QUOTIENTOPES

VINCENT PILAUD AND FRANCISCO SANTOS

For any lattice congruence of the weak order on $\mathfrak{S}_n$, N. Reading proved that glueing together the cones of the braid fan that belong to the same congruence class defines a complete fan. We prove that this fan is the normal fan of a polytope.

1.

Denote by $\mathfrak{S}_n$ the set of permutations of $[n] := \{1, \dots, n\}$. We consider the classical on $\mathfrak{S}_n$ defined by inclusion of inversion sets. That is if and only if inv( ) inv( ) where inv( ) $:= \{(i, j) \mid 1 \le i < j \le n$ and $(i) > (j)\}$. The Hasse diagram of the weak order can be seen geometrically:

(1) as the dual graph of the of type A_{n-1}, *i.e.* the fan defined by the arrangement of the hyperplanes $H_{ij} := \{x \in \mathbb{R}^n \mid x_i = x_j\}$ for all $1 \le i < j \le n$, directed from the region $x_1 < \dots < x_n$ to the opposite one,

(2) or as the graph of the Perm$(n) :=$ conv $(1), \dots, (n)$ $\in \mathfrak{S}_n$, oriented in the linear direction $:= (n+1, n+3, \dots, n-3, n-1)$.

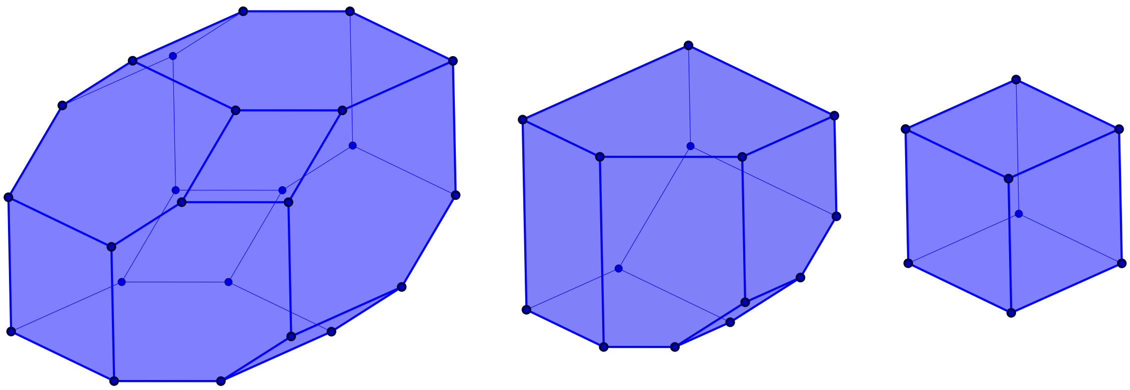
We aim at studying similar geometric realizations for lattice quotients of the weak order on $\mathfrak{S}_n$. Recall that a of a lattice $(L, \le, \wedge, \vee)$ is an equivalence relation on L that respects the meet and the join operations, *i.e.* such that $x \equiv x'$ and $y \equiv y'$ implies $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$. A lattice congruence automatically defines a $L/\equiv$ on the congruence classes of where the order relation is given by $X \le Y$ iff there exists $x \in X$ and $y \in Y$ such that $x \le y$. The meet $X \wedge Y$ (resp. the join $X \vee Y$) of two congruence classes X and Y is the congruence class of $x \wedge y$ (resp. of $x \vee y$) for arbitrary representatives $x \in X$ and $y \in Y$.

Several examples of relevant combinatorial structures arise from lattice quotients of the weak order. The fundamental example is the Tamari lattice introduced by D. Tamari in [Tam51]. It can be defined on different Catalan families (Dyck paths, binary trees, triangulations, non-crossing partitions, etc), and its cover relations correspond to local moves in these structures (exchange, rotation, flip, etc). The Tamari lattice can also be interpreted as the quotient of the weak order by the sylvester congruence on $\mathfrak{S}_n$ defined as the transitive closure of the rewriting rule $UacVbW \equiv^{\mathrm{sylv}} UcaVbW$ where $a < b < c$ are letters while U, V, W are words of $[n]$. This congruence has been widely studied in connection to geometry and algebra [Lod04, LR98, HNT05]. Among many other examples of relevant lattice quotients of the weak order, let us mention the (type A) Cambrian lattices [Rea06, CP17], the boolean lattice, the permutree lattices [PP16], the increasing flip lattice on acyclic twists [Pil15], the rotation lattice on diagonal rectangulations [LR12, Gir12], etc.

