mixed cobinary trees” in [IO13], and “Cambrian trees” in [CP14].

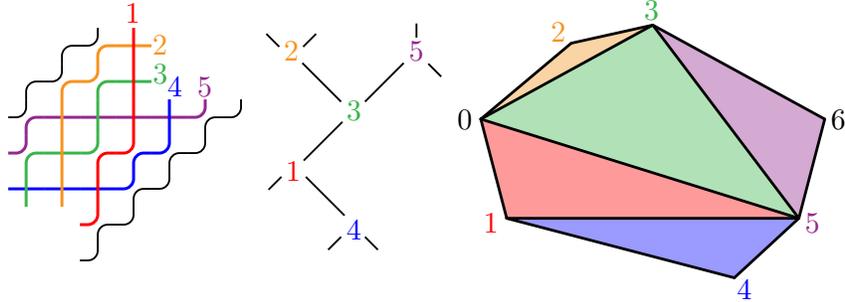


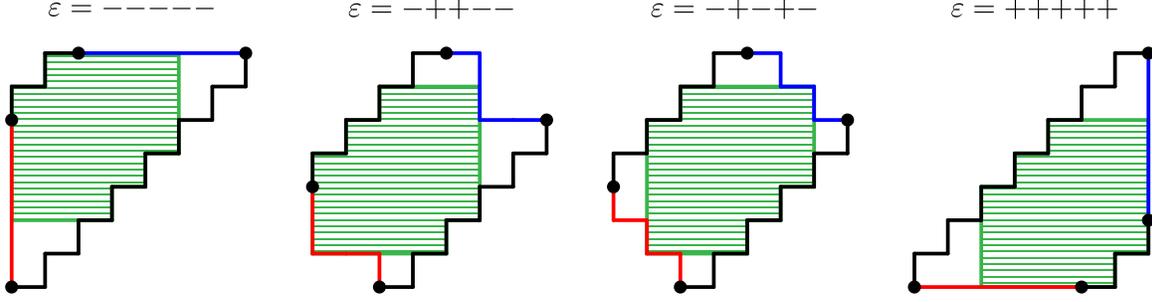
FIGURE 23. The bijection between Cambrian  $(1, \varepsilon)$ -twists  $T$  (left) and triangulations  $T^*$  of  $P^\varepsilon$  (right) sends the pipes of  $T$  to the triangles of  $T^*$ , the elbows of  $T$  to the diagonals of  $T^*$ , the contact graph of  $T$  to the dual Cambrian tree of  $T^*$  (middle), and the elbow flips in  $T$  to the diagonal flips in  $T^*$ .

**Remark 100** (Cambrian  $k$ -twists and  $k$ -triangulations). In fact, one can simultaneously as well extend the results of Section 1.1.2. Namely, set  $\bar{\varepsilon} = (-)^{k-1}\varepsilon(-)^{k-1}$ . Choosing  $\text{Sh}_\varepsilon^k$  instead of the triangular shape and  $P^{\bar{\varepsilon}}$  instead of the standard convex  $(n + 2)$ -gon, the map described in Theorem 10 defines a bijective correspondence between the Cambrian  $(k, \varepsilon)$ -twists and the  $k$ -triangulations of  $P^{\bar{\varepsilon}}$ . Details can be found in [PP12, PS12].

**Remark 101** (Numerology). According to Theorem 11 in Section 1.1.3, the previous remark ensures that the number of Cambrian  $(k, \varepsilon)$ -twists is the Hankel determinant  $\det(C_{n+2k-i-j})_{i,j \in [k]}$ , no matter the signature  $\varepsilon \in \pm^n$ . In contrast, the number of acyclic Cambrian  $(k, \varepsilon)$ -twists depends on the signature  $\varepsilon$ . For example  $|\mathcal{AT}^2(----)| = 22 \neq 24 = |\mathcal{AT}^2(++++)|$ . Table 5 gathers

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	.	2	5	14	42	132	429	1430	4862	16796
2	.	.	6	24	114	608	3532	21950	143776	982324
3	.	.	.	.	120	720	4920	37104	303072	2643156
4	.	.	.	.	.	.	5040	40320	357840	3453120
5	.	.	.	.	.	.	.	.	362880	3628800

TABLE 5. The number  $|\mathcal{AT}^k((+-)^{n/2})|$  of acyclic Cambrian  $(k, (+-)^{n/2})$ -twists for  $2k \leq n \leq 10$ . Dots indicate that the value remains constant (equal to  $n!$ ) in the column.

FIGURE 24. The zone of the third pipe the shapes  $\text{Sh}_\varepsilon^2$  of Figure 21.

the numbers of acyclic Cambrian  $(k, (+-)^{n/2})$ -twists for  $2k \leq n \leq 10$ . Comparing this table to Table 2 raises the question to determine which are the numbers of Cambrian  $(k, \varepsilon)$ -twists for all possible signatures  $\varepsilon \in \pm^n$ . We conjecture in particular that for any signature  $\varepsilon \in \pm^n$ , the number of acyclic Cambrian  $(k, \varepsilon)$ -twists is always bounded between the number of acyclic  $(k, n)$ -twists and the number of acyclic Cambrian  $(k, (+-)^{n/2})$ -twists.

As in Section 1.1.4, we need to establish some elementary properties of pipes in Cambrian twists. The properties of Lemma 13 still hold for Cambrian twists, except that a negative pipe has  $k$  SE-elbows and  $k + 1$  WN-elbows, while a positive pipe has  $k + 1$  SE-elbows and  $k$  WN-elbows. In contrast, Lemma 14 does not hold for Cambrian twists: for example, in the Cambrian 1-twist of Figure 22, the 2nd and 5th pipe are not comparable even if the 5th pipe has an elbow south-east of an elbow of the 2nd pipe. We need to refine the statement as follows. We call *zone* of a pipe  $p$  the boxes located in between the entry and exit points of  $p$ . In other words, it is the zone of the possible elbows of  $p$ . For example, Figure 24 illustrates the zone of the third pipe in the shapes  $\text{Sh}_\varepsilon^2$  of Figure 21.

**Lemma 102.** *Consider two pipes  $p, p'$  in a Cambrian  $k$ -twist  $T$ . If there is an elbow of  $p$  below  $p'$  (resp. above  $p'$ ) and in the zone of  $p'$ , then there is a path from  $p$  to  $p'$  (resp. from  $p'$  to  $p$ ) in the contact graph  $T^\#$  of  $T$ .*

*Proof.* The proof is essentially the same as that of Lemma 14. Up to adding one more edge in the beginning of the path, we can assume that  $p$  has a WN-elbow  $e$  below and in the zone of  $p'$ . We claim that there exists a sequence  $e = e_0, e_1, \dots, e_\ell$  of elbows, all located below and in the zone of  $p'$ , such that the elbows  $e_i$  and  $e_{i+1}$  are connected by a pipe  $p_{i+1}$ , and where  $e_\ell$  is an elbow of  $p'$ . Suppose that the elbows  $e = e_0, e_1, \dots, e_i$  are constructed and consider the WN-pipe  $p_{i+1}$  at  $e_i$ . Either the previous or the next elbow along  $p_{i+1}$  is still below and in the zone of  $p'$ . Otherwise, the pipe  $p_{i+1}$  would cross the pipe  $p'$  twice, since  $e_i$  is in the zone of  $p'$ . Choose  $e_{i+1}$  accordingly (pick arbitrarily one if the two options are possible). The process ends on an elbow  $e_\ell$  of  $p'$  since the distance to  $p'$  decreases at each step. Finally, the sequence of pipes  $p = p_1, \dots, p_\ell = p'$  gives a path from  $p$  to  $p'$  in the contact graph  $T^\#$ . The proof is similar if  $p$  has an elbow above and in the zone of  $p'$ .  $\square$

Similar to Section 1.1.5, there is a natural insertion (resp. deletion) operation which transforms a Cambrian  $(k, \varepsilon)$ -twist  $T$  into a Cambrian  $(k, \varepsilon')$ -twist  $T'$  where  $\varepsilon$  and  $\varepsilon'$  differ by the insertion (resp. deletion) of a single sign. The insertion is defined as in Definition 16, except that it now depends on whether the inserted pipe is negative or positive. Namely, the inserted pipe becomes a source in  $T'^\#$  when the new sign is negative, but a sink in  $T'^\#$  when the new sign is positive. These insertions are illustrated in Figure 25. Conversely, the deletion is defined as in Definition 18 except that a negative (resp. positive) pipe of a  $(k, \varepsilon)$ -twist  $T$  can be deleted only when it is a source (resp. sink) of the contact graph  $T^\#$ .

We will use these insertion and deletion operations to simplify some proofs in this section. However, we cannot use repeated insertions to define the Cambrian  $k$ -twist correspondence below.

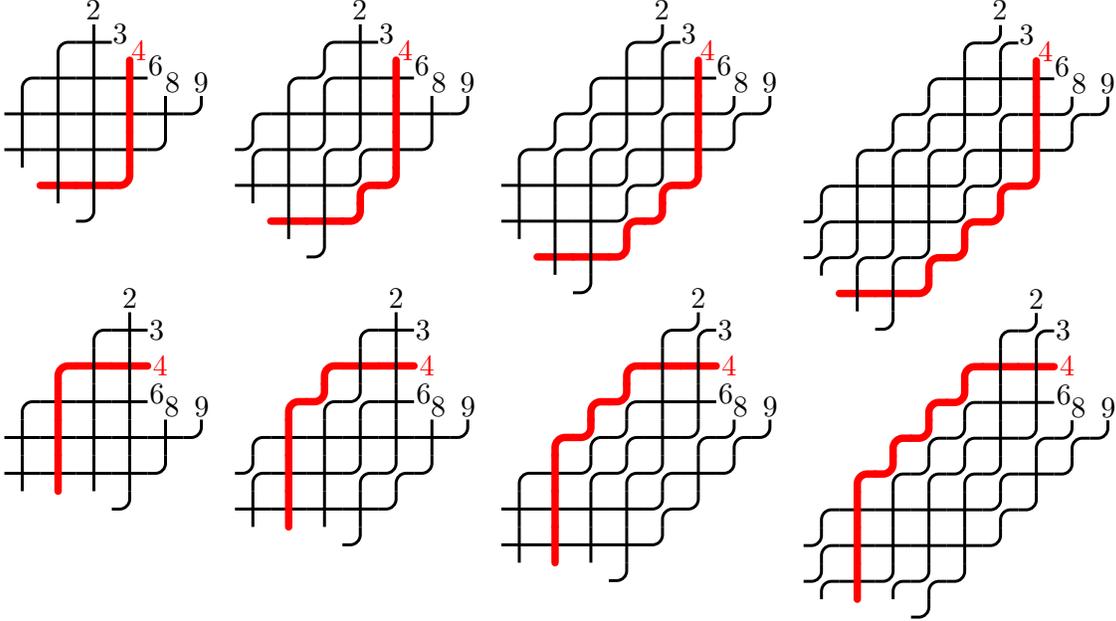


FIGURE 25. Inserting 4 as a negative (top) or positive (bottom) pipe in the  $(k, -++--)$ -twists of Figure 22. The inserted pipe is in bold red.

Indeed, we would need an insertion operation where the inserted pipe is always a source in the new twist, and such an operation is not easy to describe when the inserted pipe is positive. We will thus prefer to avoid repeated single insertions and to insert pipes directly in the final shape.

Similar to Section 1.1.6, we now define the *Cambrian  $k$ -twist correspondence*. Following the presentation in [CP14], we work with signed permutations. A *signed permutation* is a permutation  $\tau \in \mathfrak{S}_n$  endowed with a signature. It is convenient to think about these signs to be associated either to the positions or to the values of  $\tau$ . We call  *$p$ -signature* (resp.  *$v$ -signature*) of  $\tau$  the sequence  $\varepsilon_p(\tau)$  (resp.  $\varepsilon_v(\tau)$ ) of signs of  $\tau$  ordered by positions (resp. values). In concrete examples, we underline negative positions/values while we overline positive positions/values. For example, we write  $\overline{3}1\underline{5}4\underline{2}$  for the permutation  $\tau = 31542$  endowed with the signatures  $\varepsilon_p = +----+$  and  $\varepsilon_v = -++---$ . For a signature  $\varepsilon \in \pm^n$ , we denote by  $\mathfrak{S}_\varepsilon$  (resp. by  $\mathfrak{S}^\varepsilon$ ) the set of signed permutations  $\tau$  with  $p$ -signature  $\varepsilon_p(\tau) = \varepsilon$  (resp. with  $v$ -signature  $\varepsilon_v(\tau) = \varepsilon$ ).

We now define a surjection from the signed permutations of  $\mathfrak{S}^\varepsilon$  to the Cambrian  $(k, \varepsilon)$ -twists. Consider a permutation  $\tau \in \mathfrak{S}^\varepsilon$ . Starting from the empty shape  $\text{Sh}_\varepsilon^k$ , we construct a Cambrian  $(k, \varepsilon)$ -twist  $\text{ins}^k(\tau)$  by inserting the pipes  $\tau_n, \dots, \tau_1$  one by one such that each new pipe is as northwest as possible in the space left by the pipes already inserted. This procedure is illustrated in Figure 26 for the signed permutation  $\overline{3}1\underline{5}4\underline{2}$  and different values of  $k$ . Observe that  $\varepsilon$  does not appear in the notation  $\text{ins}^k(\tau)$  as this information is already carried by the signed permutation  $\tau \in \mathfrak{S}^\varepsilon$ .

**Proposition 103.** *For any Cambrian  $(k, \varepsilon)$ -twist  $T$ , the permutations  $\tau \in \mathfrak{S}^\varepsilon$  such that  $\text{ins}^k(\tau) = T$  are precisely the linear extensions of the contact graph of  $T$ . In particular,  $\text{ins}^k$  is a surjection from the permutations of  $\mathfrak{S}^\varepsilon$  to the acyclic Cambrian  $(k, \varepsilon)$ -twists.*

*Proof.* During the insertion algorithm, the last inserted pipe is always a source of the contact graph of the pipes already inserted. Otherwise, there would be an elbow  $e$  where the last inserted pipe  $p$  is above a pipe  $q$  already inserted. Flipping this elbow  $e$  would transform these two pipes  $p$  and  $q$  into two new pipes  $p'$  and  $q'$  compatible with all already inserted pipes, and the pipe  $q'$  would be northwest of  $q$ , contradicting the definition of  $q$ . Now, since each insertion creates a



source of the contact graph, it is immediate that any signed permutation  $\tau$  is a linear extension of the contact graph of  $\text{ins}^k(\tau)$ .

