QUASI-CONFIGURATIONS: BUILDING BLOCKS FOR POINT – LINE CONFIGURATIONS

JÜRGEN BOKOWSKI AND VINCENT PILAUD

Abstract. We study generalized point – line configurations and their properties in the projective plane. These generalized configurations can serve as building blocks for (n_4) configurations. In this way, we construct (37_4) and (43_4) configurations. The existence problem of finding such configurations for the remaining cases (22_4), (23_4), and (26_4) remains open.

Keywords. projective arrangements, regular point – line configurations

MSC Classes. 52C30

1. Introduction

Before we fix our notation, we cite Grünbaum's book on point – line configurations [Grü09, p. 1]:

By a k-configuration, specifically an (n_k) configuration, we shall always mean a set of n points and n lines such that every point lies on precisely k of these lines and every line contains precisely k of these points.

We recommend our reader to have a look at this detailed historical outline about the problem to determine for a given k those numbers n for which there exists (n_k) configurations. For $k = 3$ the answer is known, for $k > 4$ the problem is wide open. Our contribution concerns $k = 4$ with its small finite set of missing examples. We provide solutions for two former open cases: there does exist (37_4) and (43_4) configurations. Moreover, we study building blocks for constructing (n_k) configurations that might be of some help for clarifying the final open cases (22_4), (23_4), and (26_4). Many aspects of our presentation appeared during our investigation of the case (19_4) in which there is no (geometric) (19_4) configuration, see [BP14a, BP14b].

The approach of this paper is to construct (n_4) configurations from smaller building blocks. For example, Grünbaum's (20_4) configuration can be constructed by superposition of two (10_3) configurations as illustrated in Figure 1. To extend this kind of constructions, we study a generalized version of point – line configurations, where incidences are not regular but still prescribed.

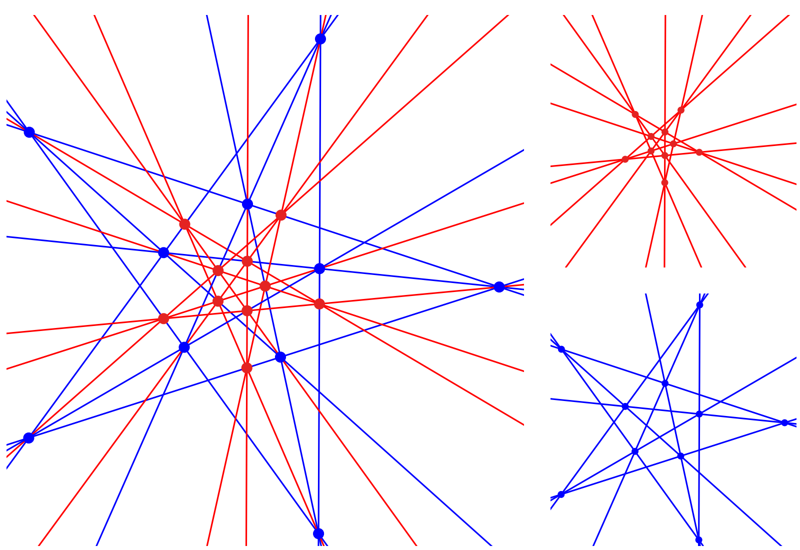

Figure 1. Splitting Grünbaum's geometric (20_4) configuration.

VP was partially supported by the spanish MICINN grant MTM2011-22792 and by the French ANR grant EGOS 12 JS02 002 01.

1.1. **General point – line configurations.** We define a *general point – line configuration* as a set P of *points* and a set L of *lines* together with a point – line *incidence relation*, where two points of P can be incident with at most one line of L and two lines of L can be incident with at most one point of P. Throughout the paper, we only consider *connected* general point – line configurations, where any two elements of $P \sqcup L$ are connected via a path of incident elements.

For a general point – line configuration (P, L), we denote by p_i the number of points of P contained in i lines of L and similarly by ℓ_j the number of lines of L containing j points of P. We find it convenient to encode these incidence numbers into a pair of polynomials $(\mathbf{P}(x), \mathbf{L}(x))$, called the *signature* of (P, L), and defined by

$$\mathbf{P}(x) := \sum\nolimits_i p_i x^i \qquad \text{and} \qquad \mathbf{L}(y) := \sum\nolimits_j \ell_j y^j.$$

With these notations, the number of points and lines are given by $|P| = \mathbf{P}(1)$ and $|L| = \mathbf{L}(1)$, and the number of point – line incidences is $|\{(p, \ell) \in P \times L \mid p \in \ell\}| = \mathbf{P}'(1) = \mathbf{L}'(1)$.

We distinguish three different levels of point – line configurations, in increasing generality:

Geometric: Points and lines are ordinary points and lines in the real projective plane $\mathbb{P}$.

Topological: Points are ordinary points in $\mathbb{P}$, but lines are *pseudolines*, *i.e.*, non-separating simple closed curves of $\mathbb{P}$ which cross pairwise precisely once.

Combinatorial: Just an abstract incidence structure (P, L) as described above, with no additional geometric structure.

In this paper, we are mainly interested in the geometric level. We therefore omit the word geometric in what follows unless we have to distinguish different levels.

