Analytic combinatorics of chord and hyperchord diagrams with k crossings

Vincent Pilaud
CNRS & LIX,
École Polytechnique

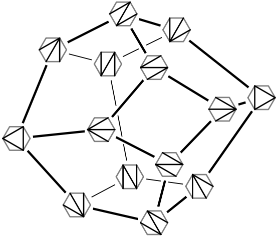
Juanjo Rué
FU Berlin

AofA'14, Paris

Planar chord configurations

Structural properties

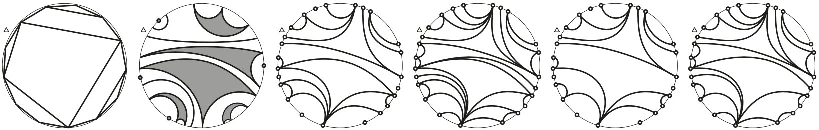
The simplicial complex of crossing-free chord diagrams is the boundary complex of the associahedron



Enumerative properties

Theorem [Flajolet & Noy '99]

chord configurations in the following families $\underset{n\to\infty}{\sim} \frac{\Lambda}{\sqrt{\pi}}\, n^{-3/2}\, \rho^{-n}$.

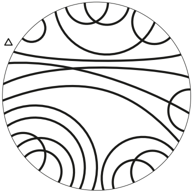


	dissections	partitions	graphs	conn. graphs	forests	trees
ρ^{-1}	$3+2\sqrt{2}$	4	$6+4\sqrt{2}$	$6\sqrt{3}$	8.2246	$\frac{27}{4}$
Λ	$\frac{\sqrt{-140+99\sqrt{2}}}{4}$	1	$\frac{\sqrt{-140+99\sqrt{2}}}{4}$	$\frac{\sqrt{6}}{9}-\frac{\sqrt{2}}{6}$	0.07465	$\frac{\sqrt{3}}{27}$

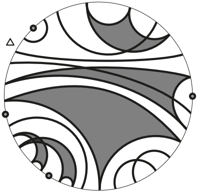
Nearly-planar chord configurations

Crossing-free chord configurations have relevant enumerative and structural properties

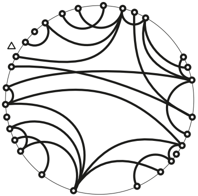
Enumerative/structural properties of nearly planar chord configurations?



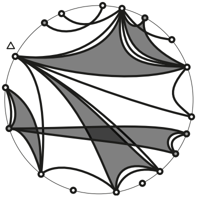
matchings



partitions



chord diagrams

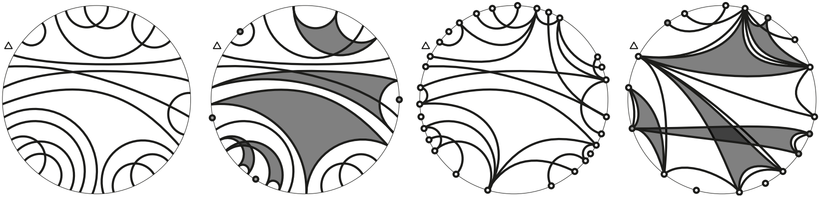


hyperchord diagrams

Nearly-planar chord configurations

Crossing-free chord configurations have relevant enumerative and structural properties

Enumerative/structural properties of nearly planar chord configurations?



matchings | partitions | chord diagrams | hyperchord diagrams

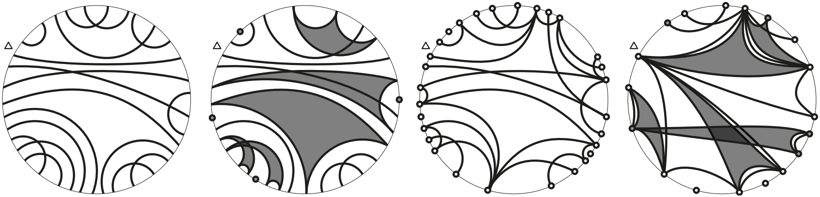
Possible constraints...

