MULTI-TRIANGULATIONS AS COMPLEXES OF STAR POLYGONS

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ABSTRACT

The maximum possible number of diagonals that can be drawn in a convex polygon with no $k+1$ of them mutually crossing is $k(2n-2k-1)$ ([CP92]). Maximal such subsets of edges (called here k-triangulations) are known to generalize nicely some properties of triangulations of a convex n-gon ([NAK00],[DKM03],[JON05]).

In this poster, we present proofs of basic properties of k-triangulations (number of edges, flip), using the new tool of stars that generalize triangles for multi-triangulations. We also discuss open problems that may hopefully be easier to analyze using this new tool.

DEFINITIONS

A k-triangulation of a convex n-gon is a maximal subset of (diagonal) edges without any $(k+1)$-crossing (that is, subset of k+1 mutually intersecting edges).

A k-star is a polygon formed by connecting 2k+1 vertices $s_0, \dots, s_{2k}$ (cyclically ordered) with the edges $[s_0, s_k], [s_1, s_{1+k}], \dots, [s_k, s_{2k}], [s_{k+1}, s_0], \dots, [s_{2k}, s_{k-1}]$.

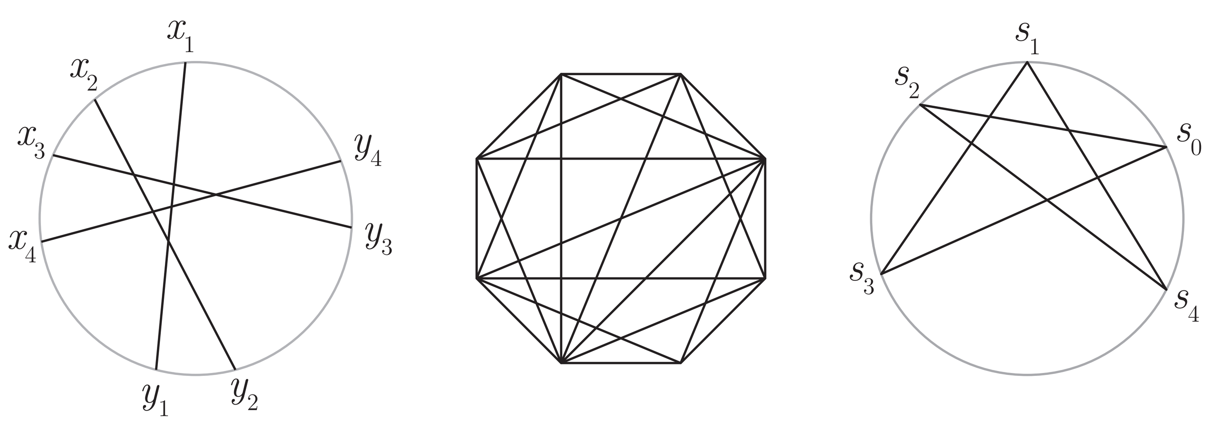

FIG. 1: A 4-crossing – a 2-triangulation of the octagon – a 2-star.

COMPLEXES OF k-STARS

THEOREM 1. An edge of a k-triangulation T is contained in zero, one or two k-stars of T, depending on whether its length is smaller, equal or greater than k.