1.2. (n_k) **configurations.** A main problem in the theory of point – line configurations is to clarify the existence of configurations with a regular point – line incidence. A *k-configuration* is a point – line configuration (P, L) where each point of P is contained in k lines of L and each line of L contains k points of P. In such a configuration, the number of points equals the number of lines, and thus it has signature (nx^k, ny^k). If we want to specify the number of points and lines, we call it an *(n_k) configuration*. We refer to the recent monographs of Grünbaum [Grü09] and Pisanski and Servatius [PS13] for comprehensive presentations of these objects. Classical examples of regular configurations are Pappus' and Desargues' configurations, which are respectively (9_3) and (10_3) configurations. In the study of the existence of (n_4) configurations there are still a few open cases. Namely, it is known that (geometric) (n_4)-configurations exists if and only if $n = 18$ or $n \geq 20$, with the possible exceptions of $n = 22, 23, 26, 37$ and 43 [Grü00, BS11, BP14b]. Different methods have been used to obtain the current results on the existence of 4-configurations:

(i) For $n \leq 16$, Bokowski and Schewe [BS05] used a counting argument based on Euler's formula to prove that there exist no (n_4) configuration, even topological ones.

(ii) For small values of n, one can search for all possible (n_4) configurations. For $n = 17$ or 18, one can first enumerate all combinatorial (n_4) configurations and search for geometric realizations among them. This approach was used by Bokowski and Schewe [BS11] to show that there is no (17_4) configuration and to produce a first (18_4) configuration. Another approach, proposed in [BP14a], is to enumerate directly all topological (n_4) configurations, and to search for geometric realizations among this restricted family. We showed this way that there are precisely two (18_4) configurations, that of [BS11] and another one [BP14a], see Figure 3. For $n = 19$, we obtained in [BP14a] all 4 028 topological (19_4) configurations and the study of their realizability has led to the result that there is no geometric (19_4) configuration [BP14b].

(iii) For larger values of n, one cannot expect anymore a complete classification of (n_4) configurations. However, one can construct families of examples of 4-configurations. One of the key ingredients for such constructions is the use of symmetries. We refer to the detailed presentation in Grünbaum's recent monograph [Grü09].

(iv) Finally, Bokowski and Schewe introduced in [BS11] a method to produce (n_4) configurations from deficient configurations. It consists in finding two point – line configurations (P_1, L_1) and (P_2, L_2) of respective signatures $(ax^3 + bx^4, cy^3 + dy^4)$ and $(cx^3 + ex^4, ay^3 + fy^4)$, where

$a+b+c+e=a+c+d+f=n$, and a projective transformation which sends the 3-valent points of P_1 to points contained in a 3-valent line of L_2, and the same time the 3-valent lines of L_1 to lines containing a 3-valent point of P_2. This method was used to obtain the first examples of (29_4) and (31_4) configurations.

In this paper, we are interested in this very last method described above. We are going to study deficient configurations (see the notion of quasi-configuration in the next subsection) for the use of them as building blocks for configurations. Our study has led in particular to first examples of (37_4) and (43_4) configurations. Thus the remaining undecided cases for the existence of (n_4) configurations are now only the cases $n = 22, 23$, and 26.

1.3. **Quasi-configurations.** A *quasi-configuration* (P, L) is a point – line configuration in which each point is contained in more than 2 lines and each line contains more than 2 points of P. In other words, the signature $(\mathbf{P}, \mathbf{L})$ of (P, L) satisfies $x^3 \mid \mathbf{P}(x)$ and $y^3 \mid \mathbf{L}(y)$. The term "quasi-configuration" for this concept was suggested by Grünbaum to the authors. As observed above, these configurations can sometimes be used as building blocks for larger configurations.

In this paper, we investigate in particular *$3|4$-configurations*, where each point of P is contained in 3 or 4 lines of L and each line of L contains 3 or 4 points of P. In other words, generalized configurations whose signature is of the form $(ax^3 + bx^4, cx^3 + dx^4)$ for some $a, b, c, d \in \mathbb{N}$ satisfying $3a + 4b = 3c + 4d$. Note that their number of points and lines do not necessarily coincide. If it is the case, *i.e.*, if $a + b = c + d = n$, we speak of an *$(n_{3|4})$ configuration*. In this case, $a = c$ and $b = d$, the number of points and lines is $n = a + b = c + d$ and the number of incidences is $3a + 4b = 3c + 4d$.

A good measure on $(n_{3|4})$ configurations is the number of missing incidences a. We say that an $(n_{3|4})$ configuration is *optimal* if it contains the maximal number of point – line incidences among all $(n_{3|4})$ configurations. One objective is to study and classify optimal $(n_{3|4})$ configurations for small values of n.

Example 1. Figure 2 represents a generalized configuration with signature $(8x^3 + 2x^4, 8y^3 + 2y^4)$. It is a self-dual quasi-configuration, even a $10_{3|4}$-configuration. 3-valent elements are colored in red while 4-valent elements are colored in blue. We will see in Section 3 that this configuration is not optimal (see Figure 6).

