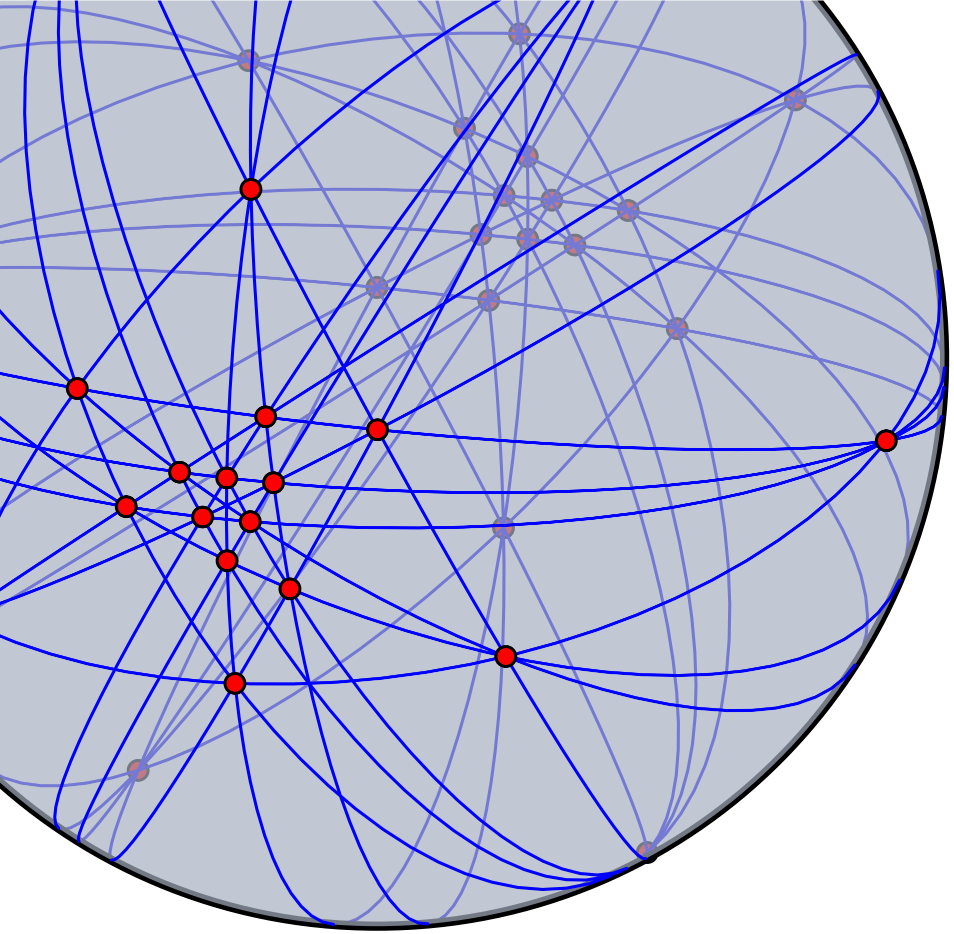
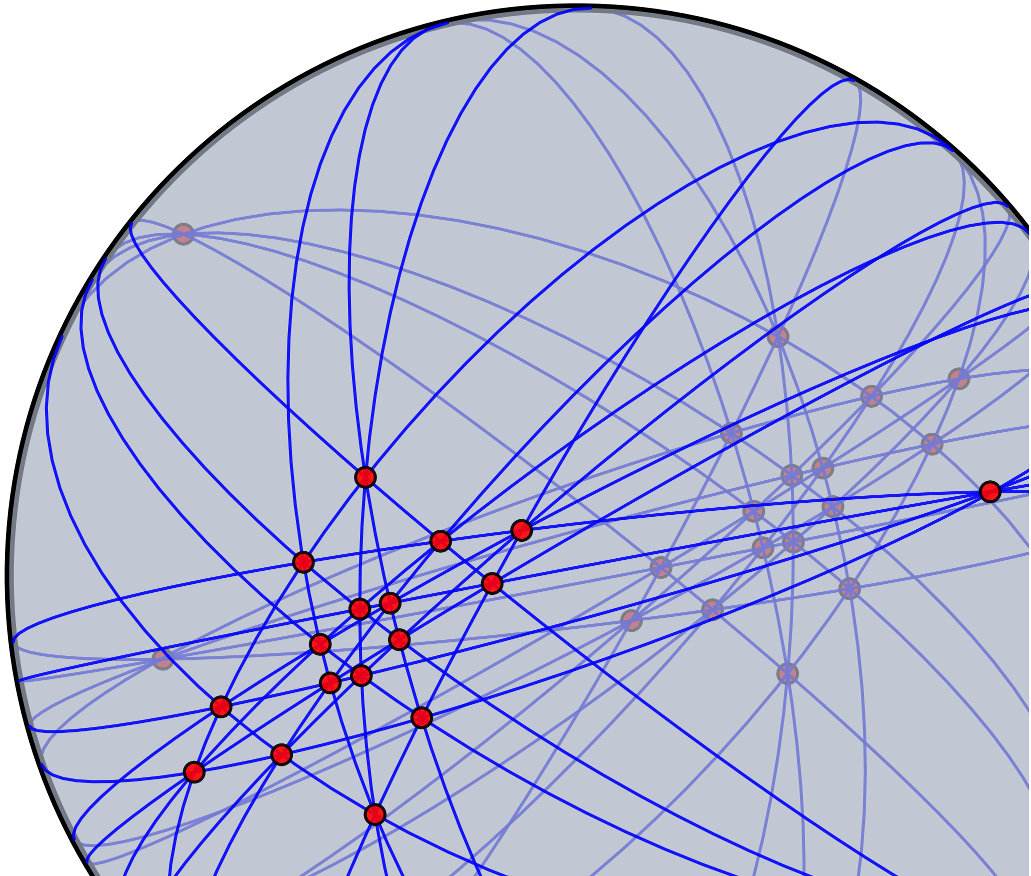


THE QUEST FOR GEOMETRIC CONFIGURATIONS

METHODS, LIMITS & BY-PRODUCTS

Jürgen BOKOWSKI
Technische Universität Darmstadt

Vincent PILAUD
CNRS & LIX, École Polytechnique

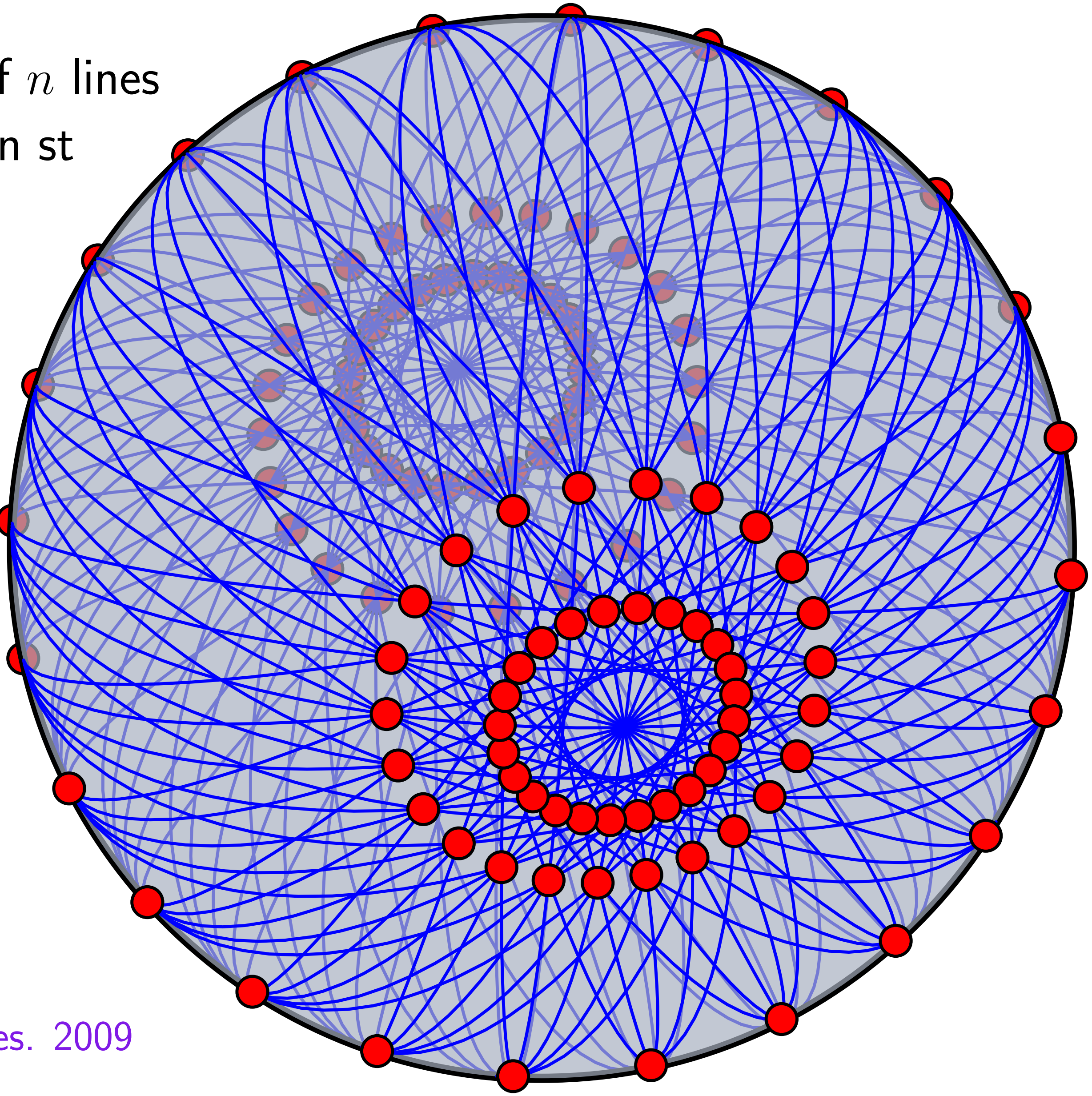


POINT-LINE CONFIGURATIONS

(n_k)-CONFIGURATIONS

(n_k)-configuration $=$

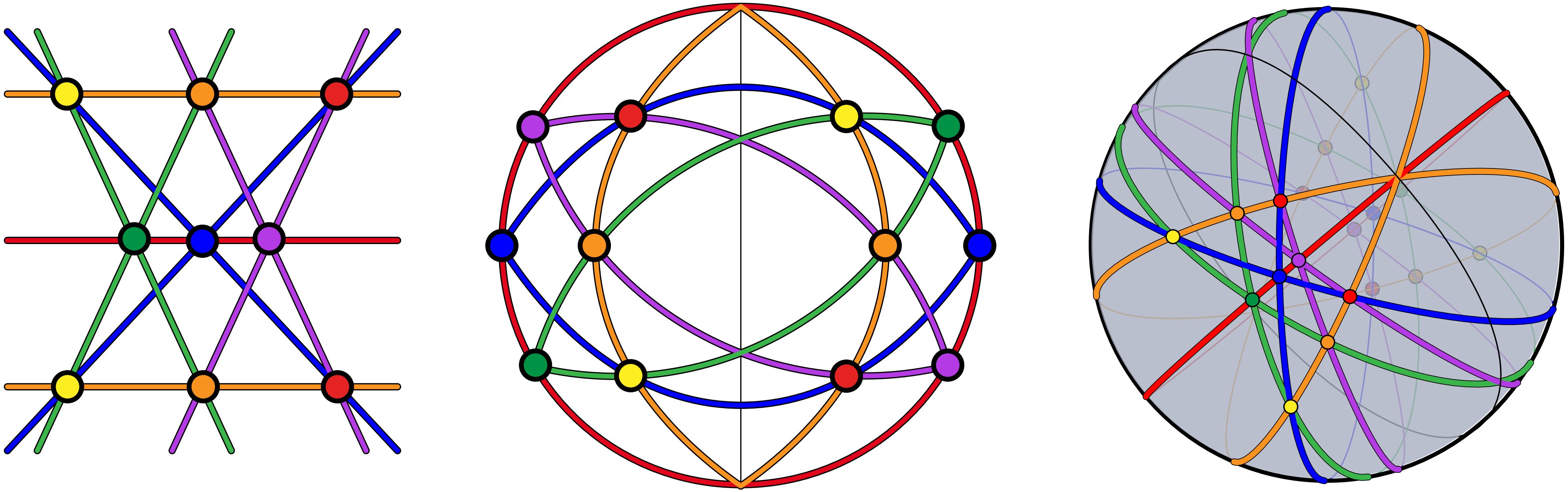
a set P of n points and a set L of n lines
with a point–line incidence relation st
each point is contained in k lines
and each line contains k points



Grünbaum. Configurations of points and lines. 2009

GEOMETRIC CONFIGURATIONS

$\mathbb{P}$ = projective plane
= space of vectorial lines of $\mathbb{R}^3$
= unit 2-sphere with antipodal points identified
= unit disk with antipodal boundary points identified

