EL-labelings and canonical spanning trees for subword complexes

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SUBWORD COMPLEXES

(W, S) finite Coxeter system, $\mathrm{Q} = \mathrm{q}_1\mathrm{q}_2\cdots\mathrm{q}_m \in S^*$, and $\rho \in W$.

Subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ = simplicial complex with

- vertices $= [m]$ = positions in Q,
- facets $= \mathcal{F}(\mathrm{Q}, \rho)$ = complements of reduced expressions of ρ in Q.

Exm. $\mathrm{Q}^{\mathrm{ex}} = \tau_2\tau_3\tau_1\tau_3\tau_2\tau_1\tau_2\tau_3\tau_1$ in $(\mathfrak{S}_4, \{(i\ \ i+1)\})$

$\rho^{\mathrm{ex}} = [4,1,3,2] = \tau_2\tau_3\tau_2\tau_1 = \tau_3\tau_2\tau_3\tau_1 = \tau_3\tau_2\tau_1\tau_3$

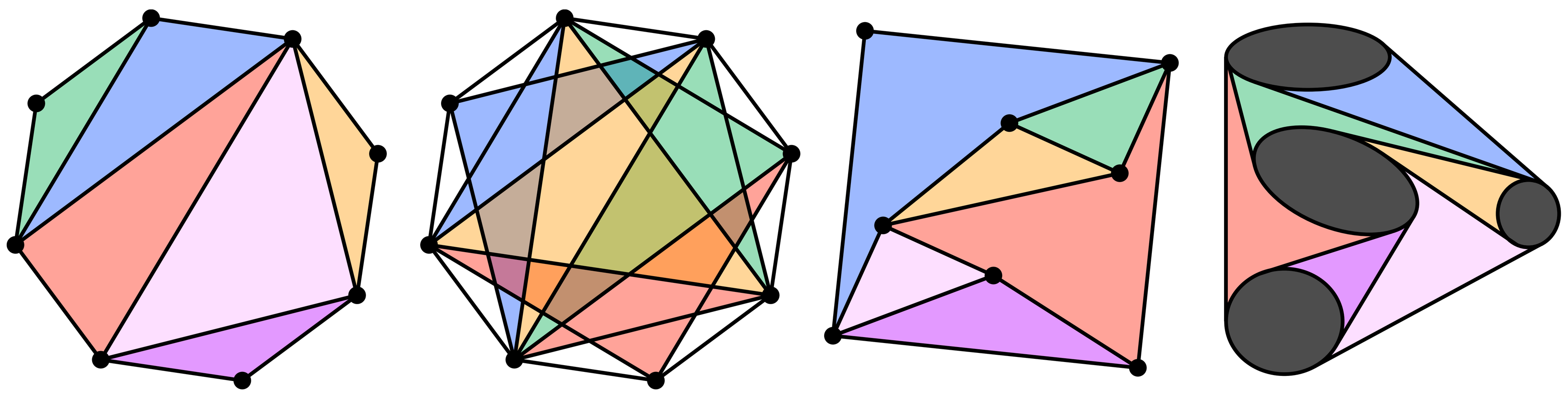
$\mathcal{F}(\mathrm{Q}^{\mathrm{ex}}, \rho^{\mathrm{ex}}) = \{1,2,3,5,6\}, \{1,2,3,6,7\}, \{1,2,3,7,9\}, \{1,3,4,5,6\}, \{1,3,4,6,7\}, \{1,3,4,7,9\}, \dots$

Inductive structure: if $\mathrm{Q}_{\dashv} = \mathrm{q}_1 \cdots \mathrm{q}_{m-1}$, then

$$\mathcal{F}(\mathrm{Q}, \rho) = \mathcal{F}(\mathrm{Q}_{\dashv}, \rho q_m) \sqcup \big(\mathcal{F}(\mathrm{Q}_{\dashv}, \rho) \star m\big).$$

Theo. [KM04] *The subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ is either a simplicial sphere or a simplicial ball.*

Type A spherical subword complexes provide combinatorial models for families of geometric objects:



A. Knutson and E. Miller. Subword complexes in Coxeter groups. 2004.

EL-LABELINGS OF GRAPHS AND POSETS

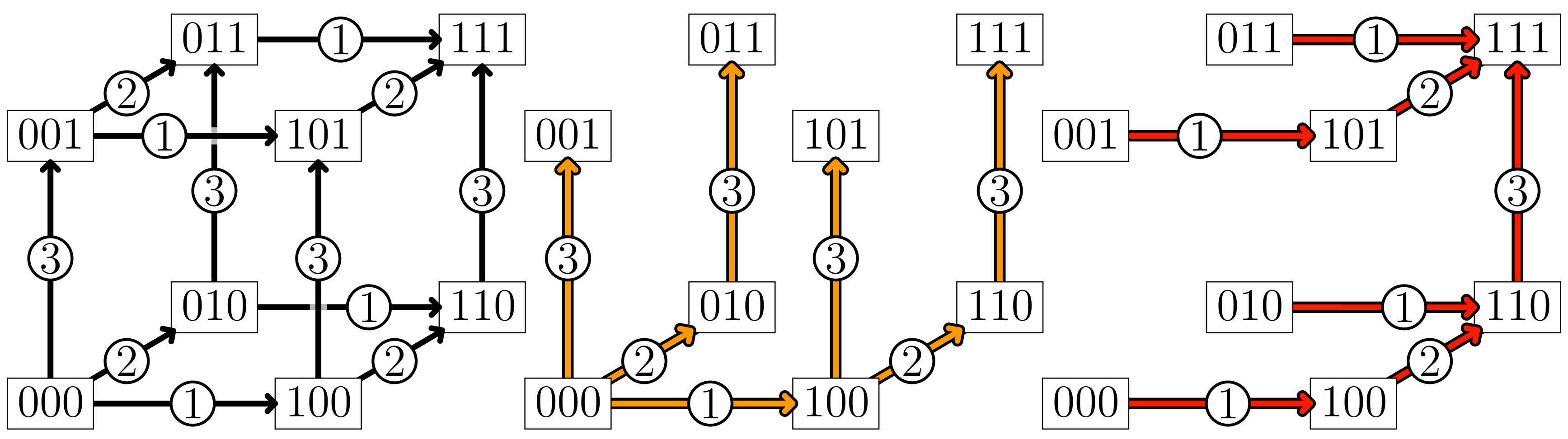
$G = (V, E)$ finite, acyclic, directed graph.

EL-labeling of G = edge labeling $\lambda : E \to \mathbb{N}$ of G such that

- there is a unique λ-rising path p between any $u \dashrightarrow v$ in G,
- $\lambda(p)$ lexicographically first among the $\lambda(p')$ for $p' : u \dashrightarrow v$.

Defines two canonical spanning trees on any interval $[u, v]$ of G:

- *λ-source tree of $[u, v]$* = union of all λ-rising paths from u,
- *λ-sink tree of $[u, v]$* = union of all λ-rising paths towards v.



If G is the Hasse diagram of a poset P, EL-labelings carry information on its Möbius function μ and the topology of its order complex.

Prop. [BW96] *For an EL-labeling λ of P, and $u \le_P v$ in P,*

$$\mu(u, v) = \mathrm{even}_\lambda(u, v) - \mathrm{odd}_\lambda(u, v),$$

where $\mathrm{even}_\lambda(u, v)$ *and* $\mathrm{odd}_\lambda(u, v)$ = *numbers of even and odd length λ-falling paths from u to v in the Hasse diagram of P.*

A. Björner and M. Wachs. Shellable nonpure complexes and posets I. 1996.

RESULTS

1. EL-labelings of the increasing flip graph

$\mathrm{Q}^{\mathrm{ex}} = \tau_2\tau_3\tau_1\tau_3\tau_2\tau_1\tau_2\tau_3\tau_1$, $\rho^{\mathrm{ex}} = [4,1,3,2]$

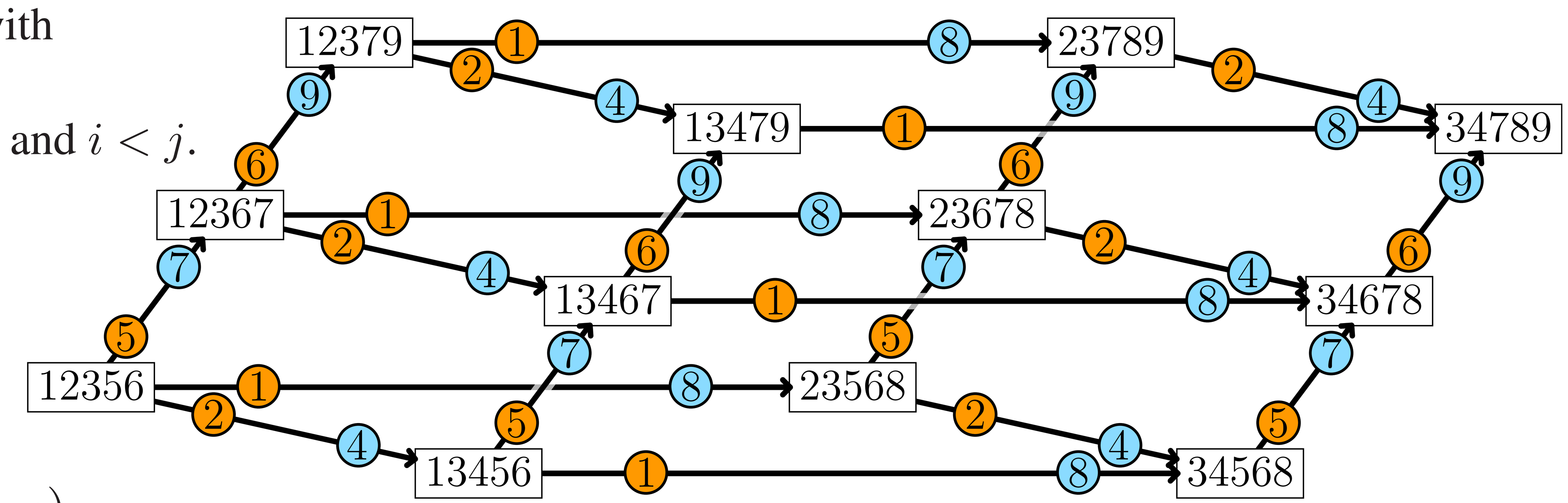
Increasing flip graph $\mathcal{G}(\mathrm{Q}, \rho)$ = directed labeled graph with

