

GEOMETRIC REALIZATIONS OF THE ACCORDION COMPLEX OF A DISSECTION

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ABSTRACT. Consider $2n$ points on the unit circle and a reference dissection D_\circ of the convex hull of the odd points. The accordion complex of D_\circ is the simplicial complex of non-crossing subsets of the diagonals with even endpoints that cross a connected subset of diagonals of D_\circ . In particular, this complex is an associahedron when D_\circ is a triangulation and a Stokes complex when D_\circ is a quadrangulation. In this paper, we provide geometric realizations (by polytopes and fans) of the accordion complex of any reference dissection D_\circ , generalizing known constructions arising from cluster algebras.

KEYWORDS. Permutahedra · Zonotopes · Associahedra · \mathbf{g} -, \mathbf{c} - and \mathbf{d} -vectors.

The $(n-3)$ -dimensional *associahedron* is a simple polytope whose face poset is isomorphic to the reverse inclusion poset of non-crossing subsets of diagonals of a convex n -gon. Introduced in early works of D. Tamari [Tam51] and J. Stasheff [Sta63], it was first realized as a convex polytope by M. Haiman [Hai84] and C. Lee [Lee89], and later constructed by more systematic methods developed by several authors, in particular [GKZ08, Lod04, HL07, CSZ15]. Various relevant generalizations of the associahedron were introduced and studied, in particular secondary polytopes and fiber polytopes [GKZ08, BFS90], generalized associahedra [FZ03b, CFZ02, HLT11, Ste13, Hoh] in connection to cluster algebras [FZ02, FZ03a], graph associahedra [CD06, Pos09, FS05, Zel06, Pil13, MP17], or brick polytopes [PS12, PS15].

In a different context, Y. Baryshnikov [Bar01] introduced the simplicial complex of crossing-free subsets of the set of diagonals of a polygon that are in some sense compatible with a reference quadrangulation Q_\circ . Although the precise definition of compatibility is a bit technical in [Bar01], it turns out that a diagonal is compatible with Q_\circ if and only if it crosses a connected subset of diagonals of Q_\circ that we call *accordion* of Q_\circ . We thus call Y. Baryshnikov's simplicial complex the *accordion complex* $\mathcal{AC}(Q_\circ)$. A polytopal realization of $\mathcal{AC}(Q_\circ)$ was announced in [Bar01], but the explicit construction and its proof were never published as far as we know. Revisiting some combinatorial and algebraic properties of $\mathcal{AC}(Q_\circ)$, F. Chapoton [Cha16, Intro.p.4] raised three explicit challenges: first prove that the oriented dual graph of $\mathcal{AC}(Q_\circ)$ has a lattice structure extending the Tamari and Cambrian lattices [MHPS12, Rea06]; second construct geometric realizations of $\mathcal{AC}(Q_\circ)$ as fans and polytopes generalizing the known constructions of the associahedron; third show that the facets of $\mathcal{AC}(Q_\circ)$ are in bijection with other combinatorial objects called serpent nests [Cha16, Sect. 4].

In [GM16], A. Garver and T. McConville defined and studied the accordion complex $\mathcal{AC}(D_\circ)$ of any reference dissection D_\circ (their presentation slightly differs as they use a compatibility condition on the dual tree of the dissection D_\circ , but the simplicial complex is the same). In this context, they settled F. Chapoton's lattice question, using lattice quotients of a lattice of biclosed sets. In this paper, we present geometric realizations of $\mathcal{AC}(D_\circ)$ for any reference dissection D_\circ , providing in particular an answer to F. Chapoton's geometric question. In fact, we present three methods to realize $\mathcal{AC}(D_\circ)$ based on constructions of the classical associahedron.

Our first method is based on the \mathbf{g} -vector fan. It belongs to a series of constructions of the (generalized) associahedra initiated by S. Shnider and S. Sternberg [SS93], popularised by J.-L. Loday [Lod04], developed by C. Hohlweg, C. Lange and H. Thomas [HL07, HLT11] using works of N. Reading and D. Speyer [Rea06, Rea07, RS09], and revisited by S. Stella [Ste13] and by V. Pilaud, F. Santos, and C. Stump [PS12, PS15]. It was recently extended by C. Hohlweg, V. Pilaud, and S. Stella [HPS17] to construct an associahedron parametrized by any initial triangulation. Here, we first extend to the D_\circ -accordion complex $\mathcal{AC}(D_\circ)$ the \mathbf{g} -vectors and \mathbf{c} -vectors

defined in the context of cluster algebras by S. Fomin and A. Zelevinski [FZ07]. Note that \mathbf{c} -vectors were already implicitly considered in [GM16], while \mathbf{g} -vectors are new in this context. When D_\circ is a triangulation, our definitions coincide with those given in terms of triangulations and laminations for cluster algebras from surfaces by S. Fomin and D. Thurston [FT12]. We then show that the \mathbf{g} -vectors with respect to the dissection D_\circ support a complete simplicial fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$ realizing the D_\circ -accordion complex $\mathcal{AC}(D_\circ)$. Finally, we construct a D_\circ -accordiohedron $\text{Acco}(D_\circ)$ realizing the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$ by deleting inequalities from the facet description of the D_\circ -zonotope $\text{Zono}(D_\circ)$ obtained as the Minkowski sum of all \mathbf{c} -vectors. See Figure 7 for an illustration of D_\circ -accordiohedra.

Our second method is based on the \mathbf{d} -vector fan. This construction is inspired from the original cluster fan of S. Fomin and A. Zelevinsky [FZ03a] later realized as a polytope by F. Chapoton, S. Fomin and A. Zelevinsky [CFZ02], and from the generalization of C. Ceballos, F. Santos and G. Ziegler [CSZ15] to construct a compatibility fan and an associahedron from any initial triangulation. For any reference dissection D_\circ , we associate to each diagonal a \mathbf{d} -vector which records the crossings of this diagonal with those of D_\circ . We show that the \mathbf{d} -vectors support a complete simplicial fan realizing the D_\circ -accordion complex $\mathcal{AC}(D_\circ)$ if and only if D_\circ contains no even interior cell. The polytopality of the resulting fan remains open in general, but was shown for arbitrary triangulations in [CSZ15].

Finally, our third method is based on projections of associahedra. Namely, for any dissection D_\circ and triangulation T_\circ such that $D_\circ \subseteq T_\circ$, the accordion complex $\mathcal{AC}(D_\circ)$ is a subcomplex of the simplicial associahedron $\mathcal{AC}(T_\circ)$. It turns out that the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$ is then a section of the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(T_\circ)$ by a coordinate subspace. Therefore, the accordion complex $\mathcal{AC}(D_\circ)$ is realized by a projection of the associahedron $\text{Asso}(T_\circ)$ of [HPS17]. This point of view provides a complementary perspective on accordion complexes that leads on the one hand to more concise but less instructive proofs of combinatorial and geometric properties of the accordion complex (pseudomanifold, \mathbf{g} -vector fan, accordiohedron), and on the other hand to natural extensions to coordinate sections of the \mathbf{g} -vector fan in arbitrary cluster algebras.

As recently observed in [GM16, PPP17, PPS17, BDM⁺17], accordion complexes are prototypes of support τ -tilting complexes introduced in [AIR14], for certain associative algebras called gentle algebras. In this context, \mathbf{g} -vectors have a deep algebraic meaning and still define a \mathbf{g} -vector fan. Although this fan is still polytopal for finite support τ -tilting complexes, it is not in general obtained by deleting inequalities in the facet description of a zonotope. We refer to [PPP17, Part 4] for details.

The paper is organized as follows. Section 1 introduces the accordion complex and accordion lattice of a dissection D_\circ . We essentially follow the definitions and arguments of A. Garver and T. McConville [GM16], except that we prefer to work on the dissection D_\circ rather than on its dual graph. Section 2 is devoted to the generalization of the \mathbf{g} -vector fan and the associahedra of [HL07, HPS17]. Section 3 discusses the generalization of the construction of the \mathbf{d} -vector fan and associahedra of [FZ03a, CSZ15]. Finally, Section 4 shows that the accordion complex is realized by a projection of a well-chosen associahedron and presents related questions on cluster algebras, subcomplexes of the cluster complex, and sections of the \mathbf{g} -vector fan.

1. THE ACCORDION COMPLEX AND THE ACCORDION LATTICE

In this section, we define the accordion complex $\mathcal{AC}(D_\circ)$ of a dissection D_\circ , show that it is a pseudomanifold, and define an orientation of its dual graph. Our definitions and proofs are essentially translations of the arguments of A. Garver and T. McConville [GM16] given in terms of the dual tree of the dissection D_\circ . However our presentation in terms of dissections is more convenient for our purposes.

1.1. The accordion complex. Let P be a convex polygon. We call *diagonals* of P the segments connecting two vertices of P . This includes both the internal diagonals and the external diagonals (or boundary edges) of P . A *dissection* of P is a set D of non-crossing internal diagonals of P . The *cells* of D are the closures of the connected components of P minus the diagonals of D . A *triangulation* (resp. *quadrangulation*) is a dissection whose cells are all triangles (resp. quadrangles).

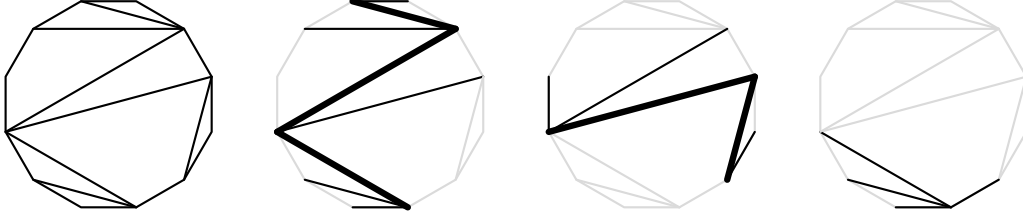


FIGURE 1. A dissection D (left) and three accordions whose zigzags are bolded (middle and right).

We denote by \bar{D} the dissection D together with all boundary edges of P . A *cut* of D is the subset of \bar{D} intersected by a line crossing two boundary edges of P . An *accordion* is a connected cut. By definition, an accordion is a tree and contains two boundary edges of P . The *zigzag* of an accordion A is the chain obtained by deleting all degree 1 vertices of A . A *subaccordion* of D is a connected subset of D intersected by a segment in the interior of P . Note that any subaccordion of an accordion A consists of the diagonals of A between two internal diagonals of A . Note that we include boundary edges of P in the accordions of D , but not in the subaccordions nor in the zigzags of D . See Figure 1.

We consider $2n$ points on the unit circle labeled clockwise by $1_\circ, 2_\bullet, 3_\circ, 4_\bullet, \dots, (2n-1)_\circ, (2n)_\bullet$. We say that $1_\circ, \dots, (2n-1)_\circ$ are the *hollow vertices* while $2_\bullet, \dots, (2n)_\bullet$ are the *solid vertices*. The *hollow polygon* is the convex hull P_\circ of $1_\circ, \dots, (2n-1)_\circ$ while the *solid polygon* is the convex hull P_\bullet of $2_\bullet, \dots, (2n)_\bullet$. We simultaneously consider *hollow diagonals* δ_\circ (with two hollow vertices) and *solid diagonals* δ_\bullet (with two solid vertices), but we never consider diagonals with one hollow vertex and one solid vertex. Similarly, we consider *hollow dissections* D_\circ (of the hollow polygon, with only hollow diagonals) and *solid dissections* D_\bullet (of the solid polygon, with only solid diagonals), but never mix hollow and solid diagonals in a dissection. To help distinguish them, hollow (resp. solid) vertices and diagonals appear red (resp. blue) in all pictures.

We fix an arbitrary reference hollow dissection D_\circ . A solid diagonal δ_\bullet is a *D_\circ -accordion diagonal* if the hollow diagonals of \bar{D}_\circ crossed by δ_\bullet form an accordion of D_\circ . In other words, δ_\bullet cannot enter and exit a cell of D_\circ using two non-incident diagonals. For example, note that for any hollow diagonal $i_\circ j_\circ \in \bar{D}_\circ$, the solid diagonals $(i-1)_\bullet(j-1)_\bullet$ and $(i+1)_\bullet(j+1)_\bullet$ are D_\circ -accordion diagonals (here and throughout, labels are considered modulo $2n$). In particular, all boundary edges of the solid polygon are D_\circ -accordion diagonals. A *D_\circ -accordion dissection* is a set of non-crossing internal D_\circ -accordion diagonals. We define the *D_\circ -accordion complex* to be the simplicial complex $\mathcal{AC}(D_\circ)$ of D_\circ -accordion dissections.

Example 1. As a running example, we consider the reference dissection D_\circ^{ex} of Figure 2 (left). Examples of maximal D_\circ^{ex} -accordion dissections are given in Figure 2 (right). The D_\circ^{ex} -accordion complex is illustrated in Figure 3 (left).

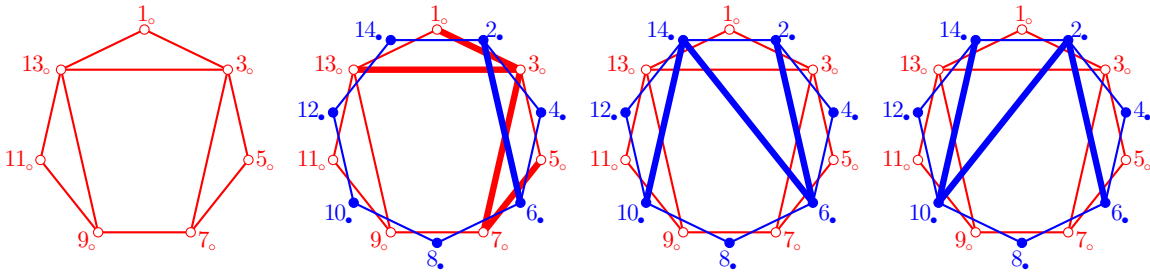


FIGURE 2. A hollow dissection D_\circ^{ex} , a solid D_\circ^{ex} -accordion diagonal whose corresponding hollow accordion is bolded, and two maximal solid D_\circ^{ex} -accordion dissections.

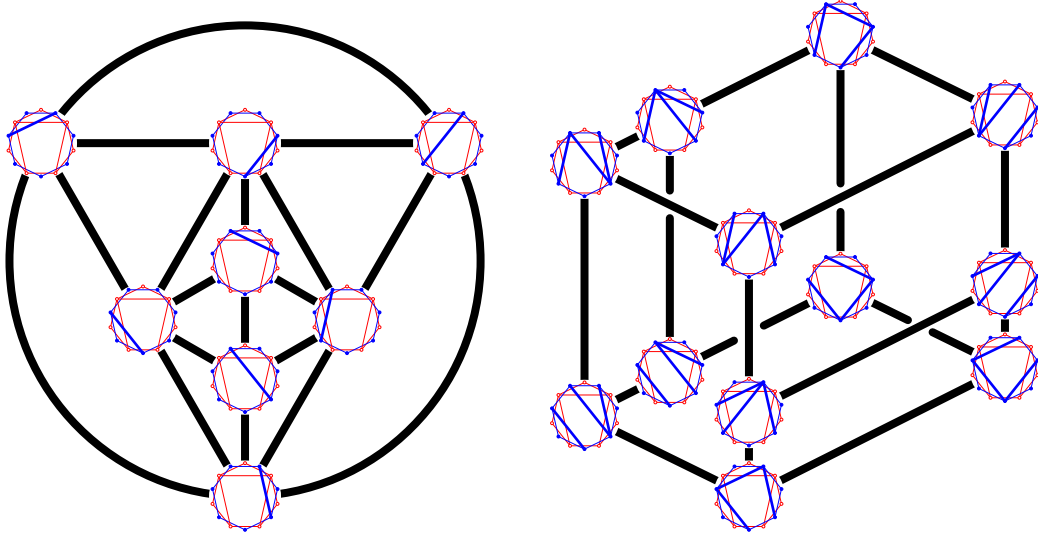


FIGURE 3. The D_0^{ex} -accordion complex (left) and the D_0^{ex} -accordion lattice (right), oriented from bottom to top, for the reference hollow dissection D_0^{ex} of Figure 2 (left).

Example 2. Special reference hollow dissections D_0 give rise to special accordion complexes $\mathcal{AC}(D_0)$:

- ◊ If D_0 is the empty dissection with the whole hollow polygon as unique cell, then the D_0 -accordion complex $\mathcal{AC}(D_0)$ is reduced to the empty D_0 -accordion dissection.
- ◊ If D_0 has a unique internal diagonal, then the D_0 -accordion complex $\mathcal{AC}(D_0)$ consists of only two points.
- ◊ For a hollow triangulation T_0 , all solid diagonals are T_0 -accordions, so that the T_0 -accordion complex $\mathcal{AC}(T_0)$ is the simplicial associahedron.
- ◊ For a hollow quadrangulation Q_0 , a solid diagonal is a Q_0 -accordion if and only if it does not cross two opposite edges of a quadrangle of Q_0 . The Q_0 -accordion complex $\mathcal{AC}(Q_0)$ is thus the Stokes complex defined by Y. Baryshnikov [Bar01] and studied by F. Chapoton [Cha16].

