On symmetric realizations of the simplicial complex of 3-crossing-free sets of diagonals of the octagon*

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Abstract

Motivated by the question of the polytopal realizability of the simplicial complex $\Gamma_{n,k}$ of $(k+1)$-crossing-free sets of diagonals of the convex n-gon, we study the first open case, namely when $n = 8$ and $k = 2$. We give a complete description of the space of symmetric realizations of $\Gamma_{8,2}$, that is, of the polytopes P whose boundary complex is isomorphic to $\Gamma_{8,2}$, and such that the natural action of the dihedral group on $\Gamma_{8,2}$ defines an action on P by isometry.

1 Introduction

Let $k \geq 1$ and $n \geq 2k+1$ denote two integers. A set of ℓ mutually crossing diagonals of the convex n-gon is called an *ℓ-crossing* (see Fig. 2). We consider the abstract simplicial complex $\Gamma_{n,k}$ of $(k+1)$-crossing-free sets of diagonals of the convex n-gon.

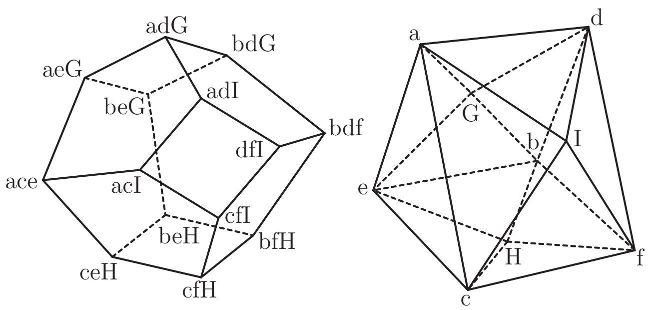

Figure 1: The 3-dimensional associahedron (vertices are labeled by triangulations of the hexagon) and its polar (vertices are labeled by diagonals of the hexagon).

When $k = 1$, the complex $\Gamma_{n,1}$ is the simplicial complex of non-crossing sets of diagonals, and has one maximal face for each *triangulation* of the convex n-gon. This simplicial complex is realized by the $(n-3)$-dimensional *associahedron* (more precisely, by an n-fold cone over the polar of the associahedron). The associahedron can be constructed in different ways (see [10, 6, 11] and the references therein), and in particular, in a very symmetric way, as the *secondary polytope* of the convex regular n-gon (see [1] and Fig. 1).

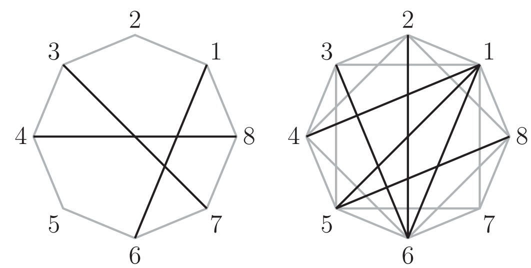

Figure 2: A 3-crossing (left) and a 2-triangulation of the octagon (right).

The maximal faces of $\Gamma_{n,k}$ are called *k-triangulations* (see Fig. 2) and were recently studied in the literature under various aspects [3, 12, 5, 8, 9, 13]. It turns out that the complex $\Gamma_{n,k}$ is a topological sphere [8, 4] (more precisely, a kn-fold cone over a sphere). But it remains an open question to know whether $\Gamma_{n,k}$ is *polytopal*, that is, whether it is the boundary complex of a polytope. So far, we only know that it holds for $k = 1$ (associahedron) and $n \leq 2k+3$ (cyclic polytopes).

Let $\mathbb{D}_n$ denote the *dihedral group* of isometries of the convex regular n-gon. There is a natural action of $\mathbb{D}_n$ on the complex $\Gamma_{n,k}$ defined by $\rho E = \{\rho(e) \,|\, e \in E\}$. When $k = 1$, the associahedron obtained as the secondary polytope of the convex regular n-gon is *symmetric under* $\mathbb{D}_n$, meaning that $\mathbb{D}_n$ acts on its vertices by isometry. For general n and k, we are also interested in realizing $\Gamma_{n,k}$ as a polytope symmetric under $\mathbb{D}_n$, not only because of the nice aspect of the result, but also because it is actually much easier to restrict to symmetric realizations.