In his vast study of lattice congruences of the weak order, N. Reading observed that "*lattice congruences on the weak order know a lot of combinatorics and geometry*" [Rea16a, Sect. 10.7]. Geometrically, he showed that each lattice congruence of the weak order is realized by a complete fan $\mathcal{F}_\equiv$ that we call . Its maximal cones correspond to the congruence classes of and are just obtained by glueing together the cones of the braid fan corresponding to permutations that belong to the same congruence class of . Although this result was stated in a much more general context (that of lattice congruences on lattice of regions of hyperplane arrangements), we restrict our discussion to lattice quotients of the weak order on $\mathfrak{S}_n$.

Theorem 1 ([Rea05]). *For any lattice congruence of the weak order on $\mathfrak{S}_n$, the cones obtained by glueing together the cones of the braid fan that belong to the same congruence class of form a complete fan $\mathcal{F}_\equiv$ whose dual graph coincides with the Hasse diagram of the quotient of the weak order by .*

VP was partially supported by the French ANR grant SC3A (15 CE40 0004 01).
FS was partially supported by the Spanish Ministry of Science through grant MTM2014-54207-P, by the Einstein Foundation Berlin, and by the NSF while he was in residence at MSRI Berkeley in the fall of 2017.



[illegible] 1. Permutahedra (left), associahedra (middle) and cubes (right) as quotientopes.

However, as observed by N. Reading in [Rea05], "*this theorem gives no means of knowing when $\mathcal{F}_\equiv$ is the normal fan of a polytope*". For the above-mentioned examples of lattice congruences, this problem was settled by specific constructions of polytopes realizing the quotient fan $\mathcal{F}_\equiv$: J.-L. Loday's associahedron [Lod04] for the Tamari lattice, C. Hohlweg and C. Lange's associahedra [HL07, LP13] for the Cambrian lattices, cubes for the boolean lattices, permutreehedra [PP16] for the permutree lattices, brick polytopes [PS12] for increasing flip lattices on acyclic twists, Minkowski sums of opposite associahedra for rotation lattices on diagonal rectangulations [LR12], etc. Although these realizations have similarities, each requires an independent construction and proof. In particular, the intersection of the half-spaces defining facets of the classical permutahedron normal to the rays of $\mathcal{F}_\equiv$ does not realize $\mathcal{F}_\equiv$ in general, in contrast to the specific situation of [Lod04, HL07, LP13, PP16]. Our contribution is to provide a generic method to construct a polytope $P_\equiv$ whose normal fan is the quotient fan $\mathcal{F}_\equiv$. We therefore prove the following statement.

Theorem 2. *For any lattice congruence of the weak order on $\mathfrak{S}_n$, the fan $\mathcal{F}_\equiv$ obtained by glueing the braid fan according to the congruence classes of is the normal fan of a polytope.*

We call [illegible] the resulting polytopes. Some examples are illustrated in Figures 1 and 2.

2. [illegible]

2.1. Polyhedral geometry. We briefly recall basic definitions and properties of polyhedral fans and polytopes, and refer to [Zie98] for a classical textbook on this topic.

We denote by $\mathbb{R}_{\geq 0}R := \big\{ \sum_{r \in R} \lambda_r r \;\big|\; \lambda_r \in \mathbb{R}_{\geq 0} \big\}$ the [illegible] of a set R of vectors of $\mathbb{R}^d$. A [illegible] is a subset of $\mathbb{R}^d$ defined equivalently as the positive span of finitely many vectors or as the intersection of finitely many closed linear halfspaces. The [illegible] of a cone C are the intersections of C with the supporting hyperplanes of C. The 1-dimensional (resp. codimension 1) faces of C are called [illegible] (resp. [illegible]) of C. A cone is [illegible] if it is generated by a set of independent vectors.

A [illegible] is a collection $\mathcal{F}$ of polyhedral cones such that

- if $C \in \mathcal{F}$ and F is a face of C, then $F \in \mathcal{F}$,
- the intersection of any two cones of $\mathcal{F}$ is a face of both.

