THE DIAMETER OF TYPE $D$ ASSOCIAHEDRA
AND THE NON-LEAVING-FACE PROPERTY

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Abstract. We prove that the graph diameter of the $n$-dimensional associahedron of type $D$ is precisely $2n - 2$ for all $n$. Furthermore, we show that all type $ABCD$ associahedra have the non-leaving-face property, that is, any minimal path connecting two vertices in the graph of the polytope stays in the minimal face containing both. In contrast, we present relevant examples related to the associahedron that do not always satisfy this property.

Keywords. Associahedron – graph diameter – pseudotriangulation – flip graph.

1. Introduction

The associahedron is a convex polytope whose vertices are in correspondence with triangulations of a convex polygon and whose edges are flips among them. Motivated by efficiency of repeated access and information update in binary search trees, D. Sleator, R. Tarjan and W. Thurston [STT88] showed that the diameter of the $n$-dimensional associahedron is at most $2n - 4$ for $n$ greater than 9, and used arguments in hyperbolic geometry to prove that this bound is tight when $n$ is large enough. They also conjectured that the diameter is $2n - 4$ for all $n$ greater than 9. This conjecture was recently settled using purely combinatorial arguments by L. Pournin [Pou14], who explicitly exhibited two triangulations realizing this maximal distance.

Associahedra are considered as one of the most important families of examples in polytope theory [DRS10, Zie95]. Besides their combinatorial beauty, they are of great importance in diverse areas in mathematics, computer science and physics [Sta63, Sta97, MHPS12]. One of the most significant appearances of associahedra is in the theory of cluster algebras initiated by S. Fomin and A. Zelevinsky [FZ02, FZ03a]. They introduced a notion of generalized associahedra which extends the concept of associahedra to arbitrary finite Coxeter groups [FZ03b]. These essential objects encode the flip graphs of cluster algebras of finite type, and were realized as polytopes for the first time by F. Chapoton, S. Fomin and A. Zelevinsky [CFZ02]. Three interesting cases are the infinite families of generalized associahedra of type $A$ (classical associahedra), of types $B$ or $C$ (cyclohedra), and of type $D$.

In this paper, we show that the diameter of the $n$-dimensional associahedron of type $D$ is precisely $2n - 2$ for all $n$. This is done using a convenient combinatorial model for type $D$ associahedra in terms of centrally symmetric pseudotriangulations of a regular $2n$-gon with a small hole in the center. Our methods are purely combinatorial and we explicitly describe two vertices of the polytope which are at maximal distance. In connection to this graph diameter question, we also show that all infinite families of associahedra of finite types have the non-leaving-face property, namely every geodesic connecting two vertices in the graph of the polytope stays in the minimal face containing both. This generalizes a known result of D. Sleator, R. Tarjan and W. Thurston [STT88, Lemma 3]. This property is also satisfied by the exceptional types $H_3$, $H_4$, $F_4$ and $E_6$. The remaining types $E_7$ and $E_8$ are still to be checked. In contrast, we present three remarkable examples related to the associahedron which do not always satisfy this property: the pseudotriangulation polytopes, the multiassociahedra, and the graph associahedra.

In contrast to the situation in type $A$, the proof of the diameter formula in type $D$ is very short and can be easily deduced from the combinatorial model. The proof of the non-leaving-face property uses similar normalization ideas to the ones used in type $A$ [STT88], however it becomes more involved in type $D$. The verification for the exceptional types mentioned above was done using the computer software Sage [S+12].

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2. **Pseudotriangulation model for type D associahedra**

In this section, we present a combinatorial model for the type $D_n$ associahedra $\text{Asso}(D_n)$ in terms of pseudotriangulations of a geometric configuration $\mathbb{D}_n$. The vertices of $\text{Asso}(D_n)$ correspond to centrally symmetric pseudotriangulations, and its edges to *flips* between them. In Remark 1, we compare our model to the geometric interpretation of type $D$ cluster algebras presented by S. Fomin and A. Zelevinsky in [FZ03b, Section 3.5][FZ03a, Section 12.4].

We consider a regular convex $2n$-gon, together with a disk $D$ placed at its center, whose radius is small enough such that $D$ only intersects the long diagonals of the $2n$-gon. We denote by $\mathbb{D}_n$ the resulting configuration, see Figure 1. The *chords* of $\mathbb{D}_n$ are all the diagonals of the $2n$-gon, except the long ones, plus all the segments tangent to the disk $D$ and with one endpoint among the vertices of the $2n$-gon. Note that each vertex $p$ is adjacent to two of the latter chords; we denote by $p^l$ (resp. by $p^r$) the chord emanating from $p$ and tangent on the left (resp. right) to the disk $D$, and we call these chords *central*. The faces of the type $D_n$ associahedron can be interpreted geometrically on the configuration $\mathbb{D}_n$ as follows:

(i) Facets correspond to *centrally symmetric pairs of (internal) chords* of the geometric configuration $\mathbb{D}_n$, see Figure 1 (left).