- at most k crossings
- no $(k+1)$-crossings
- each chord crosses at most k others
- become crossing-free when removing at most k chords

Nearly-planar chord configurations

Crossing-free chord configurations have relevant enumerative and structural properties

Enumerative/structural properties of nearly planar chord configurations?



matchings	partitions	chord diagrams	hyperchord diagrams

Possible constraints... ... on the crossing graph

- at most k crossings — edges
- no $(k+1)$-crossings — cliques
- each chord crosses at most k others — degrees
- become crossing-free when removing at most k chords — covers

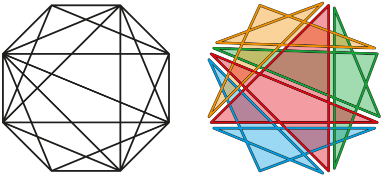
A zoom on $(k+1)$-crossing-free chord diagrams

chord diagrams with no $k+1$ mutually crossing chords have a rich combinatorial structure

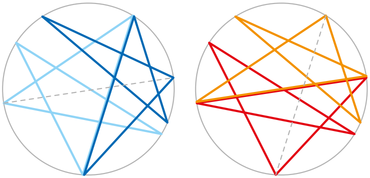
Theorem [Jonsson '03]

The simplicial complex of $(k+1)$-crossing-free chord diagrams is a sphere.

Maximal $(k+1)$-crossing-free chord diagrams are k-triangulations
They can be decomposed into a complex of k-stars [P. & Santos '09]



star decomposition

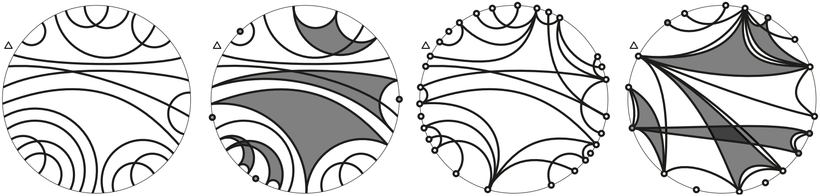


flip

k-triangulations are counted by a Hankel determinant of Catalan numbers [Jonsson '05]

Our results on configurations with k crossings

$\mathcal{C}$ family of configurations among



matchings | partitions | chord diagrams | hyperchord diagrams

$\mathcal{C}(n, m, k)$ = # confs with n vertices, m (hyper)chords, and k crossings
generating function $\mathbf{C}_k(x, y) = \sum_{n,m \in \mathbb{N}} |\mathcal{C}(n, m, k)| \, x^n \, y^m$

Our results on configurations with k crossings

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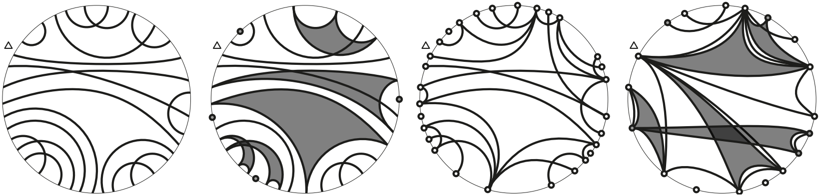


matchings | partitions | chord diagrams | hyperchord diagrams

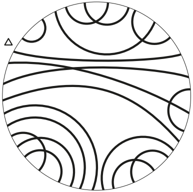
Theorem (Rationality)

The generating function $\mathbf{C}_k(x, y)$ of configurations in $\mathcal{C}$ with exactly k crossings is a rational function of the generating function $\mathbf{C}_0(x, y)$ of planar configurations in $\mathcal{C}$ and of the variables x and y.

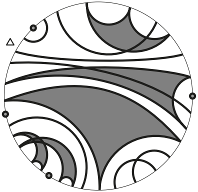
partial results in [Bona, Partitions with k crossings, '00]

Our results on configurations with k crossings

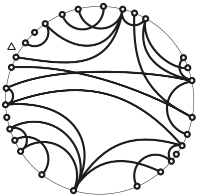
$\mathcal{C}$ family of configurations among



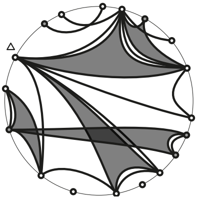
matchings



partitions



chord diagrams



hyperchord diagrams

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The generating function $\mathbf{C}_k(x, y)$ of configurations in $\mathcal{C}$ with exactly k crossings is a rational function of the generating function $\mathbf{C}_0(x, y)$ of planar configurations in $\mathcal{C}$ and of the variables x and y.

Theorem (Asymptotics)

For $k \geq 1$, the number of conf. in $\mathcal{C}$ with k crossings and n vertices is

$$[x^n]\,\mathbf{C}_k(x, 1) \underset{n\to\infty}{=} \Lambda\, n^{\alpha}\, \rho^{-n}\,(1 + o(1)),$$

for certain constants $\Lambda, \alpha, \rho \in \mathbb{R}$ depending on $\mathcal{C}$ and k.