A fan is [illegible] if all its cones are, and [illegible] if the union of its cones covers the ambient space $\mathbb{R}^d$. For two fans $\mathcal{F}, \mathcal{G}$ in $\mathbb{R}^d$, we say that $\mathcal{F}$ [illegible] $\mathcal{G}$ (and that $\mathcal{G}$ [illegible] $\mathcal{F}$) if every cone of $\mathcal{F}$ is contained in a cone of $\mathcal{G}$.

A [illegible] is a subset P of $\mathbb{R}^d$ defined equivalently as the convex hull of finitely many points or as a bounded intersection of finitely many closed affine halfspaces. The [illegible] $\dim(P)$ is the dimension of the affine hull of P. The [illegible] of P are the intersections of P with its supporting hyperplanes. The dimension 0 (resp. dimension 1, resp. codimension 1) faces are called [illegible] (resp. *edges*, resp. *facets*) of P. A polytope is *simple* if its supporting hyperplanes are in general position, meaning that each vertex is incident to $\dim(P)$ facets (or equivalently to $\dim(P)$ edges).

The (outer) *normal cone* of a face F of P is the cone generated by the outer normal vectors of the facets of P containing F. The (outer) *normal fan* of P is the collection of the (outer) normal cones of all its faces. We say that a complete polyhedral fan in $\mathbb{R}^d$ is *polytopal* when it is the normal fan of a polytope of $\mathbb{R}^d$. There is a classical characterization of polytopality of complete simplicial fans. It is a reformulation of regularity of triangulations of vector configurations, introduced in the theory of secondary polytopes [GKZ08], see also [DRS10]. Here, we present a reformulation of this condition to deal with (not necessarily simplicial) fans that coarsen a complete simplicial fan.

Proposition 3. *Consider two fans $\mathcal{F}, \mathcal{G}$ of $\mathbb{R}^d$, and let $\mathbf{R} \subset \mathbb{R}^d$ be a set of representative vectors for the rays of $\mathcal{F}$. Assume that $\mathcal{F}$ is complete and simplicial, and that $\mathcal{F}$ refines $\mathcal{G}$. Then the following assertions are equivalent:*

1. *$\mathcal{G}$ is the normal fan of a polytope in $\mathbb{R}^d$.*
2. *There exists a map $h : \mathbf{R} \to \mathbb{R}_{>0}$ with the property that for any $\mathbf{r}, \mathbf{r}' \in \mathbf{R}$ and $\mathbf{S} \subseteq \mathbf{R}$ for which $C := \mathbb{R}_{\ge 0}(\mathbf{S} \cup \{\mathbf{r}\})$ and $C' := \mathbb{R}_{\ge 0}(\mathbf{S} \cup \{\mathbf{r}'\})$ are two adjacent maximal cones of $\mathcal{F}$, if*
$$\alpha\,\mathbf{r} + \alpha'\,\mathbf{r}' + \sum_{\mathbf{s} \in \mathbf{S}} \beta_{\mathbf{s}}\,\mathbf{s} = 0$$
is the unique (up to rescaling) linear dependence with $\alpha, \alpha' > 0$ among $\{\mathbf{r}, \mathbf{r}'\} \cup \mathbf{S}$, then
$$\alpha\, h(\mathbf{r}) + \alpha'\, h(\mathbf{r}') + \sum_{\mathbf{s} \in \mathbf{S}} \beta_{\mathbf{s}}\, h(\mathbf{s}) \ge 0$$
with equality if and only the cone of $\mathcal{G}$ containing C and C' is the same.

Under these conditions, $\mathcal{G}$ is the normal fan of the polytope defined by
$$\big\{ \mathbf{x} \in \mathbb{R}^d \;\big|\; \langle \mathbf{r} \,|\, \mathbf{x} \rangle \le h(\mathbf{r}) \text{ for all } \mathbf{r} \in \mathbf{R} \big\}.$$

2.2. Braid fan. We consider the *braid arrangement* $\mathcal{H}_n := \{\mathbf{H}_{ij} \mid 1 \le i < j \le n\}$ consisting of the hyperplanes of the form $\mathbf{H}_{ij} := \{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}$. The closures of the connected components of $\mathbb{R}^n \smallsetminus \bigcup \mathcal{H}_n$ (together with all their faces) form a fan. This fan is complete and simplicial, but not essential (all its cones contain the line $\mathbb{R}\mathbb{1}$). We thus consider its intersection $\mathcal{F}_n$ with the hyperplane $\mathbb{H} := \big\{\mathbf{x} \in \mathbb{R}^n \;\big|\; \sum_{i \in [n]} x_i = 0\big\}$, that we call *braid fan*.