Conversely, consider a Cambrian  $(k, \varepsilon)$ -twist  $T$  and a linear extension  $\tau$  of  $T^\#$ . We show by induction that the pipe inserted at step  $n+1-i$  in  $\text{ins}^k(\tau)$  coincide with the  $\tau_i$ th pipe of  $T$ . Indeed, since  $\tau$  is a linear extension of  $T^\#$ , no SE-elbow of the  $\tau_i$ th pipe of  $T$  can touch a WN-elbow of the  $\tau_j$ th pipe of  $T$  for  $j < i$ . In other words, the  $\tau_i$ th pipe of  $T$  is the most possible northwest pipe in the space left by the pipes  $\tau_n, \dots, \tau_{i+1}$  of  $T$ . The latter coincide by induction with the first  $n-i$  inserted pipes in  $\text{ins}^k(\tau)$ , so that the  $\tau_i$ th pipe of  $T$  indeed coincides with the pipe inserted at step  $n+1-i$  in  $\text{ins}^k(\tau)$ . We conclude that  $\text{ins}^k(\tau) = T$ .  $\square$

**Example 104** (Cambrian 1-twist correspondence and Cambrian correspondence). When  $k = 1$ , the map  $\text{ins}^1$  can also be described directly on Cambrian trees as in [CP14]. This description is even more visual since the nodes of the Cambrian tree  $\text{ins}^1(\tau)$  form the permutation table of  $\tau$ .

Our next statement extends Proposition 32 to Cambrian  $k$ -twists, using again the characterization of weak order intervals of Definition 30 and Proposition 31 given in Section 1.1.8.

**Proposition 105.** *The fiber of any Cambrian  $k$ -twist under  $\text{ins}^k$  is an interval of the weak order.*

*Proof.* The proof is essentially the same as that of Proposition 32. We consider a Cambrian  $(k, \varepsilon)$ -twist  $T$  and check that it fulfills the WOIP criterion of Definition 30. Assume that  $a < b < c$  are such that  $a \triangleleft_T c$ . Since  $a \triangleleft_T c$ , there is a path  $\pi$  from the  $a$ th entry to the  $c$ th exit of  $\text{Sh}_\varepsilon^k$  which travels along pipes and elbows of  $T$ . Since  $a < b < c$ , the path  $\pi$  starts (weakly) south and ends (weakly) west of the zone of the  $b$ th pipe. Therefore, it must have an elbow  $e$  in the zone of the  $b$ th pipe, which ensures by Lemma 102 that either  $a \triangleleft_T b$  (if  $e$  is above the  $b$ th pipe), or  $b \triangleleft_T c$  (if  $e$  is below the  $b$ th pipe). The proof is similar if  $a \triangleright_T c$ .  $\square$

Our next step is to extend the  $k$ -twist congruence of Section 1.1.7. In other words, we characterize the fibers of the insertion map  $\text{ins}^k$  by rewriting rules.

**Definition 106.** *The Cambrian  $(k, \varepsilon)$ -twist congruence is the equivalence relation  $\equiv_\varepsilon^k$  on  $\mathfrak{S}^\varepsilon$  defined as the transitive closure of the rewriting rule  $UacV \equiv_\varepsilon^k UcaV$  if there exist  $a < a' < c' < c$  such that*

$$\begin{aligned} & |\{u \in U \mid a' \leq u \leq c', \varepsilon_u = +\}| - |\{v \in V \mid a' \leq v \leq c', \varepsilon_v = -\}| \geq k, \\ \text{or } & |\{v \in V \mid a' \leq v \leq c', \varepsilon_v = -\}| - |\{u \in U \mid a' \leq u \leq c', \varepsilon_u = +\}| \geq k, \end{aligned}$$

where  $a, c$  are elements of  $[n]$  while  $U, V$  are (possibly empty) words on  $[n]$ .

This condition can as well be reformulated in terms of congruence witnesses. Namely, we have  $UacV \equiv_\varepsilon^k UcaV$  if their exist  $p \geq 0$  and  $k+p$  witnesses  $a < b_1 < \dots < b_{k+p} < c$  such that

- either  $\varepsilon_{b_i} = +$ ,  $b_i \in U$  and  $|\{v \in V \mid b_1 \leq v \leq b_{k+p}, \varepsilon_v = -\}| = p$ ,
- or  $\varepsilon_{b_i} = -$ ,  $b_i \in V$  and  $|\{u \in U \mid b_1 \leq u \leq b_{k+p}, \varepsilon_u = +\}| = p$ .

With this formulation, the main differences between the  $k$ -twist congruence  $\equiv^k$  of Definition 26 and the Cambrian  $(k, \varepsilon)$ -twist congruence  $\equiv_\varepsilon^k$  of Definition 106 are that:

- the signs of the congruence witnesses matters: e.g.  $\underline{132} \equiv^1 \underline{312}$  while  $\underline{13\bar{2}} \not\equiv^1 \underline{31\bar{2}}$ ,
- we have positive witnesses in  $U$  or negative witnesses in  $V$ : e.g.  $\underline{132} \equiv^1 \underline{312}$  and  $\underline{213} \equiv^1 \underline{231}$ ,
- the remoteness of the witnesses matters: e.g.  $\underline{21534} \equiv^2 \underline{25134}$  while  $\underline{31524} \not\equiv^2 \underline{35124}$ .
- the number of witnesses is not always  $k$  as it depends on their remoteness: e.g. we really need  $\underline{2}$ ,  $\underline{3}$ ,  $\underline{5}$  and  $\underline{6}$  to witness the congruence  $\underline{4172356} \equiv^3 \underline{4712356}$ .

**Example 107** (Cambrian  $(1, \varepsilon)$ -twist congruence and Cambrian congruence). When  $k = 1$  and  $\varepsilon \in \pm^n$ , the Cambrian  $(1, \varepsilon)$ -twist congruence is the Cambrian congruence defined by N. Reading [Rea06] as the transitive closure of  $UacV\underline{b}W \equiv_\varepsilon UcaV\underline{b}W$  and  $U\underline{b}VacW \equiv_\varepsilon U\underline{b}VcaW$ .

**Proposition 108.** *For any  $\tau, \tau' \in \mathfrak{S}^\varepsilon$ , we have  $\tau \equiv_\varepsilon^k \tau' \iff \text{ins}^k(\tau) = \text{ins}^k(\tau')$ . In other words, the fibers of the Cambrian  $(k, \varepsilon)$ -twists by  $\text{ins}^k$  are precisely the Cambrian  $(k, \varepsilon)$ -twist congruence classes.*

*Proof.* By Proposition 103, each fiber of  $\text{ins}^k$  gathers the linear extensions of a  $(k, \varepsilon)$ -twist, so that it is connected by simple transpositions. Therefore, we just need to show the proposition for any two signed permutations  $\tau = UacV$  and  $\tau' = UcaV$  of  $\mathfrak{S}_\varepsilon$  which differ by a simple transposition. By definition of  $\text{ins}^k$ , we have  $\text{ins}^k(\tau) = \text{ins}^k(\tau')$  if and only if  $a$ th and  $c$ th pipes have no common elbow. Observe that exchanging two consecutive letters in  $U$  or in  $V$  just flips the corresponding pipes in  $\text{ins}^k(\tau)$ , and thus does not perturb the relative positions of the  $a$ th and  $c$ th pipes in  $\text{ins}^k(\tau)$ . By such exchanges, we can therefore assume that all negative letters of  $U$  appear at the beginning of  $U$  while all positive letters of  $V$  appear at the end of  $V$ . Using the deletion operation discussed earlier (see Figure 25), we thus reduce the study to the situation where  $U$  has only positive letters while  $V$  has only negative letters. With some additional exchanges, we can moreover assume that  $U = u_1 \cdots u_p$  with  $u_1 < \cdots < u_p$  while  $V = v_1 \cdots v_q$  with  $v_1 < \cdots < v_q$ . We now show the equivalence in this restricted situation.

Suppose that there exist  $i < j$  such that the letters  $v_i, \dots, v_j$  in  $V$  are  $\equiv_\varepsilon^k$ -congruence witnesses for the equivalence  $\tau \equiv_\varepsilon^k \tau'$ . In our specific situation, this means that  $1 + 2(j - i) - (v_j - v_i) \geq k$  since  $|\{u \in U \mid v_i \leq u \leq v_j, \varepsilon_u = +\}| = (v_j - v_i) - (j - i)$  and  $|\{v \in V \mid v_i \leq v \leq v_j, \varepsilon_v = -\}| = j - i + 1$ . Let  $\lambda$  denote the column of the  $a$ th exit and  $\mu$  denote the row of the  $c$ th entry of  $\text{Sh}_\varepsilon^k$ . A simple induction on  $\ell \in [i, j]$  shows that the  $v_\ell$ th pipe of  $\text{ins}^k(\tau)$  has  $1 + 2(\ell - i) - (v_\ell - v_i)$  SE-elbows to the right of column  $\lambda$ . It follows that the  $v_j$ th pipe of  $\text{ins}^k(\tau)$  has  $1 + 2(j - i) - (v_j - v_i) \geq k$  SE-elbows to the right of column  $\lambda$ , and thus it has only one SE-elbow  $e$  to the left of column  $\lambda$ . Now, if the  $a$ th and  $c$ th pipes of  $\text{ins}^k(\tau)$  shared an elbow  $e'$ , it would be to the left of column  $\lambda$ , above row  $\mu$ , and below the  $v_j$ th pipe (since the  $a$ th and  $c$ th pipes are inserted after the  $v_j$ th pipe). The elbow  $e'$  would then lie in the rectangle defined by the two steps of the  $v_j$ th pipe incident to  $e$ , a contradiction. Therefore, the  $a$ th and  $c$ th pipes share no elbows and we have  $\text{ins}^k(\tau) = \text{ins}^k(\tau')$ . The proof is similar if there exist  $i < j$  such that the letters  $u_i, \dots, u_j$  in  $U$  are  $\equiv_\varepsilon^k$ -congruence witnesses. For the converse, one easily checks that if there are neither witnesses in  $U$  nor witnesses in  $V$ , then the  $a$ th and  $c$ th pipes of  $\text{ins}^k(\tau)$  share an elbow, and thus  $\text{ins}^k(\tau) \neq \text{ins}^k(\tau')$ .  $\square$

**Remark 109** (Numerology again). Similar to Remark 29, we observe that

$$|\mathcal{AT}^k(\varepsilon)| = n! \quad \text{for any } \varepsilon \in \pm^n \text{ with } n \leq k + 1, \text{ and}$$

$$|\mathcal{AT}^k((+-)^k+)| = |\mathcal{AT}^{2k-1}(-^{2k+1})| = (2k + 1)! - (2k - 1)!$$

since the only non-trivial Cambrian  $(k, (+-)^k+)$ -twist congruence classes are the pairs of permutations  $\{1(2k + 1)(\sigma_1 + 1) \cdots (\sigma_{2k-1} + 1), (2k + 1)1(\sigma_1 + 1) \cdots (\sigma_{2k-1} + 1)\}$  for  $\sigma \in \mathfrak{S}_{2k-1}$ . The other numbers in Table 5 remain mysterious for us.

Using definitions and techniques already presented in Section 1.1.9, we show that the Cambrian  $k$ -twist congruences are lattice congruences, generalizing the result of N. Reading [Rea06].

**Theorem 110.** *For any signature  $\varepsilon \in \pm^n$  and any  $k \in \mathbb{N}$ , the Cambrian  $(k, \varepsilon)$ -twist congruence  $\equiv_\varepsilon^k$  is a lattice congruence of the weak order.*

*Proof.* We already observed in Proposition 105 that the Cambrian  $(k, \varepsilon)$ -twist congruence classes are intervals of the weak order. The proof that the up and down projection maps are order preserving is similar to the proof Theorem 37. The only delicate observation was already made in the proof of Proposition 108: namely,  $UacV \equiv_\varepsilon^k UcaV \iff U'acV' \equiv_\varepsilon^k U'caV'$  for any permutations  $U'$  of  $U$  and  $V'$  of  $V$ .  $\square$

As in Section 1.1.10, define the *increasing flip order* to be the transitive closure of the increasing flip graph on acyclic Cambrian  $(k, \varepsilon)$ -twists. See Figure 27 for an illustration when  $k = 2$  and  $n = 4$ .

**Proposition 111.** *The increasing flip order on acyclic Cambrian  $(k, \varepsilon)$ -twists is isomorphic to:*

- the quotient lattice of the weak order by the Cambrian  $(k, \varepsilon)$ -twist congruence  $\equiv_\varepsilon^k$ ,
- the subposet of the weak order induced by the permutations avoiding the increasing rewriting rules (maximums of Cambrian  $(k, \varepsilon)$ -twist congruence classes),
- the subposet of the weak order induced by the permutations avoiding the decreasing rewriting rules (minimums of Cambrian  $(k, \varepsilon)$ -twist congruence classes).