In this paper, we only tackle the problem of the first unknown case: when $n = 8$ and $k = 2$. We give a complete description of the space of symmetric realizations of the simplicial complex $\Gamma_{8,2}$. To obtain this description, we first realize $\Gamma_{8,2}$ as a "symmetric oriented matroid polytope", and then as a "symmetric polytope under the dihedral group".

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*This research was initiated when the first author was invited by the Département d'Informatique of the École Normale Supérieure de Paris. Research of the second author was also partially funded by grant MTM2008-04699-C03-02 of the Spanish Ministry of Education and Science.

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2 Preliminaries

General properties of $\Gamma_{n,k}$. Let us remind some relevant properties of the simplicial complex $\Gamma_{n,k}$:

1. Observe first that a diagonal uv can be involved in a $(k+1)$-crossing only if there remains at least k vertices on each side, *i.e.*, if $v-u>k$ and $n-v+u>k$. We call such a diagonal *k-relevant* and denote $R_{n,k}$ the set of all k-relevant diagonals of the n-gon. Let $\Delta_{n,k}$ be the simplicial complex of $(k+1)$-crossing-free subsets of $R_{n,k}$. The simpicial complex $\Gamma_{n,k}$ is just a kn-fold cone over the complex $\Delta_{n,k}$, and we will only study the latter.
2. Any k-triangulation of the n-gon contains exactly $k(n-2k-1)$ k-relevant diagonals [3, 12, 5, 13]. Thus, $\Delta_{n,k}$ is *pure* of dimension $k(n-2k-1)-1$.
3. The simplicial complex $\Delta_{n,k}$ is a *vertex decomposable piece-wise linear sphere* [8, 4].

Polytopes for small cases. Being a topological sphere is a necessary, but not sufficient condition for being polytopal and it remains open to know if $\Delta_{n,k}$ can be realized as the boundary complex of a simplicial polytope of dimension $k(n-2k-1)$. As mentioned previously, when $k=1$, it is known that $\Delta_{n,1}$ is the boundary complex of the dual of the associahedron of dimension $n-3$. Appart from this case, we only know that $\Delta_{n,k}$ is polytopal in the following cases:

1. $n=2k+1$: the complete graph is the unique k-triangulation of the convex $(2k+1)$-gon.
2. $n=2k+2$: the $k+1$ long diagonals are the only k-relevant diagonals of the $(2k+2)$-gon. Any proper subset of $R_{2k+2,k}$ is $(k+1)$-crossing-free and $\Delta_{2k+2,k}$ is the boundary complex of the k-simplex.
3. $n=2k+3$: the k-relevant diagonals of the $(2k+3)$-gon form a cycle of length $2k+3$ and k-triangulations are precisely the sets of k pairs of consecutive diagonals of this cycle. Thus, $\Delta_{2k+3,k}$ is the boundary complex of the *cyclic polytope* of dimension $2k$ with $2k+3$ vertices.

Symmetric polytope. Let Δ be an abstract simplicial complex. We say that a polytope $P\subset\mathbb{R}^d$ is a *realization* of Δ when its boundary complex $\partial(P)=\{\text{proper faces of } P\}$ is isomorphic to Δ, *i.e.*, if there is a bijection $\phi:\Delta\to\partial(P)$ which respects inclusion. Assume now that a group G acts on Δ by $G\times\Delta\to\Delta:(g,E)\mapsto gE$. Then G also acts on $\partial(P)$ by $G\times\partial(P)\to\partial(P):(g,F)\mapsto gF=\phi(g\phi^{-1}(F))$. We say that P is a *symmetric realization* (under G) if this action is *isometric*, *i.e.*, if for any $g\in G$, the application $\partial(P)\to\partial(P):F\mapsto gF$ is an isometry of P.

