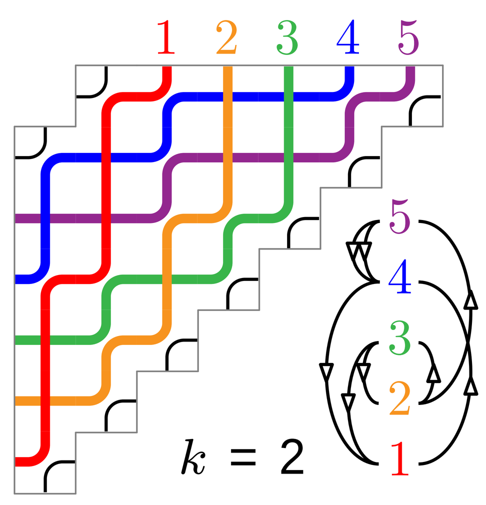
Brick polytopes — Lattices — Hopf algebras

Vincent Pilaud (CNRS & LIX, École Polytechnique)

Acyclic twists

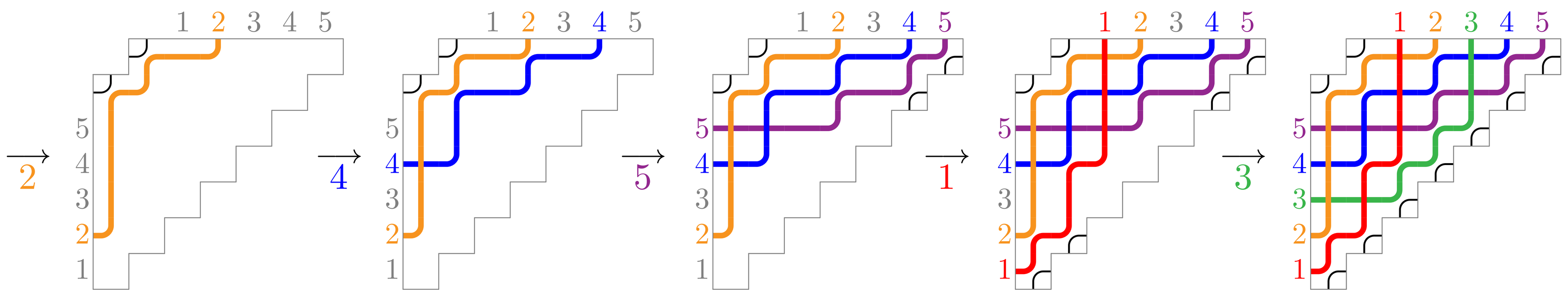
(k,n)-twist = pipe dream in the trapezoidal shape of height n and width k

Contact graph of a twist T = graph $\mathrm{T}^\#$ with
- a vertex for each pipe of T
- an arc for each contact of T from the SE pipe to the NW pipe



Acyclic twist = when its contact graph has no oriented cycle

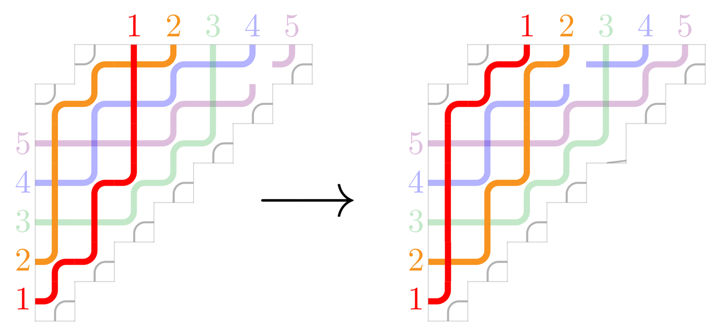
Insertion algorithm
surjection ins^k: permutations of $[n] \longrightarrow$ acyclic (k,n)-twists
algo: insert pipes from right to left as northwest as possible



k-twist congruence = transitive closure of
$$UacV_1b_1V_2b_2\cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2\cdots V_kb_kW$$
where $a < b_i < c$ for all $i \in [k]$

PROP. $\tau \equiv^k \tau' \iff \mathsf{ins}^k(\tau) = \mathsf{ins}^k(\tau') = \mathrm{T} \iff \tau, \tau' \in \mathcal{L}(\mathrm{T}^\#)$