*Proof.* Same as the proof of Proposition 40.  $\square$

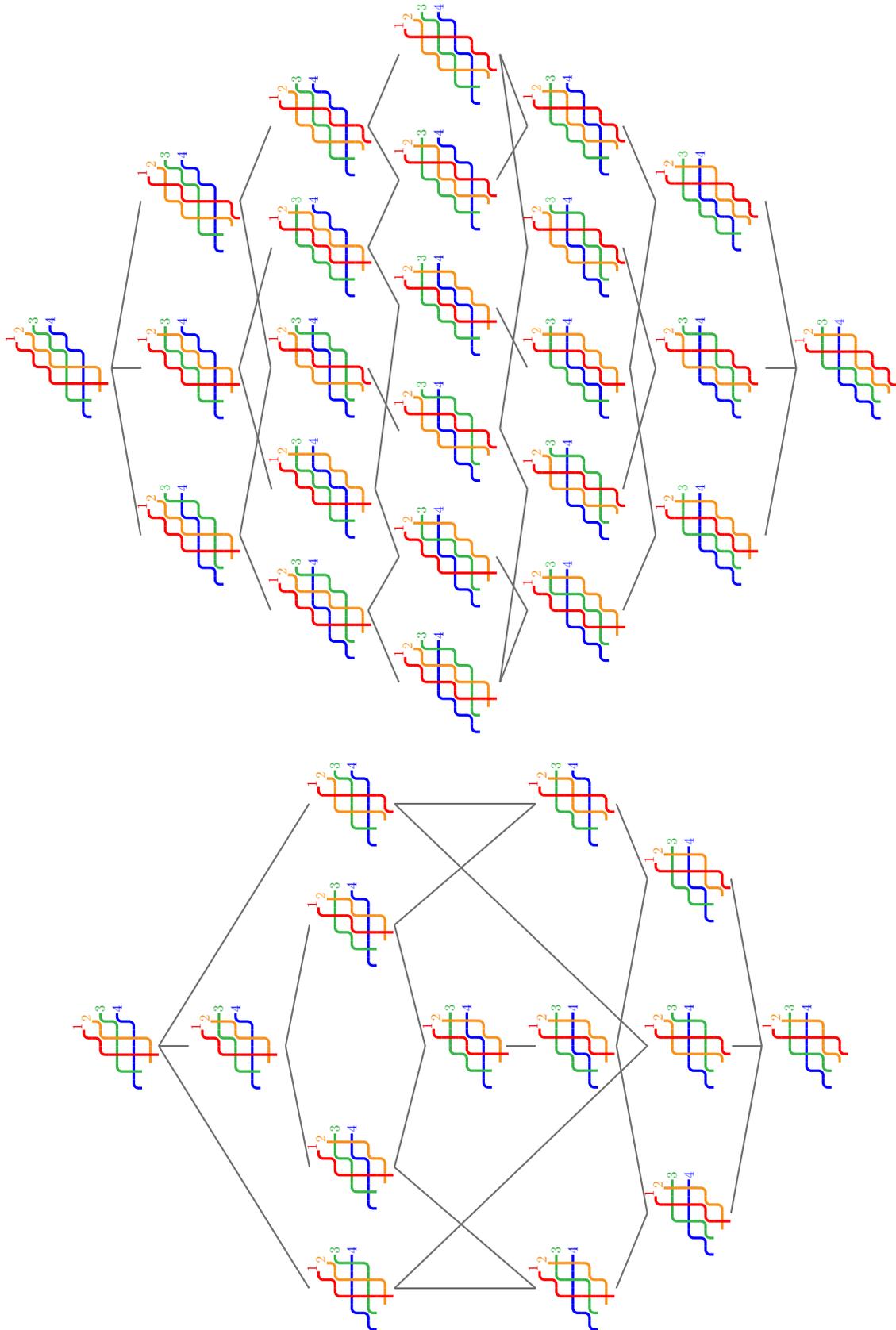


FIGURE 27. The increasing flip lattice on Cambrian  $(k, + - + +)$ -twists for  $k = 1$  (left) and  $k = 2$  (right).

**Example 112** (Cambrian lattices). When  $k = 1$ , the increasing flip lattice is the  $\varepsilon$ -Cambrian lattice of  $N$ . Reading [Rea06].

As for classical twists, the  $i$ th and  $j$ th pipes of any Cambrian  $(k, \varepsilon)$ -twist  $T$  are comparable in  $\triangleleft_T$  as soon as  $|i - j| \leq k$ . We can therefore define as in Section 1.1.12 the *canopy scheme* of  $T$  as the orientation  $\text{can}^k(T) \in \mathcal{AO}^k(n)$  with an edge  $i \rightarrow j$  for all  $i, j \in [n]$  such that  $|i - j| \leq k$  and  $i \triangleleft_T j$ . We still call *k-canopy* the map  $\text{can}^k : \mathcal{AT}^k(\varepsilon) \rightarrow \mathcal{AO}^k(n)$ . Define also the *k-recoil scheme*  $\text{rec}^k(\tau)$  of a signed permutation  $\tau$  as the  $k$ -recoil scheme of  $\tau$  when forgetting its signs.

**Proposition 113.** *The maps  $\text{ins}^k$ ,  $\text{can}^k$ , and  $\text{rec}^k$  define the following commutative diagram of lattice homomorphisms:*

$$\begin{array}{ccc} \mathfrak{S}^\varepsilon & \xrightarrow{\text{rec}^k} & \mathcal{AO}^k(n) \\ & \searrow \text{ins}^k & \nearrow \text{can}^k \\ & \mathcal{AT}^k(\varepsilon) & \end{array}$$

*Proof.* Same as the proof of Proposition 53.  $\square$

**2.1.2. Geometry of acyclic Cambrian twists.** As in Section 1.2, the geometric properties of acyclic Cambrian twists are driven by the normal fan of the brick polytope, defined as follows.

**Definition 114.** *The brick vector of a Cambrian  $(k, \varepsilon)$ -twist  $T$  is the vector  $\mathbf{x}(T) \in \mathbb{R}^n$  whose  $i$ th coordinate is the number of bricks of  $\text{Sh}_\varepsilon^k$  below the  $i$ th pipe of  $T$ , minus a constant  $\alpha_{k, \varepsilon, i}$  (independent of  $T$ ) chosen so that the origin is in the middle of the brick vectors of the minimal and maximal Cambrian  $(k, \varepsilon)$ -twists. The brick polytope  $\text{Brick}^k(\varepsilon)$  is the convex hull of the brick vectors of all Cambrian  $(k, \varepsilon)$ -twists.*

The brick polytope  $\text{Brick}^k(\varepsilon)$  is illustrated in Figure 28 for  $\varepsilon = +-++$  and  $k = 1, 2, 3$ .

**Remark 115** (Inclusions). To be consistent with Section 1.2, we have translated the brick polytope of [PS12] by the vector  $\sum_{i \in [n]} \alpha_{k, \varepsilon, i} \mathbf{e}_i$  so that the permutahedron  $\text{Perm}^k(n)$  is contained inside the brick polytope  $\text{Brick}^k(\varepsilon)$ . Nevertheless, the zonotope  $\text{Zono}^k(n)$  does not necessarily contain anymore the brick polytope  $\text{Brick}^k(\varepsilon)$  when  $k \geq 2$ , as is illustrated in the last two rows of Figure 28.

Applying the results of [PS12], the properties presented in Section 1.2.2 directly translate in the Cambrian world:

- (i) The collection of cones  $\{C^\diamond(T) \mid T \in \mathcal{AT}^k(\varepsilon)\}$ , together with all their faces, form the normal fan of the brick polytope  $\text{Brick}^k(\varepsilon)$ .
- (ii) The surjections  $\text{ins}^k : \mathfrak{S}^\varepsilon \rightarrow \mathcal{AT}^k(\varepsilon)$  and  $\text{can}^k : \mathcal{AT}^k(\varepsilon) \rightarrow \mathcal{AO}^k(n)$  are characterized by polar cone inclusions:  $C^\diamond(\tau) \subseteq C^\diamond(\text{ins}^k(\tau))$  and  $C^\diamond(T) \subseteq C^\diamond(\text{can}^k(T))$ .
- (iii) The 1-skeleton of the brick polytope  $\text{Brick}^k(\varepsilon)$ , oriented in the direction  $\sum_{i \in [n]} (n + 1 - 2i) \mathbf{e}_i$  is the Hasse diagram of the increasing flip lattice on acyclic Cambrian  $(k, \varepsilon)$ -twists.

**2.1.3. Algebra of acyclic Cambrian twists.** To extend the twist algebra to the Cambrian setting, we follow the same path as [CP14]. We first generalize the shifted shuffle and the convolution product to signed permutations. The *signed shifted shuffle product*  $\tau \sqcup \tau'$  is defined as the shifted product of the permutations where signs travel with their values, while the *signed convolution product*  $\tau \star \tau'$  is defined as the convolution product of the permutations where signs stay at their positions. For example,

$$\begin{aligned} \bar{1}2 \sqcup 23\bar{1} &= \{\bar{1}245\bar{3}, \bar{1}425\bar{3}, \bar{1}452\bar{3}, \bar{1}45\bar{3}2, 4\bar{1}25\bar{3}, 4\bar{1}52\bar{3}, 4\bar{1}5\bar{3}2, 45\bar{1}2\bar{3}, 45\bar{1}3\bar{2}, 45\bar{3}1\bar{2}\}, \\ \bar{1}2 \star 23\bar{1} &= \{\bar{1}245\bar{3}, \bar{1}345\bar{2}, \bar{1}435\bar{2}, \bar{1}534\bar{2}, \bar{2}345\bar{1}, \bar{2}435\bar{1}, \bar{2}534\bar{1}, \bar{3}425\bar{1}, \bar{3}524\bar{1}, \bar{4}523\bar{1}\}. \end{aligned}$$

Note that the shifted shuffle is compatible with signed values, while the convolution is compatible with signed positions:  $\mathfrak{S}^\varepsilon \sqcup \mathfrak{S}^{\varepsilon'} = \mathfrak{S}^{\varepsilon\varepsilon'}$  and  $\mathfrak{S}^\varepsilon \star \mathfrak{S}^{\varepsilon'} = \mathfrak{S}_{\varepsilon\varepsilon'}$ . In any case, both  $\sqcup$  and  $\star$  are

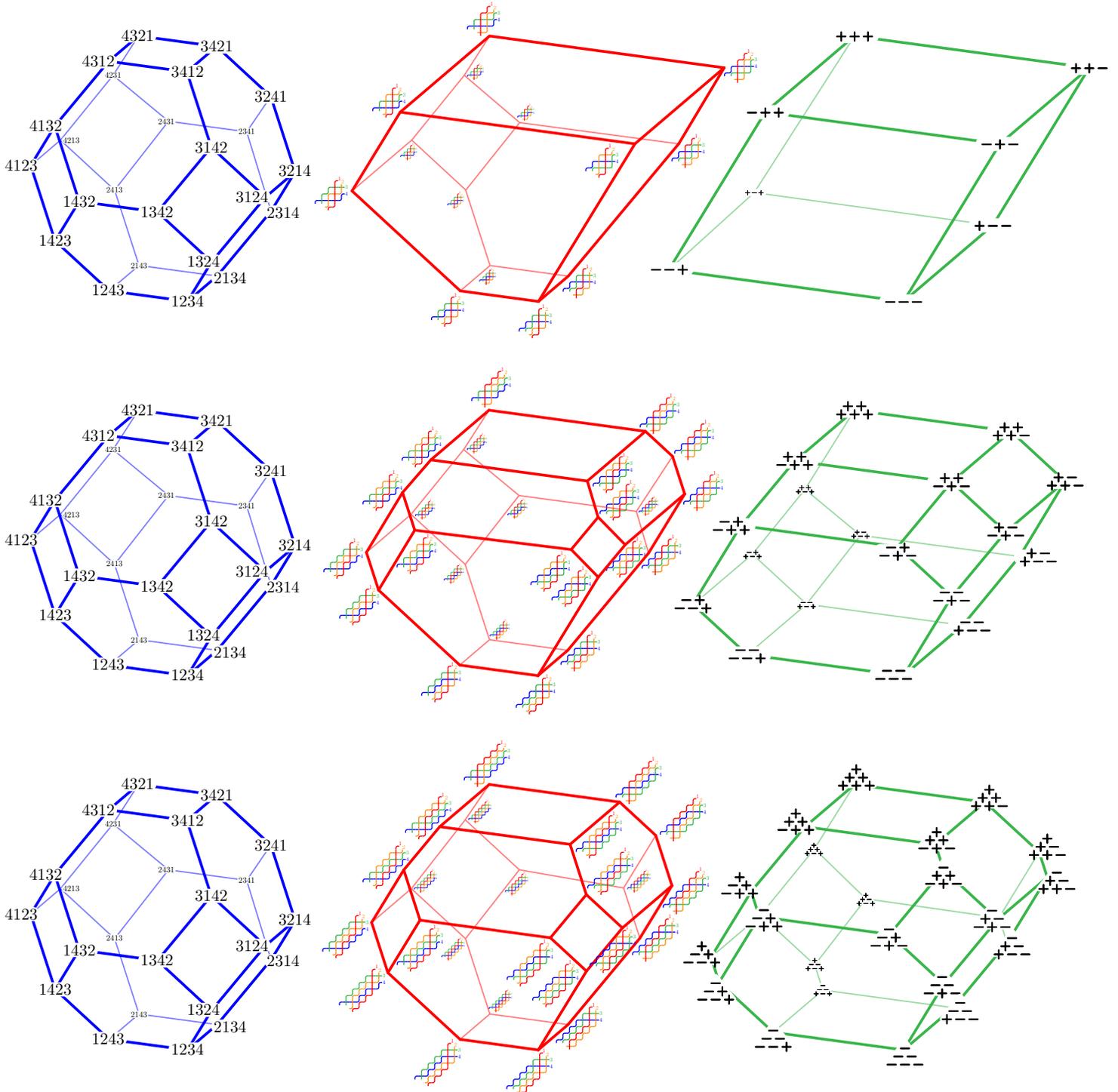


FIGURE 28. The permutahedron  $\text{Perm}^k(4)$  (left), the brick polytope  $\text{Brick}^k(++++)$  (middle) and the zonotope  $\text{Zono}^k(4)$  (right) for  $k = 1$  (top),  $k = 2$  (middle) and  $k = 3$  (bottom). For readability, we represent orientations of  $G^k(n)$  by pyramids of signs.

compatible with the distribution of positive and negative signs:  $|\tau \sqcup \tau'|_+ = |\tau|_+ + |\tau'|_+ = |\tau \star \tau'|_+$  and  $|\tau \sqcup \tau'|_- = |\tau|_- + |\tau'|_- = |\tau \star \tau'|_-$ .