Constants

Theorem (Asymptotics)

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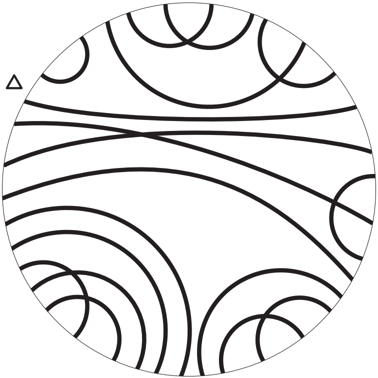
for certain constants $\Lambda, \alpha, \rho \in \mathbb{R}$ depending on $\mathcal{C}$ and k.

family	constant Λ	exp. α	sing. ρ^{-1}
matchings	$\dfrac{\sqrt{2}\,(2k-3)!!}{4^{k-1}\,k!\,\Gamma\left(k-\frac{1}{2}\right)}$	$k-\dfrac{3}{2}$	2
partitions	$\dfrac{(2k-3)!!}{2^{3k-1}\,k!\,\Gamma\left(k-\frac{1}{2}\right)}$	$k-\dfrac{3}{2}$	4
chord diagrams	$\dfrac{\left(-2+3\sqrt{2}\right)^{3k}\sqrt{-140+99\sqrt{2}}\,(2k-3)!!}{2^{3k+1}\,(3-4\sqrt{2})^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}$	$k-\dfrac{3}{2}$	$6+4\sqrt{2}$
hyperchord diagrams	$\simeq \dfrac{1.034^{3k}\;0.003655\,(2k-3)!!}{0.03078^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}$	$k-\dfrac{3}{2}$	$\simeq 64.97$

Matchings with k crossings

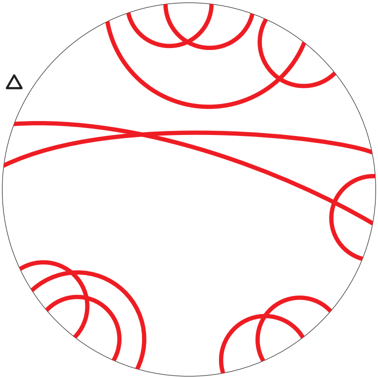
$\mathcal{M} = \{$perfect matchings with endpoints on the unit circle$\}$
All matchings are "rooted" and "up to deformation"



$\mathcal{M}(n, k) =$ number of matchings with n vertices and k crossings
generating function $\mathbf{M}_k(x) = \sum_{n \in \mathbb{N}} |\mathcal{M}(n, k)| \, x^n$

Core matchings

Core of a matching M = submatching $M^\star$ formed by all chords involved in at least one crossing

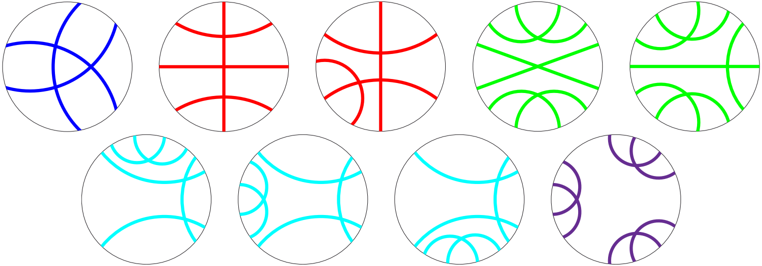


There are only finitely many core matchings with k crossings

Core matching polynomial

$$\mathbf{KM}_k(x_1, \ldots, x_k) = \sum_{\substack{K \ k\text{-core} \\ \text{matching}}} \frac{1}{n(K)} \prod_{i \in [k]} x_i^{n_i(K)}$$

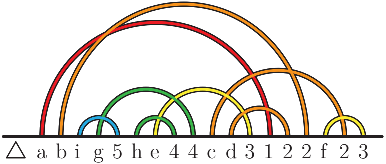
$n_i(K) = \#$ regions of the complement of K with i boundary arcs
$n(K) = \sum_i n_i(K) = \#$ of vertices of K



$$\mathbf{KM}_3(x_1, x_2, x_3) = \frac{1}{6} x_1{}^6 + \frac{3}{2} x_1{}^8 + \frac{3}{2} x_1{}^8 x_2{}^2 + 3 x_1{}^8 x_2 + \frac{1}{3} x_1{}^9 x_3$$