Flip = exchange an elbow with its corresponding crossing
Increasing flip = elbow SE crossing

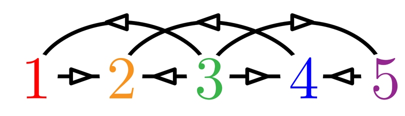


Acyclic orientations

$\mathrm{G}^k(n)$ = graph with vertices $[n]$ & edges $\{i,j\}$ for $i < j \le i+k$

k-recoil scheme of $\tau \in \mathfrak{S}_n$ = acyclic orientation $\mathsf{rec}^k(\tau)$ of $\mathrm{G}^k(n)$ with edge $i \to j$ when $|i-j| \le k$ and $\tau^{-1}(i) < \tau^{-1}(j)$

k-recoil congruence = transitive closure of
$UacV \approx^k UcaV$ if $a+k<c$



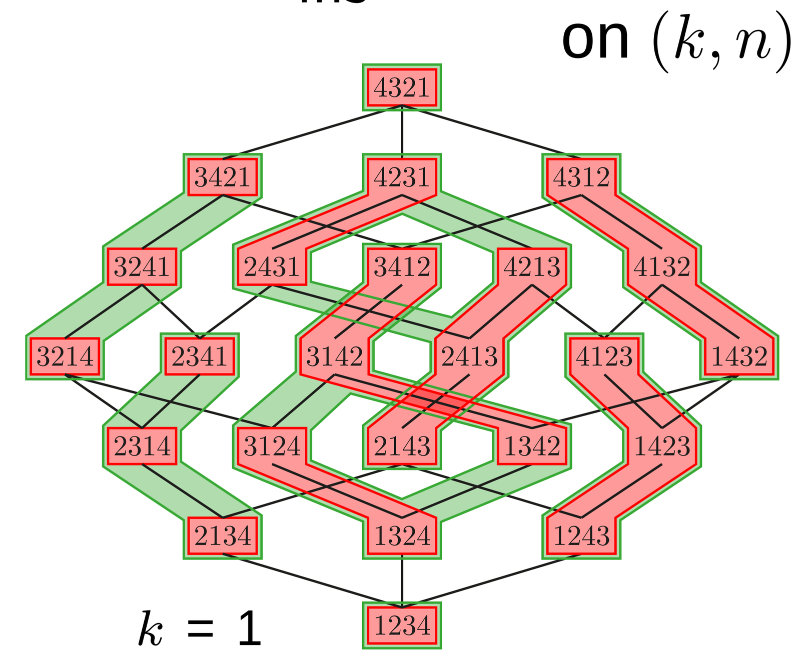
PROP. $\tau \approx^k \tau' \iff \mathsf{rec}^k(\tau) = \mathsf{rec}^k(\tau') = \mathrm{O} \iff \tau, \tau' \in \mathcal{L}(\mathrm{O})$

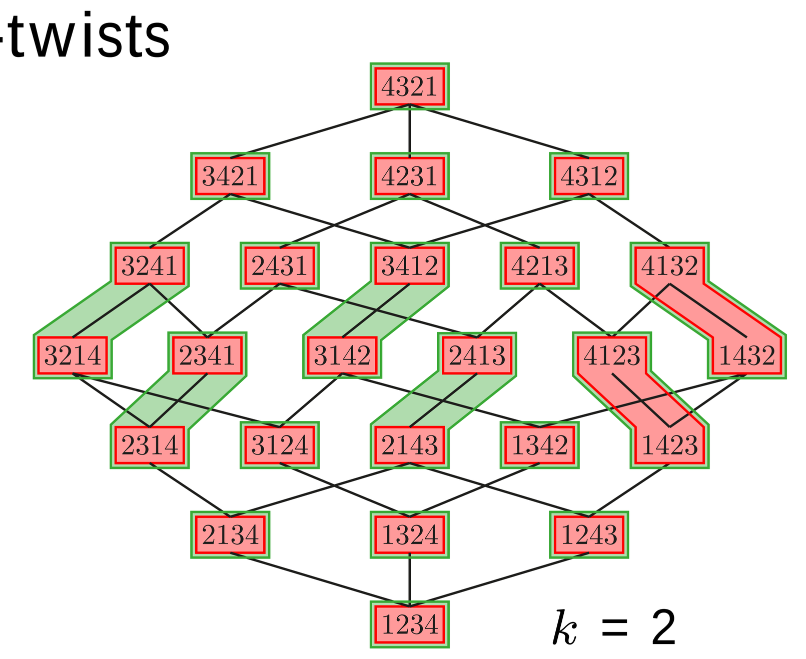
Canopy of (k,n)-twist T = acyclic orientation $\mathsf{can}^k(\mathrm{T})$ of $\mathrm{G}^k(n)$ with edge $i \to j$ when $|i-j| \le k$ and i below j in $\mathrm{T}^\#$