Remark 3. Following the original definition of the non-crossing complex of A. Garver and T. McConville [GM16], the accordion complex could equivalently be defined in terms of the dual tree D_0^* of D_0 (with one node in each cell of D and one edge connecting two adjacent cells). More precisely, the duality provides the following dictionary between the two definitions:

present paper		A. Garver and T. McConville [GM16]
reference dissection D_0	\longleftrightarrow	embedded tree D_0^*
diagonal $u_\bullet v_\bullet$ of P_\bullet	\longleftrightarrow	path connecting the leaves u_\bullet^* and v_\bullet^* of D_0^*
D_0 -accordion diagonal	\longleftrightarrow	arc (path where any two consecutive edges belong to the boundary of a face of the complement of D_0^* in the unit disk)
D_0 -subaccordion	\longleftrightarrow	segment
D_0 -accordion complex	\longleftrightarrow	non-crossing complex of D_0^*

The **g**-, **c**- and **d**-vectors defined in Section 2.1 could as well be defined in terms of D_0^* . In fact, **c**-vectors were already implicitly considered in [GM16], while **g**- and **d**-vectors are new in this context. For this paper, we find more convenient to work directly with dissections, in particular in Sections 3 and 4.

1.2. Two structural observations. Before studying the accordion complex in details in Section 1.3, we present two simple structural observations. For this, let us recall two classical notions on simplicial complexes. The *join* of two simplicial complexes Δ, Δ' with disjoint ground sets X, X' is the simplicial complex $\Delta * \Delta'$ with ground set $X \sqcup X'$ whose faces are disjoint unions of faces

of Δ with faces of Δ' . For a face D in a simplicial complex Δ on X , the *link* of D is the simplicial complex on $X \setminus D$ whose faces are the subsets D' of $X \setminus D$ such that $D \cup D'$ is a face of Δ .

Proposition 4. *If the reference hollow dissection D_\circ has a cell containing p boundary edges of the hollow polygon P_\circ , then the D_\circ -accordion complex $\mathcal{AC}(D_\circ)$ is the join of p accordion complexes.*

Proof. Assume that D_\circ has a cell C_\circ containing p boundary edges of the hollow polygon P_\circ . Let $C_\circ^1, \dots, C_\circ^p$ denote the p (possibly empty) connected components of the hollow polygon minus C_\circ . For $i \in [p] := \{1, \dots, p\}$, let D_\circ^i denote the dissection formed by the cell C_\circ together with the cells of D_\circ contained in the closure of C_\circ^i . Observe that for $i \neq j$, the internal diagonals of D_\circ^i are not incident to the internal diagonals of D_\circ^j . Thus, no D_\circ -accordion can contain internal diagonals from distinct dissections D_\circ^i and D_\circ^j . Therefore, the set of D_\circ -accordion diagonals is the union of the sets of D_\circ^i -accordion diagonals for $i \in [p]$. Moreover, for $i \neq j$, the D_\circ^i -accordion diagonals do not cross the D_\circ^j -accordion diagonals. It follows that the D_\circ -accordion complex is the join of the D_\circ^i -accordion complexes: $\mathcal{AC}(D_\circ) = \mathcal{AC}(D_\circ^1) * \dots * \mathcal{AC}(D_\circ^p)$. \square

Remark 5. In view of Proposition 4, we can do the following reductions:

- (i) If a non-triangular cell of D_\circ has two consecutive boundary edges $\gamma_\circ, \delta_\circ$ of the hollow polygon, then contracting γ_\circ and δ_\circ to a single boundary edge preserves the D_\circ -accordion complex.
- (ii) If a cell of D_\circ has two non-consecutive boundary edges of the hollow polygon, then the D_\circ -accordion complex is a join of smaller accordion complexes.

In all the examples of the paper, we therefore only consider dissections where any non-triangular cell of D_\circ has at most one boundary edge. All of our constructions work in general, but are just obtained as products or joins of the non-degenerate situation.

Proposition 6. *The links in an accordion complex are joins of accordion complexes.*

Proof. Consider a D_\circ -accordion dissection D_\bullet with cells $C_\bullet^1, \dots, C_\bullet^p$. Let D_\circ^i denote the hollow dissection obtained from D_\circ by contracting all hollow boundary edges which do not cross C_\bullet^i . Then a diagonal δ_\bullet of a cell C_\bullet^i is a D_\circ -accordion diagonal if and only if it is a D_\circ^i -accordion diagonal. Moreover, for $i \neq j$, the diagonals of C_\bullet^i do not cross the diagonals of C_\bullet^j . It follows that the link of D_\bullet in $\mathcal{AC}(D_\circ)$ is isomorphic to the join $\mathcal{AC}(D_\circ^1) * \dots * \mathcal{AC}(D_\circ^p)$. \square

1.3. Pseudo-manifold. We now prove that the accordion complex $\mathcal{AC}(D_\circ)$ is a *pseudomanifold*, *i.e.* that it is:

- (i) *pure*: all maximal D_\circ -accordion dissections have the same number of diagonals as D_\circ , and
- (ii) *thin*: any codimension 1 simplex of $\mathcal{AC}(D_\circ)$ is contained in exactly two maximal D_\circ -accordion dissections.

We follow the arguments of A. Garver and T. McConville [GM16] (except that they work on the dual tree of the dissection D_\circ). A much more concise but less instructive proof of the pseudomanifold property will be derived from geometric considerations in Remark 60.

Recall that we denote by \overline{D}_\circ the set formed by D_\circ together with all boundary edges of the hollow polygon. An *angle* $u_\circ v_\circ w_\circ$ of \overline{D}_\circ is a pair $\{u_\circ v_\circ, v_\circ w_\circ\}$ of two consecutive diagonals of \overline{D}_\circ around a common vertex v_\circ , called the *apex*. Note that \overline{D}_\circ has $2|D_\circ| + n = 2|\overline{D}_\circ| - n$ angles. Observe also that an accordion A_\circ of D_\circ can be seen as a sequence of $|A_\circ| - 1$ angles where two consecutive angles are separated by a diagonal of A_\circ . We say that a solid vertex p_\bullet belongs to an angle $u_\circ v_\circ w_\circ$ if it lies in the cone generated by the edges $v_\circ u_\circ$ and $v_\circ w_\circ$ of the angle. The main observation is given in the following statement.

Lemma 7. *Let D_\bullet be a maximal D_\circ -accordion dissection, and let $p_\bullet, q_\bullet, r_\bullet, s_\bullet$ denote four consecutive vertices of a cell C_\bullet of D_\bullet (with possibly $p_\bullet = s_\bullet$ if C_\bullet is a triangle). Then p_\bullet and s_\bullet belong to the same angle of the accordion of \overline{D}_\circ which is crossed by $q_\bullet r_\bullet$.*

Proof. Let A_\circ be the accordion of \overline{D}_\circ which is crossed by $q_\bullet r_\bullet$. Assume that p_\bullet and s_\bullet belong to distinct angles of A_\circ . Then they are separated by a diagonal ε_\circ of A_\circ . Therefore, there are two boundary edges $q_\bullet r_\bullet$ and $u_\bullet v_\bullet$ of C_\bullet with distinct vertices such that the hollow diagonal ε_\circ separates the vertices q_\bullet, u_\bullet from the vertices r_\bullet, v_\bullet . Let $\gamma_\circ^1, \dots, \gamma_\circ^i = \varepsilon_\circ, \dots, \gamma_\circ^a$ (resp. $\delta_\circ^1, \dots, \delta_\circ^j = \varepsilon_\circ, \dots, \delta_\circ^b$)

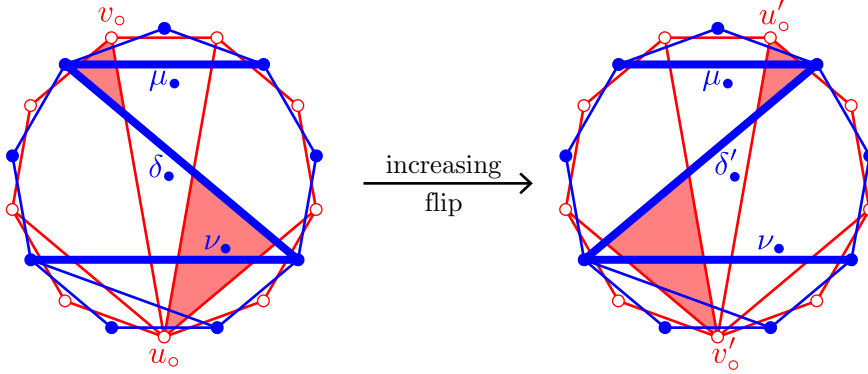


FIGURE 4. Two maximal D_0 -accordion dissection $D_•$ (left) and $D'_•$ (right) related by the flip of $\delta_•$ to $\delta'_•$. The angles of D_0 closed by $\delta_•$ and $\delta'_•$ are shaded. The flip is oriented from $D_•$ to $D'_•$.

denote the diagonals of D_0 crossed by $q_•r_•$ from $q_•$ to $r_•$ (resp. crossed by $u_•v_•$ from $u_•$ to $v_•$). Then the hollow diagonals $\gamma_0^1, \dots, \gamma_0^i = \varepsilon_0 = \delta_0^j, \dots, \delta_0^b$ which are crossed by $q_•v_•$ also form an accordion. It follows that $D_•$ is not maximal as we can still include $q_•v_•$. \square

Consider now an angle $u_0v_0w_0$ of \overline{D}_0 . In any maximal D_0 -accordion dissection $D_•$, the set $X_•$ of diagonals of $\overline{D}_•$ that cross both u_0v_0 and v_0w_0 is non-empty (since it contains the boundary edge $(v-1)_•(v+1)_•$) and totally ordered (since the diagonals of $D_•$ do not cross). Let $\delta_•$ be the largest diagonal of $X_•$ (meaning the farthest from v_0). We say that the diagonal $\delta_•$ *closes* the angle $u_0v_0w_0$. Note that each angle of \overline{D}_0 is closed by precisely one diagonal of $\overline{D}_•$. The following lemma is stated and proved in [GM16] in terms of the dual tree D_0^* of the dissection D_0 .

Lemma 8 ([GM16]). *For any maximal D_0 -accordion dissection $D_•$, each internal diagonal $\delta_•$ of $D_•$ closes two angles of \overline{D}_0 (one apex on each side of $\delta_•$) while each boundary edge of the solid polygon closes one angle of \overline{D}_0 . Therefore the accordion complex $\mathcal{AC}(D_0)$ is pure of dimension $|D_0|$.*

Proof. The first sentence is a consequence of Lemma 7: for any four consecutive vertices $p_•, q_•, r_•, s_•$ of a cell of $\overline{D}_•$, the diagonal $q_•r_•$ closes the unique angle of the accordion of \overline{D}_0 crossed by $q_•r_•$ that contains the vertices $p_•$ and $s_•$. Therefore, $q_•r_•$ closes precisely two angles (resp. one angle) of D_0 if it is an internal diagonal (resp. a boundary edge of the solid polygon). We finally obtain by double-counting that $2|D_0| + n = |\{\text{angles of } \overline{D}_0\}| = 2|D_•| + n$ and thus $|D_•| = |D_0|$ for any maximal D_0 -accordion dissection $D_•$. \square

We are now ready to prove that the D_0 -accordion complex is thin, *i.e.* that each internal diagonal of a maximal D_0 -accordion dissection can be flipped into a unique other internal diagonal to form a new maximal D_0 -accordion dissection. Here and throughout the paper, $X \triangle Y$ denotes the symmetric difference of two sets X, Y defined by $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$.

The following notations are illustrated in Figure 4. Let $D_•$ be a maximal D_0 -accordion dissection and $\delta_•$ be a diagonal of $D_•$. Let u_0 and v_0 be the apices of the angles of D_0 closed by $\delta_•$, let $\mu_•$ and $\nu_•$ denote the edges of the cells of $D_•$ containing $\delta_•$, which separate $\delta_•$ from u_0 and v_0 respectively, and let $Q_•$ denote the quadrilateral defined by the four vertices of $\mu_•$ and $\nu_•$. Note that $\delta_•$ is a diagonal of $Q_•$, and let $\delta'_•$ denote the other diagonal.

Lemma 9 ([GM16]). *With the previous notations, the collection of diagonals $D'_• := D_• \triangle \{\delta_•, \delta'_•\}$ is a maximal D_0 -accordion dissection, and $D_•$ and $D'_•$ are the only maximal D_0 -accordion dissections containing $D_• \setminus \{\delta_•\}$. In other words, the accordion complex $\mathcal{AC}(D_0)$ is thin.*

Proof. We first observe that $\delta'_•$ is a D_0 -accordion diagonal, since the edges of \overline{D}_0 crossed by $\delta'_•$ are obtained by merging three subaccordions of D_0 : the subaccordion formed by the diagonals of \overline{D}_0 crossed by $\mu_•$ but not $\delta_•$ nor $\nu_•$, the subaccordion formed by the diagonals of \overline{D}_0 crossed by $\delta_•, \mu_•$ and $\nu_•$, and the subaccordion formed by the diagonals of \overline{D}_0 crossed by $\nu_•$ but not $\delta_•$ nor $\mu_•$.

Moreover, δ_\bullet and δ'_\bullet are the only D_\circ -accordion diagonals compatible with $D_\bullet \setminus \{\delta_\bullet\}$. Indeed, any other such diagonal would cross δ_\bullet and δ'_\bullet (by maximality of D_\bullet and D'_\bullet), and thus also the subaccordion A_\circ of D_\circ crossed by δ_\bullet and δ'_\bullet (because it cannot cross μ and ν). But it would then improperly intersect the two cells of D_\circ containing precisely one diagonal of A_\circ . \square

The D_\circ -*accordion flip graph* is the dual graph $\mathcal{AFG}(D_\circ)$ of the D_\circ -accordion complex: its vertices are the maximal D_\circ -accordion dissections, and its edges are the *flips* between them, *i.e.* the pairs $\{D_\bullet, D'_\bullet\}$ of maximal D_\circ -accordion dissections with $D_\bullet \setminus \{\delta_\bullet\} = D'_\bullet \setminus \{\delta'_\bullet\}$. See Figure 3 (right).

1.4. The accordion lattice. We now define a natural orientation on the D_\circ -accordion flip graph. We use the same notations as in Lemma 9 (see also Figure 4), where $D_\bullet \setminus \{\delta_\bullet\} = D'_\bullet \setminus \{\delta'_\bullet\}$ and $\delta_\bullet, \delta'_\bullet$ are the two diagonals of the quadrilateral defined by μ_\bullet, ν_\bullet . Observe that one of the path $\mu_\bullet \delta_\bullet \nu_\bullet$ and $\mu_\bullet \delta'_\bullet \nu_\bullet$ forms a Σ while the other forms a Z , see Figure 4. We then orient the flip from the dissection containing the Σ to that containing the Z . See Figure 3 (right) for an illustration of D_\circ -accordion oriented flip graph (where the graph is oriented from bottom to top).

A. Garver and T. McConville introduced a natural closure on sets of D_\circ -subaccordions, and showed that the inclusion poset of biclosed sets of D_\circ -subaccordions is a well-behaved lattice (namely, semidistributive, congruence-uniform and polygonal). Then, they introduced a lattice quotient map from biclosed sets of D_\circ -subaccordions to maximal D_\circ -accordion dissections, which imply the following statement.

Theorem 10 ([GM16]). *The D_\circ -accordion oriented flip graph is the Hasse diagram of a lattice, that we call the D_\circ -accordion lattice and denote by $\mathcal{AL}(D_\circ)$.*

In particular, the D_\circ -accordion oriented flip graph is connected and acyclic, and has a unique source $D_\bullet^- := \{(i-1)_\bullet(j-1)_\bullet \mid i_\circ j_\circ \in D_\circ\}$ (obtained by slightly rotating D_\circ counterclockwise) and a unique sink $D_\bullet^+ := \{(i+1)_\bullet(j+1)_\bullet \mid i_\circ j_\circ \in D_\circ\}$ (obtained by slightly rotating D_\circ clockwise).

Example 11. Following Example 2, note that special reference hollow dissections D_\circ give rise to special accordion lattices $\mathcal{AL}(D_\circ)$, as it was already observed in [GM16]:

- ◊ For a fan triangulation F_\circ (*i.e.* where all internal diagonals are incident to a common vertex), the F_\circ -accordion lattice $\mathcal{AL}(F_\circ)$ is the famous Tamari lattice [Tam51, MHPS12] defined equivalently by slope increasing flips on triangulations of a convex polygon, by right rotations on binary trees, or by flips on Dyck paths.
- ◊ In general, accordion lattices of accordion triangulations (*i.e.* with no interior triangle) precisely correspond to type *A* Cambrian lattices defined by N. Reading [Rea06].
- ◊ For an arbitrary triangulation T_\circ (with or without interior triangle), the T_\circ -accordion oriented flip graph $\mathcal{AFG}(A_\circ)$ is a particular instance of the oriented exchange graphs of 2-acyclic quivers defined by T. Brüstle, G. Dupont and M. Pérotin [BDP14]. These oriented exchange graphs are far more general and their transitive closures are in general not lattices.
- ◊ For a quadrangulation Q_\circ , the Q_\circ -accordion lattice $\mathcal{AL}(Q_\circ)$ is the Stokes poset on Q_\circ -compatible quadrangulations studied by F. Chapoton [Cha16].

The following statement is a direct consequence of Proposition 4.