Computing core matching polynomials

Core matchings can be decomposed into connected matchings



level of an arc α of M = graph distance between α and the leftmost arc in the crossing graph of M

Computing core matching polynomials

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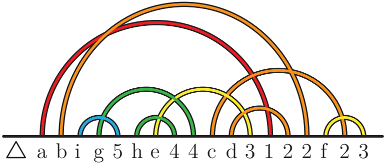

level of an arc α of M = graph distance between α and the leftmost arc in the crossing graph of M

To generate all possible connected matchings, start from a single arc and add arcs one by one. If the last constructed arc (i, j) was at level ℓ, then

(i) either add a new arc (u, v) in the current level ℓ, with $u > i$ and crossing at least one arc at level $\ell - 1$, and no arc at level $< \ell - 1$

(ii) or add an new arc (u, v) at a new level $\ell + 1$ with $u > 1$ and crossing at least one arc at level ℓ and no at level $< \ell$

Generating function of matchings with k crossings

Proposition

For $k \geq 1$, the generating function $\mathbf{M}_k(x)$ of the perfect matchings with k crossings is given by

$$\mathbf{M}_k(x) = x\frac{d}{dx}\mathbf{K}\mathbf{M}_k\left(x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}(x^{i-1}\mathbf{M}_0(x))\right)$$

In particular, $\mathbf{M}_k(x)$ is a rational function of $\mathbf{M}_0(x)$ and x

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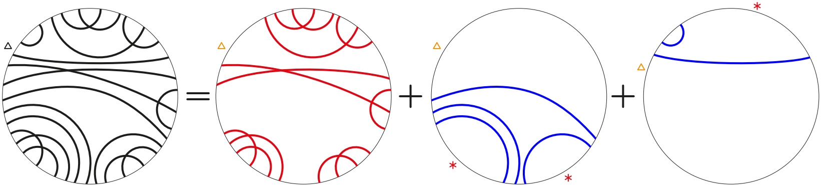

Choose a core matching with k crossings

Replace each region with i boundaries by a crossing-free matching with a root and $i-1$ additional marks

Reroot to obtain a rooted matching

Asymptotic analysis

$$\mathbf{M}_k(x) = \; x\frac{d}{dx}\sum_{\substack{K \; k\text{-core} \\ \text{matching}}} \frac{1}{n(K)} \prod_{i\geq 1}\left(\frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left(x^{i-1}\mathbf{M}_0(x)\right)\right)^{n_{\boldsymbol{i}}(K)}$$

Asymptotic analysis

$$\mathbf{M}_k(x) = x\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}}\frac{1}{n(K)}\prod_{i\geq 1}\left(\frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}(x^{i-1}\mathbf{M}_0(x))\right)^{n_i(K)}$$

$\mathbf{M}_0(x)$ has two singularities around $x = \frac{1}{2}$ and $x = -\frac{1}{2}$.
Denote $X_+ = \sqrt{1-2x}$ around $x = \frac{1}{2}$, then

$$\mathbf{M}_0(x) \underset{x\sim\frac{1}{2}}{=} 2 - 2\sqrt{2}\,X_+ + O\left({X_+}^2\right)$$

$$\frac{d^i}{dx^i}\mathbf{M}_0(x) \underset{x\sim\frac{1}{2}}{=} 2\sqrt{2}\,(2i-3)!!\,{X_+}^{1-2i} + O\left({X_+}^{2-2i}\right),$$

where $(2i-3)!! := (2i-3)\cdot(2i-5)\cdots 3\cdot 1$.

Asymptotic analysis

$$\mathbf{M}_k(x) = x\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}} \frac{1}{n(K)}\prod_{i\geq 1}\left(\frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left(x^{i-1}\mathbf{M}_0(x)\right)\right)^{n_i(K)}$$

$$\underset{x\sim\frac{1}{2}}{=} \sum_{\substack{K\ k\text{-core}\\ \text{matching}}} \frac{\phi(K)}{2n(K)}\prod_{i>1}\left(\frac{\sqrt{2}\,(2i-5)!!}{4^{i-1}\,(i-1)!}\right)^{n_i(K)} X_+^{-\phi(K)-2}(1+O(X_+)),$$

where $\phi(K) = \sum_{i>1}(2i-3)n_i(K)$

Asymptotic analysis

$$\mathbf{M}_k(x) = x\frac{d}{dx}\sum_{\substack{K\ k\text{-core}\\ \text{matching}}}\frac{1}{n(K)}\prod_{i\geq 1}\left(\frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left(x^{i-1}\mathbf{M}_0(x)\right)\right)^{n_i(K)}$$