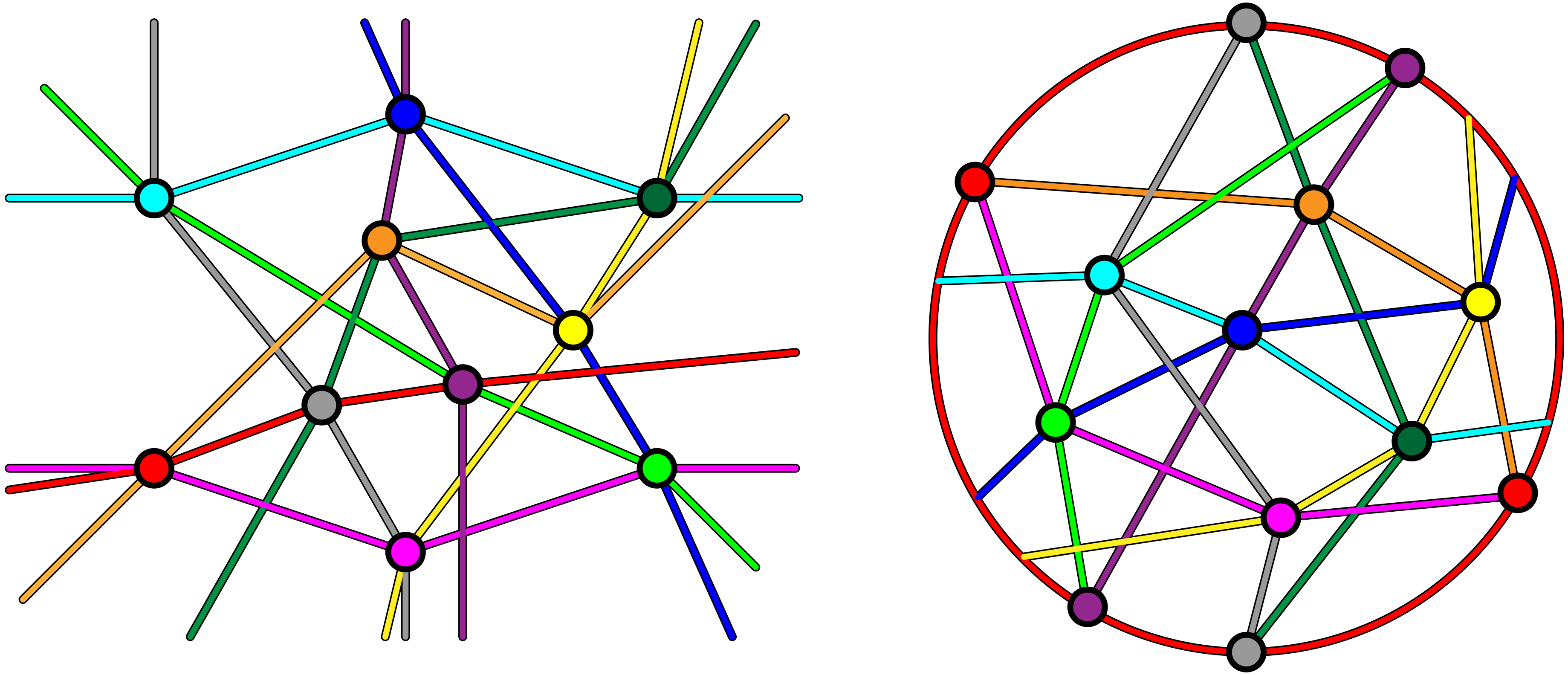


Geometric configuration = points and lines are ordinary points and lines in $\mathbb{P}$

Projective equivalence = equivalence under projective transformations

TOPOLOGICAL CONFIGURATIONS

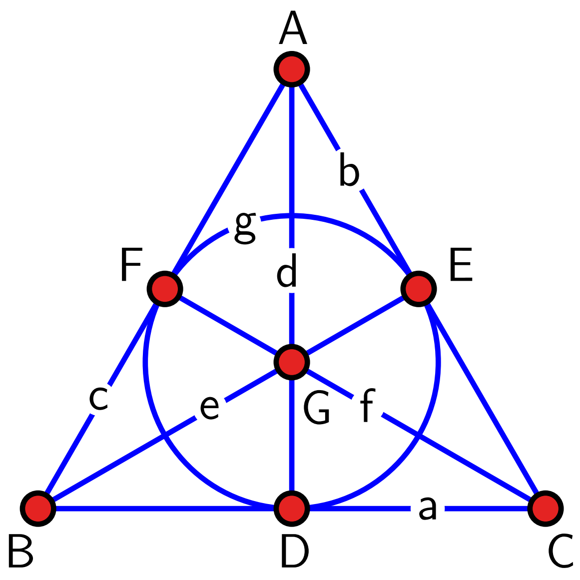
Pseudoline = non-separating simple closed curve in $\mathbb{P}$



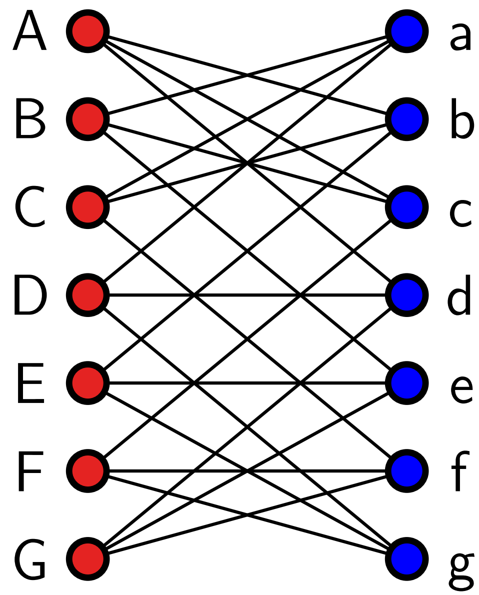
Topological configuration = points are ordinary points in $\mathbb{P}$ and lines are pseudolines in $\mathbb{P}$

Topological equivalence = equivalence under homeomorphisms of $\mathbb{P}$

COMBINATORIAL CONFIGURATIONS



	A	B	C	D	E	F	G
a		•	•	•			
b	•		•		•		
c	•	•				•	
d	•			•			•
e		•			•		•
f			•			•	•
g				•	•	•	



Combinatorial configuration $=$ k-regular bipartite graph with girth at least 6 (Levi graph)

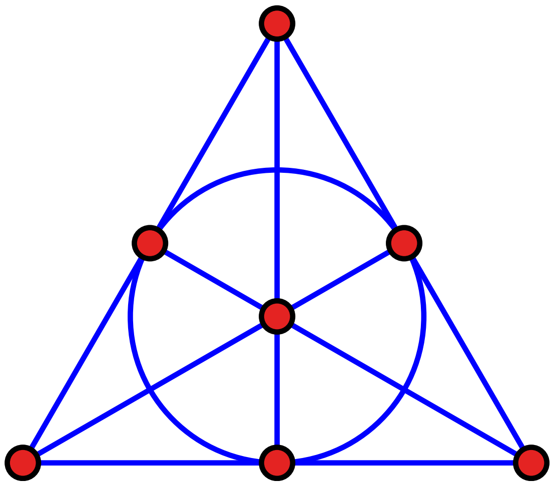
Combinatorial equivalence $=$ automorphism of the Levi graph
which sends points to points and lines to lines

Combinatorial duality $=$ automorphism of the Levi graph
which exchanges points and lines

GEOMETRIC, TOPOLOGICAL & COMBINATORIAL CONFIGURATIONS

Three different levels of configurations:

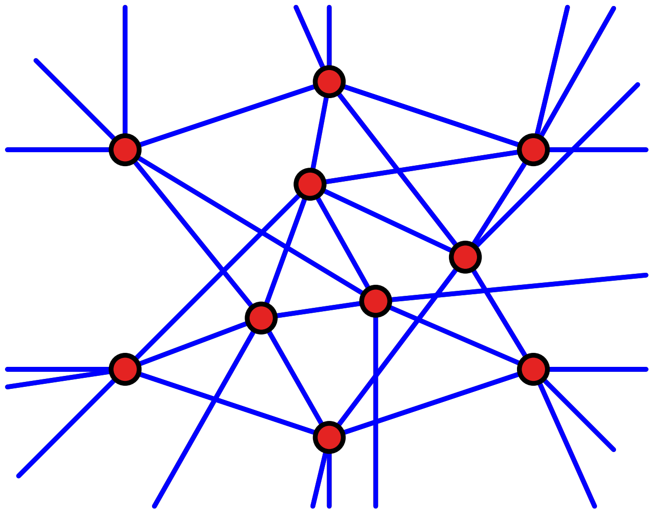
Combinatorial configuration



just an astract
incidence structure

combinatorial equivalence

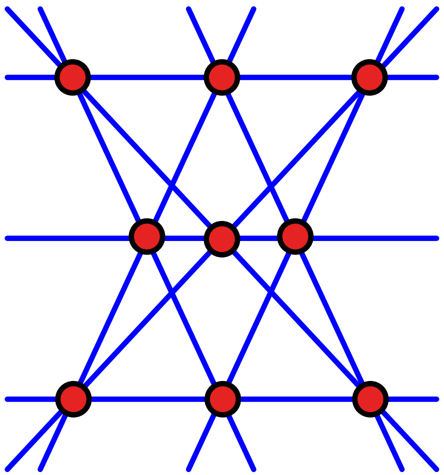
Topological configuration



ordinary points in $\mathbb{P}$
& pseudolines of $\mathbb{P}$

topological equivalence
mutation equivalence

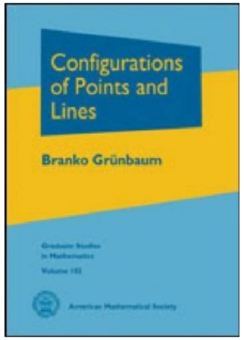
Geometric configuration



ordinary points in $\mathbb{P}$
& ordinary lines in $\mathbb{P}$

projective equivalence

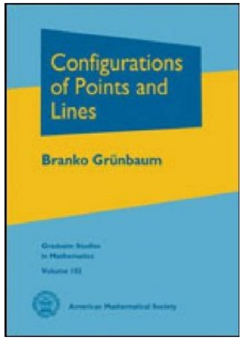
EXISTENCE & ENUMERATION OF (n_k)-CONFIGURATIONS



Two research directions on (n_k)-configurations:

1. For a given k, determine for which values of n do geometric, topological and combinatorial (n_k)-configurations exist
2. Enumerate and classify (n_k)-configurations for given k and n

EXISTENCE & ENUMERATION OF (n_k)-CONFIGURATIONS



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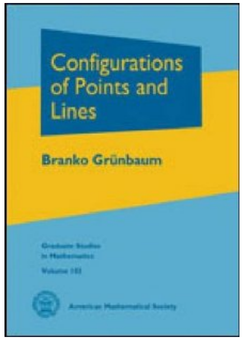
EXISTENCE OF (n_k)-CONFIGURATIONS

	Combinatorial conf.	Topological conf.	Geometric configurations
$k = 3$	exist iff $n \geq 7$	exist iff $n \geq 9$	exist iff $n \geq 9$
$k = 4$	exist iff $n \geq 13$	exist iff $n \geq 17$	exist iff $n \geq 18$ with the possible exceptions of $n = 19, 22, 23, 26, 37, 43$

Grünbaum. Connected (n_4)-configurations exist for almost all n. 2000 – 2002 – 2006

Bokowski & Schewe. On the finite set of missing geometric configurations (n_4). 2011

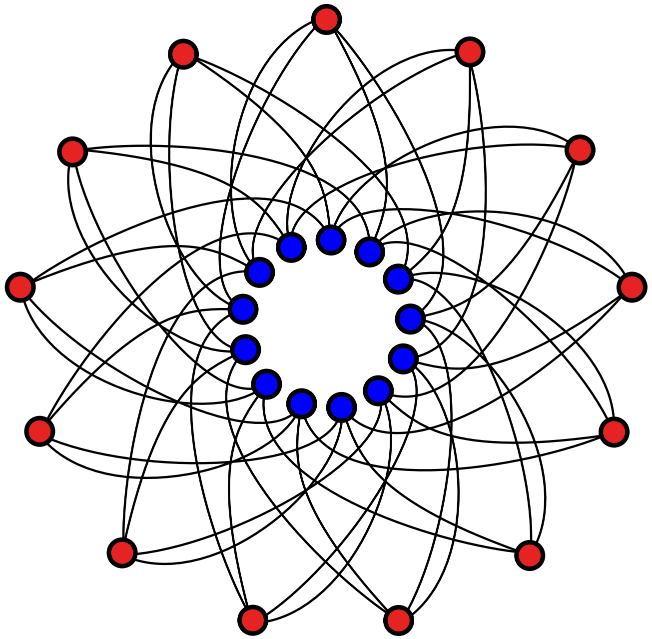
EXISTENCE & ENUMERATION OF (n_k)-CONFIGURATIONS



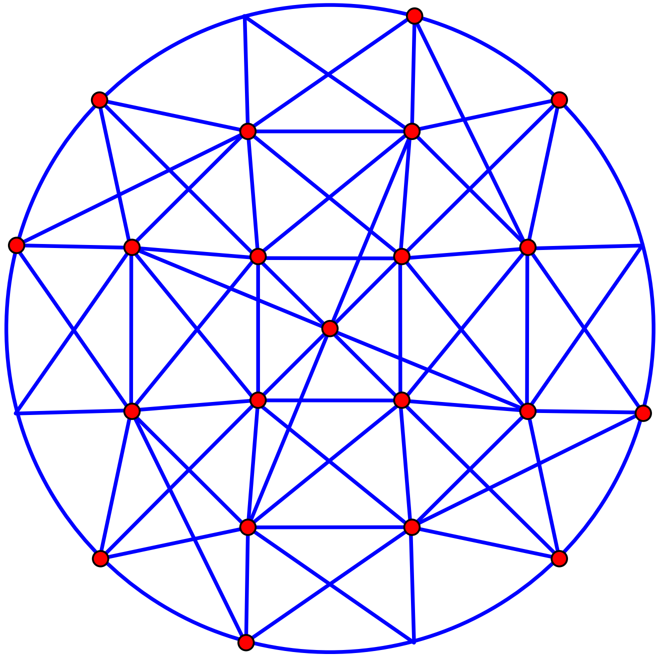
Two research directions on (n_k)-configurations:

1. For a given k, determine for which values of n do geometric, topological and combinatorial (n_k)-configurations exist
2. Enumerate and classify (n_k)-configurations for given k and n

Combinatorial (13_4)-conf.

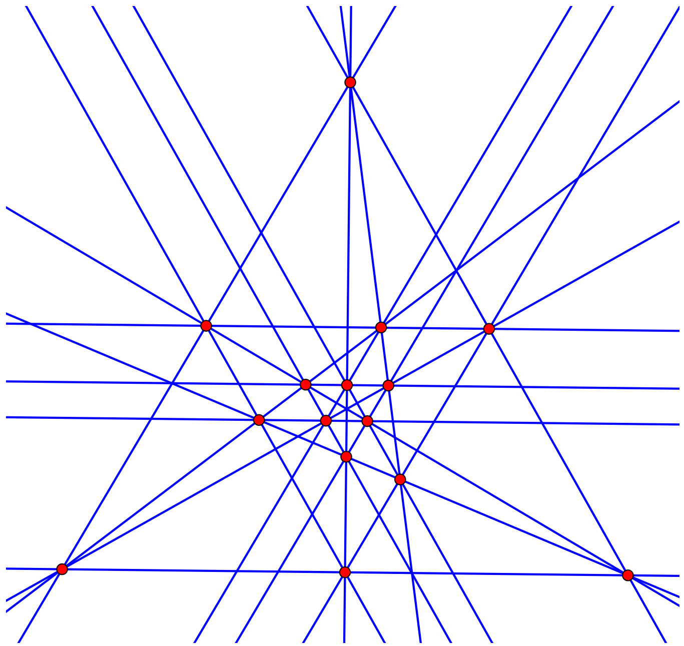


Topological (17_4)-conf.



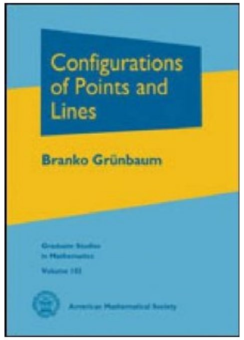
Bokowski, Grünbaum & Schewe

Geometric (18_4)-conf.