The fan $\mathcal{F}_n$ has a k-dimensional cone for each surjection from $[n]$ to $[k+1]$: namely, a surjection $\sigma : [n] \to [k+1]$ corresponds to the cone $C(\sigma) := \{\mathbf{x} \in \mathbb{H} \mid \sigma(i) \le \sigma(j) \Rightarrow x_i \le x_j \text{ for all } i, j \in [n]\}$. In particular, the fan $\mathcal{F}_n$ has a maximal cone $C(\sigma)$ for each permutation $\sigma \in \mathfrak{S}_n$, and a ray for each subset of $[n]$ distinct from $\varnothing$ and $[n]$. The fundamental chamber $C(1)$ has rays labeled by the $n-1$ subsets of the form $[k]$ with $0 < k < n$. Any other chamber $C(\sigma)$ is obtained from $C(1)$ by permutation of coordinates and has thus rays labeled by $\sigma([k])$ with $0 < k < n$. Two permutations σ, σ' are said to be *adjacent* when their cones $C(\sigma)$ and $C(\sigma')$ share a facet, or equivalently when σ and σ' differ by the exchange of two consecutive values.

To understand the geometry of $\mathcal{F}_n$, we need to choose convenient representative vectors in $\mathbb{H}$ for the rays of $\mathcal{F}_n$. We denote by $\Delta := \{\alpha_1, \dots, \alpha_{n-1}\}$ the root basis (where $\alpha_i := \mathbf{e}_{i+1} - \mathbf{e}_i$) and by $\nabla := \{\omega_1, \dots, \omega_{n-1}\}$ the fundamental weight basis (*i.e.* the dual basis of the root basis Δ). A subset $\varnothing \ne R \subsetneq [n]$ corresponds to the ray $\mathbf{r}(R)$ of $\mathcal{F}_n$ whose kth coordinate in the fundamental weight basis is $\mathbb{1}_{k+1 \in R} - \mathbb{1}_{k \in R}$. The following immediate lemma is left to the reader.

Lemma 4. *Let σ, σ' be two adjacent permutations. Let $\varnothing \ne R \subsetneq [n]$ (resp. $\varnothing \ne R' \subsetneq [n]$) be such that $\mathbf{r}(R)$ (resp. $\mathbf{r}(R')$) is the ray of $C(\sigma)$ not in $C(\sigma')$ (resp. of $C(\sigma')$ not in $C(\sigma)$). Then the linear dependence among the rays of the cones $C(\sigma)$ and $C(\sigma')$ is given by*
$$\mathbf{r}(R) + \mathbf{r}(R') = \mathbf{r}(R \cap R') + \mathbf{r}(R \cup R')$$
where we set $\mathbf{r}(\varnothing) = \mathbf{r}([n]) = 0$ by convention.

2.3. Shards. We now briefly present shards, a powerful tool to deal with lattice quotients of the weak order with a geometric perspective. Shards were introduced by N. Reading [Rea03], see also his recent survey chapters [Rea16b, Rea16a]. For any $1 \le i < j \le n$, let $[i,j] := \{i, \dots, j\}$ and $]i,j[:= \{i+1, \dots, j-1\}$. For any $S \subseteq]i,j[$, the shard $\Sigma(i,j,S)$ is the cone

$$\Sigma(i,j,S) := \{ \mathbf{x} \in \mathbb{R}^n \mid x_i = x_j, \; x_i \ge x_k \text{ for all } k \in S, \; x_i \le x_k \text{ for all } k \in]i,j[\smallsetminus S \}.$$

The hyperplane $\mathbb{H}_{ij}$ is decomposed into the 2^{j-i-1} shards $\Sigma(i,j,S)$ for all subsets $S \subseteq]i,j[$. The shards thus have to be thought of as pieces of the hyperplanes of the Coxeter arrangement. We denote by

$$\boldsymbol{\Sigma}_n := \{ \Sigma(i,j,S) \mid 1 \le i < j \le n \text{ and } S \subseteq]i,j[\}$$

the collection of all shards of the Coxeter arrangement in $\mathbb{R}^n$. Before going further, we state a small technical lemma, whose proof is left to the reader.