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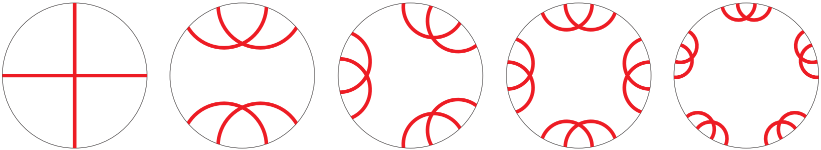
where $\phi(K)=\sum_{i>1}(2i-3)n_i(K)$ is maximized by the core matchings with $n_1(K)=3k$ and $n_k(K)=1$:



Asymptotic analysis

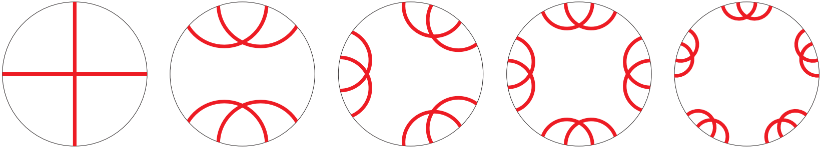
Proposition

For $k \geq 1$, the number of perfect matchings with k crossings and $n = 2m$ vertices is

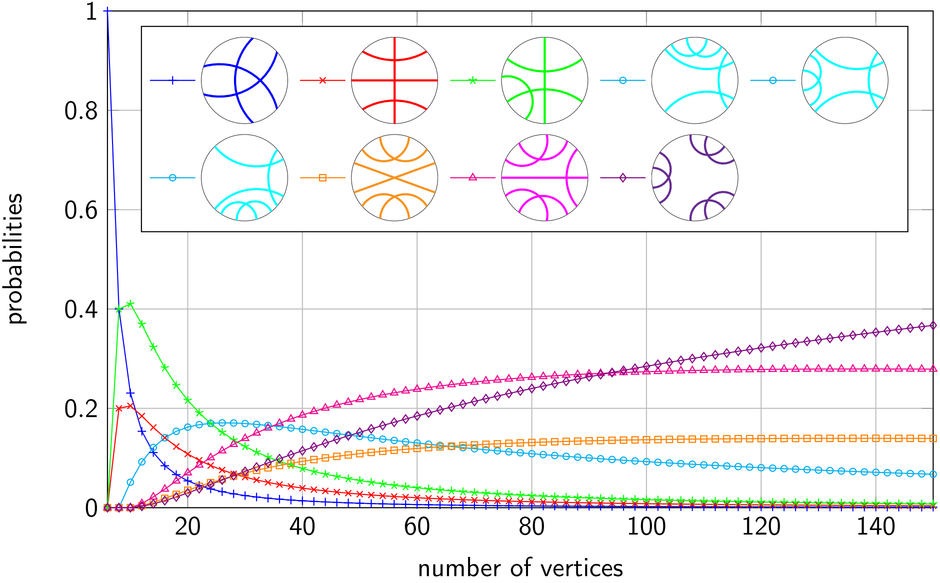
$$[x^{2m}]\,\mathbf{M}_k(x) \underset{m\to\infty}{=} \frac{(2k-3)!!}{2^{k-1}\,k!\,\Gamma\left(k-\frac{1}{2}\right)}\, m^{k-\frac{3}{2}}\, 4^m\,(1+o(1)),$$

where $(2k-3)!! := (2k-3)\cdot(2k-5)\cdots 3\cdot 1$.

