Diameters and geodesic properties of generalizations of the associahedron

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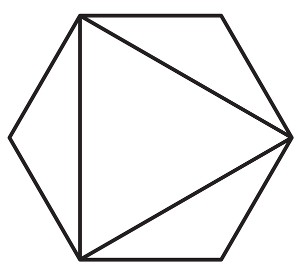
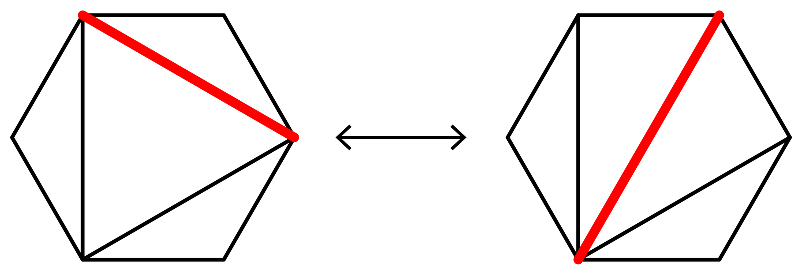
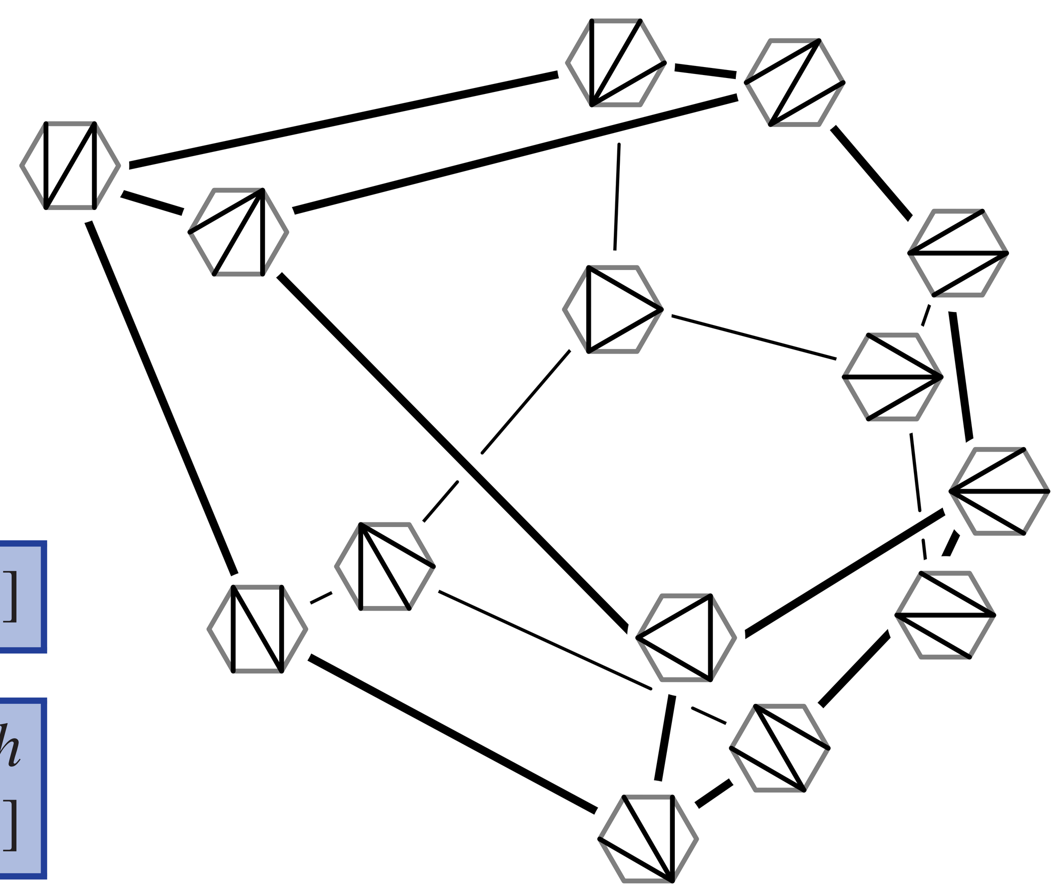
Diameter and geodesic properties of the associahedron

An n**-dimensional associahedron** $\mathrm{Asso}(n)$ is a polytope whose graph is the flip graph of triangulations of a convex $(n+3)$-gon.

vertices = triangulations: 

edges = flips: 