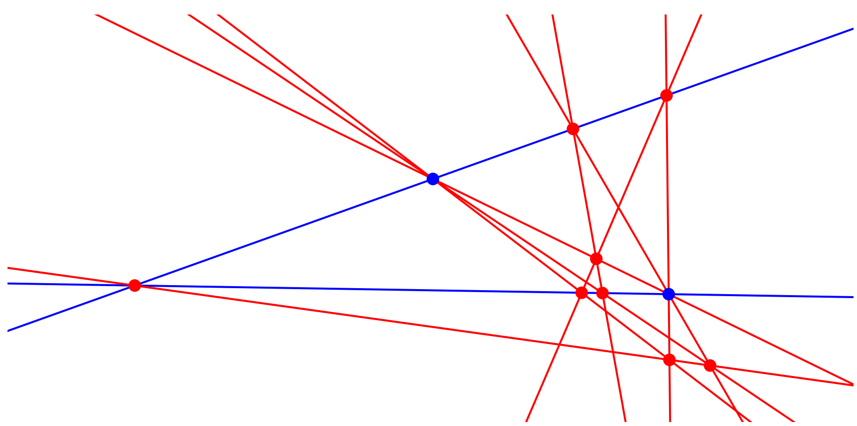

FIGURE 2. A quasi-configuration with signature $(8x^3 + 2x^4, 8y^3 + 2y^4)$.

1.4. **Overview.** The paper is divided into two parts. In Section 2, we illustrate how quasi-configurations (in particular $3|4$-configurations) can be used as building blocks to construct (n_4) configurations, and we obtain in particular examples of (37_4) and (43_4) configurations. In Section 3, we present a counting obstruction for the existence of topological quasi-configurations, and we study optimal, *i.e.*, point – line incidence maximal, $(n_{3|4})$ configurations with few points and lines.

2. Constructions

In this section, we discuss different ways to obtain new point – line configurations from old ones. We are in particular interested in the construction of new quasi-configurations from existing quasi-configurations. We use these techniques to provide the first (37_4) and (43_4) configurations.

2.1. **Operations on generalized configurations.** To construct new point – line configurations from old ones, we will use the following operations, illustrated in the next section:

Deletion: Deleting elements from a point – line configuration yields a smaller configuration. Note that deletions do not necessarily preserve connectedness neither quasi-configurations. We can however use deletions in 4-configurations to construct 3|4-configurations if no remaining element is incident to two deleted elements.

Addition: As illustrated by the example of Grünbaum's (20_4) configuration in Figure 1, certain point – line configurations can be obtained as the disjoint union of two smaller configurations (P, L) and (P', L'). In particular, we obtain an (n_4) configuration if each 3-valent elements of (P, L) is incident to precisely one 3-valent element of (P', L') and no other incidences appear.

Splitting: The reverse operation of addition is splitting: given a point – line configuration, we can split it into two smaller configurations. However, we can require additionally the two resulting configurations to be quasi-configurations or even regular configurations. For example, the two geometric (18_4) configurations as well as Grünbaum's (20_4) configuration are splittable into (n_3) configurations, see Figures 1 and 3.

Superposition: Slightly more general than addition is the superposition, where we allow the two point – line configurations (P, L) and (P', L') to share points or lines. For example, we can superpose two 2-valent vertices to make one 4-valent vertex. This idea is used in our construction of (37_4) and (43_4) configurations below.

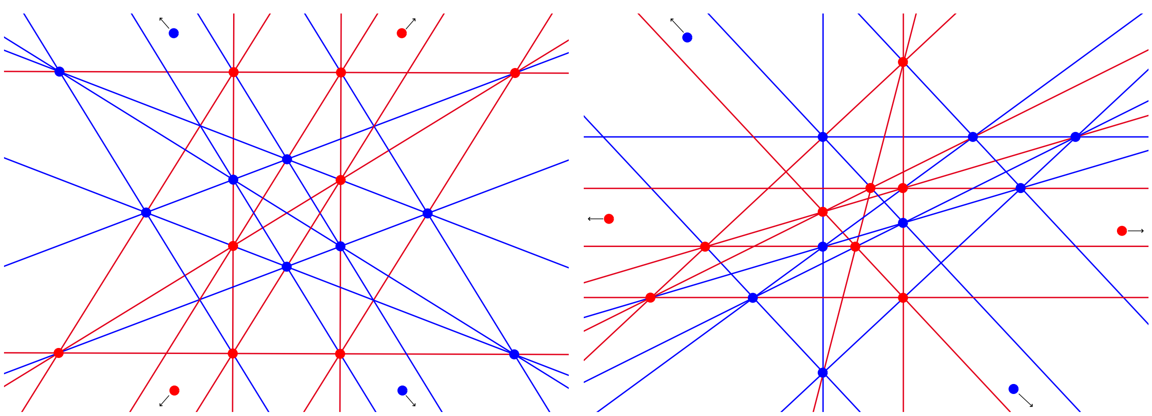

Figure 3. Splittings of the two geometric (18_4) configurations into two (9_3) configurations.

2.2. **Examples of constructions.** We now illustrate the previous operations and produce 4-configurations from smaller generalized configurations. We first observe that this is always possible to produce a 4-configuration from any 3|4-configuration.