Dominant core matchings maximize $\phi(K) = \sum_{i>1}(2i-3)n_i(K)$

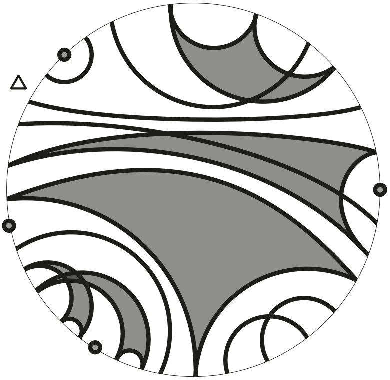


Probabilities core matchings



Extension to partitions

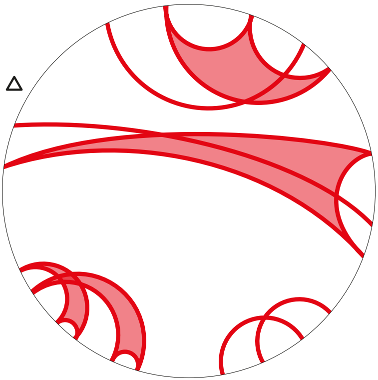
$\mathcal{S}$ = subset of $\mathbb{N}^*$ distinct from $\{1\}$
$\mathcal{P}^{\mathcal{S}}$ = {partitions with parts of size in $\mathcal{S}$}



crossing = two crossing chords that belong to distinct parts
$\mathcal{P}^{\mathcal{S}}(n, m, k)$ = # partitions with n vert., m parts, and k crossings
generating function $\mathbf{P}_k^{\mathcal{S}}(x, y) = \sum_{n,m \in \mathbb{N}} |\mathcal{P}^{\mathcal{S}}(n, m, k)| \, x^n \, y^m$

Core partitions

Core of a partition P = subpartition $P^\star$ formed by all parts involved in at least one crossing



There are only finitely many core partitions with k crossings
Encoded in the core partition polynomial $\mathbf{KP}_k^{\mathcal{S}}(x_1, \ldots, x_k)$

Generating function

Proposition

For $k \geq 1$, the generating function $\mathbf{P}_k^{\mathcal{S}}(x,y)$ of partitions with k crossings and where the size of each block belongs to $\mathcal{S}$ is

$$\mathbf{P}_k^{\mathcal{S}}(x,y) = x\frac{d}{dx}\mathbf{K}\mathbf{P}_k^{\mathcal{S}}\left(x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left(x^{i-1}\mathbf{P}_0^{\mathcal{S}}(x,y)\right), y\right).$$

If $\mathcal{S}$ is finite or ultimately periodic, then $\mathbf{P}_k^{\mathcal{S}}(x,y)$ is a rational function of $\mathbf{P}_0^{\mathcal{S}}(x,y)$ and x.

Generating function

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If $\mathcal{S}$ is finite or ultimately periodic, then $\mathbf{P}_k^{\mathcal{S}}(x,y)$ is a rational function of $\mathbf{P}_0^{\mathcal{S}}(x,y)$ and x.

Two difficulties for the asymptotic:

- minimal singularity and singular behavior of $\mathbf{P}_0^{\mathcal{S}}(x,1)$
- characterize dominant k-core partitions

Difficulty 1: Singular behavior of $\mathbf{P}_0^{\mathcal{S}}(x,1)$

Proposition

For $\mathcal{S} \neq \{1\}$, the generating function $\mathbf{P}_0^{\mathcal{S}}(x,1)$ satisfies

$$\mathbf{P}_0^{\mathcal{S}}(x,1) \underset{x\sim\rho_{\mathcal{S}}}{=} \alpha_{\mathcal{S}} - \beta_{\mathcal{S}}\sqrt{1-\frac{x}{\rho_{\mathcal{S}}}} + O\left(1-\frac{x}{\rho_{\mathcal{S}}}\right),$$

where $\rho_{\mathcal{S}}$, $\alpha_{\mathcal{S}}$ and $\beta_{\mathcal{S}}$ are defined by

$$\sum_{s\in\mathcal{S}}(s-1){\tau_{\mathcal{S}}}^s = 1, \qquad \rho_{\mathcal{S}} := \frac{\tau_{\mathcal{S}}}{\sum_{s\in\mathcal{S}} s{\tau_{\mathcal{S}}}^s},$$

$$\alpha_{\mathcal{S}} := 1 + \sum_{s\in\mathcal{S}} {\tau_{\mathcal{S}}}^s, \qquad \text{and} \qquad \beta_{\mathcal{S}} := \sqrt{\frac{2\left(\sum_{s\in\mathcal{S}} s{\tau_{\mathcal{S}}}^s\right)^3}{\sum_{s\in\mathcal{S}} s(s-1){\tau_{\mathcal{S}}}^s}}.$$

Singular behavior of generating functions defined by a smooth implicit-function schema (Meir & Moon)