Lattice homomorphisms

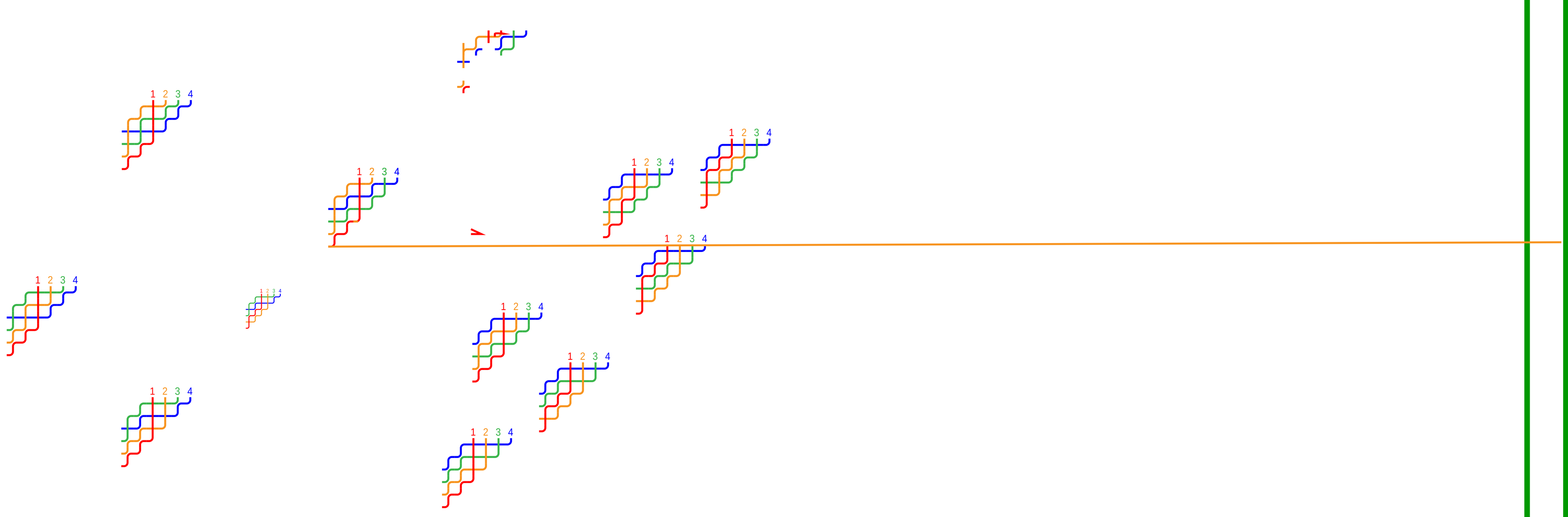
weak order on permutations of $[n]$ $\xrightarrow{\mathsf{rec}^k}$ boolean lattice on acyclic orient. of $\mathrm{G}^k(n)$