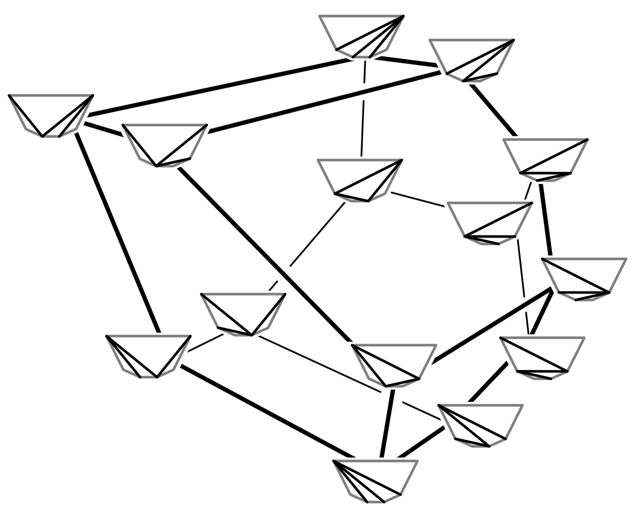


Thm. *For $n > 9$, the diameter of* $\mathrm{Asso}(n)$ *is* $2n-4$. [1, 2]

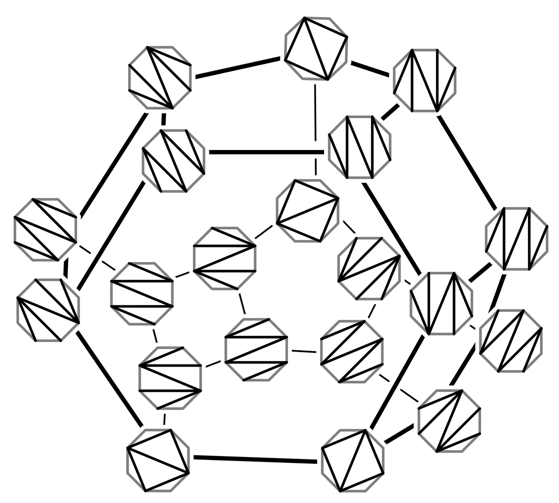
Prop. *The associahedron has the non-leaving-face property: all triangulations along a shortest path of flips between two triangulations T_1 and T_2 contain all common diagonals of T_1 and T_2.* [1]

Generalized associahedra

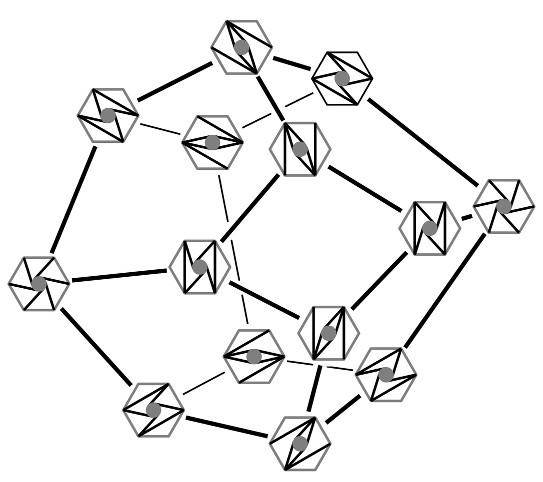
Generalized associahedra are polytopes whose graphs are **exchange graphs** of **finite type cluster algebras** (one per Dynkin diagram).



type A



type B/C



type D

Thm. *The diameter of the type B_n/C_n associahedron (cyclohedron) is asymptotically $5n/2$ up to a term of order $O(\sqrt{n})$.* [3]
The diameter of the type D_n associahedron is $2n-2$ if $n > 1$. [4]

Diameters of generalized associahedra: [1, 2, 3, 4]

type	A								
rank	2	3	4	5	6	7	8	9	$n>9$
diam.	2	4	5	7	9	11	12	15	$2n-4$

type	B/C										
rank	2	3	4	5	6	7	8	9	10	11	12	$n\sim\infty$
diam.	3	5	7	9	11	14	16	18	21	23	25	$5n/2+O(\sqrt{n})$

type	D	F	H		E			$I_2(p)$
rank	$n>1$	4	3	4	6	7	8	2
diam.	$2n-2$	8	6	10	11	14	19	$\lfloor p/2\rfloor-1$