(ii) Faces correspond to crossing-free centrally symmetric sets of chords. The face lattice corresponds to the reverse inclusion lattice on crossing-free centrally symmetric sets of chords.

(iii) Vertices correspond to *centrally symmetric pseudotriangulations* of $\mathbb{D}_n$ (*i.e.* inclusion maximal centrally symmetric crossing-free sets of chords of $\mathbb{D}_n$). Each pseudotriangulation of $\mathbb{D}_n$ contains exactly $2n$ chords, and partitions $\text{conv}(\mathbb{D}_n) \setminus D$ into *pseudotriangles* (*i.e.* interiors of simple closed curves with three convex corners related by three concave chains). See Figure 1 (middle) and (right). We refer to [RSS08] for a complete survey on pseudotriangulations, including their history, motivations, and applications.

(iv) Edges correspond to *flips* of centrally symmetric pairs of chords between centrally symmetric pseudotriangulations of $\mathbb{D}_n$. A flip in a pseudotriangulation $T$ replaces an internal chord $e$ by the unique other internal chord $f$ such that $(T \setminus e) \cup f$ is again a pseudotriangulation of $T$. To be more precise, deleting $e$ in $T$ merges the two pseudotriangles of $T$ incident to $e$ into a pseudoquadrangle $\mathbb{X}$ (*i.e.* the interior of a simple closed curve with four convex corners related by four concave chains), and adding $f$ splits the pseudoquadrangle $\mathbb{X}$ into two new pseudotriangles. The chords $e$ and $f$ are the two unique chords which lie both in the interior of $\mathbb{X}$ and on a geodesic between two opposite corners of $\mathbb{X}$. We refer again to [RSS08] for more details.

For example, the two pseudotriangulations of Figure 1 are related by a centrally symmetric pair of flips. We have represented different kinds of flips between centrally symmetric pseudotriangulations of the configuration $\mathbb{D}_n$ in Figure 2. Finally, Figure 3 and Figures 4 and 5 show the flip graph on centrally symmetric pseudotriangulations of $\mathbb{D}_3$ and $\mathbb{D}_4$, respectively.

![Figure 1](image1.png)

**Figure 1.** The configuration $\mathbb{D}_3$ has 9 centrally symmetric pairs of chords (left). A centrally symmetric pseudotriangulation $T$ of $\mathbb{D}_3$ (middle). The centrally symmetric pseudotriangulation of $\mathbb{D}_3$ obtained from $T$ by flipping the chords $2^r$ and $2^l$. 

Remark 1. Our geometric interpretation of type $D$ associahedra slightly differs from that of S. Fomin and A. Zelevinsky in [FZ03b, Section 3.5][FZ03a, Section 12.4]. Namely, to obtain their interpretation, we can just remove the disk in the configuration $D_n$ and replace the centrally symmetric pairs of chords $\{p^k, \bar{p}^k\}$ and $\{p^\ell, \bar{p}^\ell\}$ by long diagonals $[p, \bar{p}]$ colored in red and blue respectively. Long diagonals of the same color are then allowed to cross, while long diagonals of different colors cannot. Flips can then be worked out, with special rules for colored long diagonals, see [FZ03b, Section 3.5][FZ03a, Section 12.4]. Although our presentation is only slightly different from the classical presentation, we prefer it in this paper as flips are easier to visualize.

Figure 2. Different kinds of flips in type $D$.

Figure 3. The type $D_3$ flip graph interpreted geometrically with centrally symmetric pseudotriangulations of $D_3$. Note that this graph is the 1-skeleton of the 3-dimensional associahedron since $D_3 = A_3$. 

Figure 4. The type $D_4$ flip graph. We have represented some of the corresponding centrally symmetric pseudotriangulations of $\mathbb{D}_4$ on this picture, while the others can be found on Figure 5. In each pseudotriangulation, the number at the center of the disk is its label in the flip graph, and each pair of chords is labeled with the pseudotriangulation obtained when flipping it. The underlying graph used for the representation is a Schlegel diagram of the type $D_4$ associahedron [CFZ02, HLT11, PS11].
Figure 5. The remaining 42 centrally symmetric pseudotriangulations of the configuration $D_4$. See Figure 4 for the other 8 centrally symmetric pseudotriangulations, the flip graph and the explanation of the labeling conventions.
3. Diameter

**Theorem 2.** The diameter of the $n$-dimensional associahedron of type $D$ is exactly $2n-2$ for all $n$.

Define the left star $S^l$ to be the pseudotriangulation formed by all left central chords $p^l$ for $p \in [n] \cup \overline{[n]}$. Similarly, the right star $S^r$ is formed by all right central chords $p^r$ for $p \in [n] \cup \overline{[n]}$.