We denote by

$$\mathfrak{S}_\pm := \bigsqcup_{\substack{n \in \mathbb{N} \\ \varepsilon \in \pm^n}} \mathfrak{S}_\varepsilon = \bigsqcup_{\substack{n \in \mathbb{N} \\ \varepsilon \in \pm^n}} \mathfrak{S}^\varepsilon$$

the set of all signed permutations of arbitrary size and arbitrary signature. We consider the Hopf algebra  $\text{FQSym}_\pm$  with basis  $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}_\pm}$  and whose product and coproduct are defined by

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} \mathbb{F}_\sigma \quad \text{and} \quad \Delta \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}.$$

This Hopf algebra is bigraded by the number of positive and the number of negative signs of the signed permutations.

We now consider the vector subspace  $\text{Twist}_\pm^k$  of  $\text{FQSym}_\pm$  generated by the elements

$$\mathbb{P}_T := \sum_{\substack{\tau \in \mathfrak{S}_\pm \\ \text{ins}^k(\tau) = T}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(T^\#)} \mathbb{F}_\tau,$$

for all acyclic Cambrian  $(k, \varepsilon)$ -twists  $T$  for all signatures  $\varepsilon \in \pm^*$ . For example, for the signature  $\varepsilon = -+ + - -$  and the Cambrian  $(k, \varepsilon)$ -twists of Figure 26, we have

$$\begin{array}{ccc} \mathbb{P} = \sum_{\tau \in \mathfrak{S}^\varepsilon} \mathbb{F}_\tau & \mathbb{P} = \mathbb{F}_{\overline{31254}} + \mathbb{F}_{\overline{31524}} + \mathbb{F}_{\overline{35124}} & \mathbb{P} = \mathbb{F}_{\overline{31542}} \\ & + \mathbb{F}_{\overline{53124}} + \mathbb{F}_{\overline{31542}} + \mathbb{F}_{\overline{35142}} & \\ & + \mathbb{F}_{\overline{53142}} + \mathbb{F}_{\overline{35412}} + \mathbb{F}_{\overline{53412}} & \end{array} \quad \mathbb{P} = \mathbb{F}_{\overline{31542}} \quad \mathbb{P} = \mathbb{F}_{\overline{31542}}.$$

**Theorem 116.**  $\text{Twist}_\pm^k$  is a Hopf subalgebra of  $\text{FQSym}_\pm$ .

*Proof.* As in the proof of Theorem 70, we check that the subspace  $\text{Twist}_\pm^k$  is stable by the product and coproduct of  $\text{FQSym}_\pm$ . This follows from the fact that the condition of Definition 106 for two permutations  $\tau = UacV$  and  $\tau' = UcaV$  to be in the same Cambrian  $(k, \varepsilon)$ -twist congruence class only depends on the positions of the entries of  $\tau$  and  $\tau'$  between  $a$  and  $c$ . Details are left to the reader.  $\square$

**Example 117** (Cambrian algebra). The bijection given in Example 99 (see also Figure 23) defines an isomorphism from the Cambrian 1-twist algebra  $\text{Twist}_\pm^1$  to G. Chatel and V. Pilaud's Hopf algebra  $\text{Camb}$  on Cambrian trees [CP14].

As in Section 1.3.2, it is interesting to describe combinatorially the product and coproduct in  $\text{Twist}_\pm^k$  directly in terms of Cambrian  $k$ -twists:

- For  $T \in \mathcal{AT}^k(\varepsilon)$  and  $T' \in \mathcal{AT}^k(\varepsilon')$ , the product  $\mathbb{P}_T \cdot \mathbb{P}_{T'}$  in  $\text{Twist}_\pm^k$  is given by

$$\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_S \mathbb{P}_S,$$

where the sum runs over the interval of the increasing flip lattice between the Cambrian  $(k, \varepsilon\varepsilon')$ -twists  $T \setminus T'$  and  $T/T'$ , defined as in the unsigned case.

- For  $S \in \mathcal{AT}^k(\varepsilon)$ , the coproduct  $\Delta \mathbb{P}_S$  is given by

$$\Delta \mathbb{P}_S = \sum_\gamma \left( \sum_\tau \mathbb{P}_{\text{ins}^k(\tau)} \right) \otimes \left( \sum_{\tau'} \mathbb{P}_{\text{ins}^k(\tau')} \right),$$

where  $\gamma$  runs over all cuts of  $S$ , and  $\tau$  (resp.  $\tau'$ ) runs over a set of representatives of the Cambrian  $k$ -twist congruence classes of the linear extensions of  $B^\#(S, \gamma)$  (resp.  $A^\#(S, \gamma)$ ). Again, this description of the coproduct is not completely satisfactory as it essentially computes the product back in  $\text{FQSym}_\pm$ .

The reader is also invited to work out the combinatorial descriptions of the product and coproduct in the dual Hopf algebra  $\text{Twist}_\pm^{k*}$ , which are similar to that of Section 1.3.3.

## 2.2. TUPLIZATION

In this section, we extend our results to tuples of Cambrian  $k$ -twists. The ideas developed in [CP14, Section 2.3] apply verbatim to the context of  $k$ -twists. The motivation comes from S. Law and N. Reading's work on diagonal quadrangulations [LR12] and S. Giraudo's work on twin binary trees [Gir12].

**2.2.1. Cambrian  $k$ -twist  $\ell$ -tuples.** For a  $\ell$ -tuple  $\mathbf{T}$  and  $i \in [\ell]$ , we denote by  $\mathbf{T}_{[i]}$  the  $i$ th element of  $\mathbf{T}$  (this notation avoids confusion when we will use tuples of permutations:  $\tau_{[i]}$  is a permutation while  $\tau_i$  is an entry in a permutation). We focus on the following objects.

**Definition 118.** For a signature  $\ell$ -tuple  $\mathcal{E} := [\mathcal{E}_{[1]}, \dots, \mathcal{E}_{[\ell]}] \in (\pm^n)^\ell$ , a **Cambrian  $(k, \mathcal{E})$ -twist tuple** is a  $\ell$ -tuple  $\mathbf{T} := [\mathbf{T}_{[1]}, \dots, \mathbf{T}_{[\ell]}]$  where  $\mathbf{T}_{[i]}$  is a Cambrian  $(k, \mathcal{E}_{[i]})$ -twist and such that the union of the contact graphs  $\mathbf{T}_{[1]}^\#, \dots, \mathbf{T}_{[\ell]}^\#$  is acyclic. Let  $\mathcal{TT}^k(\mathcal{E})$  denote the set of Cambrian  $(k, \mathcal{E})$ -twist tuples. A **Cambrian  $k$ -twist  $\ell$ -tuple** is a Cambrian  $(k, \mathcal{E})$ -twist tuple for some signature  $\ell$ -tuple  $\mathcal{E} \in (\pm^n)^\ell$ .

To define an analogue of the Cambrian  $k$ -twist correspondence, we need permutations recording  $\ell$  signatures. Call  **$\ell$ -signed permutation** a permutation where each position/value receives an  $\ell$ -tuple of signs. For example,  $\tilde{\tilde{3}}\tilde{1}\tilde{5}\tilde{4}\tilde{2}$  is a 2-signed permutation whose signatures are marked with  $\sim/\sim$  and  $\sim/\sim$  respectively. For a  $\ell$ -signed permutation  $\tau$  and  $i \in [\ell]$ , we denote by  $\tau_{[i]}$  the signed permutation where we only keep the  $i$ th signature. For example

$$\tilde{\tilde{3}}\tilde{1}\tilde{5}\tilde{4}\tilde{2}_{[1]} = \tilde{3}\tilde{1}\tilde{5}\tilde{4}\tilde{2} \quad \text{and} \quad \tilde{\tilde{3}}\tilde{1}\tilde{5}\tilde{4}\tilde{2}_{[2]} = \tilde{3}\tilde{1}\tilde{5}\tilde{4}\tilde{2}.$$

We denote by  $\mathfrak{S}_{\pm^\ell}$  the set of all  $\ell$ -signed permutations and by  $\mathfrak{S}_{\mathcal{E}}$  (resp.  $\mathfrak{S}^{\mathcal{E}}$ ) the set of  $\ell$ -signed permutations with p-signatures (resp. v-signatures)  $\mathcal{E}$ . Applying  $\ell$  Cambrian  $k$ -twist correspondences in parallel yields a map from  $\ell$ -signed permutations to Cambrian  $\ell$ -tuples.

**Proposition 119.** The map  $\text{ins}_\ell^k$  defined by  $\text{ins}_\ell^k(\tau) = [\text{ins}^k(\tau_{[1]}), \dots, \text{ins}^k(\tau_{[\ell]})]$  defines a surjection from the  $\ell$ -signed permutations to the Cambrian  $k$ -twist  $\ell$ -tuples.

*Proof.* Consider a Cambrian  $k$ -twist  $\ell$ -tuple  $\mathbf{T}$ . For a  $\ell$ -signed permutation  $\tau$ , we have  $\text{ins}_\ell^k(\tau) = \mathbf{T}$  if and only if  $\text{ins}^k(\tau_{[i]}) = \mathbf{T}_{[i]}$  for all  $i \in [\ell]$ , or equivalently by Proposition 103, if and only if  $\tau$  is a linear extension of each Cambrian  $k$ -twist  $\mathbf{T}_{[i]}$ . We therefore obtain that the fiber of  $\mathbf{T}$  under  $\text{ins}_\ell^k$  is precisely the set of linear extensions of the union of the contact graphs  $\mathbf{T}_{[1]}^\#, \dots, \mathbf{T}_{[\ell]}^\#$ .  $\square$

We now consider the congruence on  $\mathfrak{S}_{\pm^\ell}$  recording which  $\ell$ -signed permutations have the same image by  $\text{ins}_\ell^k$ .

**Definition 120.** For a signature  $\ell$ -tuple  $\mathcal{E} \in (\pm^n)^\ell$ , the **Cambrian  $(k, \mathcal{E})$ -twist tuple congruence** on  $\mathfrak{S}^{\mathcal{E}}$  is the intersection  $\equiv_{\mathcal{E}}^k := \bigcap_{i \in [\ell]} \equiv_{\mathcal{E}_{[i]}}^k$  of all Cambrian  $(k, \mathcal{E}_{[i]})$ -twist congruences. We denote by  $\equiv_\ell^k$  the congruence relation on  $\mathfrak{S}_{\pm^\ell}$  obtained as the union of the Cambrian  $(k, \mathcal{E})$ -twist congruences  $\equiv_{\mathcal{E}}^k$  for all signature  $\ell$ -tuples  $\mathcal{E}$ .

**Proposition 121.** For any  $\tau, \tau' \in \mathfrak{S}_{\pm^\ell}$ , we have  $\tau \equiv_\ell^k \tau' \iff \text{ins}_\ell^k(\tau) = \text{ins}_\ell^k(\tau')$ . In other words, the fibers of  $\text{ins}_\ell^k$  are precisely the congruence classes of  $\equiv_\ell^k$ .

*Proof.* We have

$$\begin{aligned} \tau \equiv_\ell^k \tau' &\iff \tau_{[i]} \equiv_{\mathcal{E}_{[i]}}^k \tau'_{[i]} \text{ for all } i \in [\ell] && \text{(by definition of } \equiv_\ell^k \text{),} \\ &\iff \text{ins}^k(\tau_{[i]}) = \text{ins}^k(\tau'_{[i]}) \text{ for all } i \in [\ell] && \text{(by Proposition 108),} \\ &\iff \text{ins}_\ell^k(\tau) = \text{ins}_\ell^k(\tau') && \text{(by definition of } \text{ins}_\ell^k \text{).} \end{aligned} \quad \square$$

**Proposition 122.** For any fixed signature  $\ell$ -tuple  $\mathcal{E} \in (\pm^n)^\ell$ , the Cambrian  $(k, \mathcal{E})$ -twist tuple congruence  $\equiv_{\mathcal{E}}^k$  is a lattice congruence of the weak order on  $\mathfrak{S}_n$ .

*Proof.* An intersection of lattice congruences is a lattice congruence, see [Rea04] for details.  $\square$

We now define flips in Cambrian  $k$ -twist  $\ell$ -tuples. It is essentially an elbow flip performed in all elements of the  $\ell$ -tuple in which it is possible.

**Definition 123.** Let  $\mathbf{T}$  be a Cambrian  $k$ -twist  $\ell$ -tuple and  $p \rightarrow q$  be an arc in the union of the contact graphs  $\mathbf{T}_{[1]}^\#, \dots, \mathbf{T}_{[\ell]}^\#$ . We say that this arc is **flippable** if either  $p \rightarrow q$  is an arc or  $p$  and  $q$  are incomparable in each contact graph  $\mathbf{T}_{[i]}^\#$ . The **flip** of  $p \rightarrow q$  then transforms  $\mathbf{T}$  to the  $\ell$ -tuple  $\mathbf{T}' := [\mathbf{T}'_{[1]}, \dots, \mathbf{T}'_{[\ell]}]$  where  $\mathbf{T}'_{[i]}$  is obtained from  $\mathbf{T}_{[i]}$  by flipping the elbow between pipes  $p$  and  $q$  farthest from their crossing if  $p \rightarrow q$  is an arc in  $\mathbf{T}_{[i]}^\#$ , while  $\mathbf{T}'_{[i]} = \mathbf{T}_{[i]}$  otherwise. We say that the flip of  $p \rightarrow q$  is **increasing** if  $p < q$ .

**Proposition 124.** The increasing flip order on Cambrian  $(k, \mathcal{E})$ -twist tuples is the quotient lattice of the weak order by the Cambrian  $(k, \mathcal{E})$ -twist congruence  $\equiv_{\mathcal{E}}^k$ .