For example, the dual polytope of the $(n-3)$-dimensional associahedron obtained as the secondary polytope of the convex regular n-gon (see [1] and Fig. 1) is a symmetric realization of $\Delta_{n,1}$ under $\mathbb{D}_n$.

Symmetric oriented matroid. Let $P\subset\mathbb{R}^d$ be a symmetric realization (under G) of Δ, and V denote its vertex set. For any $v\in V$, we denote by $\vec{v}=(v,1)$ the vector of homogeneous coordinates of v. For any $v_0,\dots,v_d\in V$, we denote by $\sigma(v_0,\dots,v_d)$ the orientation of the simplex spanned by $v_0,\dots,v_d$, *i.e.*, the sign ($+1$, -1, or 0) of the determinant of the matrix $(\vec{v}_i)_{0\le i\le d}$. The application $\sigma:V^{d+1}\to\{-1,0,1\}$ is called the *oriented matroid* associated to P and satisfies the following relations (see [2] for more details):

1. *alternating relations*: for any $v_0,\dots,v_d\in V$ and any permutation π of $\{0,\dots,d\}$ of signature ε,
$$\sigma(v_{\pi(0)},\dots,v_{\pi(d)})=\varepsilon\sigma(v_0,\dots,v_d).$$
2. *Grassmann-Plucker relations*: for any $v_0,\dots,v_{d-2}$, $w_1,w_2,w_3,w_4\in V$, the set
$$\begin{aligned}\{&\sigma(v_0,\dots,v_{d-2},w_1,w_2).\sigma(v_0,\dots,v_{d-2},w_3,w_4),\\ &-\sigma(v_0,\dots,v_{d-2},w_1,w_3).\sigma(v_0,\dots,v_{d-2},w_2,w_4),\\ &\sigma(v_0,\dots,v_{d-2},w_1,w_4).\sigma(v_0,\dots,v_{d-2},w_2,w_3)\}\end{aligned}$$
either contains $\{-1,1\}$ or is contained in $\{0\}$.
3. *necessary simplex orientations*: for $v_0,\dots,v_d\in V$, if $\{v_0,\dots,v_{d-2},v_{d-1}\}$ and $\{v_0,\dots,v_{d-2},v_d\}$ are facets of P, then for any $w\in V\smallsetminus\{v_0,\dots,v_d\}$,
$$\begin{aligned}\sigma(v_0,\dots,v_{d-2},v_{d-1},v_d)&=\sigma(v_0,\dots,v_{d-2},v_{d-1},w)\\ &=\sigma(v_0,\dots,v_{d-2},w,v_d).\end{aligned}$$
4. *symmetry*: there exists a morphism $\tau:G\to\{\pm1\}$ such that for any $v_0,\dots,v_d\in V$, and any $g\in G$,
$$\sigma(gv_0,\dots,gv_d)=\tau(g)\sigma(v_0,\dots,v_d).$$

Any application σ that associates to the vertices of Δ a sign in $\{-1,0,1\}$ and satisfies these four properties will be called a *symmetric oriented matroid* realizing Δ.

3 Symmetric realizations of $\Delta_{8,2}$

In this section, we restrict our attention to the question of the realization of the simplicial complex $\Delta_{8,2}$. The polytope we want to construct would be a 6-dimensional polytope, with f-vector $(12,66,192,306,252,84)$.

It is convenient to label the 2-relevant diagonals of the octagon with letters: a = 14, b = 25, c = 36, d = 47, e = 58, f = 16, g = 27, h = 38, I = 15, J = 26, K = 37, and L = 48 (capital letters denote long diagonals).

In order to find a symmetric realization of $\Delta_{8,2}$, we first enumerate all possible symmetric oriented matroids realizing it, and then use this information to study the symmetric polytopes realizing it.

3.1 Symmetric oriented matroids

We are looking for an oriented matroid realizing $\Delta_{8,2}$, that is, for an application $\sigma : \{\mathrm{a}, \dots, \mathrm{L}\}^7 \to \{-1, 0, 1\}$ which satisfies the four properties mentioned before.