Example 2 (Any 3|4-configuration generates a 4-configuration)**.** From a 3|4-configuration with signature $(ax^3 + bx^4, cy^3 + dy^4)$, we can always construct as follows an (n_4) configuration with $n = 16a + 16b + 4c = 4a + 16c + 16d$:

(i) We take four translated copies of the 3|4-configuration and add suitable parallel lines through all 3-valent points.
(ii) We take the dual of the resulting configuration.
(iii) We take again four translated copies of this dual configuration and add suitable parallel lines through all 3-valent vertices.

Of course, we can try to obtain other 4-configurations from 3|4-configurations. This approach was used by Bokowski and Schewe [BS11] to construct (29_4) and (31_4) configurations from the $(14_{3|4})$, $(15_{3|4})$ and $(16_{3|4})$ configurations of Figure 8. We refer to their paper [BS11] for an explanation. Here, we elaborate on the same idea to construct two new relevant (n_4) configurations.

Example 3 (First (43_4) configuration)**.** To construct an $((n+m)_4)$ configuration from an (n_4) configuration and an (m_4) configuration, we proceed as follows (see also Figure 4):

(i) We delete two points not connected by a line in the (n_4) configuration and consider the eight resulting 3-valent blue lines.
(ii) We add four green points, each incident with precisely two 3-valent blue lines. All points and lines are now 4-valent again, except the four 2-valent green points.
(iii) We do the same operations in the (m_4) configuration.
(iv) Finally, we use a projective transformation that maps the set of four green points in the first configuration onto the set of four green points in the second configuration. This transformation superposes the 2-valent green points to make them 4-valent.

The result yields the desired $((n+m)_4)$ configuration. This construction is illustrated on Figure 4, where we obtain a (43_4) configuration from a (25_4) and an (18_4) configuration.

Unfortunately, the method from the previous example cannot provide a (37_4) configuration since there is no (n_4) configuration for $n \leq 17$ [BS11] and for $n = 19$ [BP14b]. We therefore need another method described in the following example.

Example 4 (First (37_4) configuration)**.** To construct an $((n+m-1)_4)$ configuration from an (n_4) configuration and an (m_4) configuration, we proceed as follows (see also Figure 5):

(i) We delete two points connected by a green line in the (n_4) configuration and consider the six resulting blue 3-valent lines.
(ii) We add three green points, each incident with precisely two 3-valent blue lines. All points and lines are now 4-valent again, except the 2-valent green line and the three 2-valent green points.
(iii) We do the same operations in the (m_4) configuration.
(iv) Finally, we use a projective transformation that maps the set of four green elements in the first configuration onto the set of four green elements in the second configuration. This transformation superposes the 2-valent green elements to make them 4-valent.

The result yields the desired $((n+m-1)_4)$ configuration. This construction is illustrated on Figure 5, where we obtain a (37_4) configuration from a (20_4) and an (18_4) configuration.

We invite the reader to try his own constructions, similar to the constructions of Examples 3 and 4, using the operations on point – line configurations described above. One can obtain this way many (n_4) configurations for various values of n. Additional features can even be imposed, such as non-trivial motions or symmetries. We have however not been able to find answers to the following question.

Question 5. *Can we create a* (22_4) *configuration by glueing two quasi-configurations with* 11 *points and lines each? More generally, can we construct* (22_4)*,* (23_4)*, or* (26_4) *configurations by superposition of smaller configurations?*

3. Obstructions and optimal 3|4-configurations

In this section, we further investigate general point – line configurations. We start with a necessary condition for the existence of topological configurations with a given signature. For this, we extend to all topological configurations an argument of Bokowski and Schewe [BS05] to prove the non-existence of (15_4) configurations. We obtain the following inequality.

Proposition 6. *If there exists a topological configuration with signature* $(\mathbf{P}, \mathbf{L})$*, then*

$$\mathbf{P}''(1) + 2\mathbf{P}'(1) - \mathbf{L}(1)^2 + \mathbf{L}(1) - 6\mathbf{P}(1) + 6 \leq 0.$$