**Lemma 3.** The flip distance between the left star $S^l$ and the right star $S^r$ is precisely $2n-2$.

**Proof.** Figure 6 illustrates a path in the flip graph that uses exactly $2n-2$ flips between $S^l$ and $S^r$. We will show that it is impossible to use less. If a centrally symmetric pseudotriangulation has more than two pairs of left central chords then none of the possible flips produces a pair of right central chords. This means that we need to apply at least $n-2$ flips to the left star until we are able to make a flip that produces a pair of right central chords. Since the right star has $n$ pairs of right central chords, we need at least $n$ additional flips to produce it. This proves that any path between $S^l$ and $S^r$ uses at least $(n-2) + n = 2n-2$ flips.

![Figure 6. The distance between the two stars $S^l$ and $S^r$ is $2n-2$.](image)

**Lemma 4.** Let $T$ be a centrally symmetric pseudotriangulation of $D_n$.

(i) If $T$ contains $\ell \geq 1$ centrally symmetric pairs of left central chords $\{p^l, \overline{p^l}\}$, then $T$ is precisely at distance $n-\ell$ from the left star $S^l$ and precisely at distance $n+\ell-2$ from the right star $S^r$.

(ii) If $T$ contains $r \geq 1$ centrally symmetric pairs of right central chords $\{p^r, \overline{p^r}\}$, then $T$ is precisely at distance $n-r$ from the right star $S^r$ and precisely at distance $n+r-2$ from the left star $S^l$.

**Proof.** We prove Point (i) of the lemma, the other point follows by symmetry. Assume that $T$ contains $\ell \geq 1$ centrally symmetric pairs of left central chords. Every non-left-central pair of chords in $T$ needs to be flipped at least once in a path connecting $T$ and $S^l$. Moreover, it is possible to flip them one at a time such that each flip produces a pair of left central chords. This shows that $T$ is precisely at distance $n-\ell$ from the left star $S^l$. The proof that $T$ is at distance $n+\ell-2$ from $S^r$ uses similar arguments to those in the proof of Lemma 3. If $\ell = 1$, then $T$ has exactly one pair of right central chords. In this case, the distance between $T$ and $S^r$ is $n+\ell-2 = n-1$ as desired. A geodesic between $T$ and $S^l$ can be obtained by successively flipping the $n-1$ non-right-central pairs of chords in $T$ such that each flip produces a pair of right central chords. If $\ell \geq 2$, then we need to apply at least $\ell-2$ flips to $T$ until we are able to make a flip that produces a pair of right central chords. Since the right star has $n$ pairs of right central chords, we need at least $n$ additional flips to produce it. Hence, the distance between $T$ and $S^r$ is $n+\ell-2$.

**Proof of Theorem 2.** By Lemma 3, the diameter of the $n$-dimensional associahedron of type $D$ is at least $2n-2$. It remains to show that the distance between any two centrally symmetric pseudotriangulations $T$ and $\overline{T}$ is at most $2n-2$. Let $\ell$ and $\overline{\ell}$ (resp. $r$ and $\overline{r}$) be the number of left (resp. right) central pairs of chords in $T$ and $\overline{T}$ respectively. If $T$ and $\overline{T}$ contain a central pair of chords of the same kind, say left, then they can be connected by a path passing through the left star $S^l$ of length

$$n-\ell + n-\overline{\ell} \leq 2n-2.$$ 

If not, we can assume without loss of generality that $\ell \geq 1$ and $\overline{r} \geq 1$. By Lemma 4, there are two paths connecting $T$ and $\overline{T}$ passing through the star triangulations $S^l$ and $S^r$ respectively of length

$$(n-\ell) + (n+\overline{r}-2) = 2n-2 - \ell + \overline{r} \quad \text{and} \quad (n+\ell-2) + (n-\overline{r}) = 2n-2 + \ell - \overline{r}.$$ 

Clearly, one of these two numbers is less than or equal to $2n-2$. 

□
4. Non-leaving-face property

This section is devoted to the following natural property related to diameter and graph distance on polytopes.

**Definition 5.** A polytope $P$ has the non-leaving-face property if any geodesic connecting two vertices in the graph of $P$ stays in the minimal face of $P$ containing both.

