Algebraic structures on integer posets

Vincent Pilaud
CNRS & LIX, École Polytechnique

Viviane Pons
LRI – Univ. Paris-Sud - Paris-Saclay

We define a lattice structure and a Hopf algebra on integer posets and use them to recover relevant structures on the elements, the intervals and the faces in the permutahedron, the associahedron, the cube and more generally all permutreehedra [2, 5, 6].

Combinatorial objects as integer posets

Many combinatorial objects can be interpreted as binary relations.

Permutations

...a...b... ...b...a...

1432 → 1 2 3 4

Binary trees

⇒

0 1 2 3 4

⇒

1 2 3 4 5

Binary sequences

⇒

1 2

⇒

1 2

Intervals of permutations

σ' = 4231

σ' = 1 2 3 4

2431 4213

δσ' = 1 2 3 4

δσ = 2413

σ = 2143

Characterization

Permutation intervals

a b c ⇒ a b c or a b c

a b c ⇒ a b c or a b c

Tamari intervals [3]

a b c ⇒ a b c

a b c ⇒ a b c

Boolean intervals

a b c ⇒ a b c

a b c ⇒ a b c

Many well known lattices can be induced from the lattice of integer posets by selecting elements satisfying specific local conditions.

Hopf algebras

Hopf algebras of integer relations and posets

We define a product of integer relations by \( R \cdot S := \sum T \) where \( T \) ranges over all integer relations \( T \in \mathcal{R}_{m+n} \) with \( T_{(m)} = R \) and \( T_{(n+m)} = S \).

\[
\begin{align*}
1 2 & \cdot 1 = 1 2 3 + 1 2 3 + 1 2 3 + \cdots + 1 2 3 + \cdots + 1 2 3 \\
1 2 3 & = 1 2 3 \otimes 1 2 3 + 1 2 3 \otimes 1 + 1 2 3 \otimes 1 2 + 1 2 3 \otimes 1 2 + 1 2 3 \otimes 1 2 + 1 2 3 \otimes 1 2 3
\end{align*}
\]

We define the coproduct of an integer relation by \( \Delta(T) := \sum X \otimes Y \) where the sum ranges over all partitions \( X \sqcup Y \subseteq [n] \) such that \( x \geq y \) and \( y \geq x \) for all \( (x, y) \in X \times Y \).

\[
\Delta(1 2 3) = 1 2 3 \otimes 1 \otimes 1 2 + 1 2 3 \otimes 1 2 + 1 2 3 \otimes 1 2 3
\]

The vector space indexed by integer relations endowed with the product and coproduct operations forms a Hopf algebra. In other words, \( \Delta(R \cdot S) = \Delta(R) \cdot \Delta(S) \).

We obtain a Hopf algebra on integer posets as a quotient of the integer relations Hopf algebra: we identify all relations which are not posets to 0. Here is an example of this product in the quotient:

\[
1 2 \cdot 1 = 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3
\]

As an integer poset cannot be cut by the coproduct into two non-posets: the coproduct stays the same as for integer relations.

Weak order

We define the weak order on binary relations by inclusion of increasing relations and inverse inclusion of decreasing relations.

\[
R \leq S \iff R^\text{Inc} \supseteq S^\text{Inc} \text{ and } R^\text{Dec} \subseteq S^\text{Dec}.
\]

We then restrict this order to relations which are transitive and antisymmetric, i.e., integer posets.

Many well known lattices can be induced from the lattice of integer posets by selecting elements satisfying specific local conditions.

Hopf algebras on other objects

By a similar quotient operation, we recover the Malvenuto–Reutenauer Hopf algebra on permutations [4] as well as the Chapoton Hopf algebra on ordered partitions [1]. We also define a new Hopf algebra on permutation intervals. As an example, here is a product of permutations in a quotient Hopf algebra:

\[
1 2 \cdot 1 = 1 2 3 + 1 2 3 + 1 2 3.
\]

Hopf algebras on binary trees, Tamari intervals and Associahedron faces can be obtained as subalgebras of the integer poset Hopf algebra.

References


