An \(n\)-dimensional associahedron \(\text{Asso}(n)\) is a polytope whose graph is the flip graph of triangulations of a convex \((n+3)\)-gon.

\[
\text{vertices} = \text{triangulations}:
\]

\[
\text{edges} = \text{flips}:
\]

**Thm.** For \(n > 9\), the diameter of \(\text{Asso}(n)\) is \(2n - 4\). \([1, 2]\)

**Prop.** The associahedron has the non-leaving-face property: all triangulations along a shortest path of flips between two triangulations \(T_1\) and \(T_2\) contain all common diagonals of \(T_1\) and \(T_2\). \([1]\)

### Generalized associahedra

**Generalized associahedra** are polytopes whose graphs are exchange graphs of finite type cluster algebras (one per Dynkin diagram).

**Thm.** The diameter of the type \(B_n/C_n\) associahedron (cyclohedron) is asymptotically \(5n/2\) up to a term of order \(O(\sqrt{n})\). \([3]\)

The diameter of the type \(D_n\) associahedron is \(2n - 2\) if \(n > 1\). \([4]\)

Diameters of generalized associahedra:

<table>
<thead>
<tr>
<th>Type</th>
<th>Rank</th>
<th>Diam.</th>
<th>(B/C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type A</td>
<td>2 3 4 5 6 7 8 9</td>
<td>(n &gt; 9)</td>
<td></td>
</tr>
<tr>
<td>Type B/C</td>
<td>2 4 5 7 9 11 12 15</td>
<td>(2n - 4)</td>
<td></td>
</tr>
<tr>
<td>Type D</td>
<td>3 5 7 9 11 14 16 18 21 23 25</td>
<td>(n \rightarrow \infty)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type</th>
<th>Rank</th>
<th>Diam.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type D</td>
<td>(n &gt; 1)</td>
<td>(2n - 2)</td>
</tr>
<tr>
<td>Type A</td>
<td>(</td>
<td>V</td>
</tr>
</tbody>
</table>

**Prop.** All generalized associahedra have the non-leaving-face property: all shortest paths between two vertices \(u\) and \(v\) stay in the minimal face containing \(u\) and \(v\). \([1, 4, 5]\)

### Graph associahedra

\(G = (V, E)\) undirected graph

- **tube** = connected subgraph of \(G\)
- **tubing** = set of pairwise nested or non-adjacent tubes
- **flip** = exchange of two tubes

**Graph associahedron** \(\text{Asso}(G)\) = polytope whose vertices are maximal tubings on \(G\) and whose edges are the flips between these tubings

**Exm.** Path

**Thm.** For any graph \(G = (V, E)\), the diameter \(\delta(\text{Asso}(G))\) of the graph associahedron \(\text{Asso}(G)\) is bounded by:

\[
\max(|E|, 2|V| - 18) \leq \delta(\text{Asso}(G)) \leq \left(\frac{|V| + 1}{2}\right). \qquad [6]
\]

**Conj.** The lower bound can be improved to \(\max(|E|, 2|V| - 4)\) and the path associahedron is the only tree achieving this bound.

**Prop.** Let \(G\) be a graph, let \(T\) be a maximal tubing on \(G\), and let \(S^+\) be an upper ideal for the inclusion poset on \(T\). Then \(S^+\) is contained in every tubing on any shortest path of flips between any two tubings on \(G\) containing \(S^+\). \([6]\)

**Rmk.** Not all faces of \(\text{Asso}(G)\) have the non-leaving-face property.

Other exms \([4, 7]\): pseudotriangulation polytopes, multiassochedra, secondary polytopes, flip graphs on all triangulations of a point set.

### References

[5] Williams, \(W\)-Associahedra are In-Your-Face, 2015