Many classical polytopes have the non-leaving-face property:

(i) the $n$-gon: proper faces are segments;
(ii) the simplex: any two vertices are at distance 1;
(iii) any simplicial polytope: any two vertices belonging to a proper face are at distance 1;
(iv) the $n$-cube: the distance between any two vertices is the Hamming distance between them;
(v) the permutahedron: the distance between two permutations $\sigma, \bar{\sigma}$ in the permutahedron is the number of inversions of $\sigma^{-1}\bar{\sigma}$, and the minimal face containing $\sigma$ and $\bar{\sigma}$ is the ordered partition $\{\sigma([n_i+1] \setminus [n_i]) \mid i \in [p]\}$, where $0 = n_0 < n_1 < \cdots < n_p = n$ is the finest subdivision of $[n]$ such that $\sigma([n_i+1] \setminus [n_i]) = \bar{\sigma}([n_i+1] \setminus [n_i])$ for all $i \in [p]$; lengthening the minimal face containing $\sigma$ and $\bar{\sigma}$ thus introduces useless inversions and therefore lengthen the way from $\sigma$ to $\bar{\sigma}$;
(vi) the associahedron: see [STT88, Lemma 3] and Proposition 6;
(vii) the cyclohedron (aka. type $B$ associahedron): see Proposition 6.

However, there are also many examples of polytopes that do not satisfy the non-leaving-face property, for example a pyramid over a hexagon. Other examples related to associahedra are presented in Section 5.

In this section, we show that all associahedra of type $A$, $B$, $C$, or $D$ have the non-leaving-face property. This property was proven in type $A$ by D. Sleator, R. Tarjan, and W. Thurston [STT88, Lemma 3], we generically refer to the geometric model for clusters in type $A$, $B$, $C$ or $D$: classical triangulations of the $(n+3)$-gon in type $A_n$, centrally symmetric triangulations of the $2n$-gon in type $B_n$ or $C_n$, and centrally symmetric pseudotriangulations of the configuration $\mathbb{D}_n$ in type $D_n$. Similarly, “diagonal” refers to a diagonal in type $A_n$, to a centrally symmetric pair of diagonals or a long diagonal in type $B_n$ or $C_n$, and to a centrally symmetric pair of chords of the configuration $\mathbb{D}_n$ in type $D_n$.

**Proposition 6.** All associahedra of type $A$, $B$, $C$, or $D$ have the non-leaving-face property. In other words, no common diagonal between two triangulations $T, \bar{T}$ is flipped in a geodesic between $T$ and $\bar{T}$.

**Remark 7.** Since the type $I_2(n)$ associahedra are all polygons, this proposition proves that all infinite families of associahedra have the non-leaving-face property. It would be interesting to know whether all associahedra of exceptional finite types also have this property. We used the computer software Sage [S+12] and a breadth first search algorithm to check this property for types $H_3$, $H_4$, $F_4$ and $E_6$. This time-consuming verification (about half a day for $E_6$) remains to be done on types $E_7$ and $E_8$.

Proposition 6 is a consequence of the following stronger statement. In fact Proposition 8 shows that any path leaving the minimal face containing two vertices has at least 2 more steps than the geodesic connecting them.

**Proposition 8.** Let $T$ and $\bar{T}$ be two triangulations of type $A$, $B$, $C$, or $D$, and $\chi$ be a diagonal in $T \setminus \bar{T}$. If the flip of $\chi$ in $T$ produces a diagonal that belongs to $\bar{T}$, then there exists a geodesic between $T$ and $\bar{T}$ which starts by the flip of $\chi$.

The proof of Proposition 8 relies on a normalization argument, generalizing the normalization of [STT88, Lemma 3] for classical triangulations. We first introduce the normalization on triangulations of types $A$, $B$, $C$ and $D$, and then return to the proof of Propositions 6 and 8 in Section 4.2.
4.1. Normalization. A normalization is a useful tool to transform a geodesic between two triangulations into a geodesic starting with a prescribed flip. In particular, it is a projection from the graph of the associahedron to the graph of one of its facets. Recall that we use the terms “triangulation” and “diagonal” generically for the corresponding geometric models in types \( A, B, C, \) and \( D. \)

**Proposition 9.** For any type \( A, B, C \) or \( D, \) and for any diagonal \( \chi, \) there exists a normalization \( N_{\chi}, \) that is, a map \( T \to N_{\chi}(T) \) satisfying the following properties:

(P0) for any triangulation \( T, \) the normalization \( N_{\chi}(T) \) is a triangulation containing \( \chi; \)

(P1) if \( \chi \in T, \) then \( N_{\chi}(T) = T; \)

(P2) if \( T, T' \) are two adjacent triangulations, then \( N_{\chi}(T) \) and \( N_{\chi}(T') \) coincide or are adjacent.