Proposition 12. *If the reference hollow dissection D_\circ has a cell containing p boundary edges of the hollow polygon P_\circ , then the D_\circ -accordion lattice $\mathcal{AL}(D_\circ)$ is a Cartesian product of p accordion lattices.*

Proof. Consider the dissections $D_\circ^1, \dots, D_\circ^p$ as in the proof of Proposition 4. Since any increasing flip in $\mathcal{AL}(D_\circ)$ is an increasing flip in one of the $\mathcal{AL}(D_\circ^i)$, we obtain that the D_\circ -accordion lattice is the Cartesian product of the D_\circ^i -accordion lattices: $\mathcal{AL}(D_\circ) = \mathcal{AL}(D_\circ^1) \times \dots \times \mathcal{AL}(D_\circ^p)$. \square

In particular, if two consecutive boundary edges $\gamma_\circ, \delta_\circ$ of the hollow polygon belong to the same non-triangular cell of D_\circ , then contracting γ_\circ and δ_\circ to a single boundary edge preserves the D_\circ -accordion lattice. This shows the following statement conjectured for quadrangulations in [Cha16] and proved in [BMP16].

Corollary 13. *Consider an accordion dissection A_\circ , i.e. a dissection where each cell has at most 2 edges which are internal diagonals of P_\circ . Then the A_\circ -accordion lattice is a Cambrian lattice.*

Remark 14. Call *cell-sequence* of a dissection the sequence whose i th entry is its number of $(i+2)$ -cells. For example, the dissection of Figure 2 (left) has cell-sequence $3, 1, 0^\infty$ and all $(p+2)$ -angulations of a $(pm+2)$ -gon have cell-sequence $0^{p-1}, m, 0^\infty$. Observe that the flip preserves the cell-sequence. Thus, all maximal D_\circ -accordion dissections have the same cell-sequence as D_\circ .

We conclude this section with a reciprocity result on accordion dissections.

Proposition 15. *Let D_\circ be a hollow dissection and D_\bullet be a solid dissection. Then D_\bullet is a maximal D_\circ -accordion dissection if and only if D_\circ is a maximal D_\bullet -accordion dissection.*

Proof. Since $D_\bullet^- := \{(i-1)\bullet(j-1)\bullet \mid i_\circ j_\circ \in D_\circ\}$ and $D_\bullet^+ := \{(i+1)\bullet(j+1)\bullet \mid i_\circ j_\circ \in D_\circ\}$ are both D_\circ -accordion dissections, we already know that D_\circ is a D_\bullet^- -accordion dissection. Observe now in Figure 4 that if D_\bullet and D'_\bullet are maximal D_\circ -accordion dissections connected by a flip, then D_\circ is a D_\bullet -accordion dissection if and only if it is a D'_\bullet -accordion dissection. Indeed, if δ_\bullet belongs to the zigzag of the D_\bullet -accordion A_\bullet of a hollow diagonal δ_\circ , then δ_\circ crosses both μ_\bullet and ν_\bullet , but then δ_\circ also crosses δ'_\bullet , and thus δ_\circ crosses the D'_\bullet -accordion $A_\bullet \triangle \{\delta_\bullet, \delta'_\bullet\}$. Since the D_\circ -accordion flip graph is connected, we obtain that D_\circ is a D_\bullet -accordion dissection for any maximal D_\circ -accordion dissection D_\bullet . Finally, maximality follows since all maximal D_\circ -accordion dissections have $|D_\circ|$ diagonals. The equivalence follows by symmetry. \square

2. THE \mathbf{g} -VECTOR FAN

In this Section, we construct accordiohedra using \mathbf{g} - and \mathbf{c} -vectors. Our construction is in the same spirit as the Cambrian fans of N. Reading and D. Speyer [Rea06, Rea07, RS09] and their polytopal realizations by C. Hohlweg, C. Lange and H. Thomas [HL07, HLT11], recently extended in [HPS17] to any initial triangulation, acyclic or not. A different approach to the \mathbf{g} -vector fan together with an alternative polytopal realization will be presented in Section 4.

2.1. \mathbf{g} - and \mathbf{c} -vectors. Consider a hollow dissection D_\circ and a solid dissection D_\bullet that are maximal accordion dissections of each other (see Proposition 15), and let $\delta_\circ \in D_\circ$ and $\delta_\bullet \in D_\bullet$. When δ_\circ crosses δ_\bullet , we let μ_\circ and ν_\circ be the other diagonals of \bar{D}_\circ crossed by δ_\bullet in the two cells of D_\circ containing δ_\circ . We say that δ_\bullet *slaloms* on δ_\circ if $\mu_\circ \delta_\circ \nu_\circ$ forms a path, and we define $\varepsilon_\circ(\delta_\circ \in D_\circ \mid \delta_\bullet)$ to be 1, -1 , or 0 depending on whether $\mu_\circ \delta_\circ \nu_\circ$ forms a \mathbf{Z} , a $\mathbf{\Sigma}$, or a \mathbf{V} . Similarly we let μ_\bullet and ν_\bullet be the other diagonals of \bar{D}_\bullet crossed by δ_\circ in the two cells of D_\bullet containing δ_\bullet , we say that δ_\circ slaloms on δ_\bullet if $\mu_\bullet \delta_\bullet \nu_\bullet$ forms a path, and we define $\varepsilon_\bullet(\delta_\circ \mid \delta_\bullet \in D_\bullet)$ to be 1, -1 , or 0 depending on whether $\mu_\bullet \delta_\bullet \nu_\bullet$ forms a $\mathbf{\Sigma}$, a \mathbf{Z} , or a \mathbf{V} . Note that the sign convention for $\varepsilon_\circ(\delta_\circ \in D_\circ \mid \delta_\bullet)$ and $\varepsilon_\bullet(\delta_\circ \mid \delta_\bullet \in D_\bullet)$ is opposite: the reciprocity already observed in Proposition 15 naturally reverses the orientation. More informally, we exchange the role of hollow and solid dissections by looking at the picture from the opposite side of the blackboard, which of course reverses the orientation. Finally, if δ_\circ and δ_\bullet do not cross, then we let $\varepsilon_\circ(\delta_\circ \in D_\circ \mid \delta_\bullet) = \varepsilon_\bullet(\delta_\circ \mid \delta_\bullet \in D_\bullet) = 0$. Let $(\mathbf{e}_{\delta_\circ})_{\delta_\circ \in D_\circ}$ denote the canonical basis of \mathbb{R}^{D_\circ} . As in [HPS17], we define the following vectors:

- (i) the **\mathbf{g} -vector** of δ_\bullet with respect to D_\circ is $\mathbf{g}(D_\circ \mid \delta_\bullet) := \sum_{\delta_\circ \in D_\circ} \varepsilon_\circ(\delta_\circ \in D_\circ \mid \delta_\bullet) \mathbf{e}_{\delta_\circ}$. We also define $\mathbf{g}(D_\circ \mid D_\bullet) := \{\mathbf{g}(D_\circ \mid \delta_\bullet) \mid \delta_\bullet \in D_\bullet\}$.
- (ii) the **\mathbf{c} -vector** of $\delta_\bullet \in D_\bullet$ with respect to D_\circ is $\mathbf{c}(D_\circ \mid \delta_\bullet \in D_\bullet) := \sum_{\delta_\circ \in D_\circ} \varepsilon_\bullet(\delta_\circ \mid \delta_\bullet \in D_\bullet) \mathbf{e}_{\delta_\circ}$. We denote by $\mathbf{c}(D_\circ \mid D_\bullet) := \{\mathbf{c}(D_\circ \mid \delta_\bullet \in D_\bullet) \mid \delta_\bullet \in D_\bullet\}$ the set of \mathbf{c} -vectors of the diagonals of D_\bullet and by $\mathbf{C}(D_\circ) := \bigcup_{D_\bullet} \mathbf{c}(D_\circ \mid D_\bullet)$ the set of all \mathbf{c} -vectors with respect to D_\circ .

Example 16. Consider the hollow dissection $D_\circ^{\text{ex}} = \{3_\circ 7_\circ, 3_\circ 13_\circ, 9_\circ 13_\circ\}$ and the rightmost solid dissection $D_\bullet^{\text{ex}} = \{2_\bullet 6_\bullet, 2_\bullet 10_\bullet, 10_\bullet 14_\bullet\}$ of Figure 2. Then we have for example

- $\diamond \varepsilon_\circ(3_\circ 13_\circ \in D_\circ^{\text{ex}} \mid 2_\bullet 10_\bullet) = 1$ since the path $1_\circ - 3_\circ - 13_\circ - 9_\circ$ forms a \mathbf{Z} ,
- $\diamond \varepsilon_\circ(9_\circ 13_\circ \in D_\circ^{\text{ex}} \mid 2_\bullet 10_\bullet) = -1$ since the path $3_\circ - 13_\circ - 9_\circ - 11_\circ$ forms a $\mathbf{\Sigma}$, and
- $\diamond \varepsilon_\circ(3_\circ 13_\circ \in D_\circ^{\text{ex}} \mid 2_\bullet 6_\bullet) = 0$ since 3_\circ connects $1_\circ, 13_\circ, 7_\circ$ as a \mathbf{V} .

Moreover, we have

$$\begin{aligned} \mathbf{g}(D_\circ^{\text{ex}} | 2_\bullet 6_\bullet) &= \mathbf{e}_{3_\circ 7_\circ}, & \mathbf{c}(D_\circ^{\text{ex}} | 2_\bullet 6_\bullet \in D_\bullet^{\text{ex}}) &= \mathbf{e}_{3_\circ 7_\circ}, \\ \mathbf{g}(D_\circ^{\text{ex}} | 2_\bullet 10_\bullet) &= \mathbf{e}_{3_\circ 13_\circ} - \mathbf{e}_{9_\circ 13_\circ}, & \mathbf{c}(D_\circ^{\text{ex}} | 2_\bullet 10_\bullet \in D_\bullet^{\text{ex}}) &= \mathbf{e}_{3_\circ 13_\circ}, \\ \mathbf{g}(D_\circ^{\text{ex}} | 10_\bullet 14_\bullet) &= -\mathbf{e}_{9_\circ 13_\circ}, & \mathbf{c}(D_\circ^{\text{ex}} | 10_\bullet 14_\bullet \in D_\bullet^{\text{ex}}) &= -\mathbf{e}_{3_\circ 13_\circ} - \mathbf{e}_{9_\circ 13_\circ}. \end{aligned}$$

Example 17. For any hollow diagonal $i_\circ j_\circ \in D_\circ$, we have

$$\begin{aligned} \mathbf{g}(D_\circ | (i-1)_\bullet (j-1)_\bullet) &= -\mathbf{e}_{i_\circ j_\circ}, & \mathbf{c}(D_\circ | (i-1)_\bullet (j-1)_\bullet \in D_\bullet^-) &= -\mathbf{e}_{i_\circ j_\circ}, \\ \mathbf{g}(D_\circ | (i+1)_\bullet (j+1)_\bullet) &= \mathbf{e}_{i_\circ j_\circ}, & \mathbf{c}(D_\circ | (i+1)_\bullet (j+1)_\bullet \in D_\bullet^+) &= \mathbf{e}_{i_\circ j_\circ}. \end{aligned}$$

Remark 18. For a hollow triangulation T_\circ , our definitions of \mathbf{g} - and \mathbf{c} -vectors coincide with the shear coordinates of S. Fomin and D. Thurston [FT12], defined in the much more general context of cluster algebras on surfaces [FST08].

Remark 19. Consider the quiver $Q(D_\circ)$ of the reference dissection D_\circ , with one node on each internal diagonal of D_\circ and one arrow between two diagonals counter-clockwise consecutive around a cell of D_\circ . Let $W(D_\circ)$ be the reflection group whose Dynkin diagram is the underlying graph of $Q(D_\circ)$. Then all \mathbf{g} -vectors of the D_\circ -accordion diagonals are weights of $W(D_\circ)$ and all \mathbf{c} -vectors of $\mathbf{C}(D_\circ)$ are roots of $W(D_\circ)$.

Remark 20. Informally, the \mathbf{g} - and \mathbf{c} -vectors can be interpreted as follows:

- (i) The \mathbf{g} -vector $\mathbf{g}(D_\circ | \delta_\bullet)$ has coordinate 1 and -1 alternating along the zigzag of the accordion crossed by δ_\bullet in D_\circ , and coordinate 0 on all other diagonals of D_\circ .
- (ii) The \mathbf{c} -vector $\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet)$ is, up to a sign, the characteristic vector of the diagonals of the subaccordion of D_\circ crossed by both diagonals μ_\bullet and ν_\bullet of Lemma 9 (see also Figure 4). Thus, any \mathbf{c} -vector is either *positive* (only non-negative coordinates) or *negative* (only non-positive coordinates).

In fact, the \mathbf{g} -vectors are clearly in bijection with the accordions and with the zigzags in D_\circ . In contrast, many pairs $(\delta_\bullet, D_\bullet)$ produce the same \mathbf{c} -vector $\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet)$. For example, if two dissections D_\bullet, D'_\bullet contain δ_\bullet and have the same cells incident to δ_\bullet , then $\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet) = \mathbf{c}(D_\circ | \delta_\bullet \in D'_\bullet)$. The set of \mathbf{c} -vectors $\mathbf{C}(D_\circ)$ without repetitions can be understood as follows.

Lemma 21. *There are bijections between:*

- ◊ the negative (resp. positive) \mathbf{c} -vectors of $\mathbf{C}(D_\circ)$,
- ◊ the subaccordions of D_\circ ,
- ◊ the D_\circ -accordion diagonals not in the source dissection $D_\bullet^- := \{(i-1)_\bullet (j-1)_\bullet | i_\circ j_\circ \in D_\circ\}$ (resp. not in the sink dissection $D_\bullet^+ := \{(i+1)_\bullet (j+1)_\bullet | i_\circ j_\circ \in D_\circ\}$).

Proof. By Remark 20 (ii), the support of any \mathbf{c} -vector is a subaccordion of D_\circ . Reciprocally, let A_\circ be a subaccordion of D_\circ , let C_\circ and C'_\circ denote the two cells of D_\circ containing exactly one diagonal of A_\circ , and let $p_\circ, q_\circ, r_\circ, s_\circ$ (resp. $p'_\circ, q'_\circ, r'_\circ, s'_\circ$) denote the four consecutive vertices in clockwise order around C_\circ (resp. around C'_\circ) such that $q_\circ r_\circ$ (resp. $q'_\circ r'_\circ$) is the diagonal of A_\circ in C_\circ (resp. in C'_\circ). Let $\delta_\bullet := (s-1)_\bullet (s'-1)_\bullet$, $\mu_\bullet := (p+1)_\bullet (s'-1)_\bullet$ and $\nu_\bullet := (p'+1)_\bullet (s-1)_\bullet$ and consider any D_\circ -accordion dissection D_\bullet containing $\{\mu_\bullet, \delta_\bullet, \nu_\bullet\}$. Then A_\circ is precisely the support of the negative \mathbf{c} -vector $\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet)$. Finally, we have associated to the subaccordion A_\circ of D_\circ a D_\circ -diagonal $\delta_\bullet = (s-1)_\bullet (s'-1)_\bullet$ which cannot be in D_\bullet^- as otherwise $s_\circ s'_\circ$ would cross $q_\circ r_\circ$. Reciprocally, A_\circ is precisely the set of diagonals of D_\circ crossed by δ_\bullet and not incident to s_\circ or s'_\circ . \square

The \mathbf{g} -vectors and \mathbf{c} -vectors are connected in the following two statements, inspired and motivated by classical analogues in cluster algebra theory.

Proposition 22. *For any maximal D_\circ -accordion dissection D_\bullet , the set of \mathbf{g} -vectors $\mathbf{g}(D_\circ | D_\bullet)$ and the set of \mathbf{c} -vectors $\mathbf{c}(D_\circ | D_\bullet)$ form dual bases.*

Proof. Let $\langle \cdot | \cdot \rangle$ denote the standard Euclidean inner product of \mathbb{R}^{D_\circ} . Given two solid diagonals $\gamma_\bullet, \delta_\bullet$ of D_\bullet , we want to compute $\langle \mathbf{g}(D_\circ | \gamma_\bullet) | \mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet) \rangle$. By Remark 20 (i), the \mathbf{g} -vector $\mathbf{g}(D_\circ | \gamma_\bullet)$ has coordinate ± 1 alternating along the zigzag Z_\circ of the accordion crossed by γ_\bullet in D_\circ , and coordinate 0 on all other diagonals of D_\circ . Moreover, by Remark 20 (ii), the

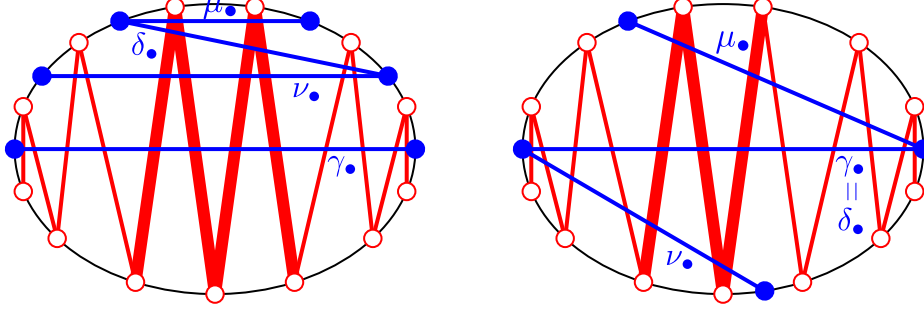


FIGURE 5. Illustration of the proof of Proposition 22. The red hollow diagonals form the zigzag of γ_\bullet , and the bolded ones are slaloming on δ_\bullet . There are an even number of bolded diagonals when $\gamma_\bullet \neq \delta_\bullet$ (left) and an odd number when $\gamma_\bullet = \delta_\bullet$ (right).