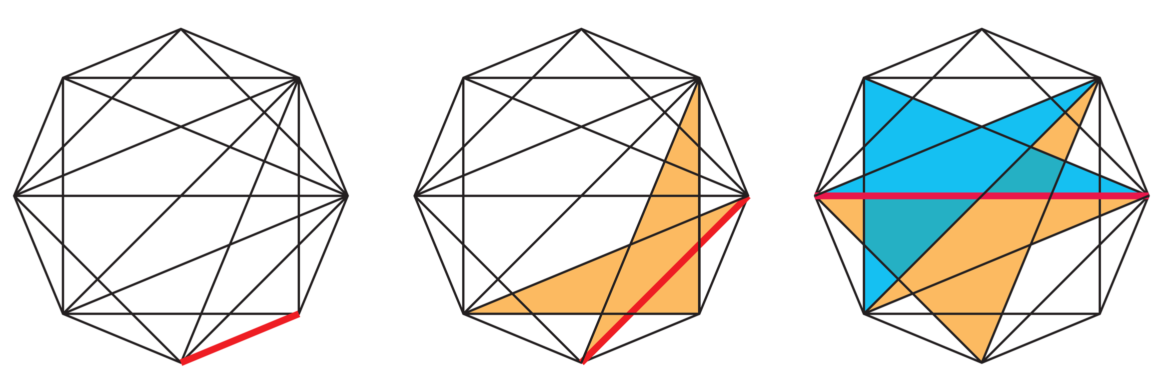

FIG. 2: 2-stars containing a given (red) edge.

THEOREM 2. Let T be a k-triangulation.
(i) Every pair of k-stars of T have a unique common bisector.
(ii) Any edge which is not in T is the common bisector of a unique pair of k-stars of T.

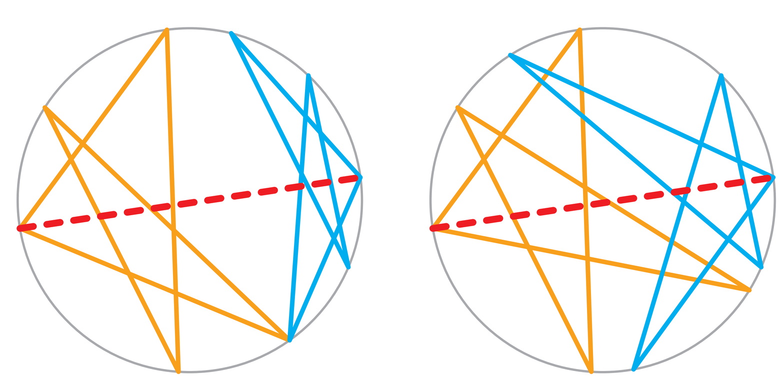

FIG. 3: Common bisector of two 2-stars.

COROLLARY. Any k-triangulation of the convex n-gon contains exactly $n-2k$ k-stars and thus $k(2n-2k-1)$ edges.

FLIPS

THEOREM 3. If e is an edge contained in two stars of T with common bisector f, then:
(i) $T \triangle \{e, f\}$ is also a k-triangulation and (ii) no other k-triangulation contains $T \setminus \{e\}$.

The k-triangulation $T \triangle \{e, f\}$ is obtained by flipping the edge e in the k-triangulation T.

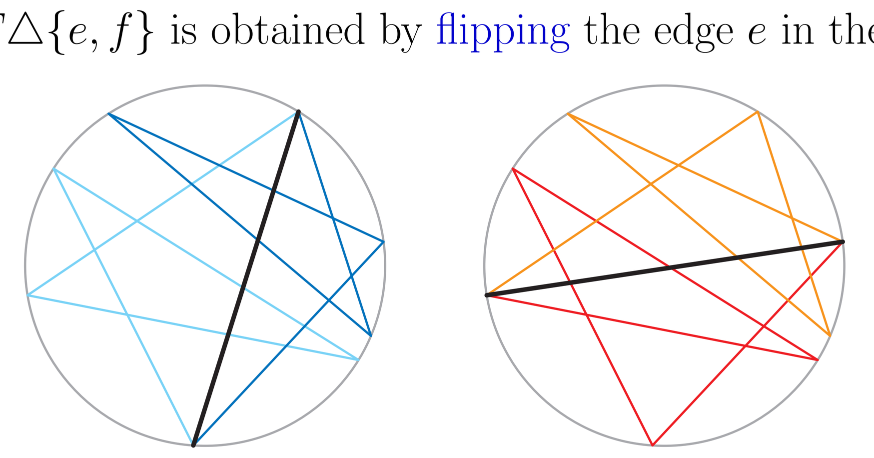

FIG. 4: The flip of an edge.

THEOREM 4. The graph of flips on the set of k-triangulations of the convex n-gon is connected, regular of degree $k(n-2k-1)$, and its diameter is at most $2k(n-2k-1)$.