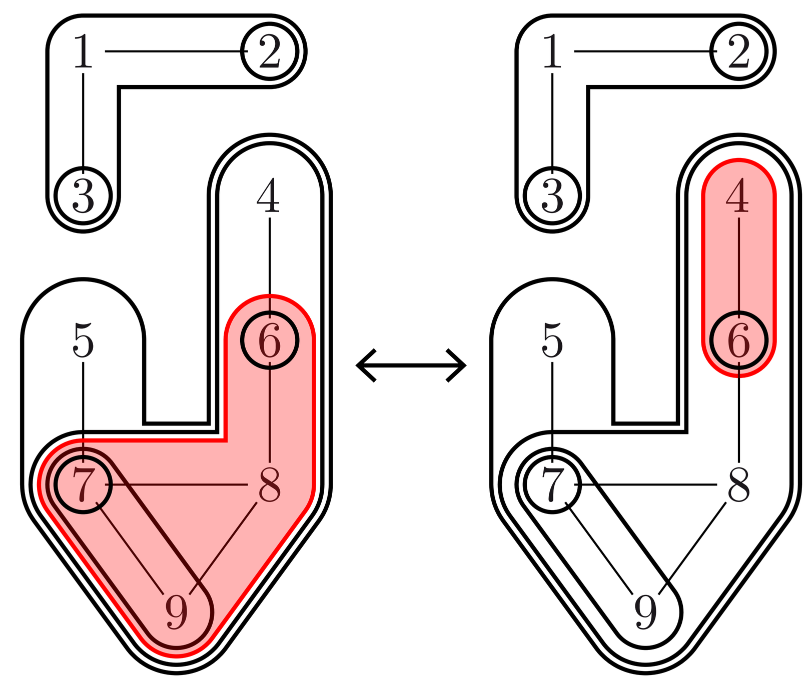
Prop. *All generalized associahedra have the non-leaving-face property: all shortest paths between two vertices u and v stay in the minimal face containing u and v.* [1, 4, 5]

References

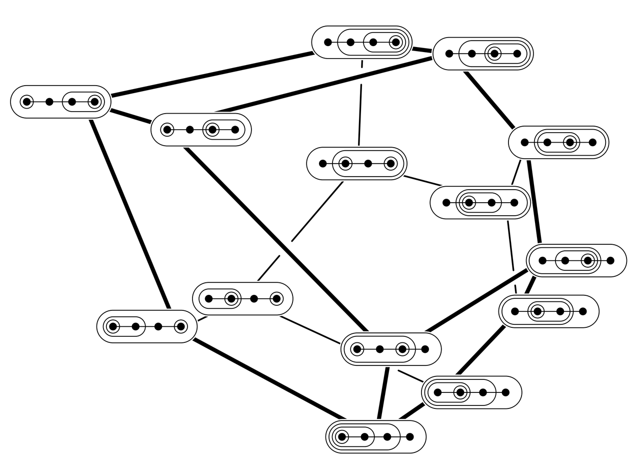
[1] Sleator – Tarjan – Thurston, Rotation distance, triangulations, and hyperbolic geometry, 1988
[2] Pournin, The diameter of associahedra, 2014
[3] Pournin, The asymptotic diameter of cyclohedra, 2014
[4] Ceballos – Pilaud, The diameter of type D associahedra and the non-leaving-face property, 2015
[5] Williams, W-Associahedra are In-Your-Face, 2015
[6] Manneville – Pilaud, Graph properties of graph associahedra, 2014
[7] Aichholzer – Santos, Personnal communications

Graph associahedra

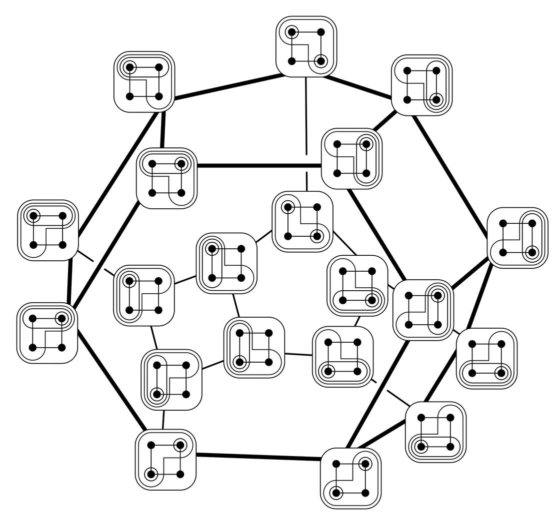
$\mathrm{G} = (V, E)$ undirected graph
tube = connected subgraph of G
tubing = set of pairwise nested or non-adjacent tubes
flip = exchange of two tubes
graph associahedron $\mathrm{Asso(G)}$ = polytope whose vertices are maximal tubings on G and whose edges are the flips between these tubings