Bokowski & Schewe

EXISTENCE & ENUMERATION OF (n_k)-CONFIGURATIONS



Two research directions on (n_k)-configurations:

1. For a given k, determine for which values of n do geometric, topological and combinatorial (n_k)-configurations exist
2. Enumerate and classify (n_k)-configurations for given k and n

ENUMERATION OF (n_k)-CONFIGURATIONS

n	$\mathrm{comb}_3(n)$	$\mathrm{topo}_3(n)$	$\mathrm{geom}_3(n)$
≤ 6	0	0	0
7	1	0	0
8	1	0	0
9	3	3	3
10	10	10	9
11	31	31	31
12	229	229	229
13	2 036	?	?
⋮	⋮	⋮	⋮
19	7 640 941 062	?	?

Betten, Brinkmann & Pisanski — Withe & Sturmfels

n	$\mathrm{comb}_4(n)$	$\mathrm{topo}_4(n)$	$\mathrm{geom}_4(n)$
≤ 12	0	0	0
13	1	0	0
14	1	0	0
15	4	0	0
16	19	0	0
17	1 972	1	0
18	971 191	16	?
19	269 224 652	?	?

Betten & Betten — Páez Osuna & San Augustín Chi

Bokowski & Schewe

CONTRIBUTION

APPROACH

1. Generate all topological (n_k)-configurations (up to combinatorial equivalence), without enumerating first combinatorial (n_k)-configurations
2. Study their geometric realizations

RESULTS

1. Confirm and complete former results on (18_4)-configurations
   In particular, discover a new geometric (18_4)-configuration
2. Enumeration of the $4\,028$ topological (19_4)-configurations, 222 of which are self-dual
3. First examples of topological (19_4)-configurations with a non-trivial symmetry group
4. There is no geometric (19_4)-configuration (to be confirmed!)
5. Study sub-configurations and quasi-configurations
   In particular, obtain the first (37_4)- and (43_4)-configurations

TOPOLOGICAL CONFIGURATIONS

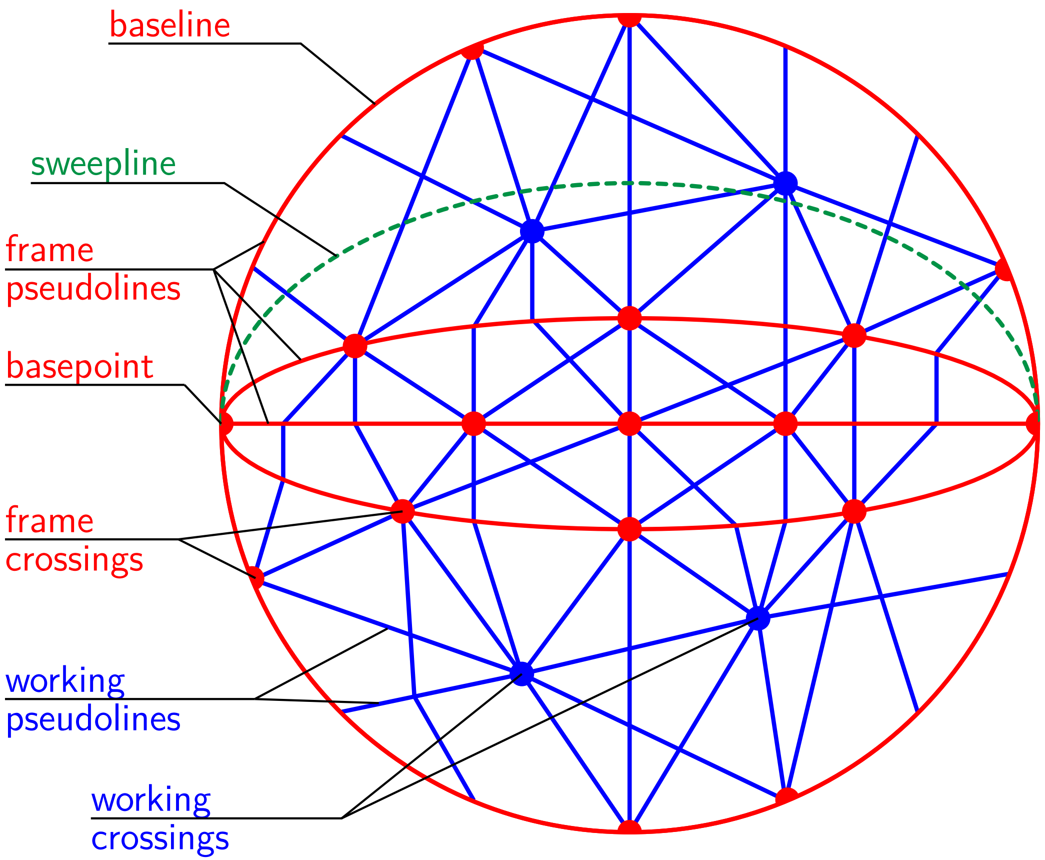
ENUMERATING TOPOLOGICAL CONFIGURATIONS

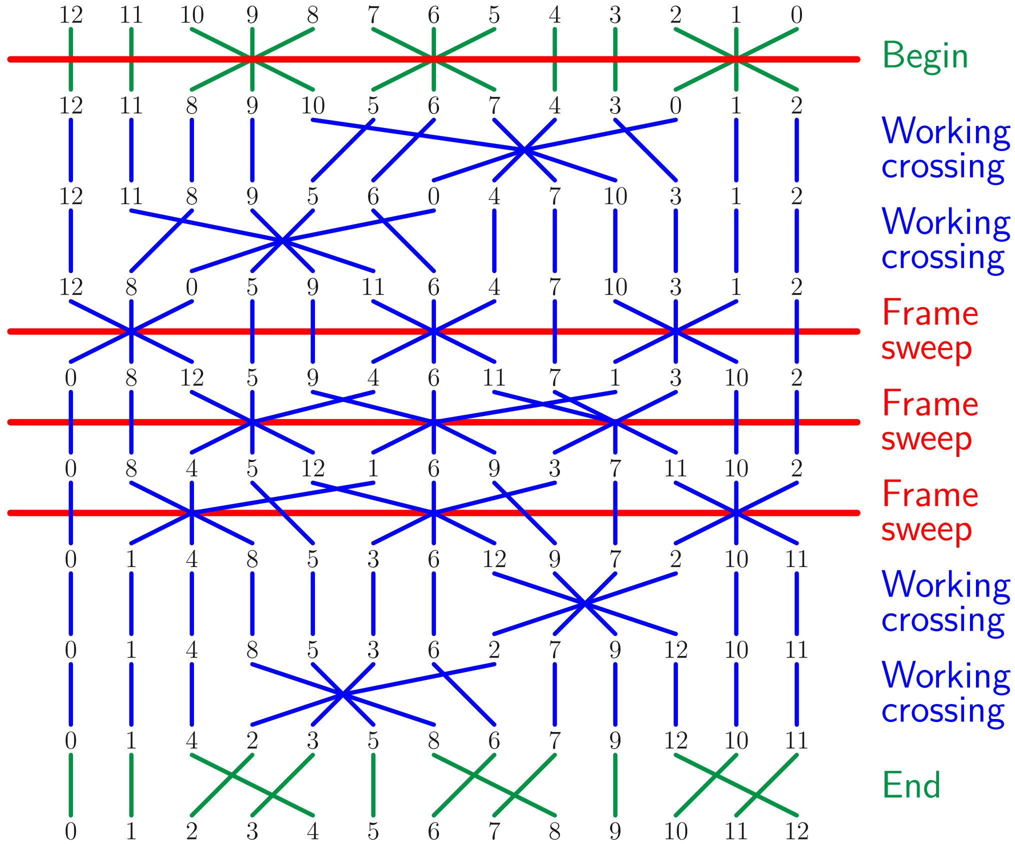
Sweeping algorithm to generate all topological (n_k)-configurations for fixed k and n

- No need to enumerate all combinatorial (n_k)-configurations
- Focus on mutation equivalence classes of topological configurations
- Requires to reduce the output up to combinatorial equivalence (multiscale invariant technique)

SWEEPING A TOPOLOGICAL CONFIGURATION

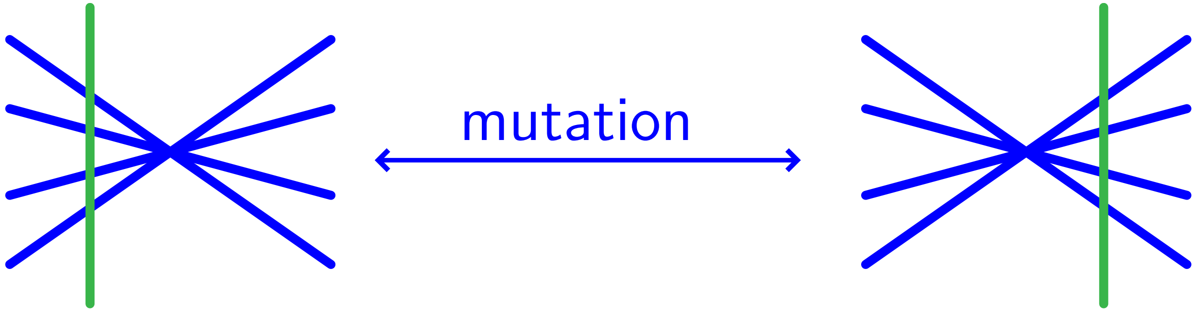
Sweeping algorithm to generate all topological (n_k)-configurations for fixed k and n





MUTATION EQUIVALENCE

mutation = local transformation where only one pseudoline moves, sweeping a single vertex of the remaining arrangement



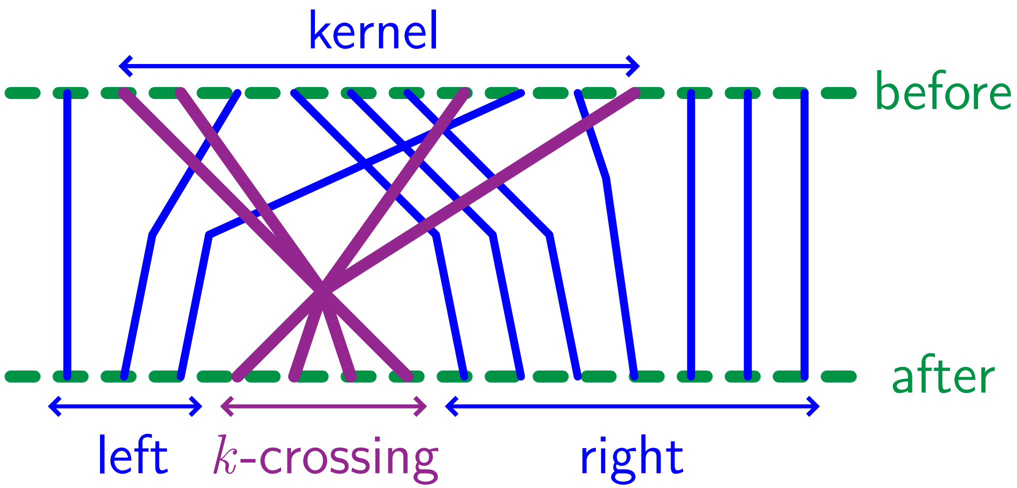
admissible mutation = a mutation where all perturbed crossings are not in P

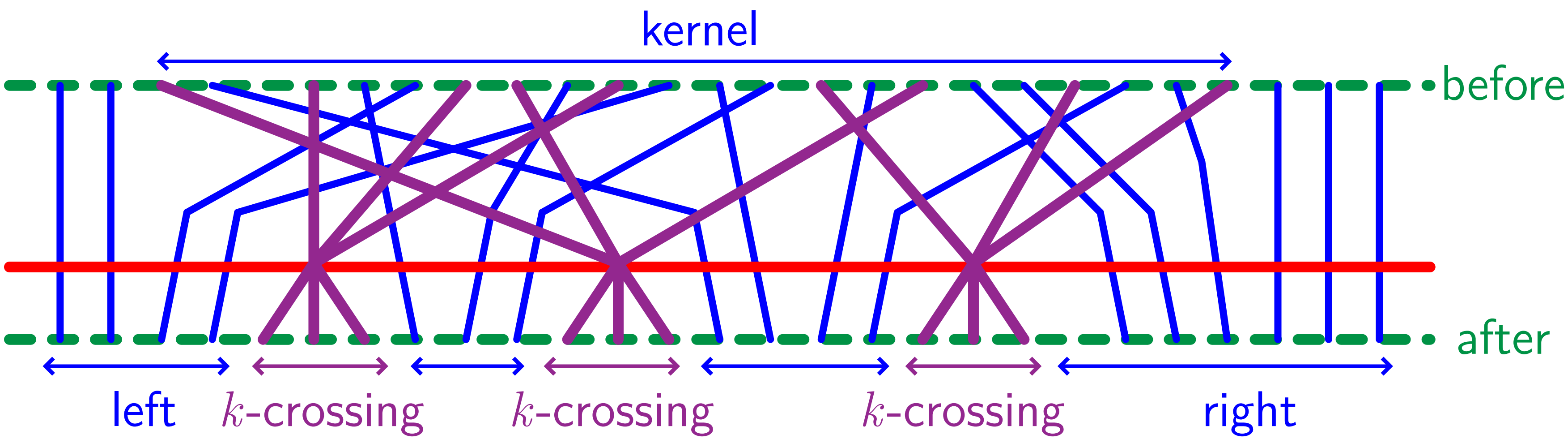
mutation equivalent configurations = configurations in the same connected component of the admissible mutations

We enumerate at least one representative in each mutation equivalence class

SWEEP EVENTS

We enumerate at least one representative in each mutation equivalence class
It enable us to assume that sweep events are of two kinds:



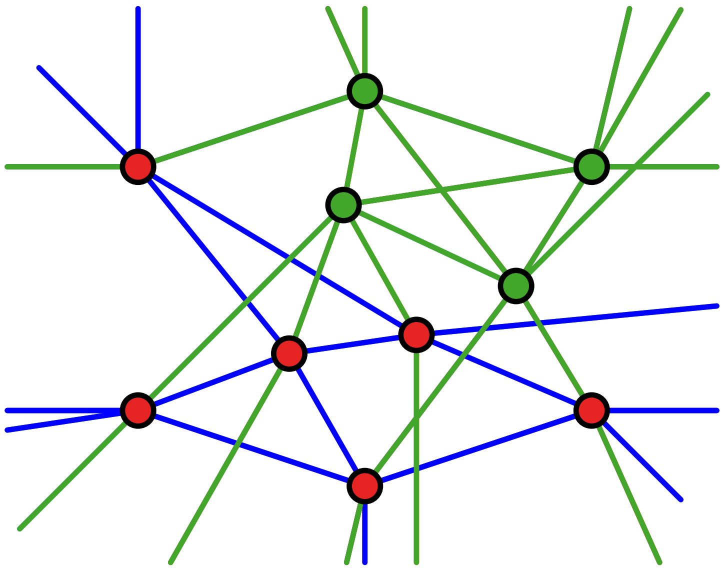


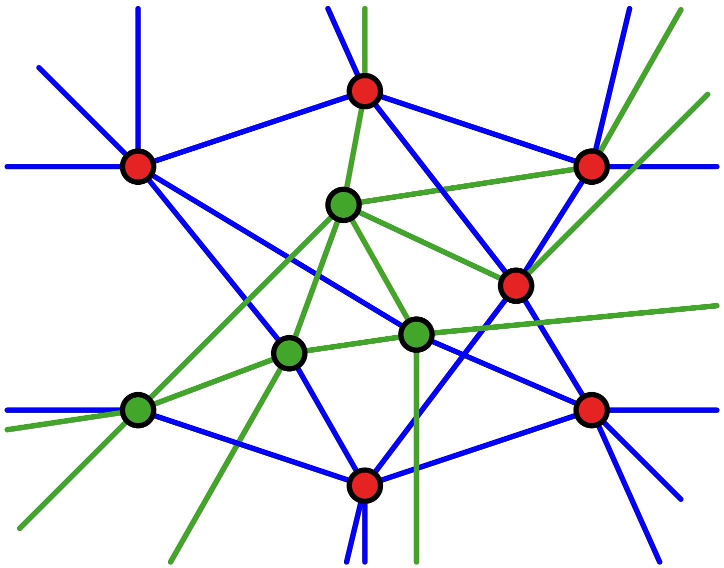
CLIQUE AND COCLIQUE DISTRIBUTIONS

(P, L) a combinatorial point-line configuration

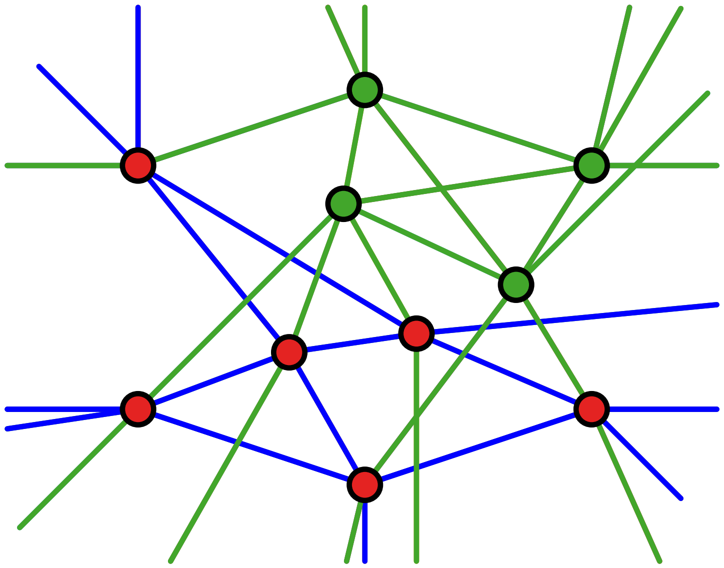
j-clique of (P, L) = set of j points of P pairwise related by lines of L
For $p \in P$, define $\gamma(p) = (\#\{j\text{-clique of } (P, L) \text{ containing } p\})_{j \geq 3}$
clique distribution of $(P, L) = \gamma(P) = \{\gamma(p) \mid p \in P\}$

j-coclique of (P, L) = set of j lines of L pairwise crossing at points of P
For $\ell \in L$, define $\delta(\ell) = (\#\{j\text{-coclique of } (P, L) \text{ containing } \ell\})_{j \geq 3}$
coclique distribution of $(P, L) = \delta(L) = \{\delta(\ell) \mid \ell \in L\}$

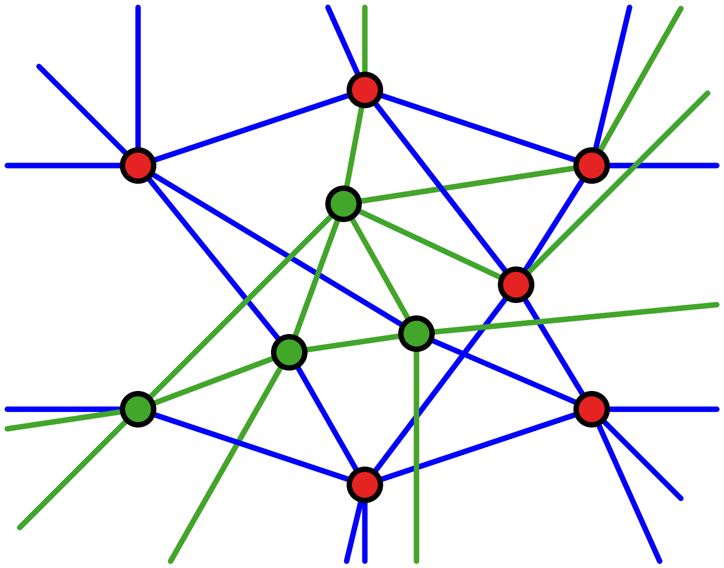

clique


coclique

COMBINATORIAL INVARIANTS



clique



coclique

Clique and coclique distributions are combinatorial invariants

Two different use:

1. either separate isomorphism classes of combinatorial configurations
   (two configurations with different invariants cannot be combinatorially equivalent)
2. or guess combinatorial isomorphisms
   (any isomorphism between two configurations respects the combinatorial invariants)

DERIVATION OF INVARIANTS

$\gamma : P \to X$
$\delta : L \to Y$ such that $\gamma(P) = \{\gamma(p) \mid p \in P\}$
$\delta(L) = \{\delta(\ell) \mid \ell \in L\}$ are combinatorial invariants of (P, L)

derivative of γ = the function $\gamma' : L \to X^k$ defined by $\gamma'(\ell) = \{\gamma(p) \mid p \in P, p \in \ell\}$
derivative of δ = the function $\delta' : P \to Y^k$ defined by $\delta'(p) = \{\delta(\ell) \mid \ell \in L, p \in \ell\}$

Then $\delta'(P)$ and $\gamma'(L)$ are still combinatorial invariants of (P, L)

They refine the initial invariants $\gamma(P)$ and $\delta(L)$

MULTISCALE TECHNIQUE

$\mathcal{C}$ a set of combinatorial configurations to be reduced up to combinatorial equivalence

$\begin{array}{l} \gamma : P \to X \\ \delta : L \to Y \end{array}$ such that $\begin{array}{l} \gamma(P) = \{\gamma(p) \mid p \in P\} \\ \delta(L) = \{\delta(\ell) \mid \ell \in L\} \end{array}$ are combinatorial invariants of (P, L)

Separate the configurations of $\mathcal{C}$ into different classes according to $(\gamma(P), \delta(L))$
Compute the derivative invariants $\delta'(P)$ and $\gamma'(L)$
In each class, we have three possibilities:

- $\delta'(P)$ and $\gamma'(L)$ are not constant
  $\Longrightarrow$ refine into subclasses according to $(\delta'(P), \gamma'(L))$ and reiterate the refinement
- $\delta'(P)$ and $\gamma'(L)$ constant, but provide more information about possible isomorphisms
  $\Longrightarrow$ reiterate the refinement
- Otherwise, $\delta'(P)$ and $\gamma'(L)$, as well as their further derivatives, provide precisely the same information about possible isomorphisms
  $\Longrightarrow$ start a brute-force search for possible isomorphisms

TOPOLOGICAL (18_4)- AND (19_4)-CONFIGURATIONS

Confirmation: 16 topological (18_4)-configurations up to combinatorial equivalence

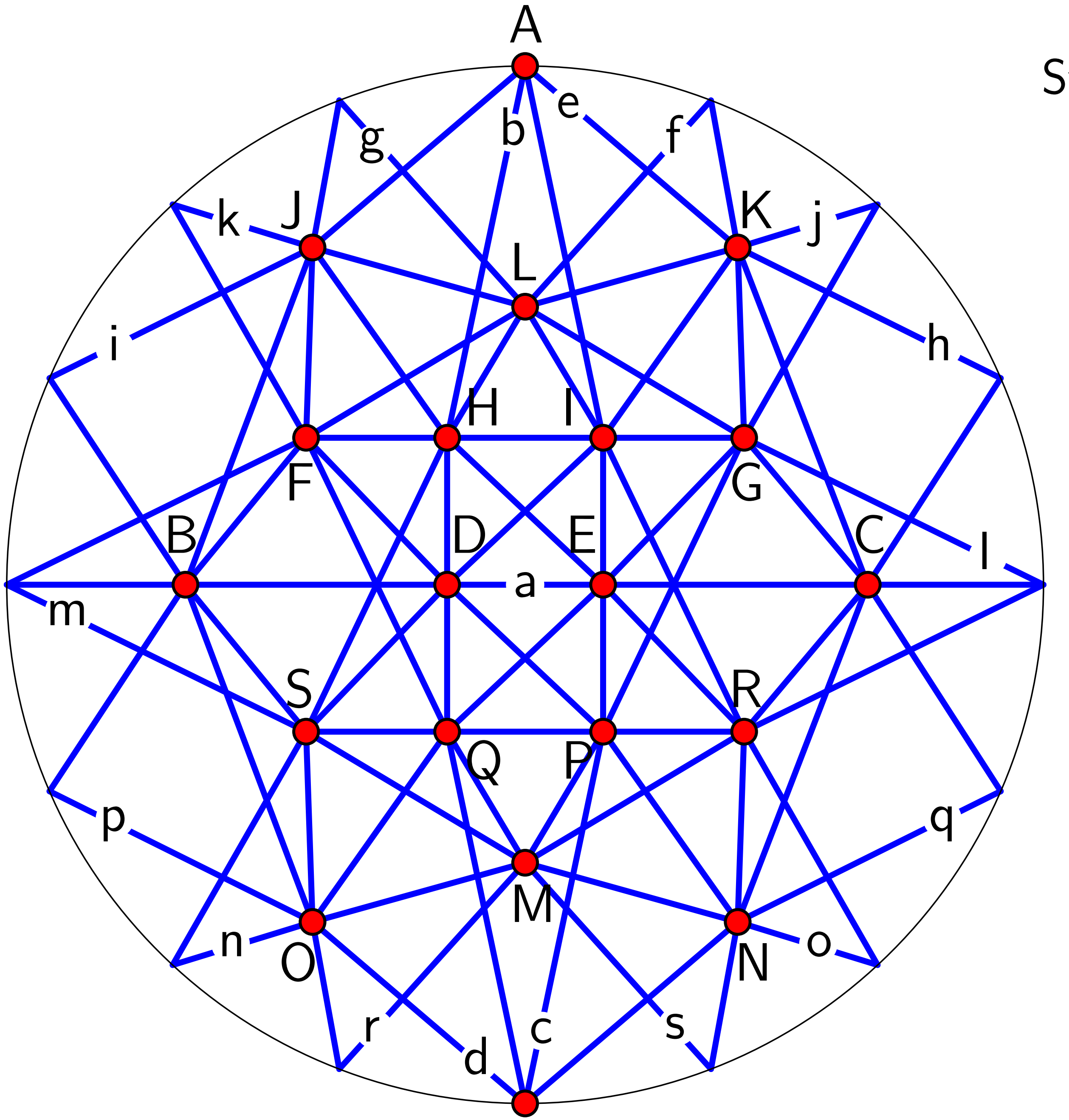
About 1 hour for the enumeration process (compared to several months of CPU time with previous methods)

New result: $4\,028$ topological (19_4)-configurations up to combinatorial equivalence, 222 of which are self-dual

The automorphism groups of the Levi graphs of these (19_4)-configurations are:

group G	1	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	D_8
# of configurations (P, L) with $\mathrm{Aut}(\mathcal{LG}(P, L)) \simeq G$	$3\,726$	283	14	2	3

SYMMETRIC TOPOLOGICAL (19_4)-CONFIGURATION



Symmetry group $\simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$:

- horizontal reflection
- vertical reflection
- self-polarity (a,A)(b,B) . . . (s,S)

GEOMETRIC CONFIGURATIONS

CONSTRUCTION SEQUENCES

INPUT: A combinatorial configuration (P, L)
OUTPUT: A system of polynomial equalities and inequalities
with a solution iff (P, L) is geometrically realizable

Choose a projective base $\{p, q, r, s\}$ in (P, L) (meaning 4 points, no 3 on a line)
Initialize the set of already constructed points $\Pi \leftarrow \{\mathbf{u}_p, \mathbf{u}_q, \mathbf{u}_r, \mathbf{u}_s\}$ and lines $\Lambda \leftarrow \varnothing$
the set of equalities $\mathbb{E} \leftarrow \varnothing$ and inequalities $\mathbb{I} \leftarrow \varnothing$

Repeat

- for each non constructed line $\ell \in L \smallsetminus \Lambda$,
  if we have already constructed at least two points p, q contained in ℓ, then

  $$\Lambda \leftarrow \Lambda \cup \{\mathbf{u}_\ell = \mathbf{u}_p \wedge \mathbf{u}_q\} \quad \mathbb{E} \leftarrow \mathbb{E} \cup \{\mathbf{u}_r \cdot \mathbf{u}_\ell = 0 \mid r \in \ell\} \quad \mathbb{I} \leftarrow \mathbb{I} \cup \{\mathbf{u}_r \cdot \mathbf{u}_\ell \neq 0 \mid r \notin \ell\}$$