RELATED TOPICS AND OPEN PROBLEMS

1. Multi-Dyck-paths

THEOREM 5.[JON05] The number of k-triangulations of the convex n-gon is $\det(C_{n-i-j})_{1\le i,j\le k}$ (where $C_m = \frac{1}{m+1}\binom{2m}{m}$ denotes the m-th Catalan number).

This determinant is also known to count k-tuples of non-crossing Dyck paths of semi-length $n-2k$ (k-Dyck-path).

PROBLEM 1. Find an explicit bijection between k-triangulations and k-Dyck-paths (done when $k=2$ in [ELI07]).

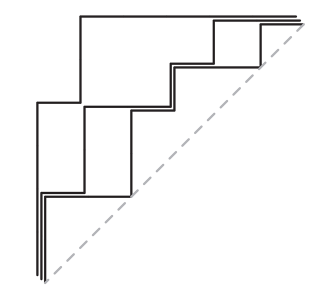

FIG. 5: 3-Dyck-path.

2. Sparsity & rigidity

A graph $G=(V,E)$ is (p,q)-sparse if for any subset F of E, $|F| \le p|V(F)| - q$ (where $V(F)$ denotes the set of vertices of F). A (p,q)-sparse graph $G=(V,E)$ is (p,q)-tight when furthermore $|E| = p|V| - q$.

Depending on the parameters (p,q), sparsity is related to different subjects ([LS07]):
(i) a (generically minimally) rigid graph in dimension d is $(d, \binom{d+1}{2})$-tight;
(ii) a graph is an ℓ-arborescence if and only if it is (ℓ,ℓ)-tight.

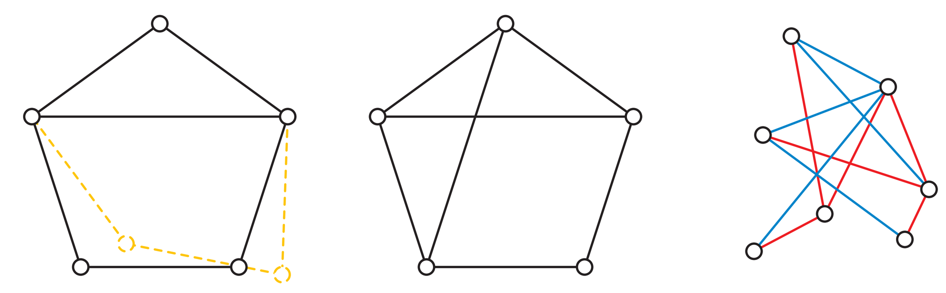

FIG. 6: A non-rigid graph – a rigid graph – a 2-arborescence.

LEMMA. (i) A k-triangulation of a convex polygon is $(2k, \binom{2k+1}{2})$-tight.
(ii) The dual graph of a k-triangulation is (k,k)-tight.

PROBLEM 2. Is a k-triangulation always (generically minimally) rigid in dimension $2k$?

3. Multi-associahedron

Let $\Delta_{n,k}$ be the complex of all subsets of edges that do not contain any $(k+1)$-crossing.

PROBLEM 3. Is there a polytope of dimension $k(n-2k-1)$ with boundary $\Delta_{n,k}$?