\mathbf{c} -vector $\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet)$ has coordinate ± 1 on the diagonals of D_\circ which slalom on δ_\bullet in D_\bullet , and coordinate 0 on all other diagonals of D_\circ . We thus need to understand how the diagonals of Z_\circ slalom on δ_\bullet in D_\bullet . See Figure 5 for a schematic illustration. Observe that there is an even (resp. odd) number of hollow diagonals of Z_\circ that slalom on δ_\bullet when $\delta_\bullet \neq \gamma_\bullet$ (resp. when $\delta_\bullet = \gamma_\bullet$). Moreover, since they are non-crossing, all hollow diagonals of Z_\circ slaloming on δ_\bullet do it the same way (either all as a Σ or all as a Z). Finally, when $\gamma_\bullet = \delta_\bullet$, consider the first hollow diagonal δ_\circ of the zigzag Z_\circ which slaloms on δ_\bullet . Then δ_\circ slaloms on δ_\bullet in the opposite way as δ_\bullet slaloms on δ_\circ . This shows that

$$\langle \mathbf{g}(D_\circ | \gamma_\bullet) | \mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet) \rangle = \sum_{\delta_\circ \in D_\circ} \varepsilon_\circ(\delta_\circ \in D_\circ | \gamma_\bullet) \cdot \varepsilon_\bullet(\delta_\circ | \delta_\bullet \in D_\bullet) = \mathbb{1}_{\gamma_\bullet = \delta_\bullet},$$

since we sum an even number of alternating ± 1 when $\gamma_\bullet \neq \delta_\bullet$, and an odd number of alternating ± 1 starting by a 1 when $\gamma_\bullet = \delta_\bullet$. In other words, $\mathbf{g}(D_\circ | D_\bullet)$ and $\mathbf{c}(D_\circ | D_\bullet)$ form dual bases. \square

Proposition 23. *Let D_\circ be a hollow dissection and D_\bullet be a solid dissection such that D_\circ and D_\bullet are maximal accordion dissections of each other (see Proposition 15). Then*

$$\mathbf{g}(D_\circ | D_\bullet) = -\mathbf{c}(D_\bullet | D_\circ)^t \quad \text{and} \quad \mathbf{c}(D_\circ | D_\bullet) = -\mathbf{g}(D_\bullet | D_\circ)^t,$$

where we consider the sets of \mathbf{g} -vectors $\mathbf{g}(D_\circ | D_\bullet)$ and \mathbf{c} -vectors $\mathbf{c}(D_\circ | D_\bullet)$ as matrices in $\mathbb{R}^{D_\circ \times D_\bullet}$, and M^t denotes the transpose of a matrix M .

Proof. We immediately derive from the definitions that for any $\delta_\circ \in D_\circ$ and $\delta_\bullet \in D_\bullet$,

$$\mathbf{g}(D_\circ | D_\bullet)_{(\delta_\circ, \delta_\bullet)} = \varepsilon_\circ(\delta_\circ \in D_\circ | \delta_\bullet) = -\varepsilon_\bullet(\delta_\bullet | \delta_\circ \in D_\circ) = -\mathbf{c}(D_\bullet | D_\circ)_{(\delta_\bullet, \delta_\circ)},$$

which shows $\mathbf{g}(D_\circ | D_\bullet) = -\mathbf{c}(D_\bullet | D_\circ)^t$. The other equality follows by exchanging D_\circ and D_\bullet . \square

Corollary 24. *For any maximal D_\circ -accordion dissection D_\bullet , we have the following [sign coherence](#):*

- (i) *for any $\delta_\bullet \in D_\bullet$, all coordinates of the \mathbf{c} -vector $\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet)$ have the same sign,*
- (ii) *for any $\delta_\circ \in D_\circ$, the δ_\circ -coordinates of all \mathbf{g} -vectors $\mathbf{g}(D_\circ | \delta_\bullet)$ for $\delta_\bullet \in D_\bullet$ have the same sign.*

Proof. Point (i) was already seen in Remark 20 (ii), and Point (ii) follows by Proposition 23. \square

2.2. \mathbf{c} -vector fan and D_\circ -zonotope. Define the \mathbf{c} -vector fan of D_\circ to be the complete polyhedral fan $\mathcal{F}^c(D_\circ)$ given by the arrangement of the linear hyperplanes orthogonal to the \mathbf{c} -vectors of $\mathbf{C}(D_\circ)$. Be careful: in contrast to the \mathbf{g} - and \mathbf{d} -vector fans defined later, the \mathbf{c} -vectors are not the rays of $\mathcal{F}^c(D_\circ)$ but the normal vectors of the hyperplanes supporting the facets of $\mathcal{F}^c(D_\circ)$.

We call D_\circ -zonotope the Minkowski sum $\text{Zono}(D_\circ)$ of all \mathbf{c} -vectors:

$$\text{Zono}(D_\circ) := \sum_{\mathbf{c} \in \mathbf{C}(D_\circ)} \mathbf{c}.$$

The normal fan of the D_\circ -zonotope $\text{Zono}(D_\circ)$ is the \mathbf{c} -vector fan $\mathcal{F}^{\mathbf{c}}(D_\circ)$. Note that the \mathbf{c} -vector fan is not always simplicial, and thus the D_\circ -zonotope $\text{Zono}(D_\circ)$ is not always simple. See Figure 7.

Example 25. Consider an accordion dissection A_\circ (where each cell has at most 2 edges which are internal diagonals of P_\circ). Label its internal diagonals by $\delta_\circ^1, \dots, \delta_\circ^{|A_\circ|}$ such that δ_\circ^k and δ_\circ^{k+1} belong to the same cell of A_\circ for all k . Identifying $\mathbf{e}_{\delta_\circ^k}$ to the simple root $\mathbf{f}_k - \mathbf{f}_{k+1}$ of type $A_{|A_\circ|}$, the \mathbf{c} -vectors of $\mathbf{C}(A_\circ)$ are all roots $\pm(\mathbf{f}_i - \mathbf{f}_j) = \pm \sum_{i \leq k \leq j} \mathbf{e}_{\delta_\circ^k}$ of type $A_{|A_\circ|}$. Therefore, the \mathbf{c} -vector fan is the type $A_{|A_\circ|}$ Coxeter fan and the A_\circ -zonotope is a permutahedron. More precisely,

$$\text{Zono}(A_\circ) = \sum_{k \in [|A_\circ|+1]} k(|A_\circ| + 1 - k) [-\mathbf{e}_{\delta_\circ^k}, \mathbf{e}_{\delta_\circ^k}] = 2 \text{Perm}(|A_\circ|) - (|A_\circ| + 2) \sum_{i \in [|A_\circ|+1]} \mathbf{f}_i,$$

where $\text{Perm}(|A_\circ|) := \text{conv} \{ \sum_{i \in [|A_\circ|+1]} \sigma(i) \mathbf{f}_i \mid \sigma \in \mathfrak{S}_{|A_\circ|+1} \}$ is the classical permutahedron.

The vertices of $\text{Zono}(D_\circ)$ correspond to *separable* subsets of $\mathbf{C}(D_\circ)$, *i.e.* those which can be strictly separated from their complement by a hyperplane. Although we could work out all facets of $\text{Zono}(D_\circ)$, we will only need the following specific inequalities.

Proposition 26. *For any D_\circ -accordion diagonal γ_\bullet , the D_\circ -zonotope $\text{Zono}(D_\circ)$ has a facet defined by the inequality*

$$\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{x} \rangle \leq \omega(D_\circ | \gamma_\bullet),$$

where $\omega(D_\circ | \gamma_\bullet)$ is the *D_\circ -height* of γ_\bullet , *i.e.* the number of D_\circ -accordion diagonals that cross γ_\bullet .

Proof. Let $\omega(D_\circ | \gamma_\bullet)$ denote the maximum of $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{x} \rangle$ over $\text{Zono}(D_\circ)$. As $\text{Zono}(D_\circ)$ is the Minkowski sum of all \mathbf{c} -vectors, we have

$$\omega(D_\circ | \gamma_\bullet) = \sum_{\substack{\mathbf{c} \in \mathbf{C}(D_\circ) \\ \langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c} \rangle > 0}} \langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c} \rangle.$$

By Remark 20, we have $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c} \rangle \in \{-1, 0, 1\}$ for any $\mathbf{c} \in \mathbf{C}(D_\circ)$. We thus just need to count the distinct \mathbf{c} -vectors \mathbf{c} such that $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c} \rangle > 0$. It turns out that it is more convenient and equivalent (since $\mathbf{C}(D_\circ) = -\mathbf{C}(D_\circ)$) to count the distinct \mathbf{c} -vectors \mathbf{c} such that $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c} \rangle < 0$. For that, let Z_\circ denote the zigzag of the accordion crossed by γ_\bullet in D_\circ , and decompose $Z_\circ = Z_\circ^- \sqcup Z_\circ^+$ such that $\mathbf{g}(D_\circ | \gamma_\bullet) = \mathbb{1}_{Z_\circ^+} - \mathbb{1}_{Z_\circ^-}$ (where $\mathbb{1}_{X_\circ} := \sum_{\delta_\circ \in X_\circ} \mathbf{e}_{\delta_\circ}$ for $X_\circ \subseteq D_\circ$).

Let δ_\bullet be a D_\circ -accordion diagonal. Let A_\circ^- (resp. A_\circ^+) denote the accordion crossed by $\delta_\bullet = u_\bullet v_\bullet$ in D_\circ and not including $(u+1)_\circ$ or $(v+1)_\circ$ (resp. $(u-1)_\circ$ or $(v-1)_\circ$). Let $\mathbf{c}^-(\delta_\bullet) := -\mathbb{1}_{A_\circ^-}$ and $\mathbf{c}^+(\delta_\bullet) := \mathbb{1}_{A_\circ^+}$. Recall from Lemma 21 that the negative (resp. positive) \mathbf{c} -vectors of $\mathbf{C}(D_\circ)$ are given by $\mathbf{c}^-(\delta_\bullet)$ (resp. $\mathbf{c}^+(\delta_\bullet)$) for all D_\circ -accordion diagonal δ_\bullet not in D_\circ^- (resp. D_\circ^+). We let the reader check that:

- ◇ If γ_\bullet and δ_\bullet do not cross and have no common endpoint, both $|Z_\circ \cap A_\circ^-|$ and $|Z_\circ \cap A_\circ^+|$ are even. Thus $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c}^-(\delta_\bullet) \rangle = \langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c}^+(\delta_\bullet) \rangle = 0$.
- ◇ If γ_\bullet and δ_\bullet have a common endpoint, and $\gamma_\bullet \delta_\bullet$ form a counterclockwise angle, then $|Z_\circ \cap A_\circ^-|$ is even while $Z_\circ \cap A_\circ^+$ is empty or starts and ends in Z_\circ^+ . Thus $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c}^-(\delta_\bullet) \rangle = 0$ while $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c}^+(\delta_\bullet) \rangle \geq 0$. The situation is similar if $\gamma_\bullet \delta_\bullet$ form a clockwise angle.
- ◇ If γ_\bullet and δ_\bullet cross, $Z_\circ \cap A_\circ^-$ and $Z_\circ \cap A_\circ^+$ are empty or start and end both in Z_\circ^- or both in Z_\circ^+ . Thus, either $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c}^-(\delta_\bullet) \rangle < 0$ and $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c}^+(\delta_\bullet) \rangle \geq 0$ or conversely.

We conclude from this case analysis that

$$\omega(D_\circ | \gamma_\bullet) = |\{ \mathbf{c} \in \mathbf{C}(D_\circ) \mid \langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{c} \rangle < 0 \}| = |\{ D_\circ\text{-accordion diagonals crossing } \gamma_\bullet \}|.$$

Finally, the inequality $\langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{x} \rangle \leq \omega(D_\circ | \gamma_\bullet)$ defines a priori a face $\mathbf{F}(\gamma_\bullet)$ of the zonotope $\text{Zono}(D_\circ)$. This face $\mathbf{F}(\gamma_\bullet)$ is the Minkowski sum of the \mathbf{c} -vectors of $\mathbf{C}(D_\circ)$ orthogonal to $\mathbf{g}(D_\circ | \gamma_\bullet)$. Proposition 22 ensures that any D_\circ -accordion dissection D_\bullet containing γ_\bullet already provides $|D_\bullet| - 1$ linearly independent such \mathbf{c} -vectors $\mathbf{c}(D_\circ | \delta_\bullet) \in D_\bullet$ for $\delta_\bullet \in D_\bullet \setminus \{\gamma_\bullet\}$. We obtain that $\mathbf{F}(\gamma_\bullet)$ has dimension $|D_\bullet| - 1 = |D_\circ| - 1$ and is therefore a facet of the zonotope $\text{Zono}(D_\circ)$. \square

Define the half-space and the hyperplane corresponding to a solid D_\circ -accordion diagonal γ_\bullet by

$$\mathbf{H}^\leq(D_\circ | \gamma_\bullet) := \{\mathbf{x} \in \mathbb{R}^{D_\circ} \mid \langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{x} \rangle \leq \omega(D_\circ | \gamma_\bullet)\},$$

and

$$\mathbf{H}^=(D_\circ | \gamma_\bullet) := \{\mathbf{x} \in \mathbb{R}^{D_\circ} \mid \langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{x} \rangle = \omega(D_\circ | \gamma_\bullet)\}.$$

2.3. \mathbf{g} -vector fan and D_\circ -accordiohedron. In this section, we give a geometric realization of the D_\circ -accordion complex. We start by realizing this simplicial complex as a complete simplicial fan in \mathbb{R}^{D_\circ} . We denote by $\mathbb{R}_{\geq 0}\mathbf{R}$ the nonnegative span of a set \mathbf{R} of vectors in \mathbb{R}^{D_\circ} .

Theorem 27. *The collection of cones*

$$\mathcal{F}^{\mathbf{g}}(D_\circ) := \{\mathbb{R}_{\geq 0}\mathbf{g}(D_\circ | D_\bullet) \mid D_\bullet \text{ any } D_\circ\text{-accordion dissection}\}$$

forms a complete simplicial fan, that we call the \mathbf{g} -vector fan of D_\circ .

The proof uses the following characterization of complete simplicial fans [DRS10, Coro. 4.5.20]. We will provide as well an alternative proof in Remark 60 based on sections of Cambrian fans.

Proposition 28. *Consider a pseudomanifold Δ on a finite vertex set X and a set of vectors $\mathbf{R} := (\mathbf{r}_x)_{x \in X}$ of \mathbb{R}^d . For $D \in \Delta$, define the cone $\mathbf{R}_D := \{\mathbf{r}_x \mid x \in D\}$. The collection of cones $\{\mathbb{R}_{\geq 0}\mathbf{R}_D \mid D \in \Delta\}$ forms a complete simplicial fan if and only if*

- (1) *there exists a facet D of Δ such that \mathbf{R}_D is a basis of \mathbb{R}^d and such that the open cones $\mathbb{R}_{> 0}\mathbf{R}_D$ and $\mathbb{R}_{> 0}\mathbf{R}_{D'}$ are disjoint for any facet D' of Δ distinct from D ;*
- (2) *for two adjacent facets D, D' of Δ with $D \setminus \{x\} = D' \setminus \{x'\}$, there is a linear dependence*

$$\alpha \mathbf{r}_x + \alpha' \mathbf{r}_{x'} + \sum_{y \in D \cap D'} \beta_y \mathbf{r}_y = 0$$

on $\mathbf{R}_{D \cup D'}$ where the coefficients α and α' have the same sign. (When these conditions hold, these coefficients do not vanish and the linear dependence is unique up to rescaling.)

Proof of Theorem 27. By Corollary 24, the cone $\mathbb{R}_{\geq 0}\mathbf{g}(D_\circ | D_\bullet^-)$ is the only cone of $\mathcal{F}^{\mathbf{g}}(D_\circ)$ intersecting the interior of the positive orthant $(\mathbb{R}_{\geq 0})^{D_\circ}$. Consider now two adjacent maximal D_\circ -accordion dissections D_\bullet, D'_\bullet . Let $\delta_\bullet \in D_\bullet$ and $\delta'_\bullet \in D'_\bullet$ be such that $D_\bullet \setminus \{\delta_\bullet\} = D'_\bullet \setminus \{\delta'_\bullet\}$, and let μ_\bullet and ν_\bullet be the other diagonals as in Lemma 9 (see also Figure 4). Note that a diagonal of D_\circ crosses none of (resp. one of, resp. both) the diagonals $\delta_\bullet, \delta'_\bullet$ if and only if it crosses none of (resp. one of, resp. both) the diagonals μ_\bullet, ν_\bullet . The same holds for a Z or a Σ of D_\circ . Therefore, we have the linear dependence $\mathbf{g}(D_\circ | \delta_\bullet) + \mathbf{g}(D_\circ | \delta'_\bullet) = \mathbf{g}(D_\circ | \mu_\bullet) + \mathbf{g}(D_\circ | \nu_\bullet)$. This shows that $\mathcal{F}^{\mathbf{g}}(D_\circ)$ satisfies the two conditions of Proposition 28, and thus concludes the proof. \square

Remark 29. The linear dependence $\mathbf{g}(D_\circ | \delta_\bullet) + \mathbf{g}(D_\circ | \delta'_\bullet) = \mathbf{g}(D_\circ | \mu_\bullet) + \mathbf{g}(D_\circ | \nu_\bullet)$ relating the \mathbf{g} -vectors of two adjacent maximal D_\circ -accordion dissections D_\bullet, D'_\bullet with $D_\bullet \setminus \{\delta_\bullet\} = D'_\bullet \setminus \{\delta'_\bullet\}$ shows that $\det(\mathbf{g}(D_\circ | D_\bullet)) = -\det(\mathbf{g}(D_\circ | D'_\bullet))$. Since the initial cone $\mathbb{R}_{\geq 0}\mathbf{g}(D_\circ | D_\bullet^-)$ is generated by the coordinate vectors (see Example 17), we obtain that $\det(\mathbf{g}(D_\circ | D_\bullet)) = \pm 1$ for all D_\circ -accordion dissection D_\bullet , so that the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$ is always *smooth*.