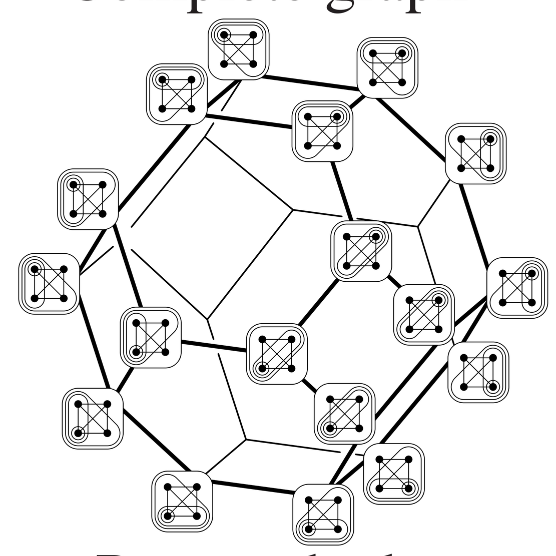
Exm. Path — Cycle — Complete graph



Associahedron



Cyclohedron



Permutahedron

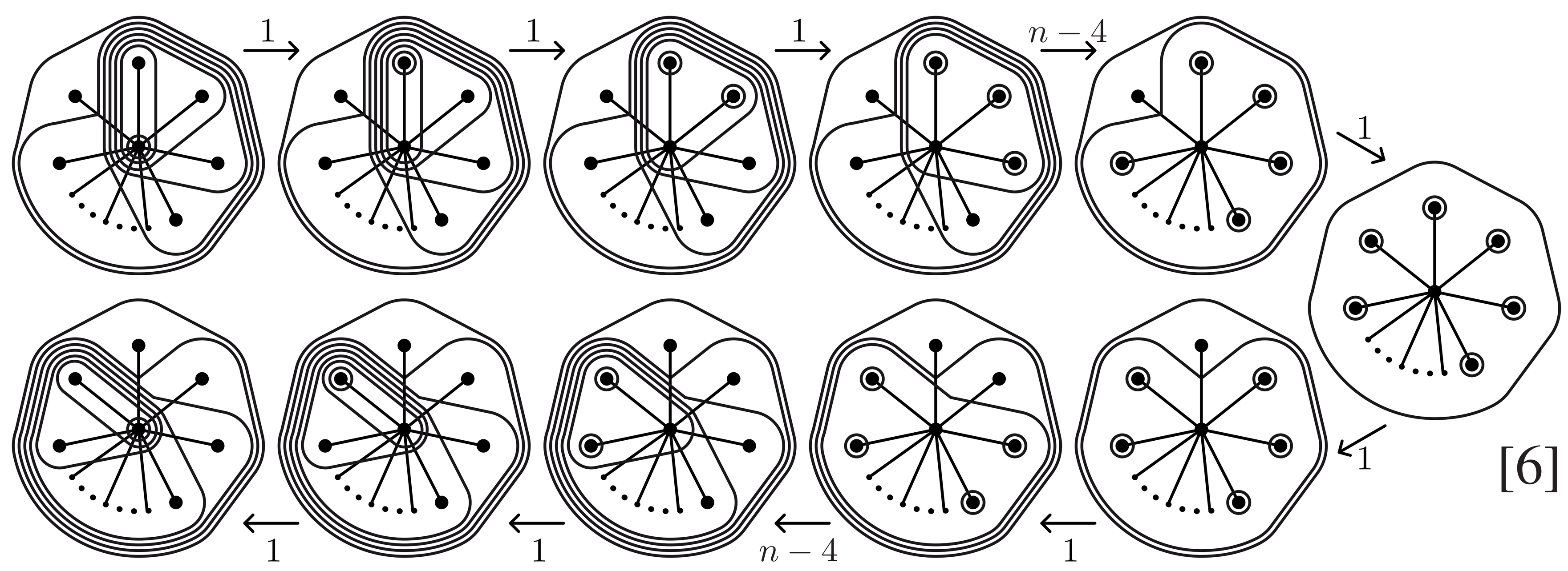
Thm. *For any graph $\mathrm{G}=(V,E)$, the diameter $\delta(\mathrm{Asso(G)})$ of the graph associahedron $\mathrm{Asso(G)}$ is bounded by:*

$$\max(|E|, 2|V|-18) \le \delta(\mathrm{Asso(G)}) \le \binom{|V|+1}{2}. \quad [6]$$

Conj. The lower bound can be improved to $\max(|E|, 2|V|-4)$ and the path associahedron is the only tree achieving this bound.

Prop. *Let G be a graph, let T be a maximal tubing on G, and let $\mathrm{S}^{\uparrow}$ be an upper ideal for the inclusion poset on T. Then $\mathrm{S}^{\uparrow}$ is contained in every tubing on any shortest path of flips between any two tubings on G containing $\mathrm{S}^{\uparrow}$.* [6]

Rmk. Not all faces of $\mathrm{Asso(G)}$ have the non-leaving-face property:



[6]

Other exms [4, 7]: pseudotriangulation polytopes, multiassociahedra, secondary polytopes, flip graphs on all triangulations of a point set.