*Proof.* Consider two distinct Cambrian  $(k, \mathcal{E})$ -twist tuples  $\mathbf{T}, \mathbf{T}'$  and their Cambrian  $(k, \mathcal{E})$ -twist tuple congruence classes  $C, C'$ . If there exists adjacent representatives of  $C$  and  $C'$ , then the Cambrian  $(k, \mathcal{E}_{[i]})$ -twists  $\mathbf{T}_{[i]}$  and  $\mathbf{T}'_{[i]}$  are either equal or differ by an elbow flip for each  $i \in [\ell]$  by Proposition 111, and thus the two distinct Cambrian  $(k, \mathcal{E})$ -twist tuples  $\mathbf{T}, \mathbf{T}'$  differ by a flip. Conversely, if  $\mathbf{T}, \mathbf{T}'$  differ by the flip of an arc  $p \rightarrow q$ , then there exists a linear extension  $\tau$  of the union of the contact graphs  $\mathbf{T}_{[1]}^\#, \dots, \mathbf{T}_{[\ell]}^\#$  where  $p$  and  $q$  are consecutive. Since  $\mathbf{T}'_{[i]}$  is obtained by flipping the elbow between  $p$  and  $q$  in  $\mathbf{T}_{[i]}$  if possible and  $\mathbf{T}'_{[i]} = \mathbf{T}_{[i]}$  otherwise, the permutation  $\tau'$  obtained by switching  $p$  and  $q$  in  $\tau$  is a linear extension of  $\mathbf{T}'_{[1]}^\#, \dots, \mathbf{T}'_{[\ell]}^\#$ . We thus obtained two adjacent permutations  $\tau \in C$  and  $\tau \in C'$ .  $\square$

In a Cambrian  $k$ -twist  $\ell$ -tuple  $\mathbf{T}$ , all Cambrian  $k$ -twists  $\mathbf{T}_{[i]}$  have the same  $k$ -canopy  $\text{can}^k(\mathbf{T}_{[i]})$ . Indeed all these Cambrian  $k$ -twists have a common linear extension and the  $k$ -canopy of a Cambrian  $k$ -twist is given by the relative order of the last  $k$  values of its linear extensions. We can thus define the  **$k$ -canopy** of the Cambrian  $k$ -twist  $\ell$ -tuple  $\mathbf{T}$  as  $\text{can}_\ell^k(\mathbf{T}) = \text{can}^k(\mathbf{T}_{[1]}) = \dots = \text{can}^k(\mathbf{T}_{[\ell]})$ . Define also the  **$k$ -recoil scheme**  $\text{rec}^k(\tau)$  of a  $\ell$ -signed permutation  $\tau$  as the  $k$ -recoil scheme of  $\tau$  when forgetting its signs. The next statement is an immediate consequence of these definitions.

**Proposition 125.** The maps  $\text{ins}_\ell^k$ ,  $\text{can}_\ell^k$ , and  $\text{rec}^k$  define a commutative diagram of lattice homomorphisms:

$$\begin{array}{ccc} \mathfrak{S}^\mathcal{E} & \xrightarrow{\text{rec}^k} & \mathcal{AO}^k(n) \\ & \searrow \text{ins}_\ell^k & \nearrow \text{can}_\ell^k \\ & \mathcal{TT}^k(\mathcal{E}) & \end{array}$$

**2.2.2. Geometric realization.** Following [LR12], we observe in this section that the increasing flip order on Cambrian  $k$ -twist  $\ell$ -tuples can be realized geometrically using Minkowski sums of brick polytopes. For a Cambrian  $(k, \mathcal{E})$ -twist tuple  $\mathbf{T}$ , we denote by  $C(\mathbf{T})$  and  $C^\diamond(\mathbf{T})$  the incidence and braid cones of the transitive closure of the union of the contact graphs  $\mathbf{T}_{[1]}^\#, \dots, \mathbf{T}_{[\ell]}^\#$ .

**Proposition 126.** The Minkowski sum  $\text{Brick}^k(\mathcal{E})$  of the brick polytopes  $\text{Brick}^k(\mathcal{E}_{[i]})$  provides a geometric realization of the increasing flip lattice Cambrian  $(k, \mathcal{E})$ -twist tuples:

- (i) The collection of cones  $\{C^\diamond(\mathbf{T}) \mid \mathbf{T} \in \mathcal{TT}^k(\mathcal{E})\}$ , together with all their faces, form the normal fan of  $\text{Brick}^k(\mathcal{E})$ .
- (ii) The surjections  $\text{ins}_\ell^k : \mathfrak{S}^\mathcal{E} \rightarrow \mathcal{TT}^k(\mathcal{E})$  and  $\text{can}^k : \mathcal{TT}^k(\mathcal{E}) \rightarrow \mathcal{AO}^k(n)$  are characterized by polar cone inclusions:  $C^\diamond(\tau) \subseteq C^\diamond(\text{ins}_\ell^k(\tau))$  and  $C^\diamond(\mathbf{T}) \subseteq C^\diamond(\text{can}_\ell^k(\mathbf{T}))$ .
- (iii) The 1-skeleton of  $\text{Brick}^k(\mathcal{E})$ , oriented in the direction  $\sum_{i \in [n]} (n+1-2i) \mathbf{e}_i$  is the Hasse diagram of the increasing flip lattice on Cambrian  $(k, \mathcal{E})$ -twist tuples.

*Proof.* The normal fan of a Minkowski sum is the common refinement of the normal fans of its summands. For  $\mathbf{T}_1 \in \mathcal{AT}^k(\mathcal{E}_{[1]}), \dots, \mathbf{T}_\ell \in \mathcal{AT}^k(\mathcal{E}_{[\ell]})$ , the normal cones  $C^\diamond(\mathbf{T}_1), \dots, C^\diamond(\mathbf{T}_\ell)$  intersect precisely when the contact graphs  $\mathbf{T}_1^\#, \dots, \mathbf{T}_\ell^\#$  have a common linear extension, and their intersection is the braid cone of the transitive closure of the union of the contact graphs  $\mathbf{T}_1^\#, \dots, \mathbf{T}_\ell^\#$ . Point (i) follows. Point (ii) follows from the fact that a poset  $\triangleleft$  is an extension of a poset  $\triangleleft'$  if

and only if  $C^\diamond(\triangleleft) \subseteq C^\diamond(\triangleleft')$ . Finally, Point (iii) is a consequence of Point (i) and the fact that increasing flips are oriented in the direction  $\sum_{i \in [n]} (n+1-2i) \mathbf{e}_i$ .  $\square$

For example, the polytope  $\text{Brick}^1(\mathcal{E})$  for  $\mathcal{E} = [-----, -++---]$  is represented in Figure 29.

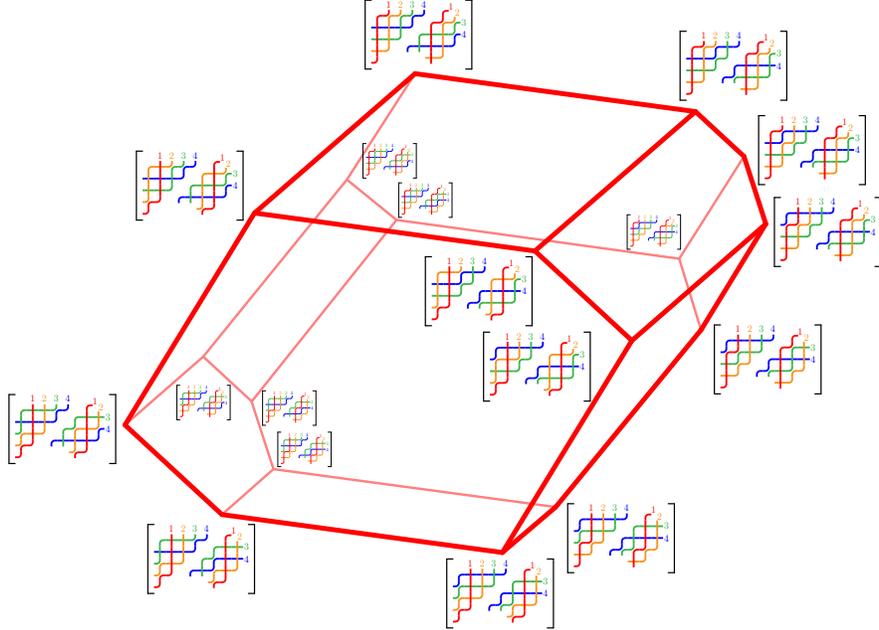


FIGURE 29. The polytope  $\text{Brick}^1(\mathcal{E})$  for the signatures  $\mathcal{E} = [-----, -++---]$ . It is the Minkowski sum of the associahedra of Figures 11 and 28 (top center).

**2.2.3. Algebra of acyclic Cambrian twist tuples.** The *shifted shuffle product*  $\tau \sqcup \tau'$  (resp. *convolution product*  $\tau \star \tau'$ ) of two  $\ell$ -signed permutations  $\tau, \tau'$  is still defined as the shifted product (resp. convolution product) where signs travel with their values (resp. stay at their positions). We denote by  $\text{FQSym}_{\pm\ell}$  the Hopf algebra with basis  $(F_\tau)_{\tau \in \mathfrak{S}_{\pm\ell}}$  indexed by  $\ell$ -signed permutations and whose product and coproduct are defined by

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} \mathbb{F}_\sigma \quad \text{and} \quad \Delta \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}.$$

We then consider the vector subspace  $\text{Twist}_\ell^k$  of  $\text{FQSym}_{\pm\ell}$  generated by the elements

$$\mathbb{P}_{\mathbf{T}} := \sum_{\substack{\tau \in \mathfrak{S}_{\pm\ell} \\ \text{ins}_\ell^k(\tau) = \mathbf{T}}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L} \left( \bigcup_{k \in [\ell]} \mathbf{T}_{[k]} \right)} \mathbb{F}_\tau,$$

for all Cambrian  $k$ -twists  $\ell$ -tuples  $\mathbf{T}$ . For example,

$$\mathbb{P}_{\left[ \begin{array}{c} 1 \ 2 \ 3 \ 4 \\ \color{red}{\curvearrowright} \color{green}{\curvearrowright} \color{blue}{\curvearrowright} \color{red}{\curvearrowright} \\ \color{red}{\curvearrowright} \color{green}{\curvearrowright} \color{blue}{\curvearrowright} \color{red}{\curvearrowright} \end{array} \right]} = \mathbb{F}_{\underline{1342}} + \mathbb{F}_{\underline{3142}} + \mathbb{F}_{\underline{3412}},$$

where the signatures are marked with  $\underline{\quad}$  and  $\sim/\sim$  respectively. The next statement is proved as usual by showing that the product and coproduct of  $\text{FQSym}_{\pm\ell}$  stabilize  $\text{Twist}_\ell^k$ .

**Theorem 127.**  *$\text{Twist}_\ell^k$  is a Hopf subalgebra of  $\text{FQSym}_{\pm\ell}$ .*

The product and coproduct of  $\mathbb{P}$ -basis elements of  $\text{Twist}_\ell^k$  can again be described directly in terms of combinatorial operations on Cambrian  $k$ -twist  $\ell$ -tuples. Details are left to the reader. See e.g. [CP14, Section 2.3] for the  $k = 1$  case.

**2.2.4. Twin twists.** When  $k = 1$ , pairs of twin Cambrian trees are particularly relevant Cambrian tuples as they have additional relevant combinatorial properties, see [CP14, 2.1] and the references therein. They extend to  $k$ -twists as follows.

**Definition 128.** A pair of twin Cambrian  $k$ -twists is a pair  $[T_{\circ}, T_{\bullet}]$  where  $T_{\circ} \in \mathcal{AT}^k(\varepsilon)$  while  $T_{\bullet} \in \mathcal{AT}^k(-\varepsilon)$  for some signature  $\varepsilon$  (where  $-\varepsilon$  is obtained by reversing all signs of  $\varepsilon$ ).

In particular, we call them pairs of twin  $k$ -twists when  $\varepsilon = -^n$  and pairs of twin alternating  $k$ -twists when  $\varepsilon = (-+)^{n/2}$ . When  $k = 1$ , these special cases have interesting combinatorial properties:

- (i) pairs of twin 1-twists are counted by Baxter numbers (see [OEIS, A001181]), and are therefore in bijection with many other Baxter families [CGHK78, DG96, DG98, YCCG03, FFNO11, BBMF11, LR12, Gir12],
- (ii) pairs of twin alternating 1-twists are counted by central binomial coefficients [CP14, Prop. 57].

For completeness, we have gathered in Tables 6 and 7 the numbers of twin  $(k, n)$ -twists which motivates the following questions.

**Question 129.** Are there nice formulas for the number of pairs of twin  $k$ -twists? What about the number of pairs of twin alternating  $k$ -twists? Are there other combinatorial families in bijection with these pairs of twin  $k$ -twists?

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	.	2	6	22	92	422	2074	10754	58202	326240
2	.	.	.	24	120	696	4512	31936	242728	1956072
3	.	.	.	.	.	720	5040	39600	341280	3175632
4	.	.	.	.	.	.	.	40320	362880	3588480
5	.	.	.	.	.	.	.	.	.	3628800

TABLE 6. The number of pairs of twin  $(k, n)$ -twists for  $2k \leq n \leq 10$ . Dots indicate that the value remains constant (equal to  $n!$ ) in the column.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	.	2	6	20	70	252	924	3432	12870	48620
2	.	.	.	24	120	672	4188	27884	197904	1462384
3	.	.	.	.	.	720	5040	38880	328560	2963376
4	.	.	.	.	.	.	.	40320	362880	3548160
5	.	.	.	.	.	.	.	.	.	3628800

TABLE 7. The number of pairs of twin alternating  $(k, n)$ -twists for  $2k \leq n \leq 10$ . Dots indicate that the value remains constant (equal to  $n!$ ) in the column.

**Remark 130** (Twin  $k$ -twists versus  $k$ -twists with opposite canopy). When  $k = 1$ , two 1-twists  $T_{\circ} \in \mathcal{AT}^1(-^n)$  and  $T_{\bullet} \in \mathcal{AT}^1(+^n)$  are twin if and only if they have the same canopy. Up to reflecting  $T_{\bullet}$  with respect to the diagonal  $x = y$ , we could equivalently consider two  $(1, n)$ -twists with opposite canopy, which would be closer to S. Giraudo's presentation [Gir12]. However, when  $k > 1$  there are Cambrian  $k$ -twists  $T_{\circ} \in \mathcal{AT}^k(-^n)$  and  $T_{\bullet} \in \mathcal{AT}^k(+^n)$  with the same canopy such that the union of their contact graphs  $T_{\circ}^{\#} \cup T_{\bullet}^{\#}$  is cyclic. For example,  $\text{ins}^2(2413)$  and  $\text{ins}^2(4213)$  have the same canopy but their contact graphs has no common linear extension.