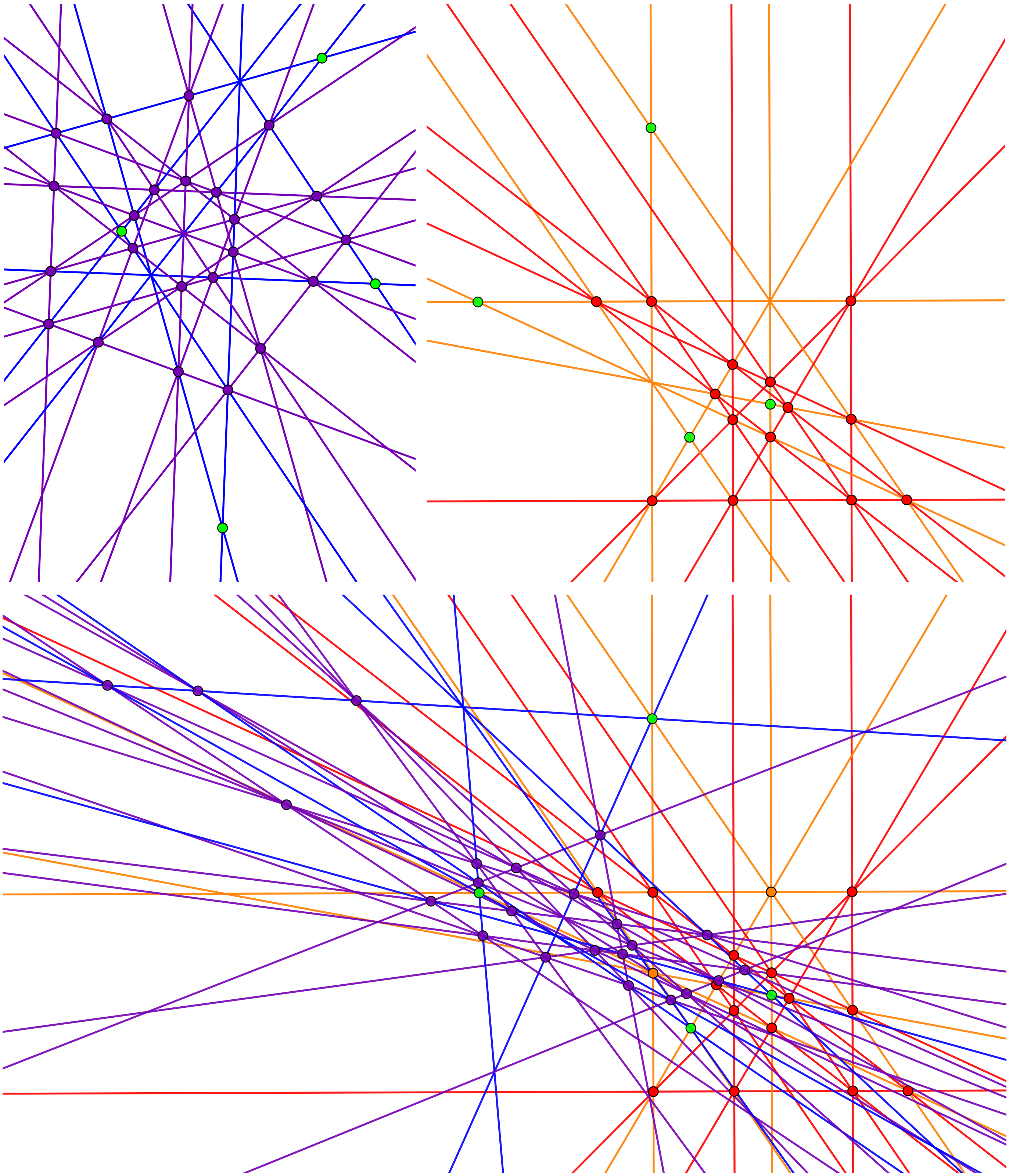

FIGURE 4. A (43_4) configuration build from deficient (25_4) and (18_4) configurations. The construction is explained in full details in Example 3.