ins^k ↘ increasing flip order on (k,n)-twists ↗ can^k





Permutahedron — Brick polytope — Zonotope



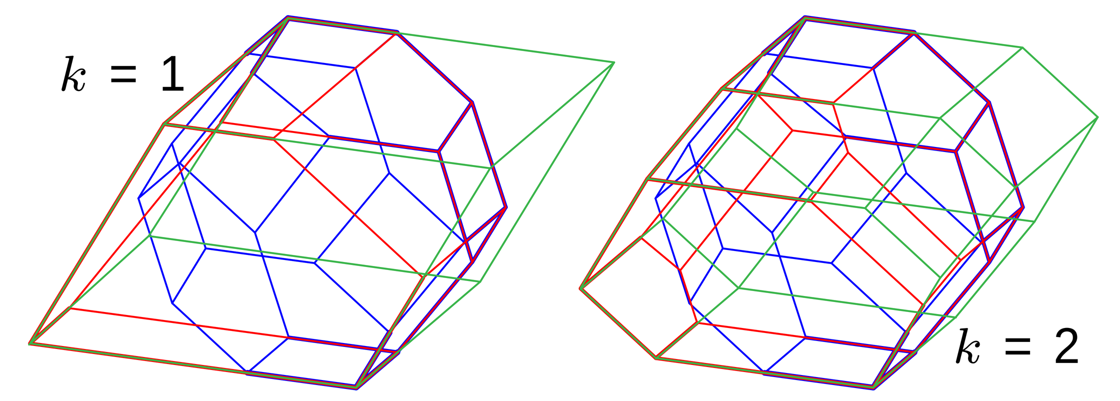
Perm(n)
conv $\{\mathsf{x}(\tau) \mid \tau \in \mathfrak{S}_n\}$
$\mathsf{x}(\tau)_i = \tau(i)$

Brick$^k(n)$
conv $\{\mathsf{b}(\mathrm{T}) \mid \mathrm{T}$ (k,n)-twist$\}$
$\mathsf{b}(\mathrm{T})_i$ = nbr bricks below pipe i

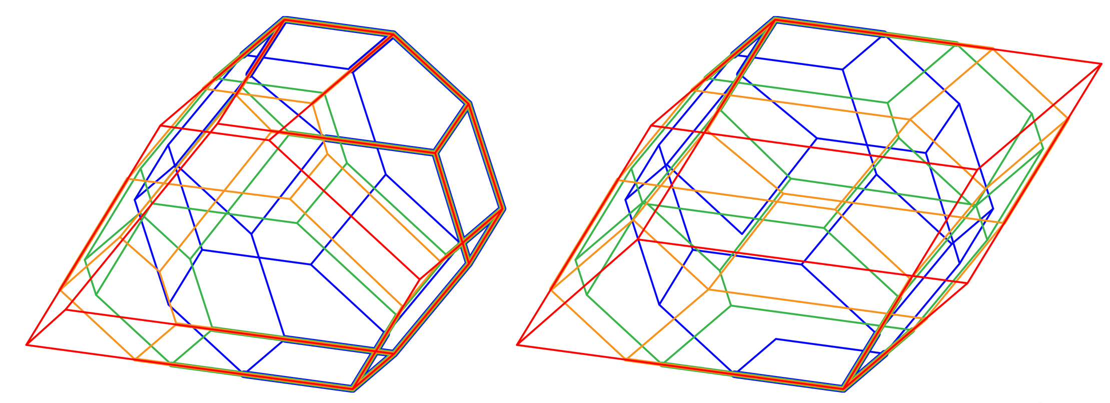
Zono$^k(n)$
$\sum_{j-i \le k,\ i<j} [\mathsf{e}_i, \mathsf{e}_j]$

Matriochka polytopes

Perm(n) ⊂ Brick$^k(n)$ ⊂ Zono$^k(n)$



Brick$^1(n)$ ⊂ Brick$^2(n)$ ⊂ Brick$^3(n)$ ⊂ ⋯ ⊂ Perm$^k(n)$



Zono$^1(n)$ ⊂ Zono$^2(n)$ ⊂ Zono$^3(n)$ ⊂ ⋯ ⊂ Perm$^k(n)$

Hopf algebras

Malvenuto-Reutenauer Hopf algebra = basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ and

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \sqcup\!\sqcup \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

$12 \sqcup\!\sqcup 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$
$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$

k-twist algebra = subalgebra of MR algebra generated by
$\mathbb{P}_\mathrm{T} := \sum_{\mathsf{ins}^k(\tau)=\mathrm{T}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T}^\#)} \mathbb{F}_\tau$ for all acyclic k-twists T