## 2.3. SCHRÖDERIZATION

Our last extension deals with all faces of the permutahedra, brick polytopes, and zonotopes. It follows F. Chapoton's ideas [Cha00] with some reformulations already used in [CP14, Part 3].

**2.3.1. Ordered partitions, hypertwists, and partial orientations.** We first recall combinatorial descriptions of the faces of  $\text{Perm}^k(n)$ ,  $\text{Brick}^k(n)$  and  $\text{Zono}^k(n)$  and describe lattice structures on these families extending the weak order on permutations on  $\mathfrak{S}_n$ , the increasing flip lattice on acyclic  $(k, n)$ -twists, and the increasing flip lattice on acyclic orientations of  $\mathbb{G}^k(n)$ .

**ORDERED PARTITIONS** Remember first that the faces of the permutahedron  $\text{Perm}^k(n)$  correspond to the *ordered partitions* of  $[n]$ . Namely, the face of  $\text{Perm}^k(n)$  corresponding to the ordered partition  $\lambda$  of  $[n]$  is the convex hull of the vertices  $\mathbf{x}(\tau)$  of  $\text{Perm}^k(n)$  given by the permutations  $\tau$  refining  $\lambda$ . We denote by  $\mathfrak{P}_n$  the set of ordered partitions of  $[n]$ .

The weak order on permutations was extended to all ordered partitions by D. Krob, M. Latapy, J.-C. Novelli, H. D. Phan and S. Schwer in [KLN<sup>+</sup>01]. See also [PR06, DHP15] for an extension to all finite Coxeter systems.

**Definition 131.** The *coinversion map*  $\text{coinv}(\lambda) : \binom{[n]}{2} \rightarrow \{-1, 0, 1\}$  of an ordered partition  $\lambda \in \mathfrak{P}_n$  is the map defined for  $i < j$  by

$$\text{coinv}(\lambda)(i, j) = \begin{cases} -1 & \text{if } \lambda^{-1}(i) < \lambda^{-1}(j), \\ 0 & \text{if } \lambda^{-1}(i) = \lambda^{-1}(j), \\ 1 & \text{if } \lambda^{-1}(i) > \lambda^{-1}(j). \end{cases}$$

The *weak order*  $\leq$  on  $\mathfrak{P}_n$  is defined by  $\lambda \leq \lambda'$  if  $\text{coinv}(\lambda)(i, j) \leq \text{coinv}(\lambda')(i, j)$  for all  $i < j$ .

The weak order on  $\mathfrak{P}_3$  is represented in Figure 31 (left). Note that the restriction of the weak order on  $\mathfrak{P}_n$  to  $\mathfrak{S}_n$  is the classical weak order on permutations, which is a lattice. This property was extended to the weak order on  $\mathfrak{P}_n$  in [KLN<sup>+</sup>01].

**Proposition 132** ([KLN<sup>+</sup>01]). *The weak order  $\leq$  on the set of ordered partitions  $\mathfrak{P}_n$  is a lattice.*

For  $X, Y \subset \mathbb{N}$ , we define the notation

$$X \ll Y \iff \max(X) < \min(Y) \iff x < y \text{ for all } x \in X \text{ and } y \in Y.$$

**Proposition 133** ([KLN<sup>+</sup>01]). *The cover relations of the weak order  $<$  on  $\mathfrak{P}_n$  are given by*

$$\begin{aligned} \lambda_1 | \cdots | \lambda_i | \lambda_{i+1} | \cdots | \lambda_k &< \lambda_1 | \cdots | \lambda_i \lambda_{i+1} | \cdots | \lambda_k && \text{if } \lambda_i \ll \lambda_{i+1}, \\ \lambda_1 | \cdots | \lambda_i \lambda_{i+1} | \cdots | \lambda_k &< \lambda_1 | \cdots | \lambda_i | \lambda_{i+1} | \cdots | \lambda_k && \text{if } \lambda_{i+1} \ll \lambda_i. \end{aligned}$$

**HYPERTWISTS** The main characters of this section are the following objects.

**Definition 134.** A *hyperpipe* is the union of pipes of a twist, where we have additionally changed their common elbows into crossings. A  *$(k, n)$ -hypertwist* is a collection of hyperpipes obtained by merging some subsets of pipes of a  $(k, n)$ -twist  $\mathbb{T}$  which are connected in the contact graph  $\mathbb{T}^\#$ . The contact graph of a  $(k, n)$ -hyperpipe  $\mathbb{H}$  is the directed multigraph  $\mathbb{H}^\#$  whose nodes are labeled by the partition of  $[n]$  induced by the hyperpipes of  $\mathbb{H}$  and with an arc from the SE-hyperpipe to the WN-hyperpipe of each elbow of  $\mathbb{H}$ . An hypertwist is *acyclic* if its contact graph is. We denote by  $\mathcal{AHT}^k(n)$  the set of acyclic  $(k, n)$ -hypertwists.

Examples of acyclic  $(k, n)$ -hypertwists appear in Figures 30, 31 and 32. Note that we could have equivalently defined the contact graph of the hypertwist  $\mathbb{H}$  as the graph obtained by contracting in  $\mathbb{T}^\#$  the subgraphs induced by the merged subsets of pipes. Observe also that each hypertwist  $\mathbb{H}$  can be obtained by merging pipes in various twists. In fact, the face of the brick polytope  $\text{Brick}^k(n)$  corresponding to  $\mathbb{H}$  is the convex hull of the brick vectors  $\mathbf{x}(\mathbb{T})$  of the  $k$ -twists  $\mathbb{T}$  refining  $\mathbb{H}$ .

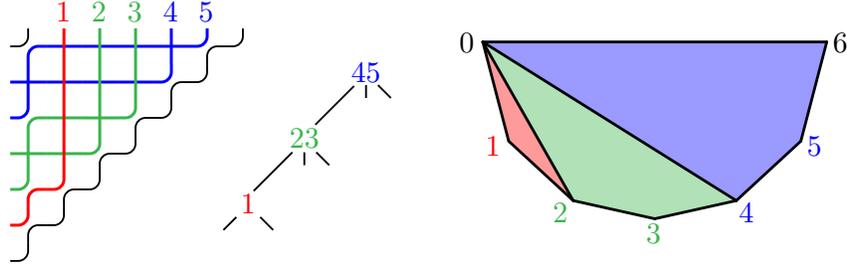


FIGURE 30. The bijection between  $(1, n)$ -hypertwists  $H$  (left) and dissections  $H^*$  of the  $(n + 2)$ -gon (right) sends the hyperpipes of  $H$  to the polygonal cells of  $H^*$ , the elbows of  $H$  to the diagonals of  $H^*$ , and the contact graph of  $H$  to the dual Schröder tree of  $H^*$  (middle).

**Example 135** (1-hypertwists, dissections and Schröder trees). The map defined in Example 5 — which sends an elbow in row  $i$  and column  $j$  of the triangular shape to the diagonal  $[i, j]$  of a convex  $(n + 2)$ -gon — defines a bijective correspondence between the  $(1, n)$ -hypertwists and the dissections of a convex  $(n + 2)$ -gon (*i.e.* the crossing-free sets of diagonals). The contact graph of a 1-hypertwist  $H$  is sent to the dual Schröder tree of  $H^*$ , as illustrated in Figure 30. As a corollary, the number of  $(1, n)$ -hypertwists with  $e$  elbows is the Schröder number

$$\frac{1}{e + 1} \binom{n + 2 + e}{e + 1} \binom{n - 1}{e + 1},$$

see [OEIS, A033282].

**Remark 136** ( $k$ -hypertwists and  $(k + 1)$ -crossing free sets of diagonals). Similarly to Example 135, the map defined in Theorem 10 — which sends an elbow in row  $i$  and column  $j$  of the  $(n + 2k) \times (n + 2k)$  triangular shape to the diagonal  $[i, j]$  of a convex  $(n + 2k)$ -gon — defines a bijective correspondence between the  $(k, n)$ -hypertwists and the subsets of  $k$ -relevant diagonals of the  $(n + 2k)$ -gon with no  $k + 1$  pairwise crossing diagonals.

**Definition 137.** Consider a  $k$ -hypertwist  $H$  and an arc  $u \rightarrow v$  of its contact graph  $H^\#$  (thus,  $u$  and  $v$  are subsets of  $[n]$ ). We denote by  $H/u \rightarrow v$  the  $k$ -hypertwist obtained by merging the hyperpipes  $u$  and  $v$ . We define the **Schröder poset**  $\leq$  on acyclic  $(k, n)$ -hypertwist as the transitive closure of the relations  $H < H/u \rightarrow v$  (resp.  $H/u \rightarrow v < H$ ) for any acyclic  $(k, n)$ -hypertwist and any arc  $u \rightarrow v \in H^\#$  such that  $u \ll v$  (resp.  $u \gg v$ ).

The Schröder poset on  $(1, 3)$ -hypertwists is represented in Figure 31. Note that the  $k$ -hypertwists  $H$  and  $H/u \rightarrow v$  are not comparable when  $u \not\ll v$  and  $u \not\gg v$ . Observe also that the restriction of the Schröder poset to the acyclic  $(k, n)$ -twists is the increasing flip lattice on acyclic  $(k, n)$ -twists: indeed,  $T < T'$  is a cover relation in the increasing flip lattice when  $T'$  is obtained by flipping an elbow  $i \rightarrow j$  in  $T$ , *i.e.* when  $T < T/i \rightarrow j = T'/j \rightarrow i < T'$  are cover relations in the Schröder poset.

We now define the insertion map  $\text{ins}^k : \mathfrak{P}_n \rightarrow \mathcal{AHT}^k(n)$  which sends ordered partitions of  $[n]$  to acyclic  $(k, n)$ -hypertwists. Given an ordered partition  $\lambda := \lambda_1 | \cdots | \lambda_p \in \mathfrak{P}_n$ , we construct a  $(k, n)$ -hypertwist  $\text{ins}^k(\lambda)$  obtained from the  $(k, 0)$ -hypertwist by inserting successively  $\lambda_p, \dots, \lambda_1$ . When we insert a new part  $\lambda_i$ , we first insert its elements in an arbitrary order and then merge the subsets of pipes of  $\lambda_i$  which induce connected subgraphs of the resulting contact graph. Note that it does not always merge all pipes of  $\lambda_i$  together ( $\lambda_i$  can have more than one connected component after the insertion step). In particular, the number of hyperpipes of  $\text{ins}^k(\lambda)$  is at least the number of parts of  $\lambda$ , but could be strictly larger. This insertion algorithm is illustrated in Figure 32.

**Proposition 138.** The map  $\text{ins}^k$  is a surjection from the ordered partitions of  $[n]$  to the acyclic  $(k, n)$ -hypertwists.

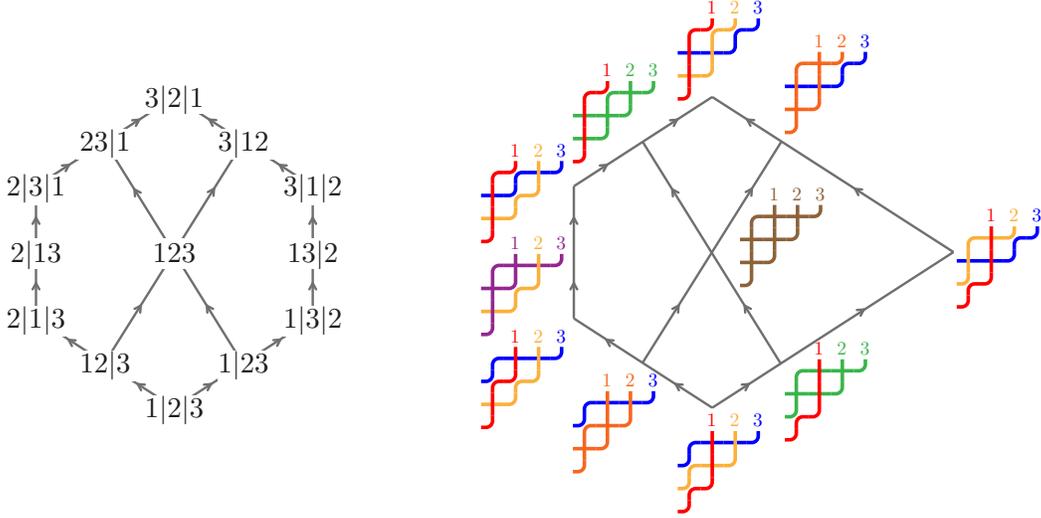


FIGURE 31. The weak order on ordered partitions of  $[3]$  (left) and the Schröder lattice on  $(1,3)$ -hypertwists (right).

*Proof.* Consider an ordered partition  $\lambda$  and a linear extension  $\tau$  of  $\lambda$ . By definition of the map  $\text{ins}^k$  on ordered partitions,  $\text{ins}^k(\lambda)$  is obtained from  $\text{ins}^k(\tau)$  by merging subsets of pipes which are connected in  $\text{ins}^k(\tau)^\#$ . Since  $\text{ins}^k(\tau)$  is acyclic by Proposition 23, it implies that  $\text{ins}^k(\lambda)$  is also acyclic. Conversely, since the new inserted nodes are sources of the contact graph at each step, any acyclic  $(k, n)$ -hypertwist  $H$  is the image of any linear extension of its contact graph  $H^\#$ .  $\square$

In fact, the fiber of a  $(k, n)$ -hypertwist  $H$  by  $\text{ins}^k$  is the set of all ordered partitions obtained from a linear extension of the contact graph  $H^\#$  by merging parts which label incomparable vertices of  $H^\#$ . We now extend the  $k$ -twist congruence on all ordered partitions.