Lemma 5. *Let σ, σ' be two adjacent permutations, let $\varnothing \ne R \subsetneq [n]$ (resp. $\varnothing \ne R' \subsetneq [n]$) be such that $\mathbf{r}(R)$ (resp. $\mathbf{r}(R')$) is the ray of $C(\sigma)$ not in $C(\sigma')$ (resp. of $C(\sigma')$ not in $C(\sigma)$), and let k, k' be such that $R \smallsetminus \{k\} = R' \smallsetminus \{k'\}$. Assume without loss of generality that $k < k'$. Then the common facet of $C(\sigma)$ and $C(\sigma')$ belongs to the shard $\Sigma(k, k', R \cap R' \cap]k,k'[)$.*

It turns out that the shards are precisely the right pieces of the hyperplanes of $\mathcal{H}_n$ to delimit the cones of the quotient fan $\mathcal{F}_\equiv$ for any lattice congruence $\equiv$ of the weak order on $\mathfrak{S}_n$. Conversely, to understand which sets of shards can be used to define a quotient fan, we need the forcing order between shards. A shard $\Sigma(i,j,S)$ is said to force a shard $\Sigma(k,\ell,T)$ if $k \le i < j \le \ell$ and $S = T \cap]i,j[$. We denote by $\Sigma(i,j,S) \prec \Sigma(k,\ell,T)$ the forcing order. The following statement uses shards to describe the lattice quotients of the weak order on $\mathfrak{S}_n$.

Theorem 6 ([Rea16a, Sect. 10.5]). *For any lattice congruence $\equiv$ of the weak order on $\mathfrak{S}_n$, there is a subset $\boldsymbol{\Sigma}_\equiv$ of the shards of $\boldsymbol{\Sigma}_n$ such that the interior of the maximal cones of the fan $\mathcal{F}_\equiv$ are precisely the connected components of $\mathbb{H} \smallsetminus \bigcup \boldsymbol{\Sigma}_\equiv$. Moreover, the map $\equiv \mapsto \boldsymbol{\Sigma}_\equiv$ is a bijection between the lattice congruences of the weak order on $\mathfrak{S}_n$ and the upper ideals of the forcing order $\prec$.*

Remark 7. It is often convenient to represent shards by arcs: the shard $\Sigma(i,j,S)$ corresponds to the arc with endpoints i and j and passing above the vertices of S and below those of $]i,j[\smallsetminus S$. Each region C of $\mathcal{F}_\equiv$ then corresponds to a unique noncrossing arc diagram [Rea15] (given by the shards containing a down facet of C). This correspondence provides the canonical join representation of C. See [Rea15] for precise definitions and details. We also refer to N. Reading's surveys [Rea16b, Rea16a] for further technology on the geometry of lattice quotients (see also Remark 12).

3. Proof

This section is devoted to the proof of Theorem 2. We say that a function $f : \boldsymbol{\Sigma}_n \to \mathbb{R}_{>0}$ is *forcing dominant* if

$$f(\Sigma) > \sum_{\Sigma' \prec \Sigma} f(\Sigma')$$

for any shard $\Sigma \in \boldsymbol{\Sigma}_n$. Such a function clearly exists, take for example $f(\Sigma(i,j,S))$ to be $n^{-(j-i)^2}$. For the remaining of the paper, we fix a forcing dominant function f.

For a shard $\Sigma(i,j,S) \in \boldsymbol{\Sigma}_n$ and a subset $\varnothing \ne R \subsetneq [n]$, we define the *contribution* $\kappa(\Sigma, R)$ of Σ to R to be 1 if $|R \cap \{i,j\}| = 1$ and $S = R \cap]i,j[$, and 0 otherwise. For a geometric interpretation of this definition, let $\mathcal{H}_n^{ij}$ denote the arrangement of the hyperplanes $\mathbb{H}_{ik}$ and $\mathbb{H}_{kj}$ for all $k \in]i,j[$. Then $\Sigma(i,j,S)$ contributes to R if the ray $\mathbf{r}(R)$ lies in the (closed) region of $\mathcal{H}_n^{ij}$ containing $\Sigma(i,j,S)$, but not on $\Sigma(i,j,S)$.

We consider a lattice congruence $\equiv$ of the weak order on $\mathfrak{S}_n$. For a subset $\varnothing \ne R \subsetneq [n]$, we define the *height* $h^f_\equiv(R) \in \mathbb{R}_+$ to be

$$h^f_\equiv(R) := \sum_{\Sigma \in \boldsymbol{\Sigma}_\equiv} f(\Sigma)\, \kappa(\Sigma, R).$$

We set also $h^f_\equiv(\varnothing) = h^f_\equiv([n]) = 0$ by convention. This height function fulfills the following property.