By Proposition 22, any non-maximal cone of $\mathcal{F}^{\mathbf{g}}(D_\circ)$ is supported by a hyperplane orthogonal to a \mathbf{c} -vector of $\mathbf{C}(D_\circ)$. We thus obtain the following consequence.

Corollary 30. *The \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$ coarsens the \mathbf{c} -vector fan $\mathcal{F}^{\mathbf{c}}(D_\circ)$.*

Example 31. Following Example 2, we observe that special reference dissections give rise to the following relevant fans:

- ◊ For an accordion triangulation A_\circ (*i.e.* with no interior triangle), the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(A_\circ)$ coincides with a type A Cambrian fan of N . Reading and D. Speyer [RS09].
- ◊ For an arbitrary triangulation T_\circ (with or without interior triangle), the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(T_\circ)$ was recently constructed in [HPS17].

Example 32. Figure 6 illustrates the \mathbf{g} -vector fans $\mathcal{F}^{\mathbf{g}}(D_\circ)$ for various reference dissections D_\circ : the fan, the snake, and the cyclic triangulation of the hexagon, and a dissection of the heptagon. More precisely, we have represented the stereographic projection of the fans from the point $[1, 1, 1]$.

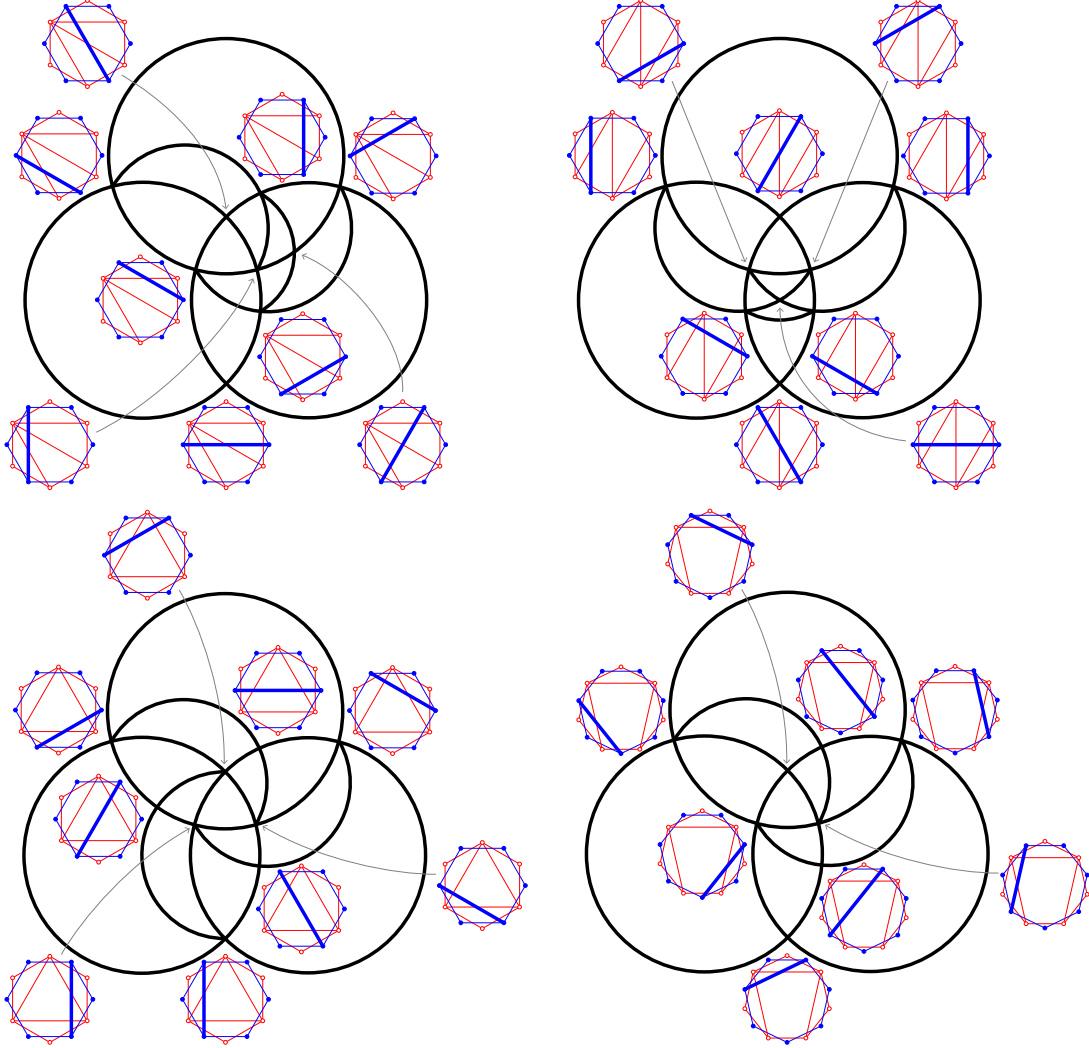


FIGURE 6. Stereographic projections of the \mathbf{g} -vector fans $\mathcal{F}^{\mathbf{g}}(D_{\circ})$ for various reference hollow dissections D_{\circ} . See Figure 9 for alternative simplicial fan realizations of these accordion complexes.

Therefore, the external face of the projection corresponds to the D_{\circ} -accordion dissection D_{\bullet}^{-} . We have labeled all vertices of the projection (*i.e.* the rays of the fan) by the corresponding D_{\circ} -accordion diagonals.

We now provide a first polytopal realization of the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_{\circ})$ (see also Section 4). This fan has a maximal cone for each maximal D_{\circ} -accordion dissection and a ray for each D_{\circ} -accordion diagonal. For a maximal D_{\circ} -accordion dissection D_{\bullet} , we define a point $\mathbf{p}(D_{\circ} | D_{\bullet}) \in \mathbb{R}^{D_{\circ}}$ by

$$\mathbf{p}(D_{\circ} | D_{\bullet}) := \sum_{\delta_{\bullet} \in D_{\bullet}} \omega(D_{\circ} | \delta_{\bullet}) \cdot \mathbf{c}(D_{\circ} | \delta_{\bullet} \in D_{\bullet}),$$

where $\omega(D_{\circ} | \delta_{\bullet})$ still denotes the D_{\circ} -height of δ_{\bullet} defined as the number of D_{\circ} -accordion diagonals that cross δ_{\bullet} . We will need the following two technical lemmas in the proof of Theorem 35.

Lemma 33. *For any maximal D_{\circ} -accordion dissection D_{\bullet} , the point $\mathbf{p}(D_{\circ} | D_{\bullet})$ is the intersection of all hyperplanes $\mathbf{H}^{\circ}(D_{\circ} | \delta_{\bullet})$ with $\delta_{\bullet} \in D_{\bullet}$.*

Proof. Observe first that the hyperplanes $\mathbf{H}^=(D_\circ | \delta_\bullet)$ with $\delta_\bullet \in D_\bullet$ have a unique intersection point, since $\mathbf{g}(D_\circ | D_\bullet)$ is a basis. Moreover, since $\mathbf{g}(D_\circ | D_\bullet)$ and $\mathbf{c}(D_\circ | D_\bullet)$ form dual bases by Proposition 22, we have for any $\gamma_\bullet \in D_\bullet$:

$$\begin{aligned} \langle \mathbf{g}(D_\circ | \gamma_\bullet) | \mathbf{p}(D_\circ | D_\bullet) \rangle &= \sum_{\delta_\bullet \in D_\bullet} \omega(D_\circ | \delta_\bullet) \cdot \langle \mathbf{g}(D_\circ | \gamma_\bullet) | \mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet) \rangle \\ &= \sum_{\delta_\bullet \in D_\bullet} \omega(D_\circ | \delta_\bullet) \cdot \mathbb{1}_{\gamma_\bullet = \delta_\bullet} = \omega(D_\circ | \gamma_\bullet). \quad \square \end{aligned}$$

Lemma 34. *If D_\bullet, D'_\bullet are two adjacent maximal D_\circ -accordion dissections, and $\delta_\bullet \in D_\bullet$ and $\delta'_\bullet \in D'_\bullet$ are such that $D_\bullet \setminus \{\delta_\bullet\} = D'_\bullet \setminus \{\delta'_\bullet\}$, then*

$$\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet) = -\mathbf{c}(D_\circ | \delta'_\bullet \in D'_\bullet) \quad \text{and} \quad \mathbf{p}(D_\circ | D'_\bullet) - \mathbf{p}(D_\circ | D_\bullet) \in \mathbb{Z}_{<0} \cdot \mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet).$$

Proof. Let D_\bullet, D'_\bullet be two adjacent maximal D_\circ -accordion dissections, let $\delta_\bullet \in D_\bullet$ and $\delta'_\bullet \in D'_\bullet$ be such that $D_\bullet \setminus \{\delta_\bullet\} = D'_\bullet \setminus \{\delta'_\bullet\}$, and let μ_\bullet and ν_\bullet be the other diagonals as in Lemma 9 (see also Figure 4). A quick case analysis then shows that

$$\mathbf{c}(D_\circ | \gamma_\bullet \in D'_\bullet) = \begin{cases} \mathbf{c}(D_\circ | \gamma_\bullet \in D_\bullet) & \text{for all diagonal } \gamma_\bullet \in D_\bullet \setminus \{\delta_\bullet, \mu_\bullet, \nu_\bullet\}, \\ -\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet) & \text{if } \gamma_\bullet = \delta'_\bullet, \\ \mathbf{c}(D_\circ | \gamma_\bullet \in D_\bullet) + \mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet) & \text{if } \gamma_\bullet \in \{\mu_\bullet, \nu_\bullet\}. \end{cases}$$

Summing the contribution of all \mathbf{c} -vectors with their coefficients $\omega(D_\circ | \gamma_\bullet)$, we obtain

$$\mathbf{p}(D_\circ | D'_\bullet) - \mathbf{p}(D_\circ | D_\bullet) = (\omega(D_\circ | \mu_\bullet) + \omega(D_\circ | \nu_\bullet) - \omega(D_\circ | \delta_\bullet) - \omega(D_\circ | \delta'_\bullet)) \cdot \mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet).$$

Finally, note that any diagonal of P_\bullet that crosses one of (resp. both) the diagonals μ_\bullet, ν_\bullet also crosses one of (resp. both) the diagonals $\delta_\bullet, \delta'_\bullet$. Moreover, δ_\bullet and δ'_\bullet cross each other but do not cross μ_\bullet and ν_\bullet . It follows that $\omega(D_\circ | \mu_\bullet) + \omega(D_\circ | \nu_\bullet) - \omega(D_\circ | \delta_\bullet) - \omega(D_\circ | \delta'_\bullet) \leq -2 < 0$. \square

Theorem 35. *The \mathbf{g} -vector fan is the normal fan of the D_\circ -accordiohedron $\text{Acco}(D_\circ)$ defined equivalently as*

- \diamond the convex hull of the points $\mathbf{p}(D_\circ | D_\bullet)$ for all maximal D_\circ -accordion dissection D_\bullet , or
- \diamond the intersection of the half-spaces $\mathbf{H}^\leq(D_\circ | \gamma_\bullet)$ for all D_\circ -accordion diagonals γ_\bullet .

Thus, the polar dual of $\text{Acco}(D_\circ)$ is a polytopal realization of the D_\circ -accordion complex $\mathcal{AC}(D_\circ)$.

The proof of Theorem 35 is based on the following characterization of polytopal realizations of a complete simplicial fan, whose proof can be found *e.g.* in [HLT11, Thm. 4.1].

Theorem 36. *Given a complete simplicial fan \mathcal{F} in \mathbb{R}^d , consider for each ray \mathbf{r} of \mathcal{F} a half-space $\mathbf{H}_\mathbf{r}^\leq$ of \mathbb{R}^d containing the origin and defined by a hyperplane $\mathbf{H}_\mathbf{r}^\perp$ orthogonal to \mathbf{r} . For each maximal cone C of \mathcal{F} , let $\mathbf{a}(C) \in \mathbb{R}^d$ be the intersection of all hyperplanes $\mathbf{H}_\mathbf{r}^\perp$ with $\mathbf{r} \in C$. Then the following assertions are equivalent:*

- (i) *The vector $\mathbf{a}(C') - \mathbf{a}(C)$ points from C to C' for any two adjacent maximal cones C, C' of \mathcal{F} .*
- (ii) *The polytopes*

$$\text{conv}\{\mathbf{a}(C) \mid C \text{ maximal cone of } \mathcal{F}\} \quad \text{and} \quad \bigcap_{\mathbf{r} \text{ ray of } \mathcal{F}} \mathbf{H}_\mathbf{r}^\leq$$

coincide and their normal fan is \mathcal{F} .

Proof of Theorem 35. The \mathbf{g} -vector fan $\mathcal{F}^\mathbf{g}(D_\circ)$ has a ray $\mathbf{g}(D_\circ | \delta_\bullet)$ for each D_\circ -accordion diagonal δ_\bullet and a maximal cone $C(D_\bullet) = \mathbb{R}_{\geq 0}\mathbf{g}(D_\circ | D_\bullet)$ for each maximal D_\circ -accordion dissection D_\bullet . Consider the half-spaces $\mathbf{H}^\leq(D_\circ | \gamma_\bullet)$ for all D_\circ -accordion diagonals γ_\bullet . Lemma 33 ensures that the point $\mathbf{a}(C(D_\bullet))$ coincides with $\mathbf{p}(D_\circ | D_\bullet)$ for each maximal D_\circ -accordion dissection D_\bullet . Finally, Lemma 34 shows that the conditions of application of Theorem 36 are fulfilled. \square

Example 37. Following Example 2, observe that special reference hollow dissections give rise to the following relevant polytopes, illustrated in Figure 7:

- \diamond For a fan triangulation T_\circ , the T_\circ -accordiohedron $\text{Acco}(T_\circ)$ is the classical associahedron constructed by S. Shnider and S. Sternberg [SS93] and J.-L. Loday [Lod04].

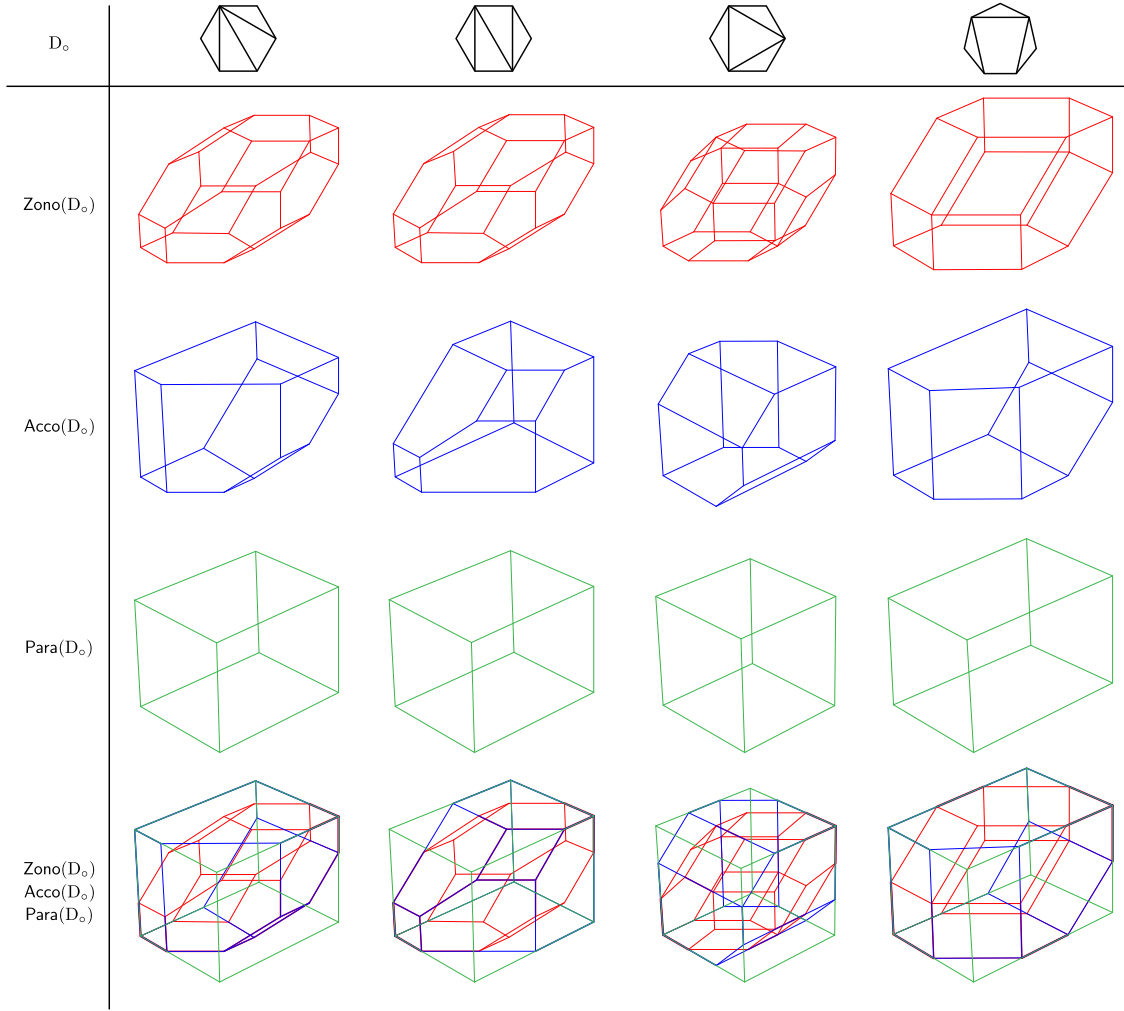


FIGURE 7. The zonotope $\text{Zono}(D_o)$, D_o -accordiohedron $\text{Acco}(D_o)$ and parallelepiped $\text{Para}(D_o)$ for different reference dissections D_o . The first column is J.-L. Loday's associahedron [Lod04], the second column is one of C. Hohlweg and C. Lange's associahedra [HL07], the third column appeared in a discussion in C. Ceballos, F. Santos and G. Ziegler's survey on associahedra [CSZ15, Fig. 3] and was explained in C. Hohlweg, V. Pilaud and S. Stella's recent paper [HPS17], and the last column is a Stokes complex discussed by F. Chapoton in [Cha16] and illustrated in Figure 3.