**Definition 139.** The  $k$ -hypertwist congruence is the equivalence relation  $\equiv^k$  on  $\mathfrak{P}_n$  defined as the transitive closure of the rewriting rules

$$U|\mathbf{a}|\mathbf{c}|V \equiv^k U|\mathbf{ac}|V \equiv^k U|\mathbf{c}|\mathbf{a}|V$$

where  $\mathbf{a}, \mathbf{c}$  are parts while  $U, V$  are sequences of parts of  $[n]$ , and there exist  $b_1, \dots, b_k$  in  $\bigcup V$  such that  $\max(\mathbf{a}) < b_i < \min(\mathbf{c})$  for all  $i \in [\ell]$ .

**Proposition 140.** For any  $\lambda, \lambda' \in \mathfrak{P}_n$ , we have  $\lambda \equiv^k \lambda' \iff \text{ins}^k(\lambda) = \text{ins}^k(\lambda')$ . In other words, the fibers of  $\text{ins}^k$  are precisely the congruence classes of  $\equiv^k$ .

*Proof.* It is sufficient to prove that  $\lambda \equiv^k \lambda' \iff \text{ins}^k(\lambda) = \text{ins}^k(\lambda')$  for any two ordered partitions  $\lambda = U|\mathbf{a}|\mathbf{c}|V$  and  $\lambda' = U|\mathbf{ac}|V$  which differ by merging two consecutive parts  $\mathbf{a} \ll \mathbf{c}$  (the other case is similar). As in the proof of Proposition 27, the hyperpipes labeled by  $\mathbf{a}$  and  $\mathbf{c}$  in the insertion will share an elbow if and only if there is no  $b_1, \dots, b_k$  in  $\bigcup V$  such that  $\max(\mathbf{a}) < b_i < \min(\mathbf{c})$  for all  $i \in [\ell]$ . The result follows immediately.  $\square$

**Theorem 141.** The  $k$ -hypertwist congruence  $\equiv^k$  is a lattice congruence of the weak order on  $\mathfrak{P}_n$ .

*Proof.* We only sketch the proof as it is similar to that of Theorem 37 and of [CP14, Prop. 105]. There are two steps:

- The congruence classes of  $\equiv^k$  are intervals of the weak order on  $\mathfrak{P}_n$ . To see it, consider a  $k$ -hypertwist  $H$ . Since incomparable nodes in  $H^\#$  have comparable label sets by Proposition 140, there is a minimal  $\mu$  and a maximal  $\omega$  linear extension of  $H^\#$ . The fiber of  $H$  under  $\text{ins}^k$  is then the weak order interval  $[\mu, \omega]$  of  $\mathfrak{P}_n$ .

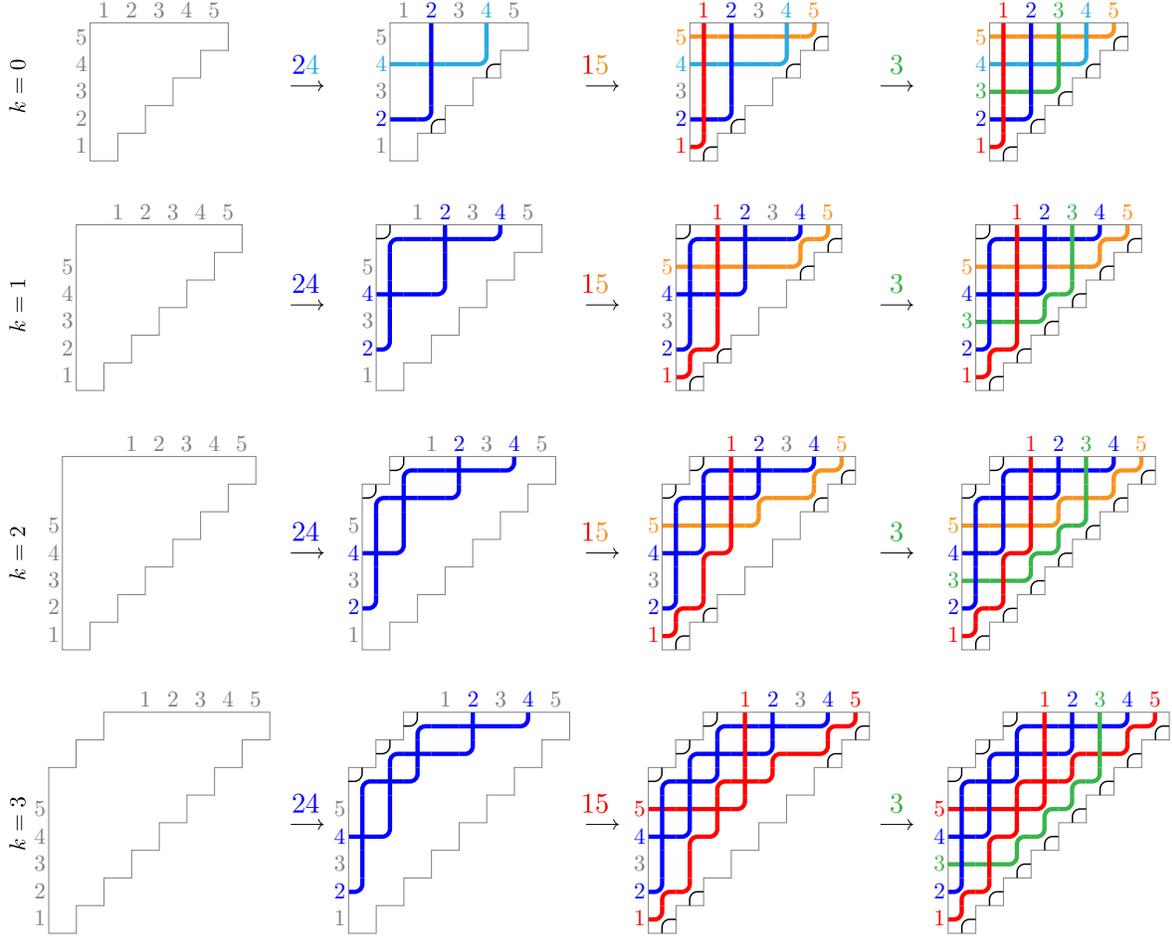


FIGURE 32. Insertion of the ordered partition  $3|15|24$  in a  $k$ -hypertwist for  $k = 0, 1, 2, 3$ .

- The up and down projection maps of the congruence classes are order preserving. The proof is very similar to that of Theorem 37, except that there are height instead of four cases. These cases are described in details in the proof of [CP14, Lemma 107] for the case  $k = 1$  in the Cambrian setting. It then suffices to consider only the signature  $-n$  and to replace the Schröder Cambrian witness  $b$  by  $k$ -twist congruence witnesses  $b_1, \dots, b_k$ .  $\square$

**Proposition 142.** *The map  $\text{ins}^k : \mathfrak{P}_n \rightarrow \mathcal{AHT}^k(n)$  defines an isomorphism from the lattice quotient of the weak order on  $\mathfrak{P}_n$  by the  $k$ -hypertwist congruence  $\equiv^k$  to the Schröder poset on acyclic  $(k, n)$ -hypertwists. In particular, the Schröder poset defines in fact a lattice.*

*Proof.* Consider a cover relation  $\lambda < \lambda'$  in the weak order on  $\mathfrak{P}_n$ . Assume that  $\lambda'$  is obtained by merging two parts  $\lambda_i \ll \lambda_{i+1}$  of  $\lambda$  (the other cover relation is treated similarly). Let  $u$  and  $v$  be the nodes of  $\text{ins}^k(\lambda)$  containing  $\max(\lambda_i)$  and  $\min(\lambda_{i+1})$  respectively. If  $u$  and  $v$  are incomparable in  $\text{ins}^k(\lambda)^\#$ , then  $\text{ins}^k(\lambda) = \text{ins}^k(\lambda')$ . Otherwise, there is an arc from  $u$  to  $v$  in  $\text{ins}^k(\lambda)^\#$ , and  $\text{ins}^k(\lambda)' = \text{ins}^k(\lambda)_{u \rightarrow v}$ .  $\square$

**ACYCLIC PARTIAL ORIENTATIONS** Finally, we recall that the faces of a graphical zonotope  $\text{Zono}(G)$  correspond to *acyclic partial orientations* of  $G$ , i.e. to partial orientations of  $G$  where

- the unoriented edges form connected induced subgraphs of  $G$ ,
- the oriented graph obtained by contracting the unoriented edges is acyclic.

The face of  $\text{Zono}(G)$  corresponding to a partial orientation  $O$  is the convex hull of the vertices  $\mathbf{x}(O)$  of  $\text{Zono}(G)$  given by the acyclic orientations of  $G$  refining  $O$ . We denote by  $\mathcal{APO}^k(n)$  the set of all acyclic partial orientations of  $G^k(n)$ . We can also think of a partial orientation of  $G^k(n)$  as a map  $O : \{(i, j) \in [n]^2 \mid i < j \leq i + k\} \rightarrow \{-1, 0, 1\}$  where  $O(i, j) = -1$  if the edge  $\{i, j\}$  is oriented from  $j$  to  $i$ ,  $O(i, j) = 0$  if the edge remains unoriented, and  $O(i, j) = 1$  if the edge is oriented from  $i$  to  $j$ . We then define a lattice structure on  $\mathcal{APO}^k(n)$  by  $O \leq O'$  if  $O(i, j) \leq O'(i, j)$  for all  $i < j \leq i + k$ .

We define the *k-recoils scheme* of an ordered partition  $\lambda \in \mathfrak{P}_n$  as the acyclic partial orientation  $\text{rec}^k(\lambda)$  defined by the restriction of the coinversions of  $\lambda$  to  $\{(i, j) \in [n]^2 \mid i < j \leq i + k\}$ , and the *k-recoils scheme* of a  $(k, n)$ -hypertwist  $H$  as the partial orientation  $\text{can}^k(H)$  with an edge  $i \rightarrow j$  if there is an oriented path in the contact graph  $H^\#$  from the node containing  $i$  to the node containing  $j$ . The next statement is an immediate consequence of these definitions.

**Proposition 143.** *The maps  $\text{ins}^k$ ,  $\text{can}^k$ , and  $\text{rec}^k$  define a commutative diagram of lattice homomorphisms:*

$$\begin{array}{ccc} \mathfrak{P}_n & \xrightarrow{\text{rec}^k} & \mathcal{APO}^k(n) \\ & \searrow \text{ins}^k & \nearrow \text{can}^k \\ & \mathcal{AHT}^k(\varepsilon) & \end{array}$$

**2.3.2. Algebra of acyclic hypertwists.** We now define Hopf algebras whose bases are indexed by ordered partitions, acyclic  $k$ -hypertwists, and acyclic partial orientations of  $G^k(n)$ . To begin with, we reformulate as in [CP14] the Hopf algebra of F. Chapoton [Cha00] indexed by the faces of the permutahedra in terms of ordered partitions instead of surjections.

We first define two restrictions on ordered partitions. Consider an ordered partition  $\mu$  of  $[n]$  into  $p$  parts. For  $I \subseteq [p]$ , we let  $n_I := |\{j \in [n] \mid \exists i \in I, j \in \mu_i\}|$  and we denote by  $\mu|_I$  the ordered partition of  $[n_I]$  into  $|I|$  parts obtained from  $\mu$  by deletion of the parts in  $[p] \setminus I$  and standardization. Similarly, for  $J \subseteq [n]$ , we let  $p_J := |\{i \in [k] \mid \exists j \in J, j \in \mu_i\}|$  and we denote by  $\mu|^{J,k}$  the ordered partition of  $[|J|]$  into  $kp_J$  parts obtained from  $\mu$  by deletion of the entries in  $[n] \setminus J$  and standardization. For example, for the ordered partition  $\mu = 16|27|4|35$  we have  $\mu|_{\{2,3\}} = 13|2$  and  $\mu|^{1,3,5} = 1|23$ .

The *shifted shuffle product*  $\lambda \sqcup \lambda'$  and the *convolution product*  $\lambda \star \lambda'$  of two ordered partitions  $\lambda \in \mathfrak{P}_n$  and  $\lambda' \in \mathfrak{P}_{n'}$  are defined by:

$$\begin{aligned} \lambda \sqcup \lambda' &:= \{\mu \in \mathfrak{P}_{n+n'} \mid \mu|_{\{1, \dots, n\}} = \lambda \text{ and } \mu|_{\{n+1, \dots, n+n'\}} = \lambda'\}, \\ \text{and } \lambda \star \lambda' &:= \{\mu \in \mathfrak{P}_{n+n'} \mid \mu|_{\{1, \dots, k\}} = \lambda \text{ and } \mu|_{\{k+1, \dots, k+k'\}} = \lambda'\}. \end{aligned}$$

Note that if  $\lambda$  has  $p$  parts and  $\lambda'$  has  $p'$  parts, then the partitions of  $\lambda \sqcup \lambda'$  have at most  $p + p'$  parts, while the partitions of  $\lambda \star \lambda'$  have precisely  $p + p'$  parts.

For example,

$$\begin{aligned} 1|2 \sqcup 2|13 &= \{1|2|4|35, 1|24|35, 1|4|2|35, 1|4|235, 1|4|35|2, 14|2|35, 14|235, \\ &\quad 14|35|2, 4|1|2|35, 4|1|235, 4|1|35|2, 4|135|2, 4|35|1|2\}, \\ 1|2 \star 2|13 &= \{1|2|4|35, 1|3|4|25, 1|4|3|25, 1|5|3|24, 2|3|4|15, \\ &\quad 2|4|3|15, 2|5|3|14, 3|4|2|15, 3|5|2|14, 4|5|2|13\}. \end{aligned}$$

We denote by  $\text{OrdPart}$  the Hopf algebra with basis  $(\mathbb{F}_\lambda)_{\lambda \in \mathfrak{P}}$  and whose product and coproduct are defined by

$$\mathbb{F}_\lambda \cdot \mathbb{F}_{\lambda'} = \sum_{\mu \in \lambda \sqcup \lambda'} \mathbb{F}_\mu \quad \text{and} \quad \Delta \mathbb{F}_\mu = \sum_{\mu \in \lambda \star \lambda'} \mathbb{F}_\lambda \otimes \mathbb{F}_{\lambda'}.$$

This indeed defines a Hopf algebra according to the work of F. Chapoton [Cha00].