Lemma 8. *Let* $\;;\;'$ *be two adjacent permutations. Let* $\varnothing \neq R \subsetneq [n]$ *(resp.* $\varnothing \neq R' \subsetneq [n]$*) be such that* $\mathrm{r}(R)$ *(resp.* $\mathrm{r}(R')$*) is the ray of* $C(\;)$ *not in* $C(\;')$ *(resp. of* $C(\;')$ *not in* $C(\;)$*). Then*

$$h^f_{\equiv}(R) + h^f_{\equiv}(R') \quad h^f_{\equiv}(R \cap R') + h^f_{\equiv}(R \cup R')$$

with equality if and only if the common facet of $C(\;)$ *and* $C(\;')$ *belongs to a shard of* $\;_{\equiv}$.

Proof. Let k, k' be such that $R \setminus \{k\} = R' \setminus \{k'\}$. Assume without loss of generality that $k < k'$. We consider a shard $= (\;i; j; \;)\in\;_{\equiv}$ and evaluate its contributions to R, R', $R \cap R'$ and $R \cup R'$. We assume that $S \setminus \{k, k'\} = R \cap R' \cap \,]i; j[$, as otherwise $(i; j; \;)$ contributes to none of R, R', $R \cap R'$ and $R \cup R'$. Under this assumption,

if $\{k, k'\} \cap [i; j] = \varnothing$, then $(\;;R) = (\;;R') = (\;;R \cap R') = (\;;R \cup R') = 1$;
if $\{k, k'\} \cap [i; j] = \{k\}$, then $(\;;R) = (\;;R \cup R')$ and $(\;;R') = (\;;R \cap R')$;
if $\{k, k'\} \cap [i; j] = \{k'\}$, then $(\;;R) = (\;;R \cap R')$ and $(\;;R') = (\;;R \cup R')$.

We conclude that

$$(\;;R) + (\;;R') = (\;;R \cap R') + (\;;R \cup R')$$

for any shard $= (\;i; j; \;)$ for which $\{k, k'\} \not\subseteq [i; j]$. To deal with the remaining shards, consider the particular shard $_{\bullet} = (\;k, k'; R \cap R' \cap \,]k, k'[)$. According to Lemma 5, $_{\bullet}$ is a shard of $_{\equiv}$ if and only if the cones $C(\;)$ and $C(\;')$ do not belong to the same cone of $F_{\equiv}$. Moreover, observe that $_{\bullet}$ forces any shard $(i; j; \;) \in\;_{\equiv}$ such that $S \setminus \{k, k'\} = R \cap R' \cap \,]i; j[$ and $\{k, k'\} \subseteq [i; j]$. Therefore,

(i) If $_{\bullet} \notin\;_{\equiv}$, then

$$h^f_{\equiv}(R) + h^f_{\equiv}(R') = h^f_{\equiv}(R \cap R') + h^f_{\equiv}(R \cup R').$$

(ii) If $_{\bullet} \in\;_{\equiv}$, then

$$h^f_{\equiv}(R) + h^f_{\equiv}(R') \quad h^f_{\equiv}(R \cap R') \quad h^f_{\equiv}(R \cup R') \quad 2f(\;_{\bullet}) \quad 2 \sum_{'\prec} f(\;') > 0.$$

since $(\;_{\bullet};R) = (\;_{\bullet};R') = 1$ while $(\;_{\bullet};R \cap R') = (\;_{\bullet};R \cup R') = 0$ and f is forcing dominant. This concludes the proof. □

Combining the polytopality criterion of Proposition 3 with the observations of Lemmas 4 and 8, we obtain the polytopality of the quotient fan $F_{\equiv}$.

Corollary 9. *For any lattice congruence* *of the weak order on* $\mathfrak{S}_n$*, and any forcing dominant function* $f : \;_n \to \mathbb{R}_{>0}$*, the quotient fan* $F_{\equiv}$ *is the normal fan of the polytope*

$$P^f_{\equiv} := \big\{ x \in \mathbb{R}^n \;\big|\; \langle \mathrm{r}(R) \mid x \rangle \quad h^f_{\equiv}(R) \text{ for all } \varnothing \neq R \subsetneq [n] \big\}.$$

In particular, the graph of $P^f_{\equiv}$ *oriented in the linear direction* $:= (\;n+1;\, n+3;\, \dots;\, n\;3;\, n\;1)$ *is the Hasse diagram of the quotient of the weak order by* .