- ◇ The A_o -accordiohedra $\text{Acco}(A_o)$ for all accordion triangulations A_o are precisely the associahedra constructed by C. Hohlweg and C. Lange in [HL07].
- ◇ For a triangulation T_o with an interior triangle, the T_o -accordiohedron $\text{Acco}(T_o)$ was recently constructed in [HPS17]. For example, for the triangulation of the hexagon with an interior triangle, this associahedron appeared as a mysterious realization in [CSZ15].
- ◇ For a quadrangulation Q_o , the Q_o -accordiohedron $\text{Acco}(Q_o)$ is a realization of the Stokes polytope announced by Y. Baryshnikov [Bar01] and discussed by F. Chapoton in [Cha16].

We conclude this section by an immediate consequence of Theorem 35. To our knowledge, this property of accordion complexes was not observed before. However, using the connection between accordion complexes and support τ -tilting complexes [GM16, PPP17, PPS17, BDM⁺17], it can also be obtained from [DIJ15, Thm. 1.7].

Corollary 38. *For any reference dissection D_o , the D_o -accordion complex $\mathcal{AC}(D_o)$ is shellable.*

2.4. Some properties of $\text{Acco}(D_\circ)$. We conclude this section by pointing out some relevant combinatorial and geometric properties and observations on the D_\circ -accordiohedron.

Proposition 39. *The graph of the D_\circ -accordiohedron $\text{Acco}(D_\circ)$ linearly oriented in the direction $-\mathbb{1} := -\sum_{\delta_\circ \in D_\circ} \mathbf{e}_{\delta_\circ}$ is the Hasse diagram of the accordion lattice $\mathcal{AL}(D_\circ)$.*

Proof. Consider two adjacent maximal D_\circ -accordion dissections D_\bullet, D'_\bullet such that the flip from D_\bullet to D'_\bullet is increasing. Let $\delta_\bullet \in D_\bullet$ and $\delta'_\bullet \in D'_\bullet$ be such that $D_\bullet \setminus \{\delta_\bullet\} = D'_\bullet \setminus \{\delta'_\bullet\}$. As observed in Remark 20 (ii), the \mathbf{c} -vector $\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet)$ is the characteristic vector $\mathbb{1}_{A_\circ}$ of the set A_\circ of diagonals of D_\circ crossed by both δ_\bullet and δ'_\bullet . Applying Lemma 34, we therefore obtain that

$$\langle -\mathbb{1} \mid \mathbf{p}(D_\circ | D'_\bullet) - \mathbf{p}(D_\circ | D_\bullet) \rangle = \langle -\mathbb{1} \mid \lambda \cdot \mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet) \rangle = \lambda \cdot \langle -\mathbb{1} \mid \mathbb{1}_{A_\circ} \rangle = -\lambda \cdot |A_\circ|,$$

for some $\lambda \in \mathbb{Z}_{<0}$. Thus, the linear functional $-\mathbb{1}$ indeed orients the edge $[\mathbf{p}(D_\circ | D_\bullet), \mathbf{p}(D_\circ | D'_\bullet)]$ from $\mathbf{p}(D_\circ | D_\bullet)$ to $\mathbf{p}(D_\circ | D'_\bullet)$. \square

Remark 40. Since the \mathbf{c} -vector fan $\mathcal{F}^{\mathbf{c}}(D_\circ)$ refines the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$, there is a natural projection π from the vertices of the D_\circ -zonotope $\text{Zono}(D_\circ)$ to that of the D_\circ -accordiohedron $\text{Acco}(D_\circ)$. In analogy to the acyclic case, one could hope to obtain the accordion lattice as a lattice quotient through this projection. However, the transitive closure of the graph of the D_\circ -zonotope $\text{Zono}(D_\circ)$ oriented in the direction $-\mathbb{1}$ is not a lattice in general (the first counter-example is the dissection with a central square surrounded by 4 triangles). As shown in [GM16], the right objects are not the separable subsets of \mathbf{c} -vectors (*i.e.* the vertices of $\text{Zono}(D_\circ)$) but the biclosed subsets of \mathbf{c} -vectors.

Proposition 41. *The accordiohedron $\text{Acco}(D_\circ)$ has precisely $|D_\circ|$ pairs of parallel facets.*

Proof. Two facets of $\text{Acco}(D_\circ)$ are parallel if and only if the corresponding \mathbf{g} -vectors are opposite. We therefore want to prove that the pairs of opposite coordinate vectors are the only pairs of opposite \mathbf{g} -vectors. Assume by contradiction that there exist two hollow diagonals $\delta_\circ, \delta'_\circ \in D_\circ$ and two solid D_\circ -diagonals $\delta_\bullet, \delta'_\bullet$ such that $\mathbf{g}(D_\circ | \delta_\bullet)$ and $\mathbf{g}(D_\circ | \delta'_\bullet)$ have non-zero opposite coordinate both on δ_\circ and δ'_\circ . Then both δ_\bullet and δ'_\bullet cross both δ_\circ and δ'_\circ . But this implies that they both slalom on δ_\circ (and on δ'_\circ) in the same way. Contradiction. \square

Recall from Example 17 that the \mathbf{g} -vectors of the diagonals of D_\bullet^- (resp. D_\bullet^+) are the coordinate vectors (resp. negative of the coordinate vectors). Consider the D_\circ -*parallelepiped*

$$\text{Para}(D_\circ) := \{ \mathbf{x} \in \mathbb{R}^{D_\circ} \mid \langle \mathbf{g}(D_\circ | \delta_\bullet) \mid \mathbf{x} \rangle \leq \omega(D_\circ | \delta_\bullet) \text{ for all } \delta_\bullet \in D_\bullet^- \cup D_\bullet^+ \}$$

defined by the inequalities of the D_\circ -zonotope $\text{Zono}(D_\circ)$ corresponding to the positive and negative basis vectors. Our next statement follows from Proposition 41 and is illustrated in Figure 7.

Corollary 42. *For any D_\circ , we have matriochka polytopes: $\text{Zono}(D_\circ) \subseteq \text{Acco}(D_\circ) \subseteq \text{Para}(D_\circ)$.*

In fact, each polytope in this chain is obtained by deleting facets from the previous one.

Consider now an isometry σ of the plane that preserves the hollow polygon P_\circ and the solid polygon P_\bullet . For any diagonals and dissections $\delta_\bullet \in D_\bullet$ and $\delta_\circ \in D_\circ$, we have

- ◇ δ_\bullet is a D_\circ -accordion diagonal $\iff \sigma(\delta_\bullet)$ is a $\sigma(D_\circ)$ -accordion diagonal,
- ◇ D_\bullet is a D_\circ -accordion dissection $\iff \sigma(D_\bullet)$ is a $\sigma(D_\circ)$ -accordion dissection,
- ◇ if $\Sigma : \mathbb{R}^{D_\circ} \rightarrow \mathbb{R}^{\sigma(D_\circ)}$ denotes the isometry defined by $(\Sigma(\mathbf{x}))_{\sigma(\delta_\circ)} := \varepsilon(\sigma) \cdot \mathbf{x}_{\delta_\circ}$, (where $\varepsilon(\sigma) = 1$ if σ is direct and -1 if σ is indirect), then we have

$$\begin{aligned} \mathbf{g}(\sigma(D_\circ) | \sigma(\delta_\bullet)) &= \Sigma(\mathbf{g}(D_\circ | \delta_\bullet)), & \mathbf{c}(\sigma(D_\circ) | \sigma(\delta_\bullet) \in \sigma(D_\bullet)) &= \Sigma(\mathbf{c}(D_\circ | \delta_\bullet \in D_\bullet)), \\ \omega(\sigma(D_\circ) | \sigma(\delta_\bullet)) &= \omega(D_\circ | \delta_\bullet), & \text{and} & \quad \mathbf{p}(\sigma(D_\circ) | \sigma(D_\bullet)) &= \Sigma(\mathbf{p}(D_\circ | D_\bullet)). \end{aligned}$$

This immediately implies the following statement.

Proposition 43. *Any P_\circ -preserving isometry $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ induces an isometry $\Sigma : \mathbb{R}^{D_\circ} \rightarrow \mathbb{R}^{\sigma(D_\circ)}$ with $\Sigma(\text{Zono}(D_\circ)) = \text{Zono}(\sigma(D_\circ))$, $\Sigma(\text{Acco}(D_\circ)) = \text{Acco}(\sigma(D_\circ))$ and $\Sigma(\text{Para}(D_\circ)) = \text{Para}(\sigma(D_\circ))$.*

We say that a dissection D is σ -invariant when $\sigma(D) = D$. Assume now that σ is a rotation and D_\circ is σ -invariant. We call σ -invariant D_\circ -accordion complex the simplicial complex $\mathcal{AC}^\sigma(D_\circ)$ whose vertices are the crossing-free σ -orbits of D_\circ -accordion diagonals, and whose faces are sets of such orbits whose union is crossing-free. In other words, the faces of $\mathcal{AC}^\sigma(D_\circ)$ are σ -invariant D_\circ -accordion dissections, seen as sets of σ -orbits of diagonals.

Lemma 44. *The σ -invariant D_\circ -accordion complex $\mathcal{AC}^\sigma(D_\circ)$ is a pseudomanifold.*

Proof. Assume first that σ is the central symmetry. In this case, there are two possible types of orbits: the long D_\circ -accordion diagonals and the centrally symmetric pairs of D_\circ -accordion diagonals. One can check that any facet of $\mathcal{AC}^\sigma(D_\circ)$ has a long diagonal if and only if D_\circ has, and has as many centrally symmetric pairs of diagonals as D_\circ . Finally, any orbit in any facet of $\mathcal{AC}^\sigma(D_\circ)$ can be flipped: long diagonals can already be flipped in $\mathcal{AC}(D_\circ)$, and a centrally symmetric pair of diagonals can be flipped by flipping one after the other its two diagonals in $\mathcal{AC}(D_\circ)$.

Finally, the general statement follows from this special case. Indeed, if σ is not a central symmetry, let C_\circ denote the cell of D_\circ containing the center of P_\circ , let u_\circ be a vertex of C_\circ , let \underline{D}_\circ be the set of diagonals of D_\circ whose endpoints are between u_\circ and $\sigma(u_\circ)$, and let ρ be the central symmetry around the middle of $u_\circ\sigma(u_\circ)$. Then $\mathcal{AC}^\sigma(D_\circ)$ is isomorphic to $\mathcal{AC}^\rho(\underline{D}_\circ \cup \rho(\underline{D}_\circ))$. \square

Let $\Sigma : \mathbb{R}^{D_\circ} \rightarrow \mathbb{R}^{D_\circ}$ denote the isometry defined by $(\Sigma(\mathbf{x}))_{\sigma(\delta_\circ)} := \mathbf{x}_{\delta_\circ}$ and $\text{Fix}(\Sigma)$ denote the linear subspace of fixed points of Σ . According to the previous discussion, a maximal D_\circ -accordion dissection D_\bullet is σ -invariant if and only if $\mathbf{p}(D_\circ | D_\bullet) \in \text{Fix}(\Sigma)$. We obtain the following statement.

Proposition 45. *For a σ -invariant dissection D_\circ , the polytope $\text{Acco}^\sigma(D_\circ)$ defined equivalently as*

- \diamond *the convex hull of $\mathbf{p}(D_\circ | D_\bullet)$ for all σ -invariant maximal D_\circ -accordion dissections D_\bullet ,*
- \diamond *the intersection of the D_\circ -accordiohedron $\text{Acco}(D_\circ)$ with the fixed space $\text{Fix}(\Sigma)$,*

is a polytopal realization of the σ -invariant accordion complex $\mathcal{AC}^\sigma(D_\circ)$.

Proof. Denote by $P = \text{conv}\{\mathbf{p}(D_\circ | D_\bullet) \mid \sigma\text{-invariant maximal } D_\circ\text{-accordion dissections } D_\bullet\}$ and by $Q = \text{Acco}(D_\circ) \cap \text{Fix}(\Sigma)$. The inclusion $P \subseteq Q$ is clear since D_\bullet is σ -invariant if and only if $\mathbf{p}(D_\circ | D_\bullet) \in \text{Fix}(\Sigma)$. We now prove the reverse inclusion. For that, consider an arbitrary σ -invariant maximal D_\circ -accordion dissection D_\bullet . Its corresponding point $\mathbf{p}(D_\circ | D_\bullet)$ is a common vertex of P and Q . Moreover, any edge e of Q incident to $\mathbf{p}(D_\circ | D_\bullet)$ is the intersection of $\text{Fix}(\Sigma)$ with a face F of $\text{Acco}(D_\circ)$ that corresponds to a σ -invariant D_\circ -dissection. Since $\mathcal{AC}^\sigma(D_\circ)$ is a pseudomanifold, this dissection can be refined into another maximal σ -invariant D_\circ -accordion dissection D'_\bullet . The point $\mathbf{p}(D_\circ | D'_\bullet)$ belongs to F and to $\text{Fix}(\Sigma)$ and thus to e . We conclude that if v is a common vertex of P and Q , then so are all neighbors of v in the graph of Q . Propagating this property, we obtain that all vertices of Q are also vertices of P , so that $P = Q$. Finally, there is a clear injection from the σ -invariant accordion complex $\mathcal{AC}^\sigma(D_\circ)$ to the boundary complex of $P = Q$, thus a bijection (since these complexes are two spheres with the same vertex set). \square

3. THE \mathbf{d} -VECTOR FAN

In this section, we discuss the generalization to the D_\circ -accordion complex of another classical geometric realization of the associahedron coming from the theory of cluster algebras [FZ02, FZ03a, CFZ02, CSZ15]. Namely, we define compatibility vectors in analogy with the denominator vectors of cluster variables, and we characterize the reference dissections D_\circ for which these vectors support a complete simplicial fan realizing the D_\circ -accordion complex.

3.1. \mathbf{d} -vectors. Fix a dissection D_\circ of the hollow n -gon. For a hollow diagonal $\delta_\circ = i_\circ j_\circ$ and a solid diagonal δ_\bullet , we denote by

$$(\delta_\circ | \delta_\bullet) := \begin{cases} -1 & \text{if } \delta_\bullet = (i-1)_\bullet(j-1)_\bullet, \\ 0 & \text{if } \delta_\bullet \text{ and } (i-1)_\bullet(j-1)_\bullet \text{ do not cross,} \\ 1 & \text{if } \delta_\bullet \text{ and } (i-1)_\bullet(j-1)_\bullet \text{ cross.} \end{cases}$$

For any D_\circ -accordion diagonal δ_\bullet , the **d-vector** of δ_\bullet with respect to D_\circ is the vector

$$\mathbf{d}(D_\circ | \delta_\bullet) = \sum_{\delta_\circ \in D_\circ} (\delta_\circ | \delta_\bullet) \mathbf{e}_{\delta_\circ}.$$

In other words, our **d-vector** $\mathbf{d}(D_\circ | \delta_\bullet)$ records the compatibility of the diagonal δ_\bullet with the dissection D_\circ^- . For a D_\circ -accordion dissection D_\bullet , we define $\mathbf{d}(D_\circ | D_\bullet) := \{\mathbf{d}(D_\circ | \delta_\bullet) \mid \delta_\bullet \in D_\bullet\}$.

Example 46. Consider the hollow dissection $D_\circ^{\text{ex}} = \{3_\circ 7_\circ, 3_\circ 13_\circ, 9_\circ 13_\circ\}$ and the rightmost solid dissection $D_\bullet^{\text{ex}} = \{2_\bullet 6_\bullet, 2_\bullet 10_\bullet, 10_\bullet 14_\bullet\}$ of Figure 2. Its **d-vectors** are given by

$$\mathbf{d}(D_\circ^{\text{ex}} | 2_\bullet 6_\bullet) = -\mathbf{e}_{3_\circ 7_\circ}, \quad \mathbf{d}(D_\circ^{\text{ex}} | 2_\bullet 10_\bullet) = \mathbf{e}_{9_\circ 13_\circ}, \quad \text{and} \quad \mathbf{d}(D_\circ^{\text{ex}} | 10_\bullet 14_\bullet) = \mathbf{e}_{3_\circ 13_\circ} + \mathbf{e}_{9_\circ 13_\circ}.$$

3.2. d-vector fan. We now consider the set of cones

$$\{\mathbb{R}_{\geq 0} \mathbf{d}(D_\circ | D_\bullet) \mid D_\bullet \text{ any } D_\circ\text{-accordion dissection}\}$$

generated by the **d-vectors** of the D_\circ -accordion dissections. We want to characterize the reference hollow dissections D_\circ for which these cones form a complete simplicial fan realizing the D_\circ -accordion complex. We start with a negative result. An *even interior cell* of a dissection D is a cell with an even number of edges which are all internal diagonals of D .