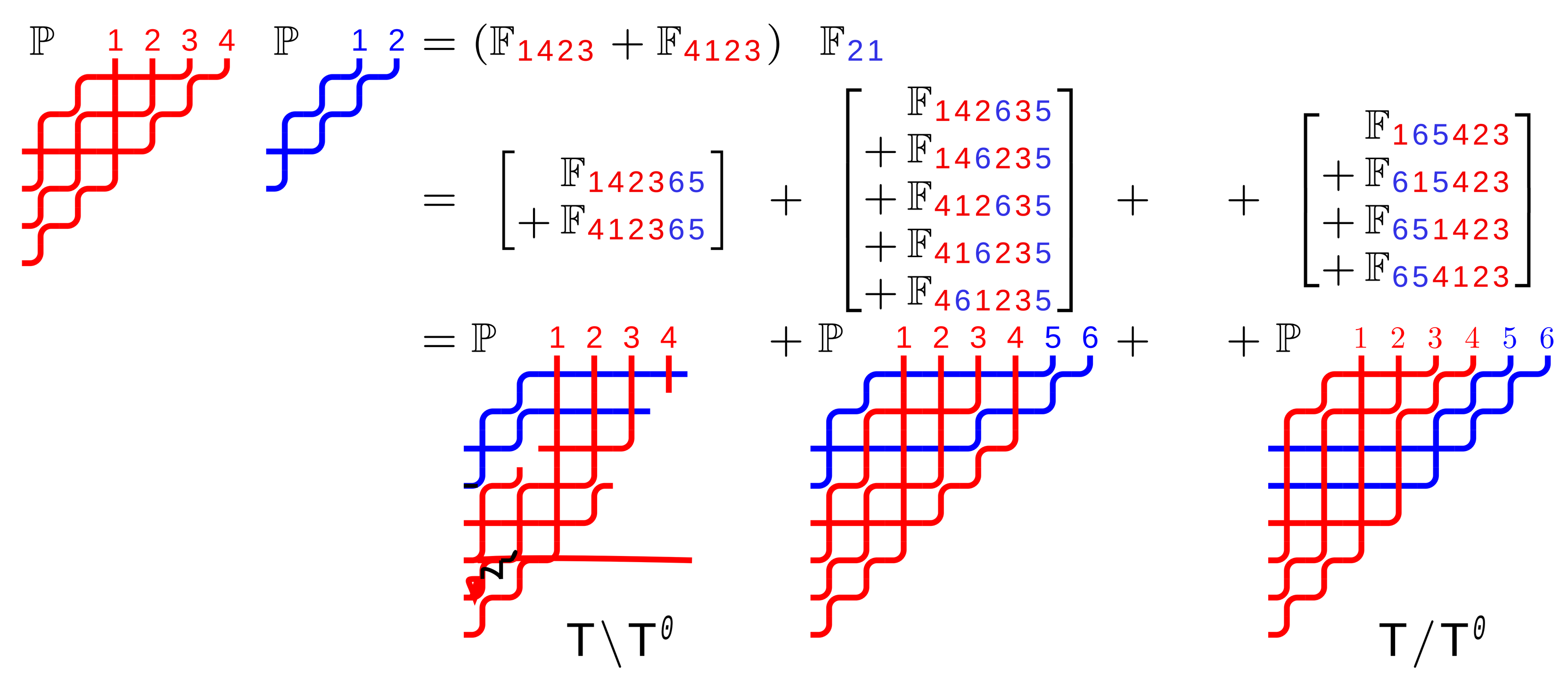
Exm: $k=1 \implies$ Loday-Ronco Hopf algebra on binary trees

k-recoil algebra = subalgebra of MR algebra generated by
$\mathbb{X}_\mathrm{O} := \sum_{\mathsf{rec}^k(\tau)=\mathrm{O}} \mathbb{F}_\tau$ for all acyclic orientations O of $\mathrm{G}^k(n)$ for $n \in \mathbb{N}$

Exm: $k=1 \implies$ Solomon descent algebra

Matriochkas: • k-recoil alg. ↪ k-twist alg. ↪ MR alg.
• if $\ell < k$, ℓ-rec ↪ k-rec and ℓ-twist ↪ k-twist

Products



PROP. $\mathbb{P}_\mathrm{T} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_\mathrm{S} \mathbb{P}_\mathrm{S}$ where S runs over the interval between $\mathrm{T}\backslash\mathrm{T}'$ and T/T' in the $(k, n+n')$-twist lattice

Further topics...

arXiv:1505.07665

Multiplicative bases, k-twistiform algebras, ...
Cambrian, tuples, Schröder extensions