Asymptotic analysis

Proposition

For $k \geq 1$, and $\mathcal{S} \neq \{1\}$, the number of partitions with k crossings, n vertices, and where the size of each block belongs to $\mathcal{S}$ is

$$[x^n]\, \mathbf{P}_k^{\mathcal{S}}(x,1) \underset{\substack{n\to\infty \\ \gcd(\mathcal{S})|n}}{=} \Lambda_{\mathcal{S}}\, n^{\frac{\psi(k,\mathcal{S})}{2}}\, {\rho_{\mathcal{S}}}^{-n}\, (1+o(1)),$$

where $\psi(k,\mathcal{S})$ = *maximum of* $\phi(K) := \sum_{i>1}(2i-3)\, n_i(K)$ *and*

$$\Lambda_{\mathcal{S}} := \frac{\gcd(\mathcal{S})\, \psi(k,\mathcal{S})}{2\,\Gamma\left(\frac{\psi(k,\mathcal{S})}{2}+1\right)} \sum_{\substack{K\in\mathcal{P}^{\mathcal{S}} \\ \phi(K)=\psi(k,\mathcal{S})}} \frac{{\tau_{\mathcal{S}}}^{n_1(K)}}{n(K)} \prod_{i>1} \left(\frac{{\rho_{\mathcal{S}}}^i\, \beta_{\mathcal{S}}\,(2i-5)!!}{2^{i-1}\,(i-1)!} \right)^{n_i(K)}.$$

Difficulty 2: Dominant k-core partitions

Only determined for specific instances:

- all partitions: $\mathcal{S} = \mathbb{N}^*$

Proposition

For $k \geq 1$, the number of partitions with k crossings and n vertices is

$$[x^n]\, \mathbf{P}_k^{\mathbb{N}^*}(x,1) \underset{n\to\infty}{=} \frac{(2k-3)!!}{2^{3k-1}\, k!\, \Gamma\left(k-\frac{1}{2}\right)}\, n^{k-\frac{3}{2}}\, 4^n\, (1+o(1)).$$

- q-uniform partitions: $\mathcal{S} = \{q\}$ and $k = k'(q-1)^2$

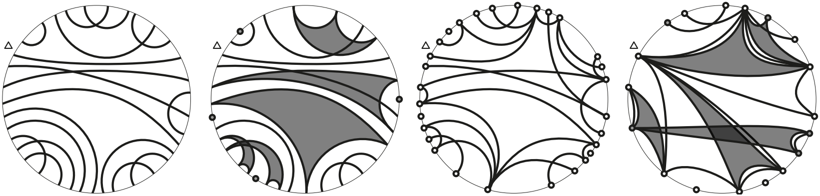
$$[x^{qm}]\, \mathbf{P}_{k'(q-1)^2}^{\{q\}}(x,1) \underset{m\to\infty}{=} \Lambda_{k'}^{\{q\}}\, m^{k'-\frac{3}{2}} \left(\frac{q^q}{(q-1)^{q-1}}\right)^m (1+o(1)).$$

- q-multiple partitions: $\mathcal{S} = q\mathbb{N}$ and $k = k'(q-1)^2$

$$[x^{qm}]\, \mathbf{P}_{k'(q-1)^2}^{q\mathbb{N}^*}(x,1) \underset{m\to\infty}{=} \Lambda_{k'}^{q\mathbb{N}}\, m^{k'-\frac{3}{2}} \left(\frac{(q+1)^{q+1}}{q^q}\right)^m (1+o(1)).$$

Our results on configurations with k crossings

$\mathcal{C}$ family of configurations among



matchings | partitions | chord diagrams | hyperchord diagrams

Theorem (Rationality)

The generating function $\mathbf{C}_k(x, y)$ of configurations in $\mathcal{C}$ with exactly k crossings is a rational function of the generating function $\mathbf{C}_0(x, y)$ of planar configurations in $\mathcal{C}$ and of the variables x and y.

Theorem (Asymptotics)

For $k \geq 1$, the number of conf. in $\mathcal{C}$ with k crossings and n vertices is

$$[x^n]\,\mathbf{C}_k(x, 1) \underset{n\to\infty}{=} \Lambda\, n^{\alpha}\, \rho^{-n}\,(1 + o(1)),$$

for certain constants $\Lambda, \alpha, \rho \in \mathbb{R}$ depending on $\mathcal{C}$ and k.