- nodes = facets of $\mathcal{SC}(\mathrm{Q}, \rho)$,
- arcs $= I \to J$ if $\exists i \in I, j \in J$ such that $I \smallsetminus i = J \smallsetminus j$ and $i < j$.

$i = \mathsf{p}(I \to J)$ = *positive edge label*

$j = \mathsf{n}(I \to J)$ = *negative edge label*

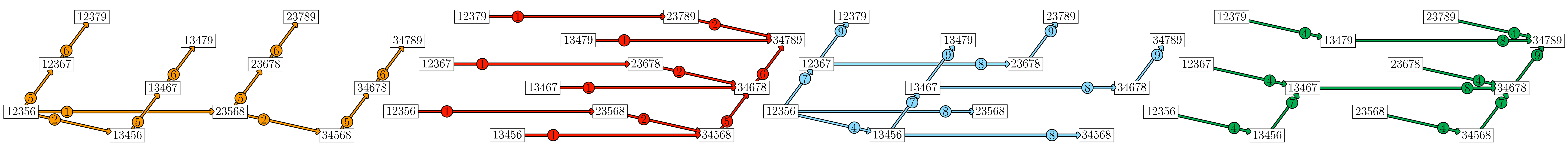
Theo. *The positive and negative edge labelings* p, n *are EL-labelings of the increasing flip graph* $\mathcal{G}(\mathrm{Q}, \rho)$.



2. Greedy facets and spanning trees of $\mathcal{SC}(\mathrm{Q}, \rho)$

Prop. *The lexicographically smallest (resp. largest) facet of $\mathcal{SC}(\mathrm{Q}, \rho)$ is the unique source (resp. sink) of $\mathcal{G}(\mathrm{Q}, \rho)$.*

Positive/negative source/sink trees of $\mathcal{SC}(\mathrm{Q}, \rho)$ = canonical spanning trees oriented from/towards the source/sink of $\mathcal{SC}(\mathrm{Q}, \rho)$.



Simple inductive descriptions of the first and last trees, and characterizations of the father of a given node in these four trees.

It yields a *greedy flip algorithm* to generate $\mathcal{F}(\mathrm{Q}, \rho)$ in polynomial running time and working space.

3. Double root free subword complexes

Increasing flip poset $\Gamma(\mathrm{Q}, \rho)$ = transitive closure of the increasing flip graph $\mathcal{G}(\mathrm{Q}, \rho)$.

Prop. *$\mathcal{SC}(\mathrm{Q}, \rho)$ is double root free $\iff$ $\mathcal{G}(\mathrm{Q}, \rho)$ coincides with the Hasse diagram of $\Gamma(\mathrm{Q}, \rho)$.*

Theo. *If $\mathcal{SC}(\mathrm{Q}, \rho)$ is double root free and I, J are facets of $\mathcal{SC}(\mathrm{Q}, \rho)$, then*

- *There is at most one* p*-falling (resp.* n*-falling) path between I and J.*
- *The Möbius function on $\Gamma(\mathrm{Q}, \rho)$ is given by $\mu(I, J) = (-1)^{|J \smallsetminus I|}$ if there is a* p*-falling (resp.* n*-falling) path from I to J, and 0 otherwise.*

Relevant Examples:

- $\mathcal{SC}(\mathrm{w}_\circ(\mathrm{c}), w_\circ)$ = Cluster complex
  $\Gamma(\mathrm{w}_\circ(\mathrm{c}), w_\circ)$ = Cambrian lattice
  see also M. Kallipoliti and H. Mühle's poster
- Duplicated words (boolean lattices)

V. Pilaud and C. Stump. EL-labelings and greedy flip trees for subword complexes. In *Discrete Geometry and Optimization*. Bezdek, Deza & Ye (eds.). 2013.