Proposition 47. *If the reference hollow dissection D_\circ contains an even interior cell, then the **d-vectors** cannot realize the D_\circ -accordion complex.*

Proof. Assume that D_\circ contains an even interior cell C_\circ . Denote its vertices by $i_\circ^1, \dots, i_\circ^{2p}$ (in clockwise order) and its edges $\delta_\circ^k := i_\circ^k i_\circ^{k+1}$ for $k \in [2p]$ (where $i_\circ^{2p+1} = i_\circ^1$ by convention). Denote by D_\circ^k the set of diagonals of D_\circ separated from C_\circ by δ_\circ^k (including δ_\circ^k itself), and let $D_\bullet^k := \{(i-1)_\bullet (j-1)_\bullet \mid i_\circ j_\circ \in D_\circ^k\}$. Consider the solid diagonals $\delta_\bullet^k := (i^k+1)_\bullet (i^{k+1}+1)_\bullet$ for $k \in [2p]$. Observe that δ_\bullet^k only crosses diagonals of D_\bullet^{k-1} and D_\bullet^k , and that δ_\bullet^k and δ_\bullet^{k+1} cross precisely the same diagonals of D_\bullet^k . Since the cell is even, it ensures that the **d-vectors** of the diagonals δ_\bullet^k for $k \in [2p]$ satisfy the linear dependence

$$\sum_{\substack{k \in [2p] \\ k \text{ even}}} \mathbf{d}(D_\circ | \delta_\bullet^k) = \sum_{\substack{k \in [2p] \\ k \text{ odd}}} \mathbf{d}(D_\circ | \delta_\bullet^k).$$

However, as already mentioned in Section 1.4, the diagonals δ_\bullet^k for $k \in [2p]$ all belong to the D_\circ -accordion dissection $D_\bullet^+ := \{(i+1)_\bullet (j+1)_\bullet \mid i_\circ j_\circ \in D_\circ\}$. Therefore, the cone $\mathbb{R}_{\geq 0} \mathbf{d}(D_\circ | D_\bullet^+)$ is degenerate, so that the **d-vectors** cannot realize the D_\circ -accordion complex. \square

Example 48. Consider a hollow octagon and the reference dissection $D_\circ := \{1_\circ 5_\circ, 5_\circ 9_\circ, 9_\circ 13_\circ, 13_\circ 1_\circ\}$ with an interior square cell $1_\circ 5_\circ 9_\circ 13_\circ$. Then we have

$$\begin{aligned} \mathbf{d}(D_\circ | 2_\bullet 6_\bullet) &= \mathbf{e}_{1_\circ 5_\circ} + \mathbf{e}_{5_\circ 9_\circ} & \mathbf{d}(D_\circ | 6_\bullet 10_\bullet) &= \mathbf{e}_{5_\circ 9_\circ} + \mathbf{e}_{9_\circ 13_\circ} \\ \mathbf{d}(D_\circ | 10_\bullet 14_\bullet) &= \mathbf{e}_{9_\circ 13_\circ} + \mathbf{e}_{13_\circ 1_\circ} & \mathbf{d}(D_\circ | 14_\bullet 2_\bullet) &= \mathbf{e}_{13_\circ 1_\circ} + \mathbf{e}_{1_\circ 5_\circ} \end{aligned}$$

so that there is already a linear dependence

$$\mathbf{d}(D_\circ | 2_\bullet 6_\bullet) + \mathbf{d}(D_\circ | 10_\bullet 14_\bullet) = \mathbf{d}(D_\circ | 6_\bullet 10_\bullet) + \mathbf{d}(D_\circ | 14_\bullet 2_\bullet)$$

among the **d-vectors** of the D_\circ -accordion dissection $D_\bullet^+ = \{2_\bullet 6_\bullet, 6_\bullet 10_\bullet, 10_\bullet 14_\bullet, 14_\bullet 2_\bullet\}$.

On the negative side, we have seen that the presence of even interior cells prohibits the **d-vectors** from forming a complete simplicial fan. The positive side is that the even interior cells are the only obstructions.

Theorem 49. *The collection of cones*

$$\mathcal{F}^{\mathbf{d}}(D_\circ) := \{\mathbb{R}_{\geq 0} \mathbf{d}(D_\circ | D_\bullet) \mid D_\bullet \text{ any } D_\circ\text{-accordion dissection}\}$$

*forms a complete simplicial fan, that we call the **d-vector fan** of D_\circ , if and only if D_\circ contains no even interior cell.*

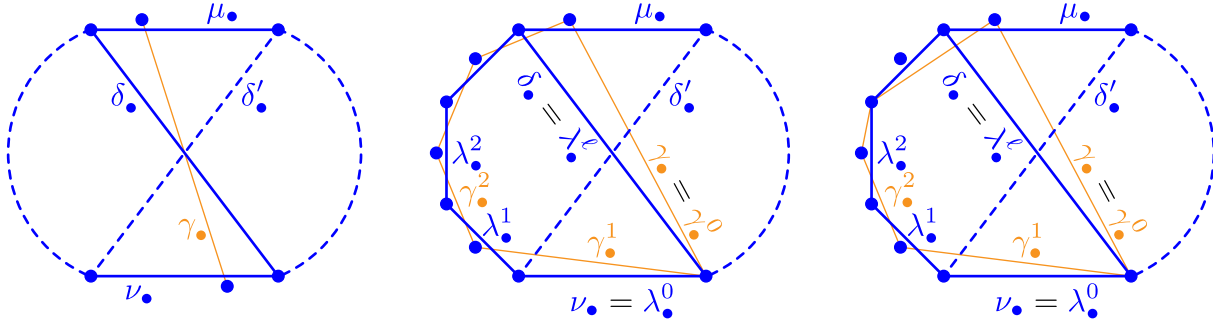


FIGURE 8. Illustration of the notations and of the different cases in the proof of Theorem 49.

Proof. We use the characterization of complete simplicial fans presented in Proposition 28.

Observe first that $\mathbf{d}(D_\circ | D_\circ^-) = (\mathbb{R}_{\leq 0})^{D_\circ}$ is the only cone of $\mathcal{F}^{\mathbf{d}}(D_\circ)$ intersecting the interior of the negative orthant $(\mathbb{R}_{\leq 0})^{D_\circ}$. Therefore, $\mathcal{F}^{\mathbf{d}}(D_\circ)$ fulfills Condition (1) in Proposition 28.

To check Condition (2), consider two adjacent maximal D_\circ -accordion dissections D_\bullet and D'_\bullet and let $\delta_\bullet \in D_\bullet$ and $\delta'_\bullet \in D'_\bullet$ be such that $D_\bullet \setminus \{\delta_\bullet\} = D'_\bullet \setminus \{\delta'_\bullet\}$. Let μ_\bullet and ν_\bullet be the diagonals of $\overline{D_\bullet} \cap \overline{D'_\bullet}$ as in Lemma 9 (see also Figure 4). In other words, μ_\bullet and ν_\bullet are incident to both δ_\bullet and δ'_\bullet , and they are crossed by the hollow diagonal which intersect δ_\bullet and δ'_\bullet . Let $\gamma_\circ = i_\circ j_\circ$ be such a hollow diagonal crossing δ_\bullet , δ'_\bullet , μ_\bullet and ν_\bullet , and let $\gamma_\bullet = (i-1)_\bullet(j-1)_\bullet$. We now distinguish three cases:

- ◊ Assume that γ_\bullet still crosses μ_\bullet and ν_\bullet . In this case, any diagonal of D_\circ^- crossing both (resp. either) δ_\bullet and (resp. or) δ'_\bullet also crosses both (resp. either) μ_\bullet and (resp. or) ν_\bullet . See Figure 8 (left). Therefore, the \mathbf{d} -vectors of $D_\bullet \cup D'_\bullet$ satisfy the linear dependence

$$\mathbf{d}(D_\circ | \delta_\bullet) + \mathbf{d}(D_\circ | \delta'_\bullet) = \mathbf{d}(D_\circ | \mu_\bullet) + \mathbf{d}(D_\circ | \nu_\bullet).$$

- ◊ Assume that γ_\bullet crosses neither μ_\bullet nor ν_\bullet . Then γ_\bullet is incident to both μ_\bullet and ν_\bullet , and therefore is either δ_\bullet or δ'_\bullet , say $\gamma_\bullet = \delta_\bullet$. Then $\mathbf{d}(\gamma_\circ | \delta_\bullet) = -1$ while $\mathbf{d}(\gamma_\circ | \delta'_\bullet) = 1$ (since δ'_\bullet crosses $\delta_\bullet = \gamma_\bullet$), so that $\mathbf{d}(\gamma_\circ | \delta_\bullet) + \mathbf{d}(\gamma_\circ | \delta'_\bullet) = 0$. Moreover, we have $\mathbf{d}(\gamma_\circ | \delta'_\bullet) = 0$ for any diagonal $\varepsilon_\bullet \in D_\bullet \cap D'_\bullet$ since $\delta_\bullet = \gamma_\bullet$ cannot cross ε_\bullet as they both belongs to D_\bullet . Therefore, the set $\{\mathbf{d}(D_\circ | \delta_\bullet) + \mathbf{d}(D_\circ | \delta'_\bullet)\} \cup \mathbf{d}(D_\circ | D_\bullet \cap D'_\bullet)$ contains $|D_\circ|$ vectors of \mathbb{R}^{D_\circ} whose γ_\circ -coordinate all vanish, so that it admits a linear dependence.
- ◊ Otherwise, we can assume that γ_\bullet crosses μ_\bullet but not ν_\bullet . Then γ_\bullet has a common endpoint with ν_\bullet and δ_\bullet (or δ'_\bullet , but we then permute notations). Changing our initial choice of γ_\circ , we can assume that no diagonal of D_\circ^- separates γ_\bullet from δ_\bullet . We now denote clockwise
 - by $\nu_\bullet =: \lambda_\bullet^0, \lambda_\bullet^1, \dots, \lambda_\bullet^\ell := \delta_\bullet$ the edges of the cell C_\bullet of D_\bullet containing ν_\bullet and δ_\bullet ,
 - by $\gamma_\bullet =: \gamma_\bullet^0, \gamma_\bullet^1, \dots, \gamma_\bullet^k$ the edges of the cell C_\bullet^- of D_\circ^- containing γ_\bullet and crossed by δ_\bullet .

These notations are illustrated on Figure 8. We still distinguish two subcases as in Figure 8:

- If γ_\bullet^i crosses λ_\bullet^i for all i as in Figure 8 (middle), then $\ell = k$ and we have the linear dependence

$$2\mathbf{d}(D_\circ | \delta_\bullet) + \mathbf{d}(D_\circ | \delta'_\bullet) = \mathbf{d}(D_\circ | \mu_\bullet) + \sum_{i \in [\ell-1]} (-1)^{(i-1)} \mathbf{d}(D_\circ | \lambda_\bullet^i).$$

It is essential here that $\ell = k$ is even. This is guarantied by the assumption that D_\circ (and thus D_\circ^-) has no even interior cell, since C_\bullet^- is an interior cell of D_\circ^- of size k .

- Otherwise, we are in a situation similar to Figure 8 (right). Considering the maximal index m such that γ_\bullet^i crosses λ_\bullet^i for all $i \leq m$, and we have the linear dependence

$$\mathbf{d}(D_\circ | \delta_\bullet) + \mathbf{d}(D_\circ | \delta'_\bullet) = \mathbf{d}(D_\circ | \mu_\bullet) + \sum_{i \in [m]} (-1)^{(i-1)} \mathbf{d}(D_\circ | \lambda_\bullet^i). \quad \square$$

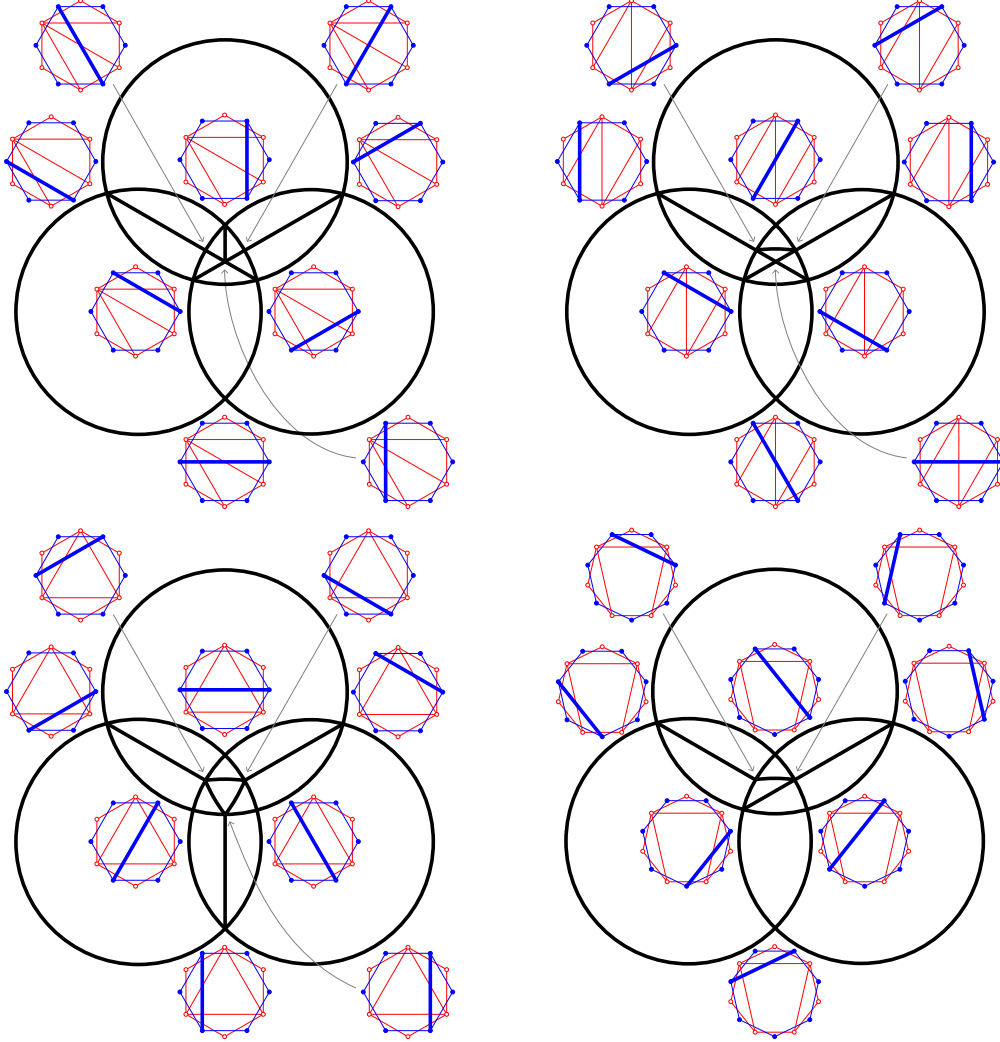


FIGURE 9. Stereographic projections of the \mathbf{d} -vector fans $\mathcal{F}^{\mathbf{d}}(D_{\circ})$ for various reference hollow dissections D_{\circ} . See Figure 6 for alternative simplicial fan realizations of these accordion complexes.

Example 50. Following Example 2, we observe that special reference dissections give rise to the following relevant fans:

- ◇ For a snake triangulation Σ_{\circ} , the \mathbf{d} -vector fan $\mathcal{F}^{\mathbf{d}}(\Sigma_{\circ})$ coincides with the type A cluster fan of S. Fomin and A. Zelevinsky [FZ03a].
- ◇ For any triangulation T_{\circ} , the \mathbf{d} -vector fan $\mathcal{F}^{\mathbf{d}}(T_{\circ})$ was already constructed in [CSZ15].
- ◇ For a quadrangulation Q_{\circ} with no interior quadrangle (equivalently, with no cross), we obtain an alternative realization of the Stokes complexes studied in [Bar01, Cha16]. This was observed by A.-H. Bateni, T. Manneville and V. Pilaud in [BMP16].

Figure 9 illustrates the \mathbf{d} -vector fans $\mathcal{F}^{\mathbf{d}}(D_{\circ})$ for the same reference dissections D_{\circ} as in Figure 6. More precisely, we have represented the stereographic projection of the fans from the point $[-1, -1, -1]$. Therefore, the external face of the projection corresponds to the D_{\circ} -accordion dissection D_{\circ}^{-} . We have labeled all vertices of the projection (*i.e.* the rays of the fan) by the corresponding D_{\circ} -accordion diagonals. Compare with Figure 6.