(P3) if \( T, T' \) are two adjacent triangulations with \( \chi \in T' \setminus T, \) then \( N_{\chi}(T) = N_{\chi}(T') = T'. \)

We prove this proposition case by case using the geometric models in types \( A, B, C, \) and \( D. \)

**Type \( A — \)** We present the normalization of [STT88, Lemma 3] described in a slightly different way which will be convenient to introduce a normalization for other types. Let \( \chi \) be a diagonal and \( S \) a subdivision (crossing-free set of diagonals) of a convex \( (n+3) \)-gon. Fix an arbitrary orientation on \( \chi. \) We imagine that all diagonals of \( S \) are rubber bands attached to their endpoints and we pull all these rubber bands along \( \chi \) towards its target endpoint. We define the normalization \( N_{\chi}(S) \) as the resulting diagonals together with \( \chi. \) Figure 7 illustrates this normalization on triangulations.

To show that this normalization suits, we first observe the following properties:

**Crossing property:** \( N_{\chi} \) sends non-crossing diagonals to a set of non-crossing diagonals.

**Polygon property:** \( N_{\chi} \) transforms a polygon \( P \) into segments or polygons, each with no more corners than \( P. \)

**Quadrangle property:** \( N_{\chi} \) cannot transform a quadrangle into two or more quadrangles.

The conditions of Proposition 9 immediately follow from these properties. Indeed \( N_{\chi} \) maps subdivisions to subdivisions (crossing property), triangulations to triangulations (polygon property), and almost triangulations to either triangulations or almost triangulations (quadrangle property). By an almost triangulation, we mean a subdivision obtained by deleting a diagonal from a triangulation. Properties (P0) and (P1) follow from the definition and the fact that \( N_{\chi}(T) \) is a triangulation. Property (P2) follows from the fact that \( N_{\chi}(T) \cap N_{\chi}(T') = N_{\chi}(T \cap T') \) is a triangulation or an almost triangulation. In fact, Property (P2) can be made more explicit:

**Flip property:** For any two adjacent triangulations \( T \) and \( T', \) the normalized triangulations \( N_{\chi}(T) \) and \( N_{\chi}(T') \) coincide if the almost triangulation \( T \cap T' \) is sent to a triangulation and are adjacent if \( T \cap T' \) is sent to an almost triangulation.

Finally, in the situation of Property (P3), we have \( N_{\chi}(T \cap T') = (T \cap T') \cup \chi = T' \) since \( \chi \) does not cross any diagonal of \( T \cap T'. \) It follows from the flip property and (P1) that \( N_{\chi}(T) = N_{\chi}(T') = T'. \)
Type B or C — Let \( \chi \) be a centrally symmetric pair of oriented diagonals or a long diagonal, and \( S \) a centrally symmetric subdivision of the \( 2n \)-gon. We imagine again that all diagonals of \( S \) are rubber bands attach to their endpoints and we pull them

(i) along \( \chi \) towards its target endpoints if \( \chi \) is a centrally symmetric pair of oriented diagonals, 
(ii) from the center of the \( 2n \)-gon towards the endpoints of \( \chi \) if \( \chi \) is a long diagonal of the \( 2n \)-gon.

The long diagonal of \( S \), if any, is duplicated and a copy is pulled towards each endpoint of \( \chi \).

We define \( N_\chi(S) \) as the resulting diagonals together with \( \chi \). Figure 8 illustrates this normalization on centrally symmetric triangulations both when \( \chi \) is a centrally symmetric pair of oriented diagonals or a long diagonal. This map \( N_\chi \) satisfies the crossing, polygon, quadrangle, and flip properties. Moreover, it maps centrally symmetric subdivisions to centrally symmetric subdivisions. Using these properties, the proof that \( N_\chi \) is a type B normalization is exactly the same as in type A.

Type D — Let \( \chi \) be a centrally symmetric pair of chords and \( S \) be a centrally symmetric pseudosubdivision (crossing-free set of chords) of the configuration \( \mathbb{D}_n \). We distinguish on whether \( \chi \) is a pair of central chords or not.

Case 1. If \( \chi \) is not a pair of central chords, the definition of \( N_\chi \) is similar to types A and B. Namely, we fix an arbitrary orientation on \( \chi \), we imagine again that all chords of \( T \) are rubber bands attached to their endpoints and that the central disk is a fixed obstacle, and we pull all these rubber bands along \( \chi \) towards its target endpoints. We define \( N_\chi(S) \) as the resulting chords together with \( \chi \). The first three columns of Figure 9 illustrate this normalization on centrally symmetric pseudotriangulations.

To prove that \( N_\chi \) satisfies the conditions of Proposition 9, we verify the type D analog of the crossing, polygon, quadrangle, and flip properties:

**Crossing property:** \( N_\chi \) sends non-crossing chords to a set of non-crossing chords. Indeed, crossings in \( N_\chi(S) \) may only occur between central chords. However, normalization maps central chords to central chords of the same type (left or right). Therefore, \( N_\chi(S) \) contains central chords of distinct types only when \( S \) contains central chords of distinct types or a chord crossing \( \chi \) twice. In both cases, the two central chords incident to the target of \( \chi \) are the only central chords in \( N_\chi(S) \).