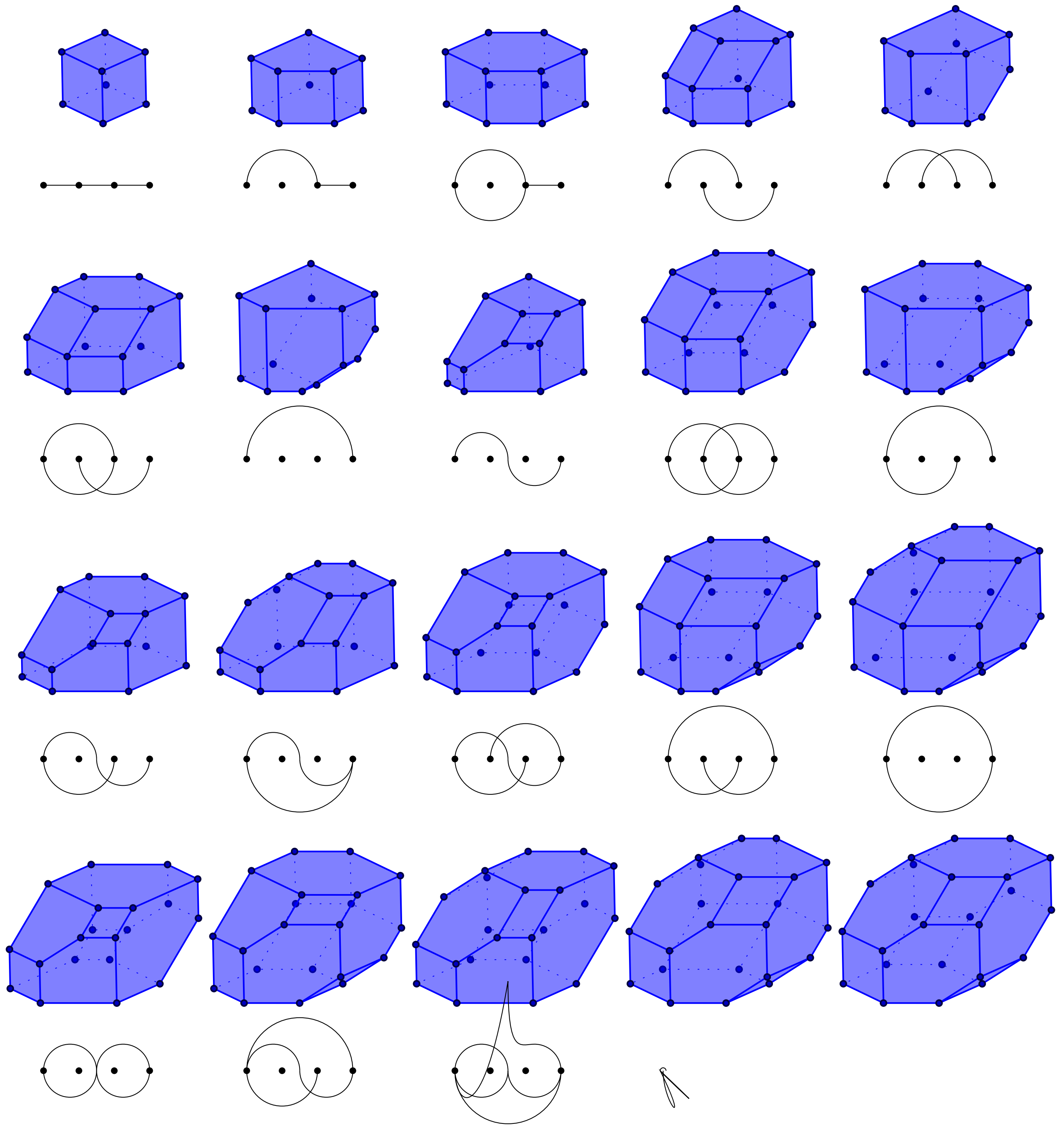
We call the resulting polytope $P^f_{\equiv}$. See Figures 1 and 2 for illustrations. Note that not all quotientopes are simple since not all quotient fans are simplicial.