Remark 51. To prove that the \mathbf{d} -vector fan $\mathcal{F}^{\mathbf{d}}(D_{\circ})$ is polytopal, we would need to find suitable hyperplanes orthogonal to their rays in order to apply Theorem 36. For the \mathbf{g} -vector fan, these

hyperplanes were defined using the height function $\omega(D_\circ | \delta_\bullet)$. It would be natural to use the same height function for the \mathbf{d} -vector fan as well. Unfortunately, for this choice of height function, we can only prove Condition (i) of Theorem 36 when D_\circ is a triangulation (see also [CSZ15]). We were not able to find suitable right hand sides for any dissection D_\circ .

Remark 52. Our \mathbf{d} -vectors record the compatibility with the dissection D_\circ^- . A priori, we could compute compatibility vectors with respect to any other maximal D_\circ -accordion dissection D_\circ^{ini} . Experiments suggest that the \mathbf{d} -vector construction provides a complete simplicial fan as long as neither D_\circ nor D_\circ^{ini} contain no even interior cell. We checked it for reference quadrangulations with at most 5 diagonals. The linear dependences involved seem however much more complicated than those of the proof of Theorem 49 (in particular, they may involve \mathbf{d} -vectors of diagonals not included in the cells containing δ_\bullet and δ'_\bullet).

4. SECTIONS AND PROJECTIONS

Recall that for a fan \mathcal{F} of \mathbb{R}^d and a linear subspace V of \mathbb{R}^d , the *section* of \mathcal{F} by V is the fan $\mathcal{F}|_V := \{C \cap V \mid C \in \mathcal{F}\}$. For a polytope $P \subseteq \mathbb{R}^d$ and a projection $\pi : \mathbb{R}^d \rightarrow V$, the normal fan of the projected polytope $\pi(P)$ is the section of the normal fan of P by V [Zie95, Lem. 7.11]. We now consider sections of the \mathbf{g} - and \mathbf{d} -vector fans by coordinate subspaces. For two dissections $D_\circ \subset D'_\circ$, we naturally identify \mathbb{R}^{D_\circ} with the subspace spanned by $\{\mathbf{e}_{\delta_\circ} \mid \delta_\circ \in D_\circ\}$ in $\mathbb{R}^{D'_\circ}$.

4.1. Coordinate sections of the \mathbf{d} -vector fan. We start by presenting sections of the \mathbf{d} -vector fan which are not very surprising. The following lemma is immediate from the definition of \mathbf{d} -vectors.

Lemma 53. *Consider two dissections $D_\circ \subset D'_\circ$, and a D'_\circ -accordion diagonal δ_\bullet . Then we have $\mathbf{d}(D_\circ | \delta_\bullet) \in \mathbb{R}^{D_\circ}$ if and only if δ_\bullet does not cross any diagonal of $\{(i-1)_\bullet(j-1)_\bullet \mid i_\circ j_\circ \in D'_\circ \setminus D_\circ\}$.*

Corollary 54. *For two dissections $D_\circ \subset D'_\circ$, the face complex of the section of the \mathbf{d} -vector fan $\mathcal{F}^{\mathbf{d}}(D'_\circ)$ by \mathbb{R}^{D_\circ} is isomorphic to the link of the dissection $\{(i-1)_\bullet(j-1)_\bullet \mid i_\circ j_\circ \in D'_\circ \setminus D_\circ\}$ in the D'_\circ -accordion complex $\mathcal{AC}(D'_\circ)$.*

4.2. Coordinate sections of the \mathbf{g} -vector fan. More relevant are the sections of the \mathbf{g} -vector fan. They provide an alternative approach to polytopal realizations of the accordion complex based on projected associahedra. This approach relies on the following crucial observation.

Lemma 55. *Consider two dissections $D_\circ \subset D'_\circ$, and a D'_\circ -accordion diagonal δ_\bullet . Then we have $\mathbf{g}(D'_\circ | \delta_\bullet) \in \mathbb{R}^{D_\circ}$ if and only if δ_\bullet is a D_\circ -accordion diagonal. Moreover, in this case, the \mathbf{g} -vectors $\mathbf{g}(D_\circ | \delta_\bullet)$ and $\mathbf{g}(D'_\circ | \delta_\bullet)$ coincide.*

Proof. Let $\delta_\circ \in D'_\circ \setminus D_\circ$. By definition, a D'_\circ -accordion diagonal δ_\bullet does not slalom on δ_\circ if and only if the δ_\circ -coordinate of $\mathbf{g}(D_\circ | \delta_\bullet)$ vanishes. Thus, δ_\bullet is a D_\circ -accordion diagonal if and only if the δ_\circ -coordinate of $\mathbf{g}(D'_\circ | \delta_\bullet)$ vanishes for all $\delta_\circ \in D'_\circ \setminus D_\circ$. \square

Based on this lemma, we obtain in the following statements an alternative realization on the \mathbf{g} -vector fan, which is illustrated on Figure 10.

Theorem 56. *For two dissections $D_\circ \subset D'_\circ$, the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$ is precisely the set of cones $\{C \in \mathcal{F}^{\mathbf{g}}(D'_\circ) \mid C \subset \mathbb{R}^{D_\circ}\}$ and coincides with the section of the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D'_\circ)$ by \mathbb{R}^{D_\circ} .*

Proof. Lemma 55 immediately implies that $\mathcal{F}^{\mathbf{g}}(D_\circ) = \{C \in \mathcal{F}^{\mathbf{g}}(D'_\circ) \mid C \subset \mathbb{R}^{D_\circ}\}$. A priori, it is a subfan of the section $\mathcal{F}^{\mathbf{g}}(D'_\circ)|_{\mathbb{R}^{D_\circ}} = \{C \cap \mathbb{R}^{D_\circ} \mid C \in \mathcal{F}^{\mathbf{g}}(D'_\circ)\}$. However, since $\mathcal{F}^{\mathbf{g}}(D_\circ)$ is already a complete simplicial fan of \mathbb{R}^{D_\circ} , it coincides with $\mathcal{F}^{\mathbf{g}}(D'_\circ)|_{\mathbb{R}^{D_\circ}}$. \square

Theorem 57. *For two dissections $D_\circ \subset D'_\circ$, the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$ is realized by the orthogonal projection of the D'_\circ -accordiohedron $\text{Acco}(D'_\circ)$ on \mathbb{R}^{D_\circ} , which is equivalently described by:*

- ◊ the convex hull of the points $\sum_{\delta_\bullet \in D_\circ} \omega(D'_\circ | \delta_\bullet) \cdot \mathbf{c}(D_\circ | \delta_\bullet)$ for all D_\circ -accordion dissections D_\circ ,
- ◊ the intersection of the half-spaces $\{\mathbf{x} \in \mathbb{R}^{D_\circ} \mid \langle \mathbf{g}(D_\circ | \gamma_\bullet) \mid \mathbf{x} \rangle \leq \omega(D'_\circ | \delta_\circ)\}$ for all D_\circ -accordion diagonals γ_\bullet .

Proof. Since $\mathcal{F}^{\mathbf{g}}(D'_\circ)$ is the normal fan of $\text{Acco}(D'_\circ)$, Theorem 56 implies that $\mathcal{F}^{\mathbf{g}}(D_\circ) = \mathcal{F}^{\mathbf{g}}(D'_\circ)|_{\mathbb{R}^{D_\circ}}$ is the normal fan of the orthogonal projection of $\text{Acco}(D'_\circ)$ on \mathbb{R}^{D_\circ} [Zie95, Lem. 7.11]. We therefore just need to prove the given vertex and facet descriptions of this projection. First, since $\mathcal{F}^{\mathbf{g}}(D_\circ) = \mathcal{F}^{\mathbf{g}}(D'_\circ)|_{\mathbb{R}^{D_\circ}}$, the inequalities of the projection of $\text{Acco}(D'_\circ)$ on \mathbb{R}^{D_\circ} are just the inequalities of $\text{Acco}(D'_\circ)$ whose normal vectors are in \mathbb{R}^{D_\circ} . Finally, the vertex description follows from the inequality description using the same argument as in Lemma 33. \square

Remark 58. The projection of the accordiohedron $\text{Acco}(D'_\circ)$ on \mathbb{R}^{D_\circ} differs from the accordiohedron $\text{Acco}(D_\circ)$: they have both $\mathcal{F}^{\mathbf{g}}(D_\circ)$ as normal fan, but their precise geometry is different.

Corollary 59. *For any hollow dissection D_\circ , the \mathbf{g} -vector fan $\mathcal{F}^{\mathbf{g}}(D_\circ)$ is realized by a projection of an associahedron of [HPS17].*

Proof. Apply Theorem 57 to any triangulation T_\circ that refines D_\circ . \square

Remark 60. Approaching accordion complexes as coordinate sections of \mathbf{g} -vector fans actually provides more concise (but also less instructive) proofs for Sections 1.3 and 2.3. Namely, consider any dissection D_\circ and let T_\circ be a triangulation that refines D_\circ . The sign coherence property for triangulations (see Corollary 24) shows that the section $\mathcal{F}^{\mathbf{g}}(T_\circ)|_{\mathbb{R}^{D_\circ}} = \{C \cap \mathbb{R}^{D_\circ} \mid C \in \mathcal{F}^{\mathbf{g}}(T_\circ)\}$ actually coincides with $\{C \in \mathcal{F}^{\mathbf{g}}(T_\circ) \mid C \subset \mathbb{R}^{D_\circ}\}$. Therefore, this gives an alternative concise proof that the collection of cones $\{C \in \mathcal{F}^{\mathbf{g}}(T_\circ) \mid C \subset \mathbb{R}^{D_\circ}\}$ forms a complete simplicial fan. Moreover, this fan has the same combinatorics as the D_\circ -accordion complex $\mathcal{AC}(D_\circ)$ by Lemma 55. We conclude directly that $\mathcal{AC}(D_\circ)$ is a pseudomanifold realized by the fan $\{C \in \mathcal{F}^{\mathbf{g}}(T_\circ) \mid C \subset \mathbb{R}^{D_\circ}\}$ and by the orthogonal projection of the associahedron $\text{Asso}(T_\circ)$ on \mathbb{R}^{D_\circ} .

4.3. Cluster algebra analogues. The perspective on accordion complexes developed in this section also opens the door to generalizations on arbitrary cluster algebras (finite type or not). Namely, consider an arbitrary cluster $X_\circ = (x_\circ^1, \dots, x_\circ^m)$ in an arbitrary cluster algebra \mathcal{A} . For any cluster variable $y \in \mathcal{A}$, we denote by $\mathbf{g}(X_\circ \mid y) \in \mathbb{R}^m$ and $\mathbf{d}(X_\circ \mid y) \in \mathbb{R}^m$ the \mathbf{g} - and \mathbf{d} -vectors of y computed with respect to X_\circ , see [FZ02, FZ07]. Fix a non-empty proper subset I of $[m]$. We consider two natural subcomplexes of the cluster complex of \mathcal{A} :

- ◊ the subcomplex $\Delta^{\mathbf{d}}(X_\circ, I)$ induced by the variables y such that $\mathbf{d}(X_\circ \mid y)_i = 0$ for all $i \in I$,
- ◊ the subcomplex $\Delta^{\mathbf{g}}(X_\circ, I)$ induced by the variables y such that $\mathbf{g}(X_\circ \mid y)_i = 0$ for all $i \in I$.

It is well-known that the subcomplex $\Delta^{\mathbf{d}}(X_\circ, I)$ is the cluster complex obtained by freezing all variables x_i for $i \in I$. For example in type A , it is a join of simplicial associahedra and it can therefore be realized by a product of smaller associahedra. In contrast, we do not know whether the subcomplex $\Delta^{\mathbf{g}}(X_\circ, I)$ has been investigated. The present paper dealt with the type A situation.

Example 61. Let T_\circ be a triangulation, with internal diagonals labeled by $1, \dots, m$. Consider the corresponding type A_m cluster X_\circ . Then for any non-empty proper subset I of $[m]$, the

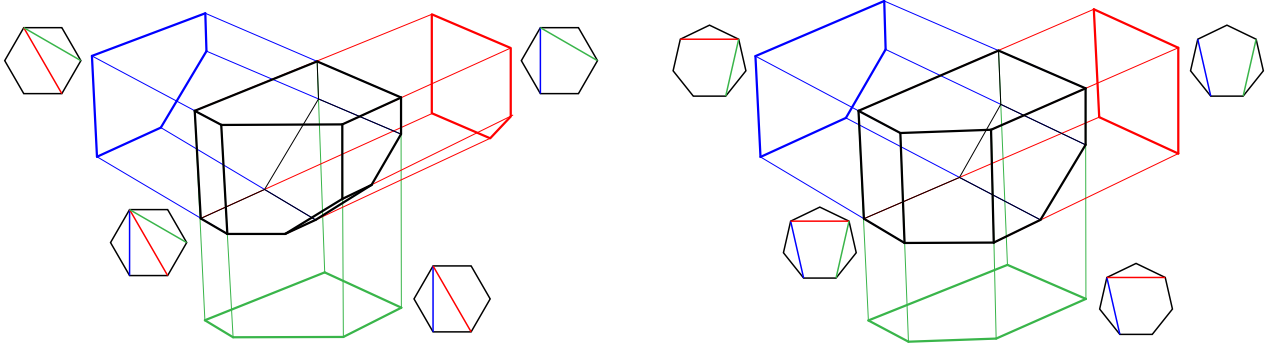


FIGURE 10. Projecting accordiohedra on coordinate planes yields smaller accordiohedra.

subcomplex $\Delta^{\mathfrak{g}}(X_{\circ}, I)$ is isomorphic to the D_{\circ} -accordion complex, where D_{\circ} is the dissection obtained by deleting in T_{\circ} the diagonals labeled by I .

Example 62. Example 61 extends to cluster algebras on surfaces [FST08, FT12], using accordions of dissections of surfaces.

The following statement extends Theorem 56 to arbitrary cluster algebras.

Theorem 63. *The subset $\{C \in \mathcal{F}^{\mathfrak{g}}(X_{\circ}) \mid C \subseteq \mathbb{R}^{[m] \setminus I}\}$ of the \mathfrak{g} -vector fan $\mathcal{F}^{\mathfrak{g}}(X_{\circ})$ of X_{\circ} coincides with the section $\mathcal{F}^{\mathfrak{g}}(X_{\circ})|_{\mathbb{R}^{[m] \setminus I}} = \{C \cap \mathbb{R}^{[m] \setminus I} \mid C \in \mathcal{F}^{\mathfrak{g}}(X_{\circ})\}$.*

Proof. The inclusion $\{C \in \mathcal{F}^{\mathfrak{g}}(X_{\circ}) \mid C \subseteq \mathbb{R}^{[m] \setminus I}\} \subseteq \mathcal{F}^{\mathfrak{g}}(X_{\circ})|_{\mathbb{R}^{[m] \setminus I}}$ is clear. For the reverse inclusion, we use the sign coherence property of \mathfrak{g} -vectors in cluster algebras, which was conjectured in [FZ07, Conj. 6.13] and proved in [GHKK14, Thm. 5.1] in general. This property implies that the coordinate plane $\mathbb{R}^{[m] \setminus I}$ intersects any cone C of $\mathcal{F}^{\mathfrak{g}}(X_{\circ})$ in a face C' . This shows that $C \cap \mathbb{R}^{[m] \setminus I} = C'$ belongs to $\{C \in \mathcal{F}^{\mathfrak{g}}(X_{\circ}) \mid C \subseteq \mathbb{R}^{[m] \setminus I}\}$. \square

Corollary 64. *The subcomplex $\Delta^{\mathfrak{g}}(X_{\circ}, I)$ induced by the variables y such that $\mathfrak{g}(X_{\circ} \mid y)_i = 0$ for all $i \in I$ is a pseudomanifold.*

Moreover, extending the result of C. Hohlweg, C. Lange and H. Thomas [HLT11] in the acyclic case, C. Hohlweg, V. Pilaud and S. Stella recently constructed a polytope $\text{Asso}(X_{\circ})$ realizing the \mathfrak{g} -vector fan $\mathcal{F}^{\mathfrak{g}}(X_{\circ})$ in [HPS17]. We can use this associahedron to realize the subcomplex $\Delta^{\mathfrak{g}}(X_{\circ}, I)$ as a convex polytope, extending Theorem 57.

Corollary 65. *The orthogonal projection of $\text{Asso}(X_{\circ})$ on $\mathbb{R}^{[m] \setminus I}$ is a realization of $\Delta^{\mathfrak{g}}(X_{\circ}, I)$.*

Finally, when oriented in the suitable direction v (the sum of the positive roots, or equivalently the sum of the fundamental weights), the graph of the generalized associahedron $\text{Asso}(X_{\circ})$ is the Hasse diagram of a Cambrian lattice [Rea06]. One can similarly orient the graph of the projection of $\text{Asso}(X_{\circ})$ on $\mathbb{R}^{[m] \setminus I}$ in the direction of the projection of v on $\mathbb{R}^{[m] \setminus I}$. Is the resulting graph the Hasse diagram of a lattice? Combining the results of [GM16] with that of the present paper shows that this property holds in type A . We also computationally verified the statement in types B_4 , B_5 , D_4 and D_5 . Following [GM16] it seems promising to construct first a lattice structure on biclosed sets of \mathfrak{c} -vectors, and to obtain then the graph of the projection of $\text{Asso}(X_{\circ})$ on $\mathbb{R}^{[m] \setminus I}$ as the Hasse diagram of a lattice quotient.

To conclude, let us mention that the ideas developed in this section have also inspired further investigation of sections of \mathfrak{g} -vector fans of support τ -tilting complexes of associative algebras, see [PPS17] and [PPP17, Sect. 4.2.6].

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