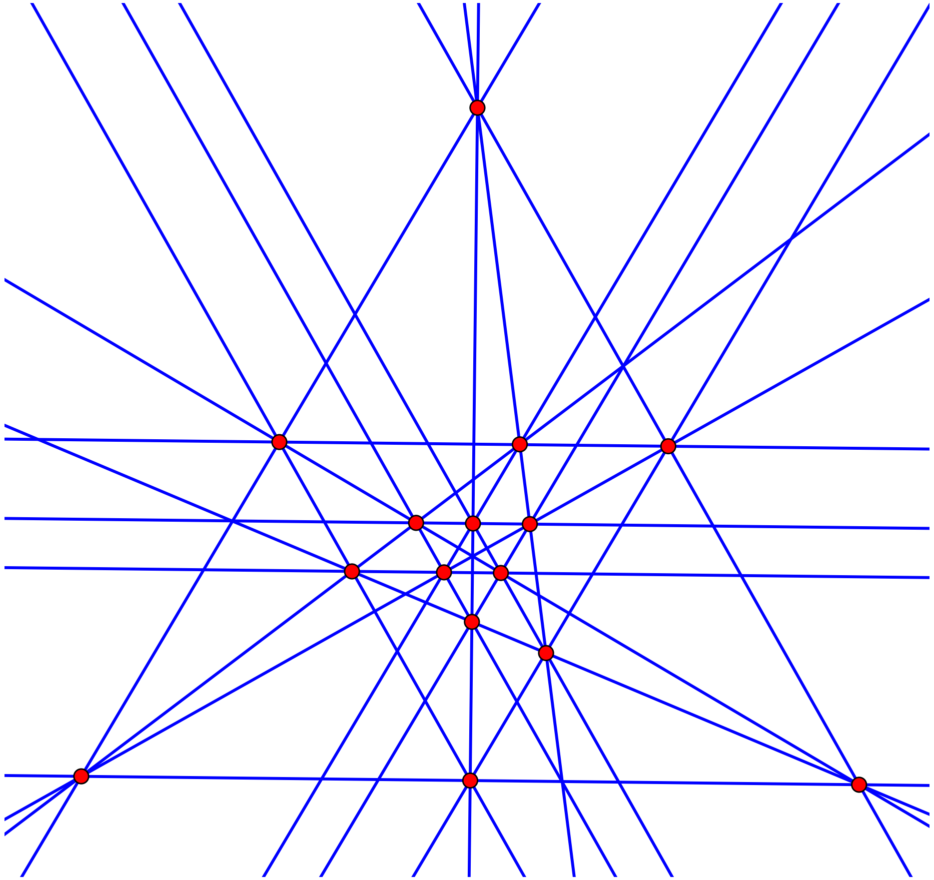
- if no new line can be added this way, then choose one arbitrary non constructed line $\ell \in L \smallsetminus \Lambda$, and set

  $$\Lambda \leftarrow \Lambda \cup \{\mathbf{u}_\ell = [x, y, z]\} \quad \mathbb{E} \leftarrow \mathbb{E} \cup \{\mathbf{u}_r \cdot \mathbf{u}_\ell = 0 \mid r \in \ell\} \quad \mathbb{I} \leftarrow \mathbb{I} \cup \{\mathbf{u}_r \cdot \mathbf{u}_\ell \neq 0 \mid r \notin \ell\}$$

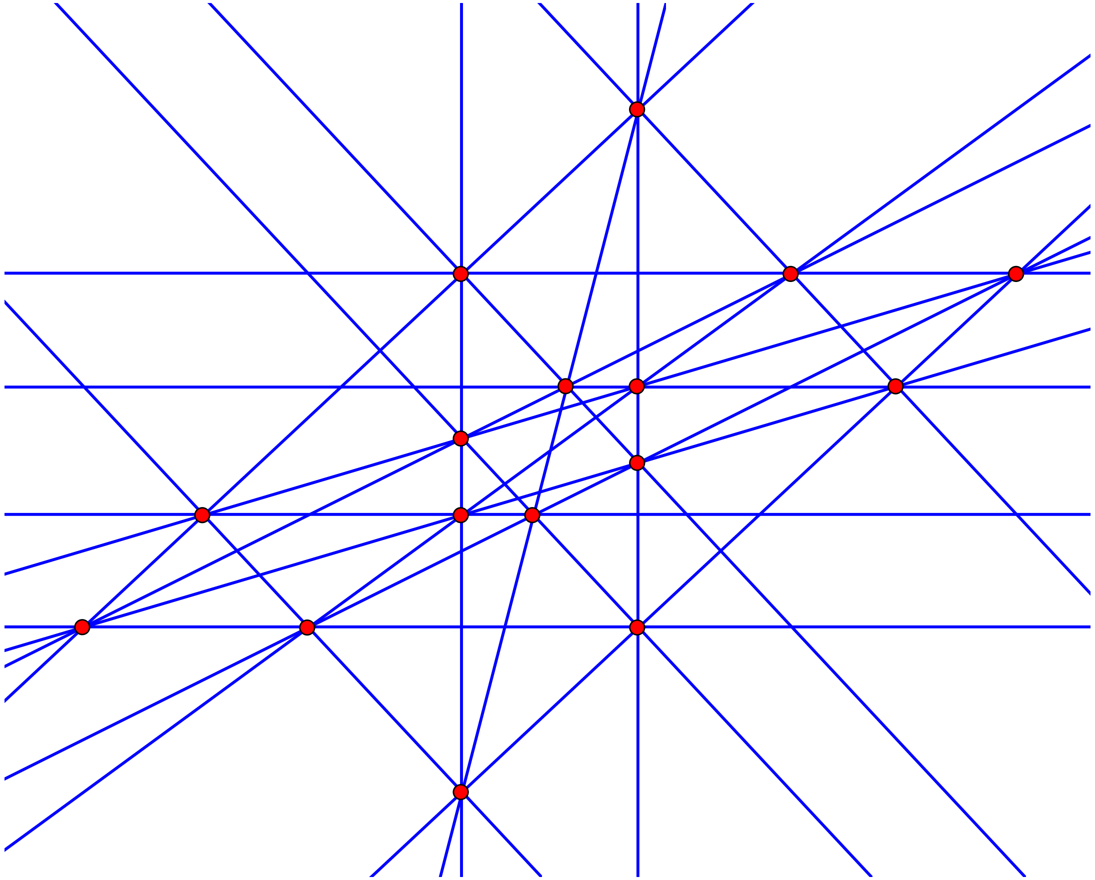
- dualize to go to the next step

until all points and lines are constructed

GEOMETRIC (18_4)-CONFIGURATIONS

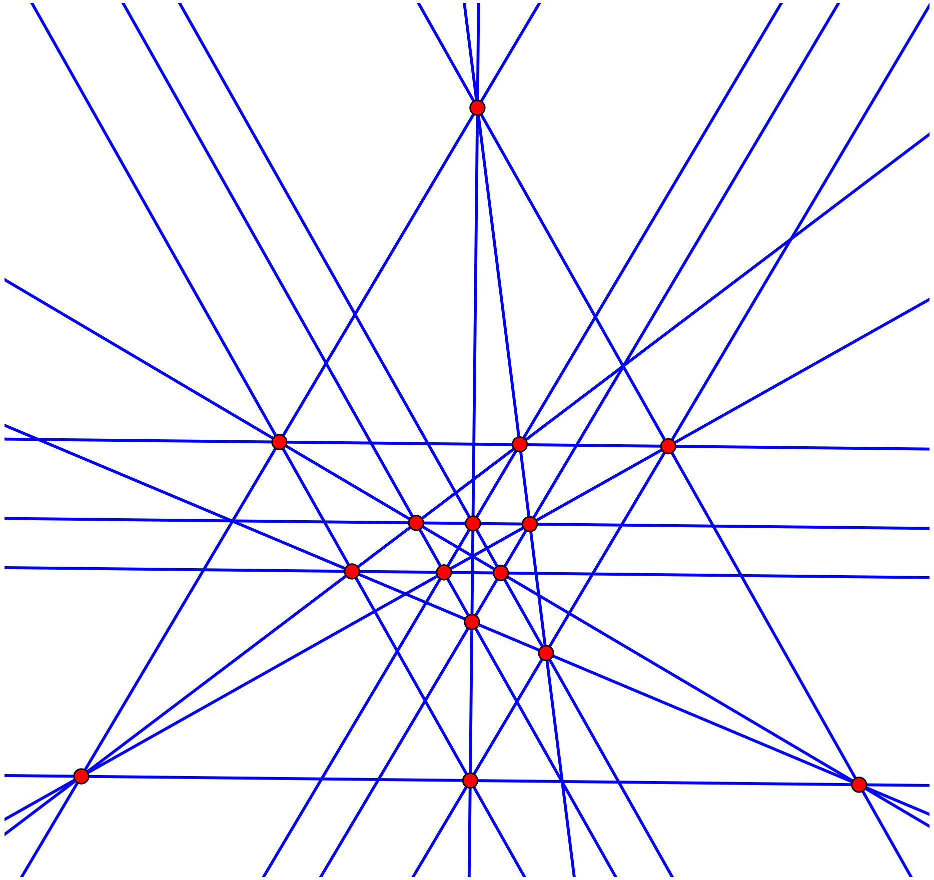


Bokowski & Schewe



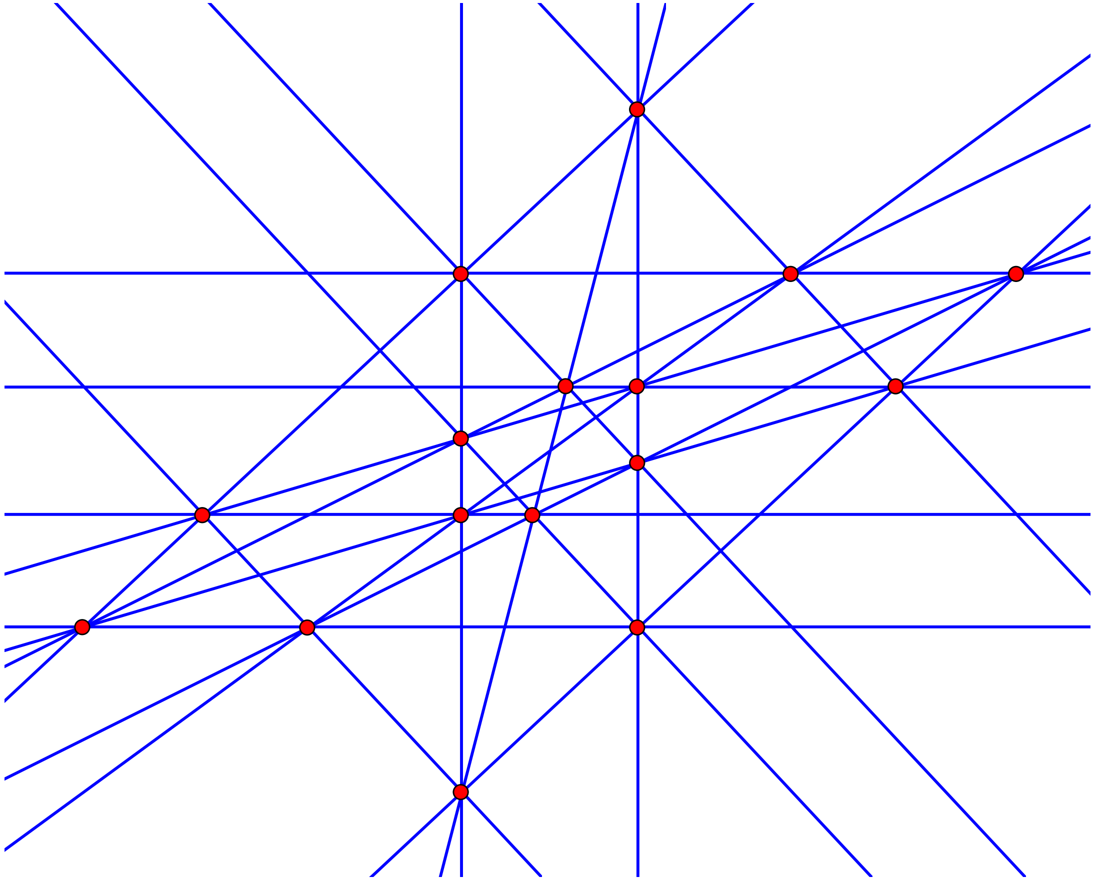
NEW!!

GEOMETRIC (18_4)-CONFIGURATIONS



Bokowski & Schewe

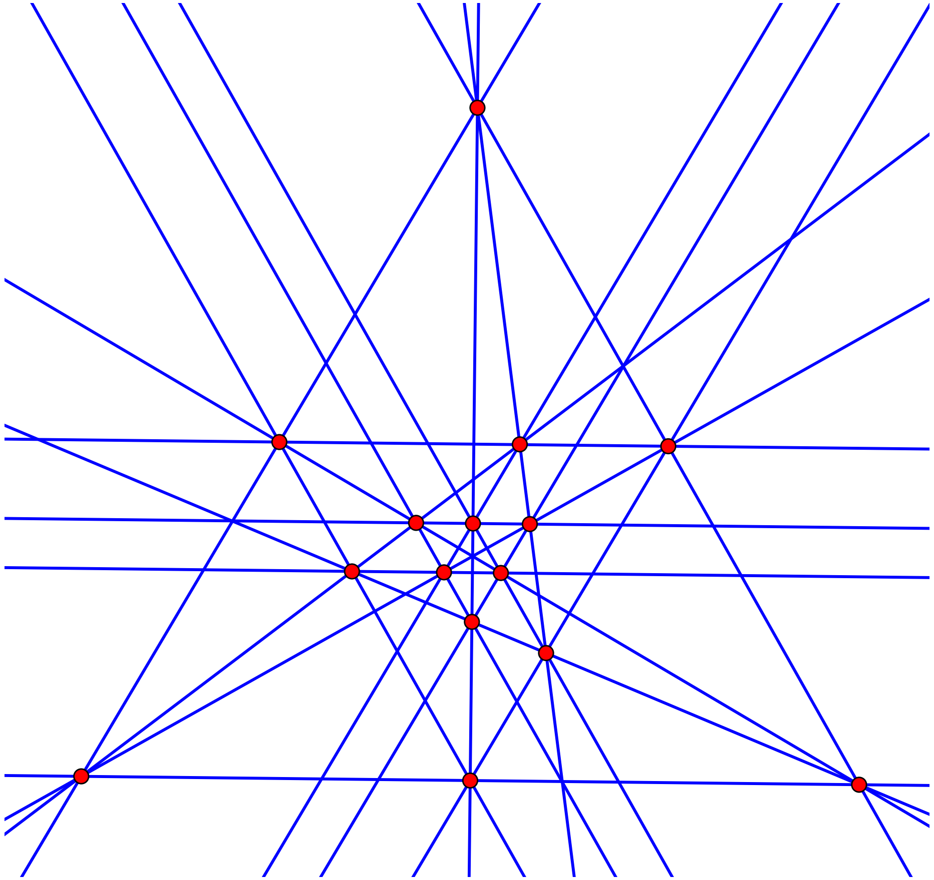
coordinates in $\mathbb{Q}\left[1+\sqrt{5}\right]$