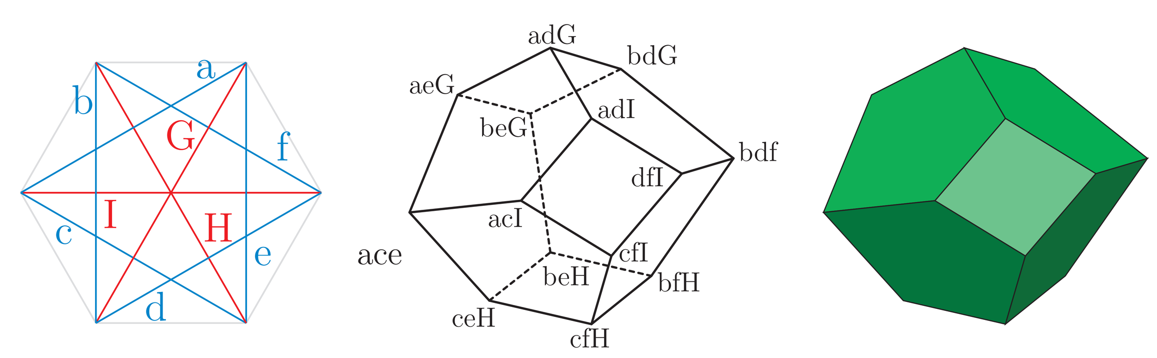

FIG. 7: A realization of the polar of $\Delta_{6,1}$ (the associahedron).

4. Surfaces

The polygonal complex $\mathcal{C}(T)$ associated to a k-triangulation T is a polygonal decomposition of an orientable surface with boundary $\mathcal{S}_{n,k}$.

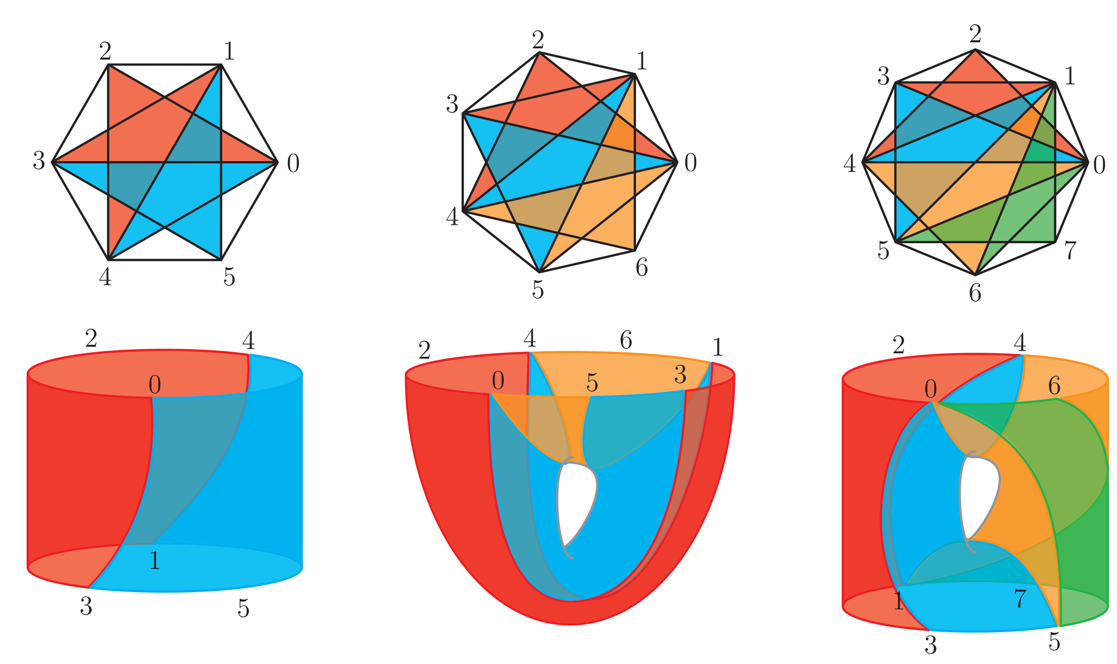

FIG. 8: Decomposition of $\mathcal{S}_{n,2}$ (n=6,7,8) associated to the greedy 2-triangulation.

PROBLEM 4. Characterize the decompositions of $\mathcal{S}_{n,k}$ that correspond to k-triangulations.

REFERENCES

V. CAPOYLEAS & J. PACH, A Turán-type theorem on chords of a convex polygon, 1992
A. DRESS, J. KOOLEN & V. MOULTON, On line arrangements in the hyperbolic plane, 2002
S. ELIZALDE, A bijection between 2-triangulations and pairs of non-crossing Dyck paths, 2007
J. JONSSON, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005
A. LEE & I. STREINU, Pebble game algorithms and sparse graphs, 2007
T. NAKAMIGAWA, A generalization of diagonal flips in a convex polygon, 2000