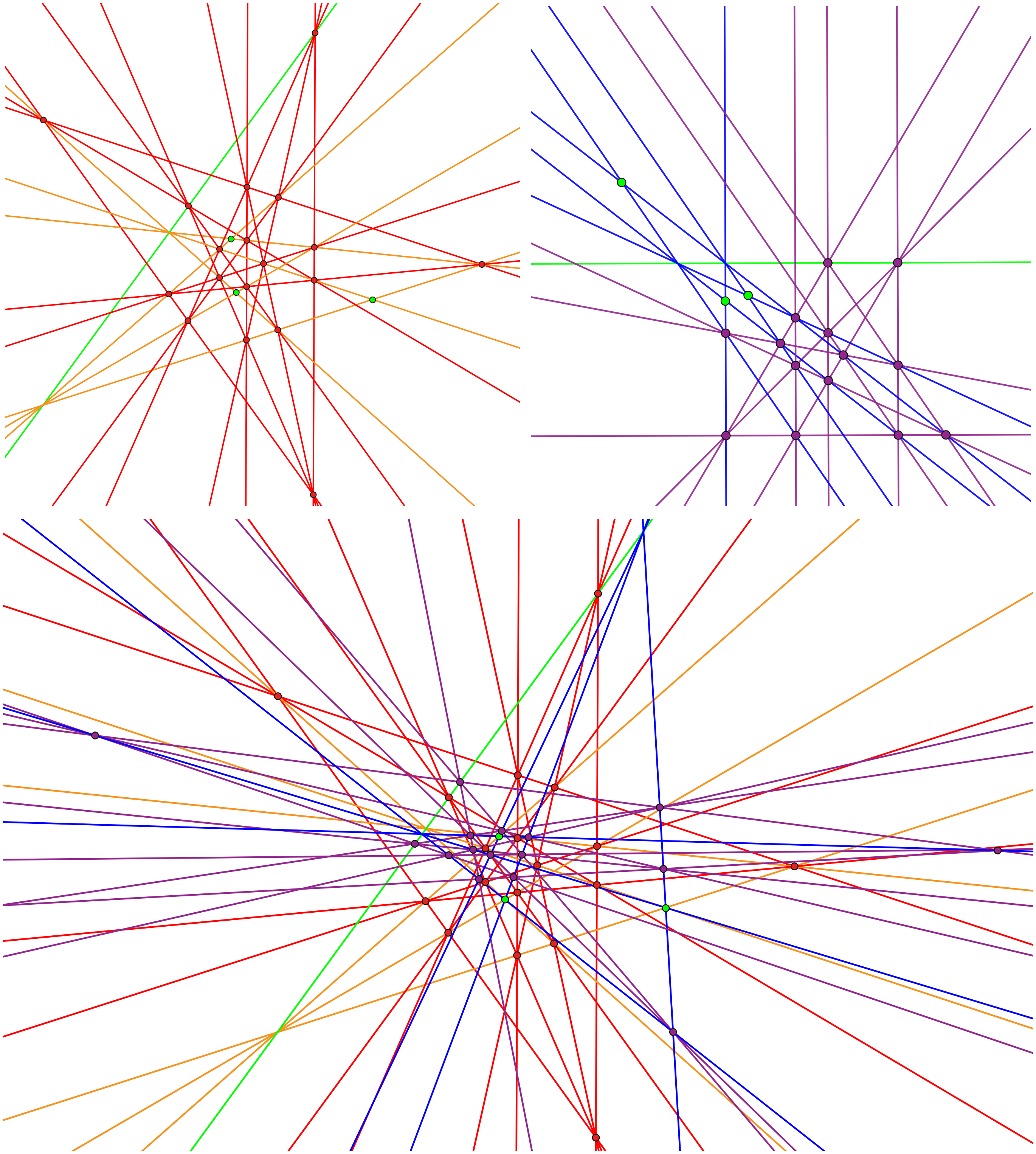

FIGURE 5. A (37_4) configuration build from deficient (20_4) and (18_4) configurations. The construction is explained in full details in Example 4.

Proof. Let p_i denote the number of i-valent points and ℓ_j the number of j-valent lines in the configuration (P, L). The signature $(\mathbf{P}, \mathbf{L})$ is given by $\mathbf{P}(x) := \sum_i p_i x^i$ and $\mathbf{L}(y) := \sum_j \ell_j y^j$. Denote by $\mathbf{p} := \mathbf{P}(1) = \sum_i p_i$ the number of points and by $\mathbf{l} := \mathbf{L}(1) = \sum_i \ell_i$ the number of lines.

Since the configuration is topological, we can draw it on the projective plane $\mathbb{P}$ such that no three pseudolines pass through a point which is not in P. We call *additional 2-crossings* the intersection points of two lines of L which are not points of P. We consider the lifting of this drawing on the 2-sphere. We obtain a graph embedded on the sphere, whose vertices are all points of P together with all additional 2-crossings, whose edges are the segments of lines of L located between two vertices, and whose faces are the connected components of the complement of L. Let f_0, f_1 and f_2 denote respectively the number of vertices, edges and faces of this map. We have

$$\begin{aligned}
f_0 &= 2\binom{\mathbf{l}}{2} - 2\sum_{p\in P}\left(\binom{\deg(p)}{2} - 1\right) = \mathbf{l}(\mathbf{l}-1) + 2\mathbf{p} - \sum_i i(i-1)p_i,\\
f_1 &= 2\sum_{\ell\in L}\deg(\ell) + 2f_0 - 2\mathbf{p} = 2\sum_j j\ell_j + 2f_0 - 2\mathbf{p} = 2\sum_i ip_i + 2f_0 - 2\mathbf{p},\\
f_2 &= f_1 - f_0 + 2.
\end{aligned}$$

Moreover, since no face is a digon, we have $3f_2 \le 2f_1$. Replacing f_2 and f_1 by the above expressions, we obtain

$$0 \ge 3f_2 - 2f_1 = f_1 - 3f_0 + 6 = 2\sum_i ip_i - 4\mathbf{p} - f_0 + 6 = \sum_i i(i+1)p_i - \mathbf{l}(\mathbf{l}-1) - 6(\mathbf{p}-1),$$

and thus the desired inequality. □

Corollary 7. *If there exists a topological configuration with signature* (ax^3+bx^4, ay^3+by^4)*, then*

$$-(a+b)^2 + 7a + 15b + 6 \le 0.$$

The following table provides the minimum value of b for which there could exists a topological configuration with signature (ax^3+bx^4, ay^3+by^4)*:*

a	0	1	2	3	4	5	6	7
$b_{\min}$	16	14	13	11	9	8	6	3

Proof. Direct application of Proposition 6 with $\mathbf{P}(x) = ax^3 + bx^4$ and $\mathbf{L}(y) = ay^3 + by^4$. □

For example, there is no topological (15_4) configuration [BS05] and no configuration with signature $(7x^3 + 2x^4, 7y^3 + 2y^4)$. Compare with Example 1.