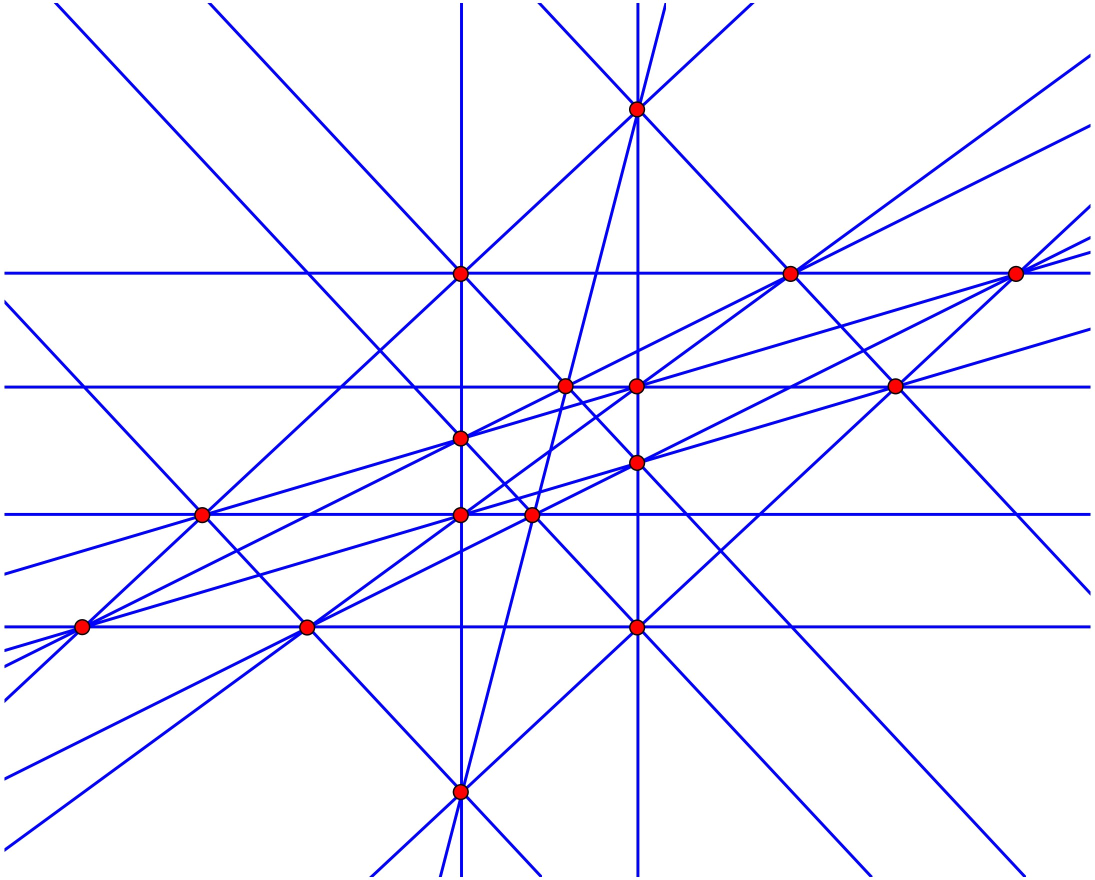


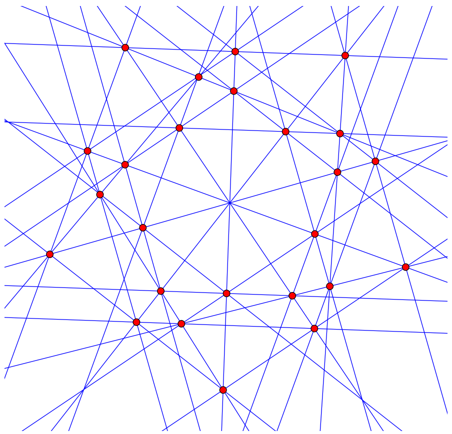
NEW!!

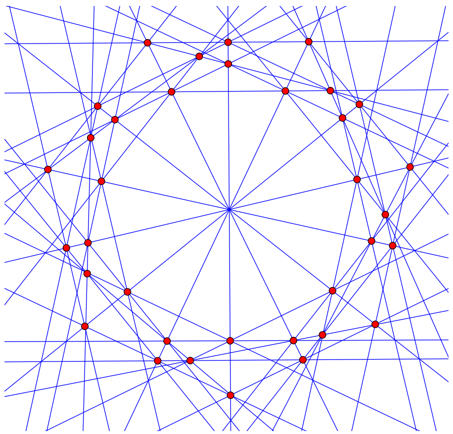
coordinates in $\mathbb{Q}\left[\sqrt[3]{108+12\sqrt{93}}\right]$

GEOMETRIC (18_4)-CONFIGURATIONS









Inspiration for a new general construction?

GEOMETRIC (19_4)-CONFIGURATIONS

There is no geometric (19_4)-configuration.

Based on the following steps:

- Enumeration of $119\,879$ topological (19_4)-configurations. (Java)
- Reduction to $4\,028$ combinatorial equivalence classes. (Haskell)
- 222 configurations are self-dual. For the other pairs, keep only one representative. Obtain $2\,125$ configurations with non-isomorphic Levi graphs. (Haskell)
- Only 512 configurations do not contradict Pappus' Theorem. (Haskell)
- For each configuration, compute an optimal construction sequence and derive a corresponding instance of the Existencial Theory of the Real. (Haskell)
- Check that this instance has no solution. (Maple)

To be confirmed: relies on Maple to solve 512 systems of equalities and inequalities on at most 2 variables with maximum degree 24.

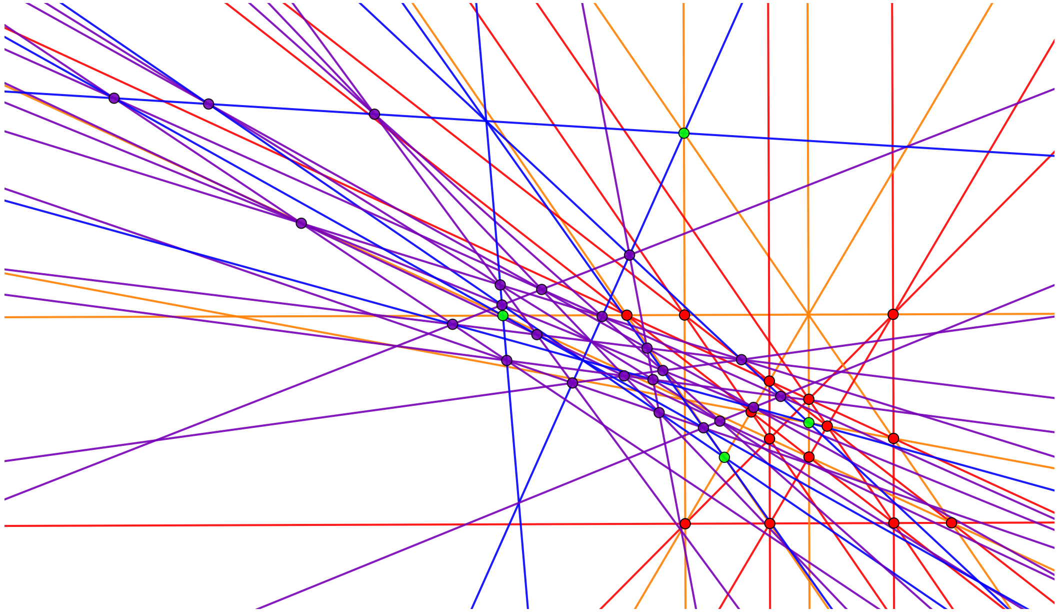
SUBCONFIGURATIONS & QUASI-CONFIGURATIONS

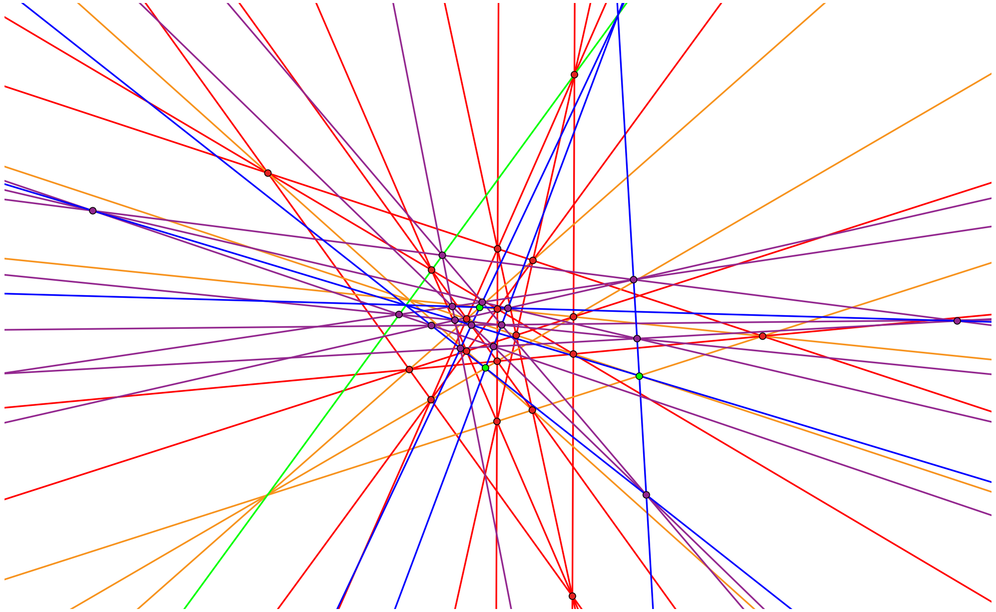
MOTIVATION

We can use smaller point-line configurations to

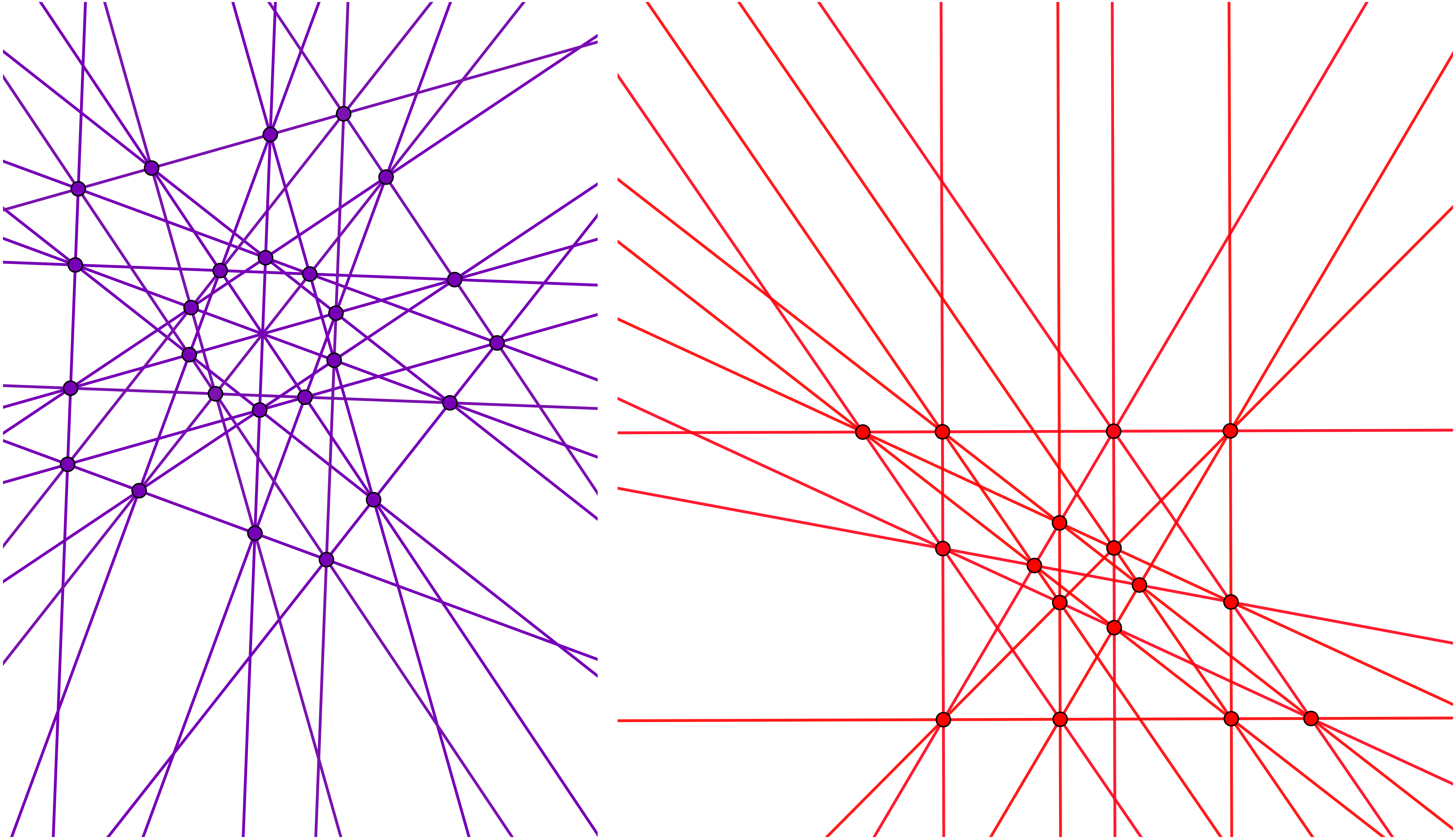
1. prove that a given large configuration is not geometrically realizable
   (example: configurations containing a non-pappus subconfiguration)

2. construct large configurations from small pieces
   (example: Jürgen's recent (37_4)- and (43_4)-configurations)