We enumerate all possibilities by computer (using the functional language HASKELL [7, 2]), starting from the set of signs given by the necessary simplex orientations, and "guessing new signs". The method is to pick one orbit whose sign remains unknown, and to try the three possibilities $-1, 0$, and 1. Applying Grassmann-Plucker relations, we may obtain:

1. either a contradiction, and we eliminate this choice of sign for this orbit;
2. or a complete oriented matroid realizing $\Delta_{8,2}$;
3. or a certain number of additional signs, but not all, and we have to iterate the guessing process until being in situation (1) or (2).

Naturally, a good choice for the orbit we want to guess should provide as much information as possible, *i.e.*, should be involved in many Grassmann-Plucker relations for which only two signs remain unknown.

This computation provides the complete list of symmetric oriented matroids realizing our complex:

Proposition 1 *There are exactly* 15 *symmetric oriented matroids realizing* $\Delta_{8,2}$.

To present them, we only give the sign of one representant for each of the 62 orbits (under alternating relations and symmetry). First, all 15 solutions have the following 59 common signs:

$\sigma(\mathrm{abcdefg}) = 0$	$\sigma(\mathrm{abcdefI}) = -1$	$\sigma(\mathrm{abcdefJ}) = 1$
$\sigma(\mathrm{abcdefK}) = -1$	$\sigma(\mathrm{abcdegI}) = 1$	$\sigma(\mathrm{abcdegJ}) = -1$
$\sigma(\mathrm{abcdeIK}) = -1$	$\sigma(\mathrm{abcdeIL}) = 1$	$\sigma(\mathrm{abcdeJK}) = 1$
$\sigma(\mathrm{abcdfgI}) = -1$	$\sigma(\mathrm{abcdfgJ}) = 1$	$\sigma(\mathrm{abcdfgL}) = 1$
$\sigma(\mathrm{abcdfIJ}) = 1$	$\sigma(\mathrm{abcdfIK}) = 1$	$\sigma(\mathrm{abcdfIL}) = -1$
$\sigma(\mathrm{abcdfJK}) = -1$	$\sigma(\mathrm{abcdfJL}) = 1$	$\sigma(\mathrm{abcdfKL}) = -1$
$\sigma(\mathrm{abcdIJK}) = -1$	$\sigma(\mathrm{abcdIJL}) = 1$	$\sigma(\mathrm{abcdIKL}) = -1$
$\sigma(\mathrm{abcefgI}) = 0$	$\sigma(\mathrm{abcefgK}) = 0$	$\sigma(\mathrm{abcefIJ}) = -1$
$\sigma(\mathrm{abcefIK}) = 1$	$\sigma(\mathrm{abcefIL}) = 1$	$\sigma(\mathrm{abcefJK}) = -1$
$\sigma(\mathrm{abcefJL}) = -1$	$\sigma(\mathrm{abcefKL}) = 1$	$\sigma(\mathrm{abcegIJ}) = 1$
$\sigma(\mathrm{abcegIK}) = -1$	$\sigma(\mathrm{abcegIL}) = -1$	$\sigma(\mathrm{abcegKL}) = -1$
$\sigma(\mathrm{abceIJL}) = -1$	$\sigma(\mathrm{abceIKL}) = 1$	$\sigma(\mathrm{abceJKL}) = -1$
$\sigma(\mathrm{abcfIJK}) = -1$	$\sigma(\mathrm{abcfIKL}) = -1$	$\sigma(\mathrm{abcIJKL}) = 1$
$\sigma(\mathrm{abdegIJ}) = 1$	$\sigma(\mathrm{abdegIK}) = 1$	$\sigma(\mathrm{abdegIL}) = 1$
$\sigma(\mathrm{abdegJK}) = -1$	$\sigma(\mathrm{abdeIJK}) = 1$	$\sigma(\mathrm{abdeIJL}) = -1$
$\sigma(\mathrm{abdfIJL}) = 1$	$\sigma(\mathrm{abdfIKL}) = 1$	$\sigma(\mathrm{abdfJKL}) = -1$
$\sigma(\mathrm{abdgIJK}) = -1$	$\sigma(\mathrm{abdgIJL}) = -1$	$\sigma(\mathrm{abdgJKL}) = 1$
$\sigma(\mathrm{abdIJKL}) = -1$	$\sigma(\mathrm{abefIJK}) = -1$	$\sigma(\mathrm{abefIJL}) = -1$
$\sigma(\mathrm{abefJKL}) = 1$	$\sigma(\mathrm{abeIJKL}) = 1$	$\sigma(\mathrm{acegIJK}) = -1$
$\sigma(\mathrm{aceIJKL}) = -1$	$\sigma(\mathrm{acfIJKL}) = 1$	