Corollary 8. *An* $(n_{3|4})$ *configuration has at most* $I_{\max} := \min\left(4n\ ,\ \left\lfloor \frac{n^2+17n-6}{8} \right\rfloor\right)$ *incidences. The values of* $I_{\max}$ *appear in the following table:*

n	7	8	9	10	11	12	13	14	15	16
$I_{\max}$	20	24	28	33	37	42	48	53	59	64

Proof. Consider an $(n_{3|4})$ configuration with signature $(ax^3 + bx^4, ay^3 + by^4)$ where $a + b = n$. The number of incidences is $I := 3a + 4b$. It can clearly not exceed $4n$. For the second term in the minimum, we apply Corollary 7 to get

$$0 \ge -(a+b)^2 + 7a + 15b + 6 = -(a+b)^2 + 8(3a+4b) - 17(a+b) + 6 = -n^2 + 8I - 17n + 6. \quad \square$$

Corollary 9. *There is no topological* $(n_{3|4})$ *configuration if* $n \le 8$*.*

Proof. If $n \le 7$, there is no topological $(n_{3|4})$ configuration since it should have at least $3n$ incidences, which is larger than the upper bound of Corollary 8. If $n = 8$, an $(8_{3|4})$ configuration should be an (8_3) configuration by Corollary 8. But there is no topological (8_3) configuration. □

To close this section, we exhibit optimal $(n_{3|4})$ configurations for small values of n, *i.e.*, $(n_{3|4})$ configurations which maximize the number of point−line incidences.

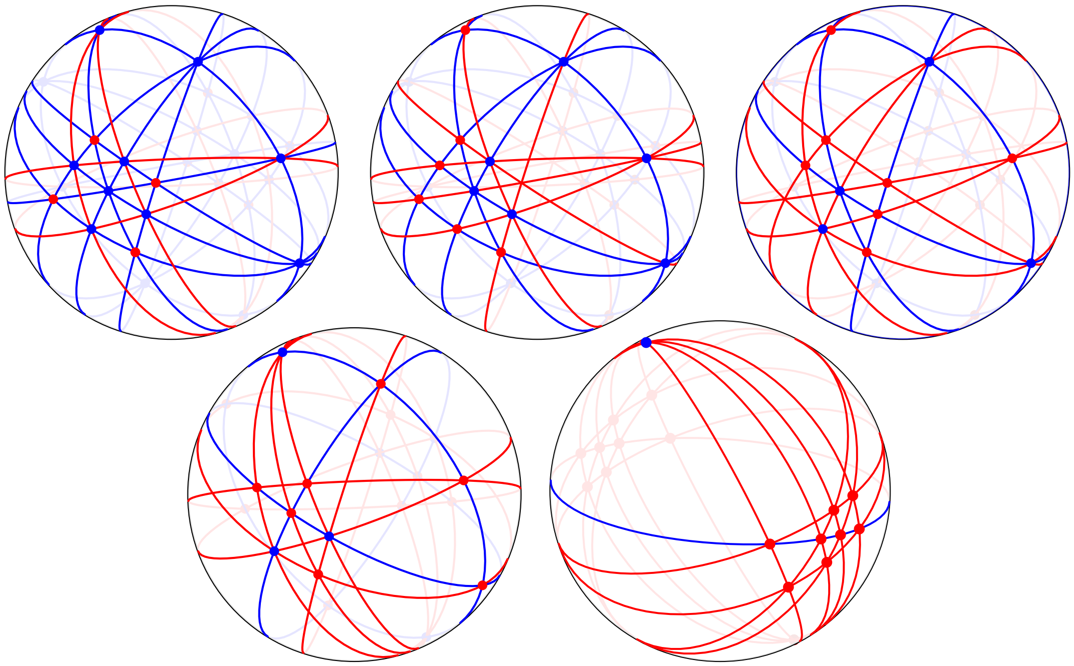

FIGURE 6. Optimal $(n_{3|4})$ configurations, for $n = 13, 12, 11, 10, 9$. They have respectively 48, 42, 37, 33, and 28 point – line incidences. 3-valent elements are colored red while 4-valent elements are colored blue.

Proposition 10. *For $9 \leq n \leq 13$, the bound of Corollary 8 is tight: there exists $(n_{3|4})$ configurations with $\left\lfloor \frac{n^2+17n-6}{8} \right\rfloor$ incidences.*

Proof. For $n = 13$, we consider the configuration of Figure 6. The homogeneous coordinates of its points and lines are given by

$$P := L := \left\{ \begin{bmatrix} i \\ j \\ 1 \end{bmatrix} \;\middle|\; i, j \in \{-1, 0, 1\} \right\} \cup \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

For $n = 10, 11$ or 12, we obtain $(n_{3|4})$ configurations removing suitable points and lines in our $(13_{3|4})$ configuration. The resulting configurations are illustrated in Figure 6. (Note that for $n = 10$, we even have two dual ways to suitably remove three points and three lines from our $(13_{3|4})$ configuration: either we remove three 3-valent points and the three 4-valent lines containing two of these points, or we remove three 3-valent lines and the three 4-valent points contained in two of these lines). Finally, for $n = 9$ we use the bottom rightmost configuration of Figure 6. □

As a curiosity, we give another example of optimal $(12_{3|4})$ configuration which contains Pappus' and Desargues' configurations simultaneously. See Figure 7.

Observe that optimal $(n_{3|4})$ configurations are given by (n_4) configurations for large n, and that the only remaining cases for optimal $(n_{3|4})$ configurations are for $n = 14, 15, 16, 17, 19, 22, 23,$ and 26. To conclude, we have represented in Figure 8 some $(15_{3|4})$ and $(16_{3|4})$ configurations which we expect to be optimal, although they do not reach the theoretical upper bound of Corollary 8. We therefore leave the following question open.