**Polygon property:** \( N_\chi \) transforms a pseudopolygon \( P \) into chords or pseudopolynomials, each with no more convex corners than \( P \). Indeed, if \( P \) touches none or one chord of \( \chi \), then the situation is similar to type A and can be handled by case analysis distinguishing whether \( P \) has a central corner or not. Otherwise, normalization maps \( P \) to two (possibly degenerate) standard polygons and one (possibly degenerate) pseudopolynomial containing the two targets of \( \chi \) and tangent to the central disk. In any case, the property is satisfied.
Quadrangle property: \( N_\chi \) cannot transform a pseudoquadrangle into two or more pseudoquadrangles. This is immediate from the cases discussed in the polygon property.

Flip property: For any two adjacent pseudotriangulations \( T \) and \( T' \), the normalized pseudotriangulations \( N_\chi(T) \) and \( N_\chi(T') \) coincide if the almost pseudotriangulation \( T \cap T' \) is sent to a pseudotriangulation and are adjacent if \( T \cap T' \) is sent to an almost pseudotriangulation. The only thing to prove is that if a pseudoquadrangle \( P \) is sent to a pseudoquadrangle \( N_\chi(P) \), then the two diagonals of \( P \) are sent to the two diagonals of \( N_\chi(P) \). Otherwise, one chord of \( \chi \) should cross the two diagonals of \( P \) and \( N_\chi(P) \) would not be a pseudoquadrangle.

Using these properties, the proof that \( N_\chi \) is a type \( D \) normalization is exactly the same as in type \( A \).

Case 2. If \( \chi \) is a pair of central chords, then the rubber band normalization strategy fails as it may produce crossing chords. We therefore use an alternative strategy. Let \( T \) be a centrally symmetric pseudotriangulation of \( D_n \). To obtain \( N_\chi(T) \), we replace in \( T \) all pseudotriangles crossing \( \chi \) by the pseudotriangles of the star containing \( \chi \) covering the same area. For later use, we denote this area by \( A \). The last two columns of Figure 9 illustrate this normalization (the area \( A \) is shaded).

We prove here directly the four properties of Proposition 9. Properties (P0) and (P1) are immediate since the normalization replaces a pseudotriangulation of \( A \) by another one containing \( \chi \). Property (P2) is ensured by the following immediate analog of the flip property:

**Flip property:** For any two adjacent pseudotriangulations \( T \) and \( T' \) with \( T \setminus t = T' \setminus t' \), the normalized pseudotriangulations \( N_\chi(T) \) and \( N_\chi(T') \) coincide if both \( t \) and \( t' \) cross \( \chi \) and are adjacent otherwise.

Finally, Property (P3) follows from the definition of \( N_\chi \).

4.2. **Proof of Proposition 6 and Proposition 8.** Using the normalization introduced in the previous section, we are ready to prove the non-leaving-face property.

*Proof of Proposition 8.* Consider two triangulations \( T \) and \( \tilde{T} \) and a diagonal \( \chi \) in \( T \setminus \tilde{T} \), such that the flip of \( \chi \) in \( T \) produces a diagonal \( \chi' \) that belongs to \( \tilde{T} \). Let \( T = T_0, T_1, \ldots, T_k = \tilde{T} \) be an arbitrary geodesic between \( T \) and \( \tilde{T} \). Consider the sequence \( T = T_0, N_{\chi'}(T_0), N_{\chi'}(T_1), \ldots, N_{\chi'}(T_k) \). By Property (P1) above, the last triangulation in this sequence is \( \tilde{T} \). By Property (P3), the first two triangulations are connected by a flip. Since there exists at least one \( i \) such that \( \chi' \in T_{i+1} \setminus T_i \), Property (P3) also ensures that \( N_{\chi'}(T_i) = N_{\chi'}(T_{i+1}) \). Finally, Property (P2) asserts that any two consecutive triangulations in the remaining sequence either coincide or are adjacent. Erasing duplicated consecutive triangulations in this sequence gives a normalized path from \( T \) to \( \tilde{T} \), which starts by the flip of \( \chi \), and is a geodesic since it is not longer than the geodesic \( T = T_0, T_1, \ldots, T_k = \tilde{T} \). □
Proof of Proposition 6. Let $T$ and $\tilde{T}$ be two triangulations. Consider a path $\pi$ from $T$ to $\tilde{T}$ which flips a common chord of $T$ and $\tilde{T}$. Let $T_1$ be the first triangulation along $\pi$ which does not contain $T \cap \tilde{T}$, and let $T_0$ the previous triangulation along $\pi$. By Proposition 8, there exists a geodesic $\pi'$ from $T_1$ to $\tilde{T}$ starting with $T_0$. Combining the subpath of $\pi$ from $T$ to $T_0$ with the subpath of $\pi'$ from $T_0$ to $\tilde{T}$ produces a path from $T$ to $\tilde{T}$ shorter than $\pi$. It follows that no common chord of $T$ and $\tilde{T}$ can be flipped along a geodesic between $T$ and $\tilde{T}$. □

5. Further examples related to the associahedron

To conclude, we survey four relevant families of examples of flip graphs closely related to the associahedron. Their diameter is not precisely determined but we present some known asymptotic bounds. Interestingly, the examples in these four families do not always satisfy the non-leaving-face property.