Remark 10 (Forcing dominance). Note that the forcing dominance condition could even be weakened to depend on the lattice congruence . More precisely, the construction and the proof still work for any function $f : \;_n \to \mathbb{R}_{>0}$ such that

$$f(\;) > \sum_{\substack{' \in \\ ' \prec}} f(\;')$$

for any shard $\in\;_{\equiv}$.

Remark 11 (, and). By definition, the quotientopes are generalized permutahedra [Pos09, PRW08] as their normal fans coarsen the braid fan. This means in particular that they are obtained by gliding inequalities of the permutahedron orthogonally to their normal vectors. Note that in our construction, the inequalities are glided the permutahedron. More precisely, if $F_{\equiv}$ refines $F_{\equiv'}$, then $P^f_{\equiv}$ contains $P^f_{\equiv'}$. For example, the cube (quotientope of the coarsest congruence so that $F_{\equiv}$ is essential) is contained in all quotientopes,



while the permutahedron (quotientope of the nest congruence) contains all quotientopes. See Figures 1 and 2 for illustration. This construction thus contrasts with the classical construction of the associahedron [Lod04] and its generalizations [HL07, LP13, Pil13, PP16], which are all obtained by gliding inequalities ts the permutahedron. More precisely, the classical associahedron is obtained by gm certain inequalities from the facet description of the classical permutahedron. Note that the similar construction does not work in general: for example, the fan $F_{\equiv}$ of the top right congruence of Figure 2 is not realized by the intersection of the half-spaces de ning facets of the classical permutahedron normal to the rays of $F_{\equiv}$.

Remark 12 (Towards quotientopes for arbitrary hyperplane arrangements?). As already mentioned, Theorem 1 actually holds in much more generality (see [Rea16b] for a detailed survey). Consider a central hyperplane arrangement H de ning a fan F, and let B be a distinguished chamber of F. For any chamber C of F, de ne its inversion set to be the set of hyperplanes of H that separate B from C. The psog Pos$(H; B)$ is the poset whose elements are the chambers of F ordered by inclusion of inversion sets. A. Björner, P. Edelman and G. Ziegler discussed in [BEZ90] some conditions for this poset of regions to be a lattice: Pos$(H; B)$ is always

[LR12] Shirley Law and Nathan Reading. The Hopf algebra of diagonal rectangulations. *J. Combin. Theory Ser. A*, 119(3):788{824, 2012.

[Pil13] Vincent Pilaud. Signed tree associahedra. Preprint, arXiv:1309.5222, 2013.

[Pil15] Vincent Pilaud. Brick polytopes, lattice quotients, and hopf algebras. Preprint, arXiv:1505.07665, 2015.

[Pos09] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026{1106, 2009.

[PP16] Vincent Pilaud and Viviane Pons. Permutrees. Preprint arXiv:1606.09643, to appear in *Algebraic Combinatorics*, 2016.

[PRW08] Alexander Postnikov, Victor Reiner, and Lauren K. Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207{273, 2008.

[PS12] Vincent Pilaud and Francisco Santos. The brick polytope of a sorting network. *European J. Combin.*, 33(4):632{662, 2012.

[Rea03] Nathan Reading. Lattice and order properties of the poset of regions in a hyperplane arrangement. *Algebra Universalis*, 50(2):179{205, 2003.

[Rea05] Nathan Reading. Lattice congruences, fans and Hopf algebras. *J. Combin. Theory Ser. A*, 110(2):237{273, 2005.

[Rea06] Nathan Reading. Cambrian lattices. *Adv. Math.*, 205(2):313{353, 2006.

[Rea15] Nathan Reading. Noncrossing arc diagrams and canonical join representations. *SIAM J. Discrete Math.*, 29(2):736{750, 2015.

[Rea16a] N. Reading. Finite Coxeter groups and the weak order. In *Lattice theory: special topics and applications. Vol. 2*, pages 489{561. Birkh auser/Springer, Cham, 2016.

[Rea16b] N. Reading. Lattice theory of the poset of regions. In *Lattice theory: special topics and applications]TJ0 g 0 344(Vo)1(l.)1(.)]19..702*

URL: http://www.lix.polytechnique.fr/~pilaud/

(FS) [illegible]
E-mail addre : francisco.santos@unican.e0 s
URL: http://personales.unican.es/0 santosf