Question 11. *What are the optimal $(14_{3|4})$ configurations? Are the $(15_{3|4})$ and $(16_{3|4})$ configurations in Figure 8 optimal?*

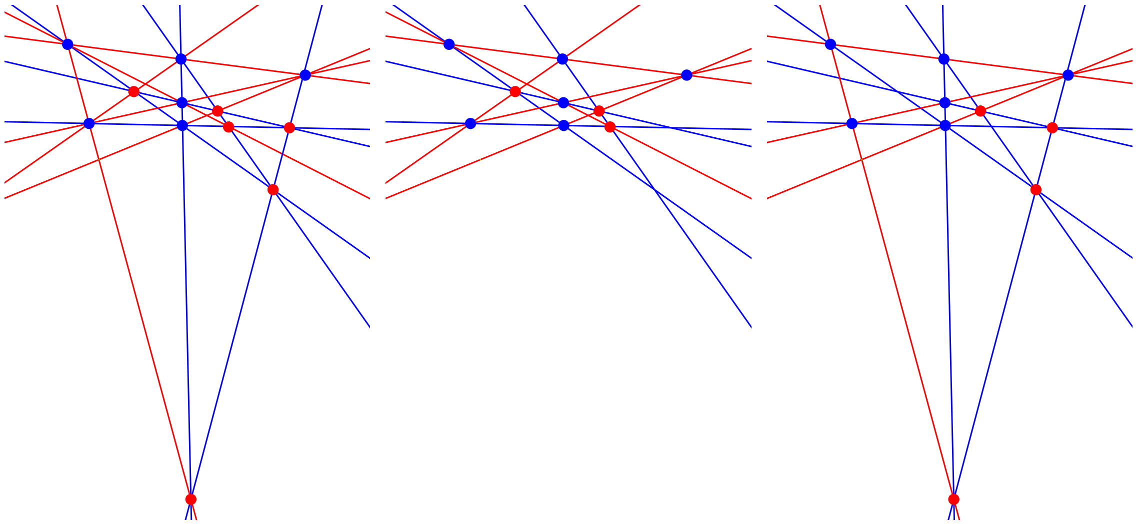

FIGURE 7. An optimal $(12_{3|4})$ configuration (left) which contains simultaneously Pappus' configuration (middle) and Desargues' configuration (right). In the $(12_{3|4})$ configuration, 3-valent elements are colored red while 4-valent elements are colored blue. In the Pappus' and Desargues' subconfigurations, all elements are 3-valent, but we keep the color to see better the correspondence.

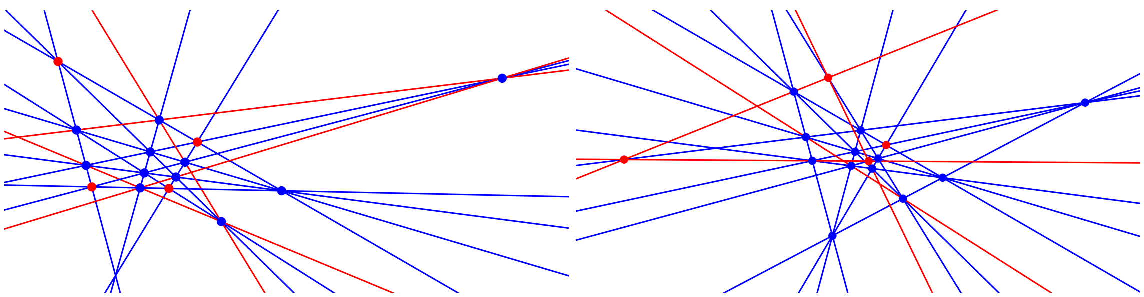

FIGURE 8. Apparently optimal $(15_{3|4})$ and $(16_{3|4})$ configurations. They have 56 and 60 point – line incidences respectively. 3-valent elements are colored red while 4-valent elements are colored blue.

References

[BP14a] Jürgen Bokowski and Vincent Pilaud. Enumerating topological (n_k)-configurations. *Comput. Geom.*, 47(2, part A):175–186, 2014.

[BP14b] Jürgen Bokowski and Vincent Pilaud. On topological and geometric (19_4) configurations. Preprint, `arXiv:1309.3201`, 2014.

[BS05] Jürgen Bokowski and Lars Schewe. There are no realizable 15_4- and 16_4-configurations. *Rev. Roumaine Math. Pures Appl.*, 50(5-6):483–493, 2005.

[BS11] Jürgen Bokowski and Lars Schewe. On the finite set of missing geometric configurations (n_4). To appear in *Computational Geometry: Theory and Applications*, 2011.

[Grü00] Branko Grünbaum. Connected (n_4) configurations exist for almost all n. *Geombinatorics*, 10(1):24–29, 2000.

[Grü09] Branko Grünbaum. *Configurations of points and lines*, volume 103 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[PS13] Tomaž Pisanski and Brigitte Servatius. *Configurations from a graphical viewpoint*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser/Springer, New York, 2013.

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