The three remaining orbits are the orbits of abcdeIJ, abceIJK and abdfIJK. The following table summarizes the possible signs for these three orbits (only 15 of the 27 possibilities are admissible):

	$\sigma(\mathrm{abcdeIJ})$	$\sigma(\mathrm{abceIJK})$	$\sigma(\mathrm{abdfIJK})$
σ_A	-1	1	-1
σ_B	-1	1	0
σ_C	-1	1	1
σ_D	0	1	-1
σ_E	0	1	0
σ_F	0	1	1
σ_G	1	-1	-1
σ_H	1	-1	0
σ_I	1	-1	1
σ_J	1	0	-1
σ_K	1	0	0
σ_L	1	0	1
σ_M	1	1	-1
σ_N	1	1	0
σ_O	1	1	1

3.2 Symmetric polytopes

We are now ready to study the symmetric polytopal realizations of $\Delta_{8,2}$. Assume that a polytope $P \subset \mathbb{R}^6$ is a polytope realizing $\Delta_{8,2}$ and symmetric under the action of $\mathbb{D}_8$. Let $\mathrm{a}, \dots, \mathrm{L}$ denote its vertices (labeled by the corresponding 2-relevant diagonals of the octagon), let $\vec{\mathrm{a}} = (\mathrm{a}, 1), \dots, \vec{\mathrm{L}} = (\mathrm{L}, 1)$ be the corresponding vectors of homogeneous coordinates, and let M denote the matrix whose columns are $\vec{\mathrm{a}}, \dots, \vec{\mathrm{L}}$. From Subsection 3.1, we know that the submatrix $N = (\vec{\mathrm{a}}, \vec{\mathrm{b}}, \vec{\mathrm{c}}, \vec{\mathrm{I}}, \vec{\mathrm{J}}, \vec{\mathrm{K}}, \vec{\mathrm{L}})$ is invertible, and we denote $\tilde{M} = N^{-1}M$. This matrix can be written:

$$\tilde{M} = \begin{bmatrix} I_3 & T & 0_{3\times4} \\ 0_{4\times3} & B & I_4 \end{bmatrix},$$

where $T \in \mathbb{R}^{3\times5}$ and $B \in \mathbb{R}^{4\times5}$ are unknown matrices. The determinants of the submatrices of size 7 of $\tilde{M}$ are symmetric under the action of the dihedral group. We use these symmetries and the oriented matroid informations to determine the matrices T and B:

Lemma 2 *The matrix T equals*

$$\begin{bmatrix} -1 & 1+\sqrt{2} & -2-\sqrt{2} & 2+\sqrt{2} & -1-\sqrt{2} \\ -1-\sqrt{2} & 2+2\sqrt{2} & -3-2\sqrt{2} & 2+2\sqrt{2} & -1-\sqrt{2} \\ -1-\sqrt{2} & 2+\sqrt{2} & -2-\sqrt{2} & 1+\sqrt{2} & -1 \end{bmatrix}$$

Lemma 3 *There exists $u \in (-1-2\sqrt{2}, -1-3\sqrt{2}/2)$ such that B equals the matrix $B(u)$ presented in Fig. 3.*