5.1. Pseudotriangulation polytope. A (pointed) pseudotriangulation of a point set $P$ in general position in the plane is a maximal set of edges of $P$ which is crossing-free and pointed (any vertex is adjacent to an angle wider than $\pi$). We refer to [RSS08] for a survey on pseudotriangulations and their properties. Using rigidity properties of pseudotriangulations and the expansive motion polyhedron, G. Rote, F. Santos and I. Streinu showed in [RSS03] that the flip graph on pseudotriangulations of $P$ can be realized as the graph of the pseudotriangulation polytope. This polytope is a realization of the type $A$ associahedron when $P$ is in convex position.

The diameter of the pseudotriangulation polytope of $P$ is known to be bounded between $|P|$ and $|P| \cdot \log(|P|)$, see [Ber05]. These are the best current bounds to our knowledge.

O. Aichholzer [Aic10] observed that not all pseudotriangulation polytopes satisfy the non-leaving-face property. His example is represented in Figure 10. The point set $P$ is formed by a small downward triangle $\triangle$ together with a big upward triangle whose vertices are replaced by convex chains with $n = 6$ points. Figure 10 shows a path of $3n + 4 = 22$ flips from the top left pseudotriangulation $T$ to the bottom left pseudotriangulation $\tilde{T}$, flipping common edges of $T$ and $\tilde{T}$. In contrast, any path of flips from $T$ to $\tilde{T}$ which preserves their common edges has length at least $4n - 1 = 23$. Indeed, if the edges of $T \cap \tilde{T}$ are preserved, no flip produces an edge of $\tilde{T}$ before all but one edge incident to one vertex of the small downward triangle $\triangle$ have been flipped. We thus need at least $n - 1$ flips before possibly creating edges of $\tilde{T} \setminus T$, and then at least $3n$ flips to create them all.

Figure 10. A geodesic (of length $3n+4 = 22$) between two pseudotriangulations that flips common edges between them.

5.2. Multiassociahedron. A $k$-triangulation of a convex $m$-gon is a maximal set of diagonals not containing a $(k+1)$-crossing, i.e. a set of $k+1$ pairwise crossing diagonals. In a $k$-triangulation, every $k$-relevant diagonal (i.e. with at least $k$ vertices of the $m$-gon on each side) can be flipped to obtain a new $k$-triangulation. We refer to [PS09] for a local description of this operation. The question whether the flip graph can be realized as the graph of a convex polytope remains open,
except for very particular cases including the classical $n$-dimensional associahedron when $k = 1$ and $m = n + 3$. We can however study the diameter of the flip graph and the properties of its geodesics.

The best known bounds for the diameter $\delta(m,k)$ of the flip graph on $k$-triangulations of the $m$-gon are given by

$$\left(k + \frac{1}{2}\right) \cdot m - (k + 1)^2 \leq \delta(m,k) \leq 2k \cdot m - 2k(4k + 1).$$

The upper bound, due to T. Nakamigawa [Nak00], holds for all $m \geq 4k^2(2k + 1)$. Observe that it is tight when $k = 1$ by the result of [STT88, Pou14] on the associahedron. The lower bound, proved in [Pil10, Lemma 2.39], holds for all $m \geq 4k + 2$. We refer to [Pil10, Section 2.3.2] for a summary of known properties on the diameter of the multiassociahedron.

We now show that the multiassociahedron does not satisfy the non-leaving-face property, i.e. that not all $k$-triangulations along a geodesic path between two $k$-triangulations always contain their common diagonals for large enough values of $k$ and $m$. We can argue using the universality property of multitriangulations [PS12, Proposition 5.6]. This property ensures in particular that the flip graph on pseudotriangulations of any planar point set in general position can be embedded as a subgraph induced by all $k$-triangulations containing a certain subset of diagonals in the flip graph on $k$-triangulations of an $m$-gon for large enough $k$ and $m$. The path of Figure 10 thus results in a path in the flip graph on $k$-triangulations of the $m$-gon which contradicts the non-leaving-face property.