We then construct two Hopf subalgebras of  $\text{OrdPart}$  indexed by acyclic  $(k, n)$ -hypertwists and acyclic partial orientations of  $G^k(n)$  for  $n \in \mathbb{N}$ . We denote by  $\text{HyperTwist}^k$  the vector subspace

of  $\text{OrdPart}$  generated by the elements

$$\mathbb{P}_H := \sum_{\substack{\lambda \in \mathfrak{P} \\ \text{ins}^k(\lambda) = H}} \mathbb{F}_\lambda,$$

for all acyclic  $k$ -hypertwists  $H$ . For example, for the  $(k, 5)$ -hypertwists of Figure 32, we have

$$\begin{aligned} \mathbb{P}_{\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}} &= \sum_{\lambda \in \mathfrak{P}_5} \mathbb{F}_\lambda & \mathbb{P}_{\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}} &= \mathbb{F}_{1|3|5|24} + \mathbb{F}_{1|5|3|24} + \mathbb{F}_{3|1|5|24} + \mathbb{F}_{5|1|3|24} + \mathbb{F}_{3|5|1|24} + \mathbb{F}_{5|3|1|24} \\ & & &+ \mathbb{F}_{1|35|24} + \mathbb{F}_{3|15|24} + \mathbb{F}_{5|13|24} + \mathbb{F}_{13|5|24} + \mathbb{F}_{15|3|24} + \mathbb{F}_{35|1|24} \\ & & &+ \mathbb{F}_{135|24} \\ \mathbb{P}_{\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}} &= \mathbb{F}_{3|1|5|24} + \mathbb{F}_{3|15|24} + \mathbb{F}_{3|5|1|24} & \text{and} & \mathbb{P}_{\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}} &= \mathbb{F}_{3|15|24}. \end{aligned}$$

Moreover, we denote by  $\text{HyperRec}^k$  the vector subspace of  $\text{OrdPart}$  generated by the elements

$$\mathbb{X}_O := \sum_{\substack{\lambda \in \mathfrak{P} \\ \text{rec}^k(\lambda) = O}} \mathbb{F}_\lambda = \sum_{\substack{H \in \mathcal{AHT}^k \\ \text{can}^k(H) = O}} \mathbb{P}_H,$$

for all acyclic partial orientations  $O$  of the graph  $G^k(n)$  (for all  $n \in \mathbb{N}$ ). The proof of the following statement is again a straightforward verification.

**Theorem 144.** *HyperTwist<sup>k</sup> and HyperRec<sup>k</sup> are both Hopf subalgebras of OrdPart, and we have the inclusions  $\text{HyperRec}^k \subset \text{HyperTwist}^k \subset \text{OrdPart}$ .*

As in Section 1.3.2, one can describe combinatorially the product and coproduct in  $\text{HyperTwist}^k_{\pm}$  directly in terms of  $k$ -hypertwists. In particular, the product  $\mathbb{P}_H \cdot \mathbb{P}_{H'}$  in  $\text{HyperTwist}^k$  of two  $k$ -hypertwists  $H \in \mathcal{AHT}^k(n)$  and  $H' \in \mathcal{AHT}^k(n')$  is given by

$$\mathbb{P}_H \cdot \mathbb{P}_{H'} = \sum_{K} \mathbb{P}_K,$$

where the sum runs over the interval of the Schröder lattice between the  $(k, n+n')$ -hypertwists  $H \setminus H'$  and  $H/H'$ , respectively obtained by inserting  $H$  in the first rows and columns of  $H'$ , and by inserting  $H'$  in the last rows and columns of  $H$ . We invite the reader to work out the combinatorial descriptions of the coproduct in  $\text{HyperTwist}^k$  and of the product and coproduct in the dual Hopf algebra  $\text{HyperTwist}^{k*}$ , which are more subtle than the product (when performing cuts, one should be careful which hyperpipes can be merge).

#### ACKNOWLEDGEMENTS

I am grateful to N. Bergeron and C. Ceballos for sharing their preliminary results on Hopf algebras on pipe dreams, to N. Reading for pointing out relevant references and sharing his expertise on lattice congruences, and to F. Hivert for very interesting discussions on monoids, Hopf algebras, and dendriform structures.

#### REFERENCES

- [AS06] Marcelo Aguiar and Frank Sottile. Structure of the Loday-Ronco Hopf algebra of trees. *J. Algebra*, 295(2):473–511, 2006.
- [BB93] Nantel Bergeron and Sara Billey. RC-graphs and Schubert polynomials. *Experiment. Math.*, 2(4):257–269, 1993.
- [BBMF11] Nicolas Bonichon, Mireille Bousquet-Mélou, and Éric Fusy. Baxter permutations and plane bipolar orientations. *Sém. Lothar. Combin.*, 61A:Art. B61Ah, 29, 2009/11.
- [BC14] Nantel Bergeron and Cesar Ceballos, 2014. Personal communication, Fields Institute, Toronto.
- [BW91] Anders Björner and Michelle L. Wachs. Permutation statistics and linear extensions of posets. *J. Combin. Theory Ser. A*, 58(1):85–114, 1991.
- [CGHK78] Fan R. K. Chung, Ronald L. Graham, Verner E. Hoggatt, Jr., and Mark Kleiman. The number of Baxter permutations. *J. Combin. Theory Ser. A*, 24(3):382–394, 1978.
- [Cha00] Frédéric Chapoton. Algèbres de Hopf des permutahédres, associahédres et hypercubes. *Adv. Math.*, 150(2):264–275, 2000.

- [CLS14] Cesar Ceballos, Jean-Philippe Labbé, and Christian Stump. Subword complexes, cluster complexes, and generalized multi-associahedra. *J. Algebraic Combin.*, 39(1):17–51, 2014.
- [CP92] Vasilis Capovleas and János Pach. A Turán-type theorem on chords of a convex polygon. *J. Combin. Theory Ser. B*, 56(1):9–15, 1992.
- [CP14] Grégory Chatel and Vincent Pilaud. Cambrian Hopf Algebras. Preprint, [arXiv:1411.3704](https://arxiv.org/abs/1411.3704), 2014.
- [DG96] Serge Dulucq and Olivier Guibert. Stack words, standard tableaux and Baxter permutations. In *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, volume 157, pages 91–106, 1996.
- [DG98] Serge Dulucq and Olivier Guibert. Baxter permutations. In *Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995)*, volume 180, pages 143–156, 1998.
- [DHP15] Aram Dermenjian, Christophe Hohlweg, and Vincent Pilaud. Lattice structures on all faces of the permutahedra and associahedra. Work in progress, 2015.
- [DKM02] Andreas W. M. Dress, Jacobus H. Koolen, and Vincent Moulton. On line arrangements in the hyperbolic plane. *European J. Combin.*, 23(5):549–557, 2002.
- [FFNO11] Stefan Felsner, Éric Fusy, Marc Noy, and David Orden. Bijections for Baxter families and related objects. *J. Combin. Theory Ser. A*, 118(3):993–1020, 2011.
- [Gir12] Samuele Giraudo. Algebraic and combinatorial structures on pairs of twin binary trees. *J. Algebra*, 360:115–157, 2012.
- [GV85] Ira Gessel and Gérard Viennot. Binomial determinants, paths, and hook length formulae. *Adv. in Math.*, 58(3):300–321, 1985.
- [Hiv04] Florent Hivert. *Combinatoire et calcul symbolique dans les algèbres de Hopf*. Habilitation à diriger des recherches, Université de Marne-la-Vallée, 2004. Available online <https://www.lri.fr/~hivert/PAPER/Habil.ps>.
- [HL07] Christophe Hohlweg and Carsten Lange. Realizations of the associahedron and cyclohedron. *Discrete Comput. Geom.*, 37(4):517–543, 2007.
- [HN07] Florent Hivert and Janvier Nzeutchap. Dual graded graphs in combinatorial hopf algebras. Unpublished, 2007.
- [HNT05] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon. The algebra of binary search trees. *Theoret. Comput. Sci.*, 339(1):129–165, 2005.
- [HP15] Florent Hivert and Vincent Pilaud. Multitwistiform algebras and operads. Work in progress, 2015.
- [IO13] Kiyoshi Igusa and Jonah Ostroff. Mixed cobinary trees. Preprint, [arXiv:1307.3587](https://arxiv.org/abs/1307.3587), 2013.
- [Jon05] Jakob Jonsson. Generalized triangulations and diagonal-free subsets of stack polyominoes. *J. Combin. Theory Ser. A*, 112(1):117–142, 2005.
- [KLN<sup>+</sup>01] Daniel Krob, Matthieu Latapy, Jean-Christophe Novelli, Ha-Duong Phan, and Sylviane Schwer. Pseudo-Permutations I: First Combinatorial and Lattice Properties. In *13th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2001)*. 2001.
- [KM04] Allen Knutson and Ezra Miller. Subword complexes in Coxeter groups. *Adv. Math.*, 184(1):161–176, 2004.
- [KM05] Allen Knutson and Ezra Miller. Gröbner geometry of Schubert polynomials. *Ann. of Math. (2)*, 161(3):1245–1318, 2005.
- [Lod01] Jean-Louis Loday. Dialgebras. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 7–66. Springer, Berlin, 2001.
- [Lod04] Jean-Louis Loday. Realization of the Stasheff polytope. *Arch. Math. (Basel)*, 83(3):267–278, 2004.
- [LP13] Carsten Lange and Vincent Pilaud. Using spines to revisit a construction of the associahedron. Preprint, [arXiv:1307.4391](https://arxiv.org/abs/1307.4391), to appear in *Combinatorica*, 2013.
- [LR98] Jean-Louis Loday and María O. Ronco. Hopf algebra of the planar binary trees. *Adv. Math.*, 139(2):293–309, 1998.
- [LR12] Shirley Law and Nathan Reading. The Hopf algebra of diagonal rectangulations. *J. Combin. Theory Ser. A*, 119(3):788–824, 2012.
- [MHPS12] Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff, editors. *Associahedra, Tamari Lattices and Related Structures. Tamari Memorial Festschrift*, volume 299 of *Progress in Mathematics*. Birkhäuser, Basel, 2012.
- [MR95] Claudia Malvenuto and Christophe Reutenauer. Duality between quasi-symmetric functions and the Solomon descent algebra. *J. Algebra*, 177(3):967–982, 1995.
- [Nak00] Tomoki Nakamigawa. A generalization of diagonal flips in a convex polygon. *Theoret. Comput. Sci.*, 235(2):271–282, 2000.
- [NRT11] Jean-Christophe Novelli, Christophe Reutenauer, and Jean-Yves Thibon. Generalized descent patterns in permutations and associated Hopf algebras. *European J. Combin.*, 32(4):618–627, 2011.
- [OEIS] The On-Line Encyclopedia of Integer Sequences. Published electronically at <http://oeis.org>, 2010.
- [Pil10] Vincent Pilaud. *Multitriangulations, pseudotriangulations and some problems of realization of polytopes*. PhD thesis, Université Paris 7 & Universidad de Cantabria, 2010. Available online [arXiv:1009.1605](https://arxiv.org/abs/1009.1605).
- [PP12] Vincent Pilaud and Michel Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. *Discrete Comput. Geom.*, 48(1):142–191, 2012.

- [PR06] Patricia Palacios and María O. Ronco. Weak Bruhat order on the set of faces of the permutohedron and the associahedron. *J. Algebra*, 299(2):648–678, 2006.
- [Pri13] Jean-Baptiste Priez. A lattice of combinatorial Hopf algebras, Application to binary trees with multiplicities. Preprint [arXiv:1303.5538](https://arxiv.org/abs/1303.5538). Preliminary version in *25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013)*, 2013.
- [PS09] Vincent Pilaud and Francisco Santos. Multitriangulations as complexes of star polygons. *Discrete Comput. Geom.*, 41(2):284–317, 2009.
- [PS12] Vincent Pilaud and Francisco Santos. The brick polytope of a sorting network. *European J. Combin.*, 33(4):632–662, 2012.
- [PS13] Vincent Pilaud and Christian Stump. EL-labelings and canonical spanning trees for subword complexes. In *Discrete geometry and optimization*, volume 69 of *Fields Inst. Commun.*, pages 213–248. Springer, New York, 2013.
- [PS15] Vincent Pilaud and Christian Stump. Brick polytopes of spherical subword complexes and generalized associahedra. *Adv. Math.*, 276:1–61, 2015.
- [Rea04] Nathan Reading. Lattice congruences of the weak order. *Order*, 21(4):315–344, 2004.
- [Rea05] Nathan Reading. Lattice congruences, fans and Hopf algebras. *J. Combin. Theory Ser. A*, 110(2):237–273, 2005.
- [Rea06] Nathan Reading. Cambrian lattices. *Adv. Math.*, 205(2):313–353, 2006.
- [Sol76] Louis Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264, 1976.
- [SS12] Luis Serrano and Christian Stump. Maximal fillings of moon polyominoes, simplicial complexes, and Schubert polynomials. *Electron. J. Combin.*, 19, 2012.
- [Stu11] Christian Stump. A new perspective on  $k$ -triangulations. *J. Combin. Theory Ser. A*, 118(6):1794–1800, 2011.
- [Vie07] Xavier Viennot. Catalan tableaux and the asymmetric exclusion process. In *19th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2007)*. 2007.
- [Woo04] Alexander Woo. Catalan numbers and schubert polynomials for  $w = 1(n + 1) \dots 2$ . Preprint, [arXiv:math/0407160](https://arxiv.org/abs/math/0407160), 2004.
- [YCCG03] Bo Yao, Hongyu Chen, Chung-Kuan Cheng, and Ronald Graham. Floorplan Representations: Complexity and connections. *ACM Transactions on Design Automation of Electronic Systems*, 8(1):55–80, 2003.
- [Zie98] Günter M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate texts in Mathematics*. Springer-Verlag, New York, 1998.

CNRS & LIX, ÉCOLE POLYTECHNIQUE, PALAISEAU  
 E-mail address: [vincent.pilaud@lix.polytechnique.fr](mailto:vincent.pilaud@lix.polytechnique.fr)  
 URL: <http://www.lix.polytechnique.fr/~pilaud/>