These lemmas imply that only three of the 15 oriented matroids realizing $\Delta_{8,2}$ are realizable by a polytope with symmetric determinants: if $-1-2\sqrt{2} < u < -2-\sqrt{2}$ we obtain σ_G, if $u = -2-\sqrt{2}$ we obtain σ_K, and if $-2-\sqrt{2} < u < -1-3\sqrt{2}/2$ we obtain σ_O.

$$B(u) = \begin{bmatrix} 1+\sqrt{2}/2 & u & -(1+\sqrt{2})(1+u) & (1+\sqrt{2})(1+u)+1 & -\sqrt{2}/2-u \\ -\sqrt{2}/2-u & (1+\sqrt{2})(1+u)+1 & -(1+\sqrt{2})(1+u) & u & 1+\sqrt{2}/2 \\ 1+\sqrt{2}/2 & -2-2\sqrt{2}-u & (1+\sqrt{2})(3+u)+2 & -(1+\sqrt{2})(3+u)-1 & 2+3\sqrt{2}/2+u \\ 2+3\sqrt{2}/2+u & -(1+\sqrt{2})(3+u)-1 & (1+\sqrt{2})(3+u)+2 & -2-2\sqrt{2}-u & 1+\sqrt{2}/2 \end{bmatrix}$$

Figure 3: The bottom matrix $B(u)$.

In order to complete our understanding of the space of symmetric realizations of $\Delta_{8,2}$, it only remains to study the possible values of the matrix N. To determine N, we again use symmetry, but this time on the length of the edges of P.

For example, we know that the vertices I, J, K, and L span a 3-dimensional simplex with |IJ| = |JK| = |KL| = |IL| and |IK| = |JL| (where |IJ| denotes the euclidean distance from I to J). Thus, since neither an orthogonal transformation, nor an homothecy destruct the symmetry, we can assume that N is of the form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & y_2 & 0 & 0 & 0 & 0 & 0 \\ x_3 & y_3 & z_3 & 0 & 0 & 0 & 0 \\ x_4 & y_4 & z_4 & 1 & -1 & 1 & -1 \\ x_5 & y_5 & z_5 & v & 0 & -v & 0 \\ x_6 & y_6 & z_6 & 0 & v & 0 & -v \end{bmatrix},$$

with $x_1 > 0$, $y_2 > 0$, $z_3 > 0$, and $v > 0$.

Using the remaining equations given by the symmetries, we obtain the following constraints:

Lemma 4 *The coefficients of the matrix N satisfy*

$$x_1 = \sqrt{\frac{(2+\sqrt{2})(2y_3+z_3\sqrt{2})(y_3+z_3)z_3}{y_3-z_3}},$$

$$x_2 = \frac{y_3(2+\sqrt{2})(y_3+z_3)}{\sqrt{z_3^2-y_3^2}}, \quad y_2 = \sqrt{z_3^2-y_3^2},$$

$$x_3 = -(2+\sqrt{2})(y_3+z_3\frac{\sqrt{2}}{2}), \quad x_4 = y_4 = z_4 = 0,$$

$$x_5 = -x_6 = y_5 = y_6 = -z_5 = z_6 = -\frac{1}{2}\sqrt{2}v(1+\sqrt{2}+u),$$

with $-z_3 < y_3 < -\frac{z_3}{\sqrt{2}}$.

Reciprocally, it is easy to check that under these conditions, the convex hull of the column vectors of NM is a symmetric realization of $\Delta_{8,2}$. Thus, we obtain our main result:

Proposition 5 *The space of symmetric realizations of $\Delta_{8,2}$ has dimension 4 (up to orthogonal tranformations and homothecies of $\mathbb{R}^6$).*

4 Conclusion

Motivated by the question of the realizability of $\Delta_{n,k}$, we solved the first open case $\Delta_{8,2}$, and completely described its space of symmetric realizations. Even if this study can not be directly generalized to any n and k, we consider that it provides new evidence and motivation to the general investigation.

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