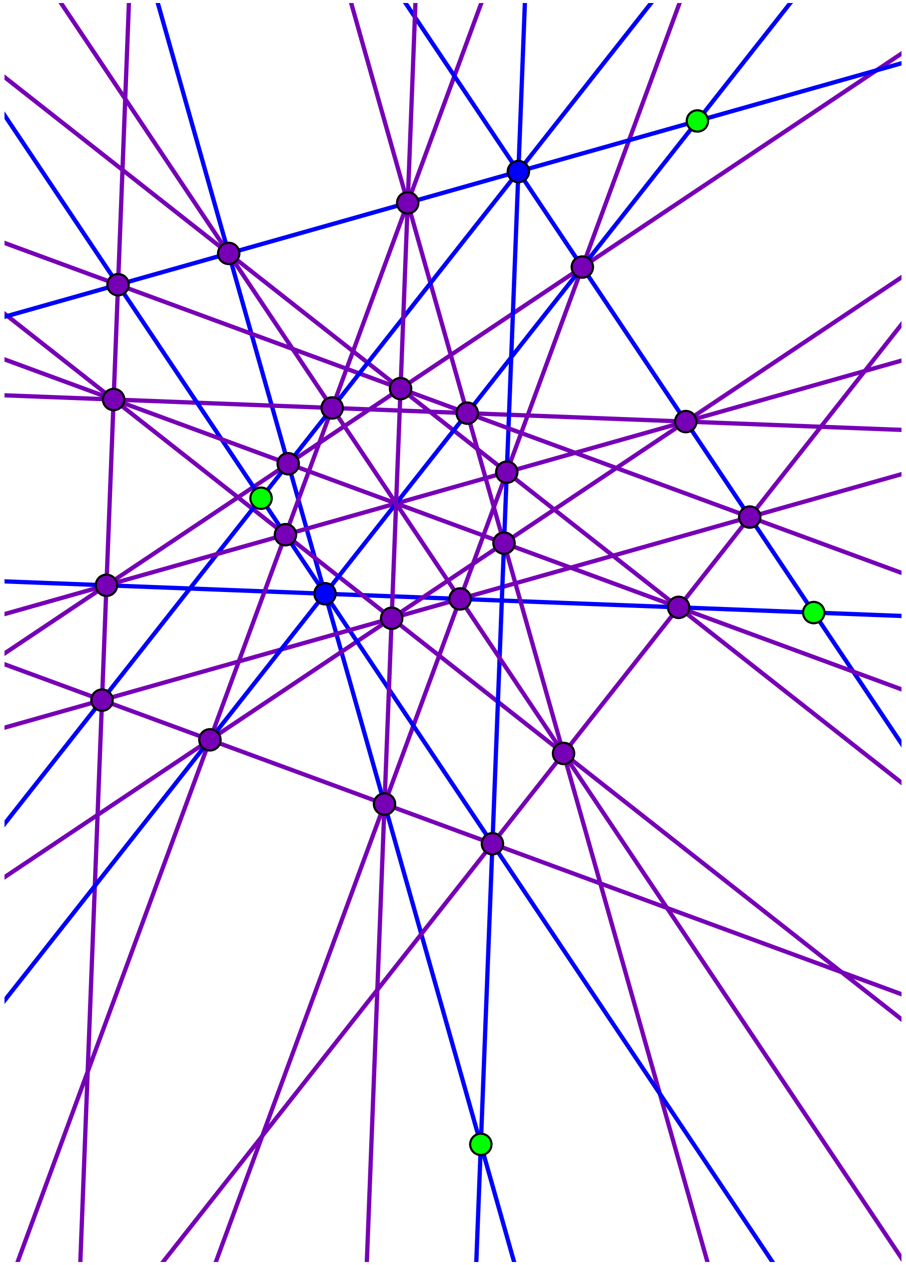


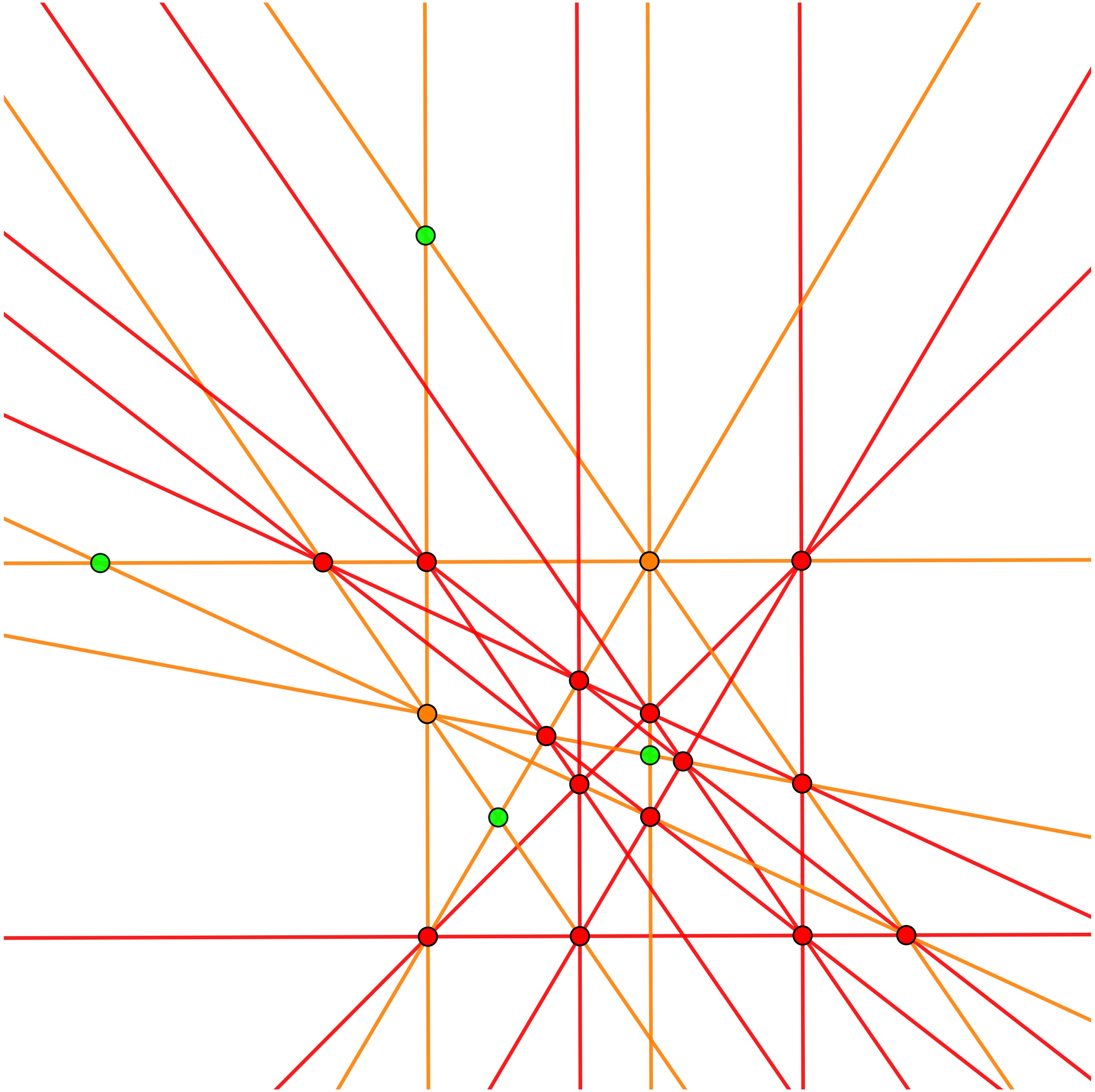


A FIRST (43_4)-CONFIGURATION

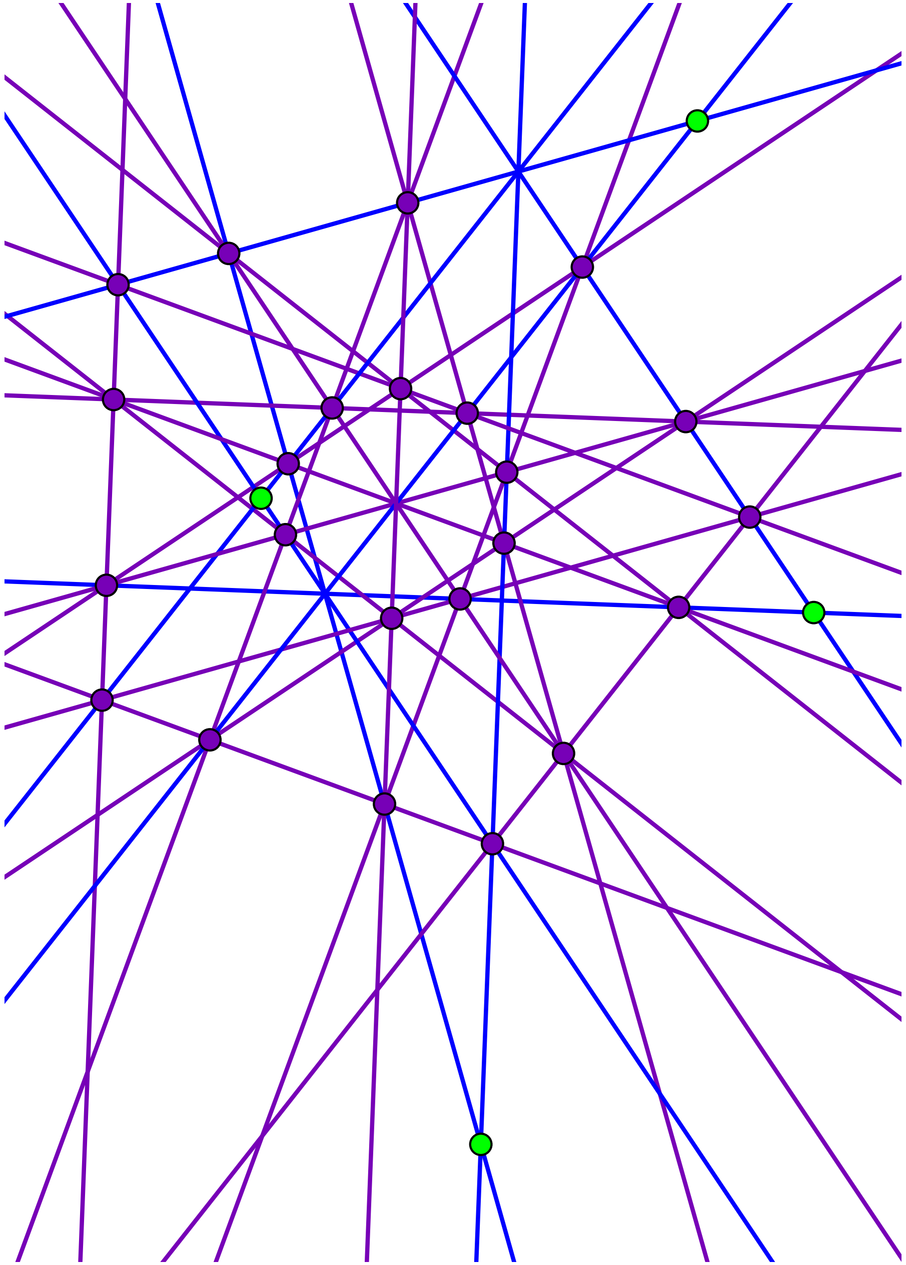


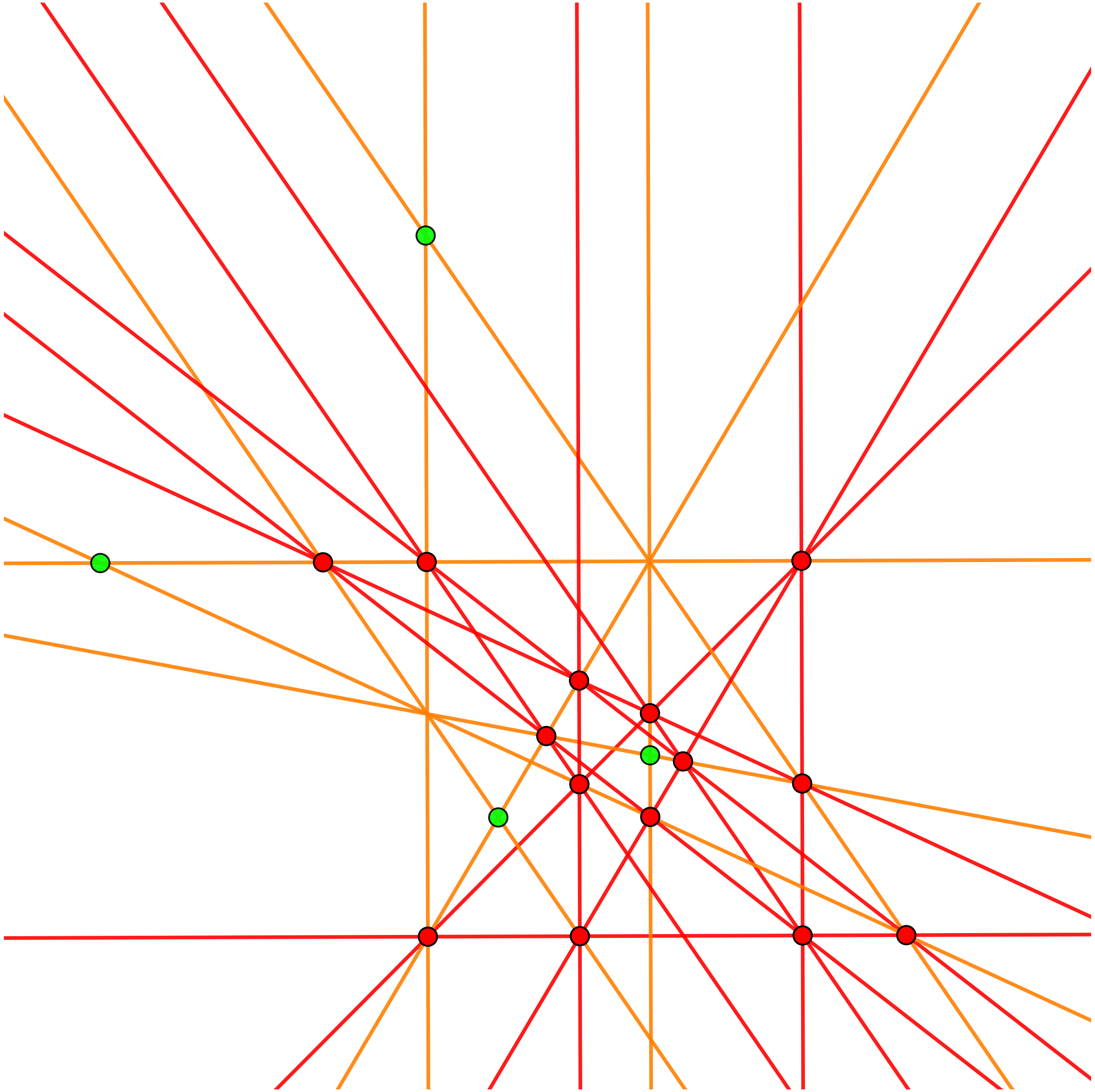
A FIRST (43_4)-CONFIGURATION



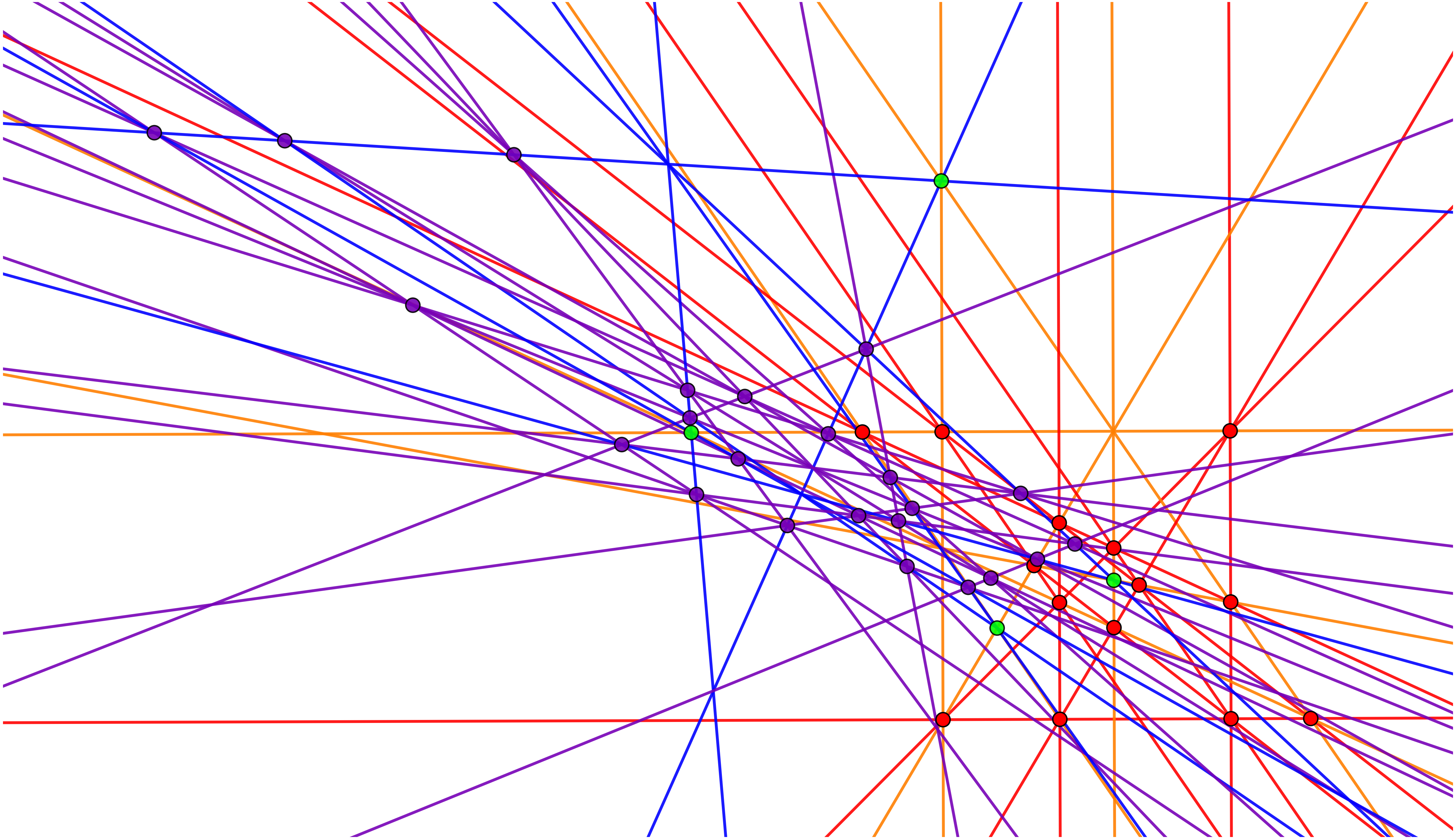


A FIRST (43_4)-CONFIGURATION