In fact, computer experiments with the software Sage [S+12] provided us with a much smaller example illustrated on Figure 11. This figure shows a path of 4 flips between the leftmost 2-triangulation $T$ and the rightmost 2-triangulation $\tilde{T}$ of the octagon, such that a common diagonal is flipped along the sequence. In each 2-triangulation, the blue dashed edge is inserted by the previous flip while the red bold edge is deleted by the next flip. Alternatively, it might help the reader to visualize these flips on the pseudoline arrangements represented below (see [PP12, Section 3.3]). To see that the path represented in this figure is a geodesic, observe that $T \setminus T'$ contains 3 edges whose flip in $T$ all produce an edge not in $T'$.

![Figure 11. A geodesic between two 2-triangulations of the octagon that flips a common edge between them.](image)

5.3. **Graph associahedron.** Fix a finite connected graph $G$. A nested set on $G$ is a set of tubes (proper connected induced subgraphs) of $G$ which are pairwise nested, or disjoint and non-adjacent in $G$. The simplicial complex of all nested sets on $G$ is called nested complex and has been realized as the boundary complex of the $G$-associahedron [CD06]. Specific families of graph-associahedra provide relevant families of polytopes: path associahedra are classical type $A$ associahedra, cycle associahedra are cyclohedra (aka. type $B$ associahedra), star associahedra are stellohedra, and complete graph associahedra are type $A$ permutahedra.
Graph properties of graph associahedra are studied by T. Manneville and V. Pilaud in [MP14]. They show that the diameter $\delta(G)$ of the $G$-associahedron is a non-decreasing function of the graph $G$, meaning that $\delta(G) \leq \delta(G')$ if $G$ is a subgraph of $G'$. Combining this non-decreasing property with the diameter of the permutahedron on the one hand, and the analysis of the diameter $\delta(T)$ for trees on the other hand (based on [Pou14]), they obtain that the diameter $\delta(G)$ is bounded by

$$2n - 20 \leq \delta(G) \leq \binom{n}{2},$$

where $n$ denotes the number of vertices of $G$.

Concerning the non-leaving-face properties, they observed the following facts:

(i) Not all graph associahedra satisfy the non-leaving-face property. Figure 12 reproduces their example. It shows a path of length $2n$ between two maximal nested sets $N, \tilde{N}$ on the star with $n$ branches. In contrast, the minimal face containing $N$ and $\tilde{N}$ is an $n$-dimensional permutahedron and the graph distance between $N$ and $\tilde{N}$ in this face is $\binom{n}{2}$. Therefore, the corresponding graph associahedron (stellohedron in this case) does not satisfy the non-leaving-face property when $n \geq 5$.

(ii) Any geodesic between two maximal nested sets $N, \tilde{N}$ on $G$ remains in the face of the $G$-associahedron corresponding to all common tubes of $N$ and $\tilde{N}$ which are not contained in tubes of $N \setminus \tilde{N}$.

![Figure 12. A geodesic (of length $2n$) between two maximal nested sets of the star that flips a common nested set.](image)

5.4. Secondary polytopes. The secondary polytope of a point set $P \subset \mathbb{R}^d$ is a polytope whose face lattice is isomorphic to the refinement poset of regular subdivisions of $P$, i.e. polyhedral subdivisions of $P$ which can be obtained as the vertical projection of the lower convex hull of the points of $P$ lifted by an arbitrary height function. In particular, the graph of the secondary polytope of $P$ has one vertex for each regular triangulation of $P$ and one edge for each regular flip. See [DRS10] for a detailed presentation of this polytope. For example, the classical associahedron is the secondary polytope of a convex point set in the plane. The secondary polytope of a $d$-dimensional configuration of $n$ points has dimension $n - d - 1$, and it is known that its diameter cannot exceed

$$\min \left( (d + 2) \left( \binom{n}{\left\lfloor \frac{d}{2} + 1 \right\rfloor}, \left( \binom{n}{d + 2} \right) \right),$$

see [DRS10, Corollary 5.3.11]. We do not know whether the secondary polytope of $P$ satisfies or not the non-leaving-face property.

Another related example is the bistellar flip graph in all triangulations (regular or not) of the point set $P$. This graph is not always connected [San00], but it is connected for point sets in the plane, in which case the diameter is at most $4n$ [DRS10, Corollary 3.4.4]. This bound assumes
that insertion and deletion flips are allowed. Otherwise, the diameter can become quadratic as illustrated by the “double chain” example in [DRS10, Example 3.4.5] reproduced in Figure 13. This example can also be used to show that geodesics between two triangulations in the bistellar flip graph delete and reinsert common edges between them. Although all triangulations along a geodesic path between these triangulations are regular, this does not provide a counterexample to the non-leaving-face property for the secondary polytope since the common edges between the two triangulations do not form a regular subdivision.

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