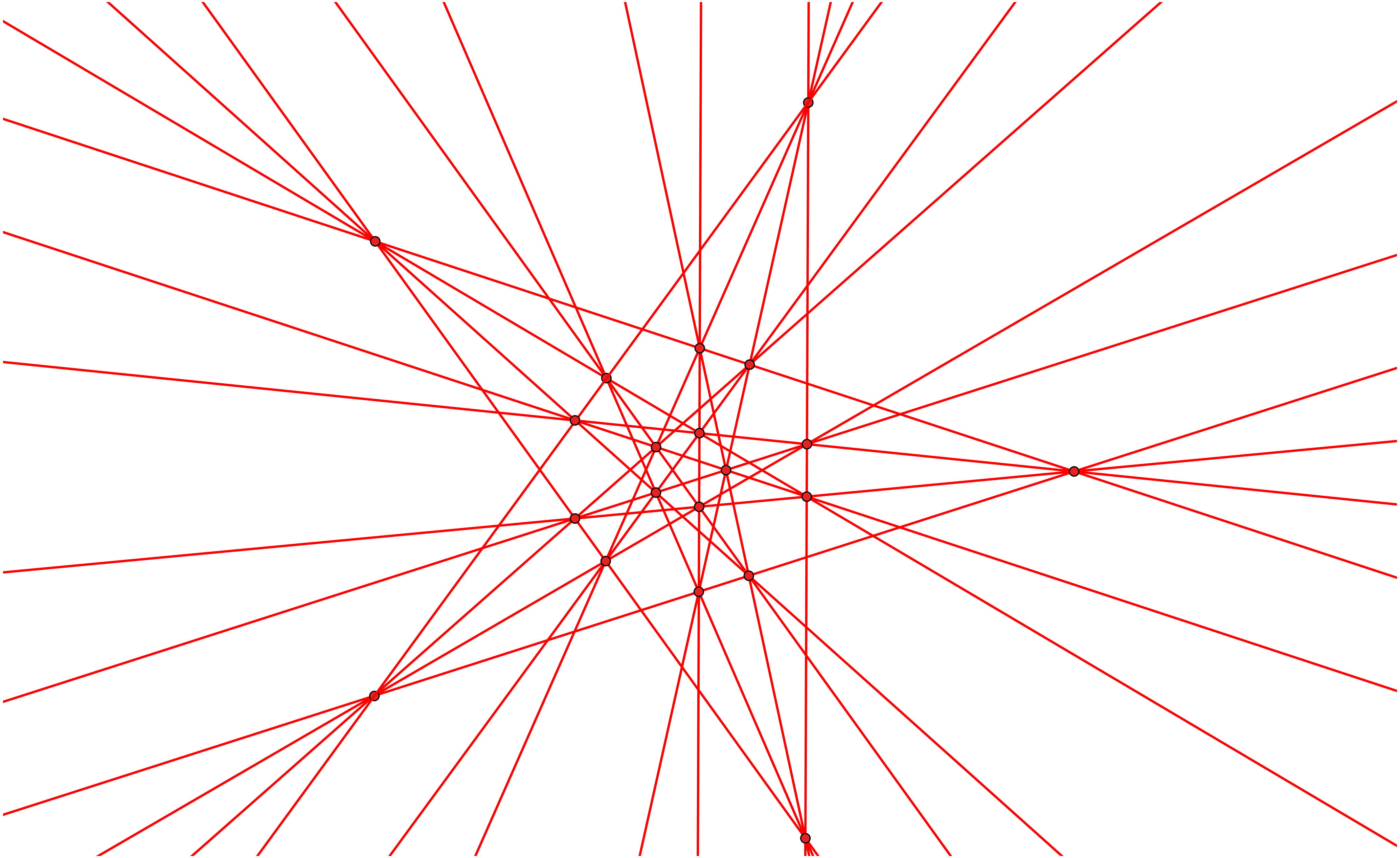




A FIRST (43_4)-CONFIGURATION



A FIRST (37_4)-CONFIGURATION



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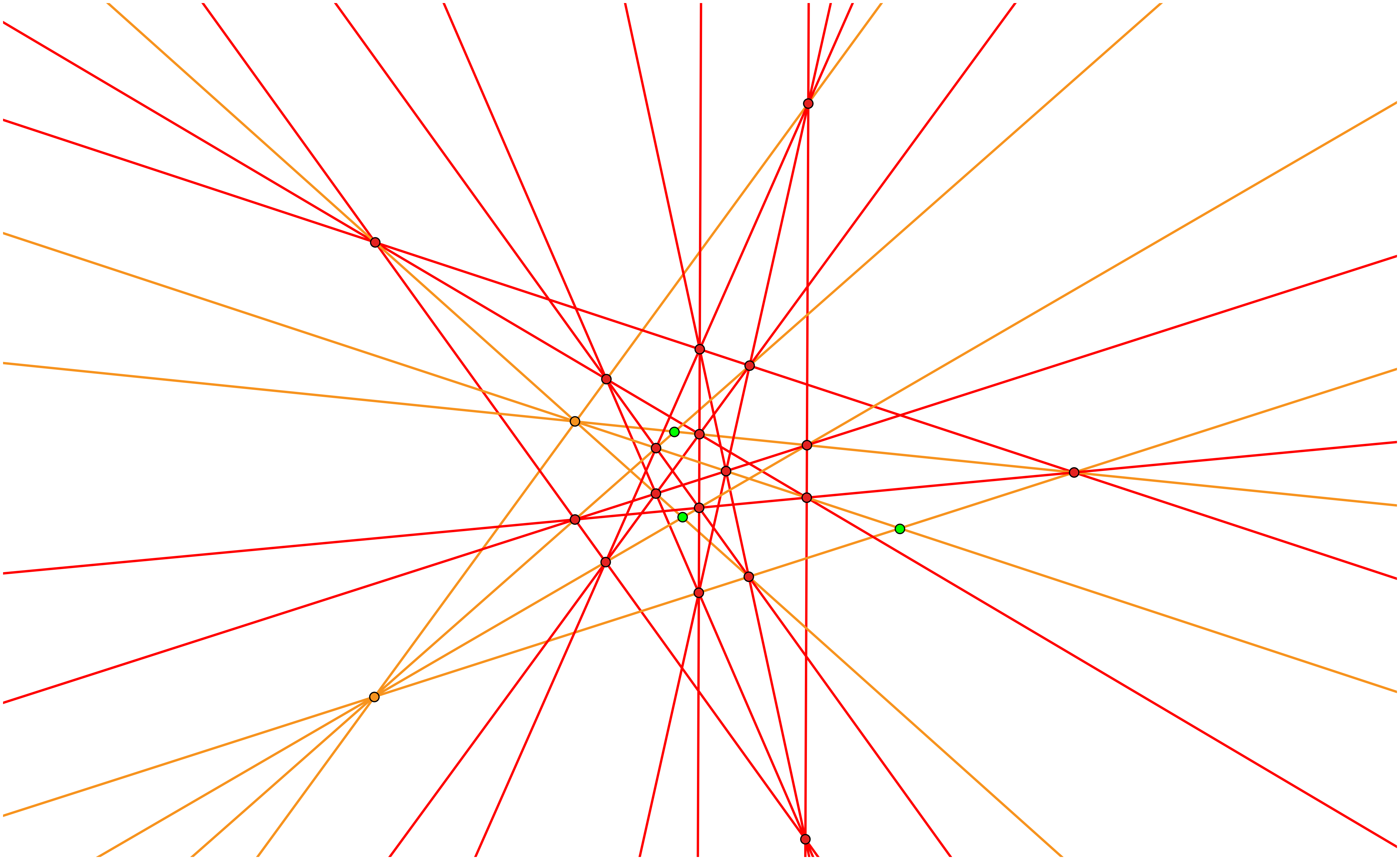

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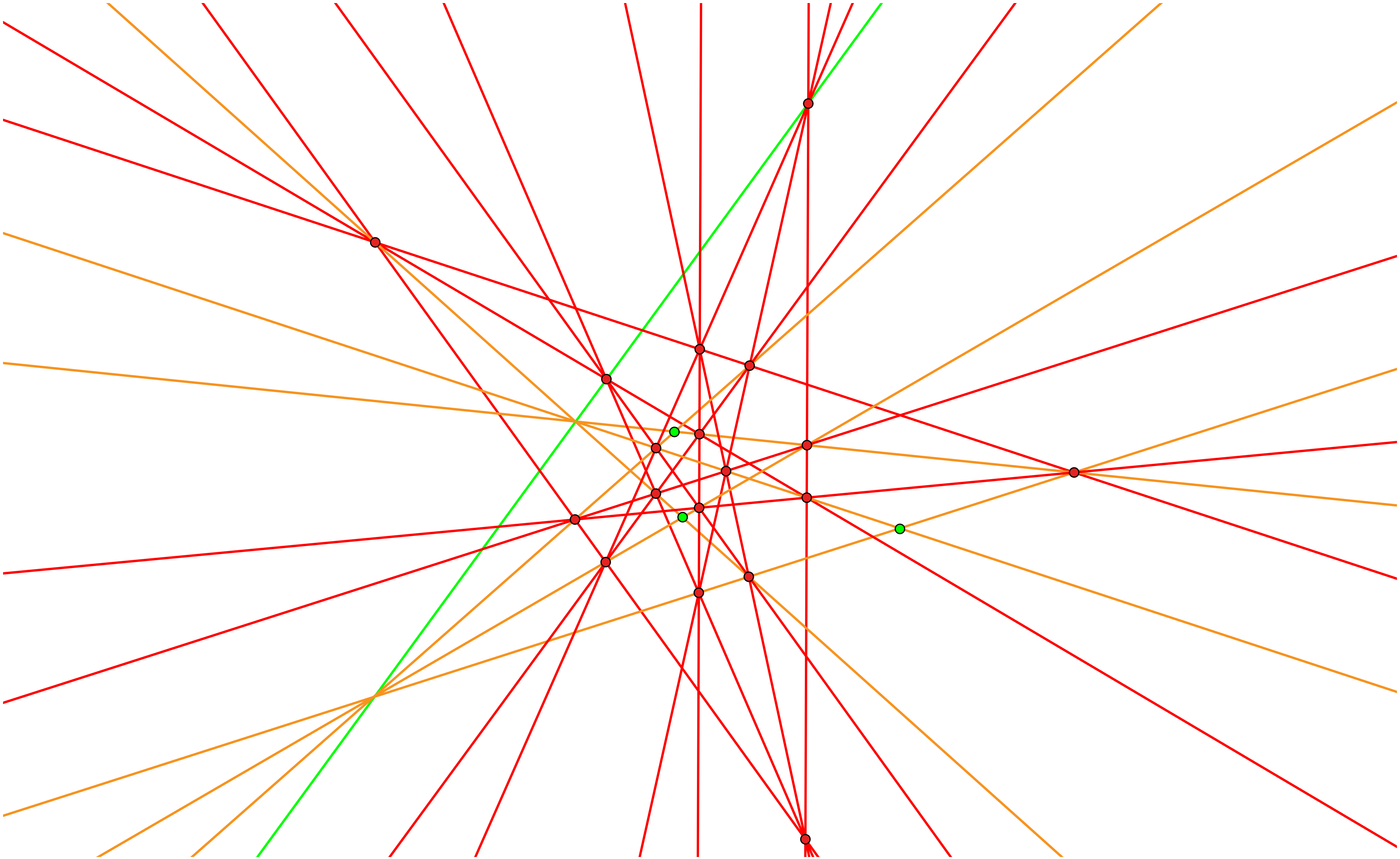
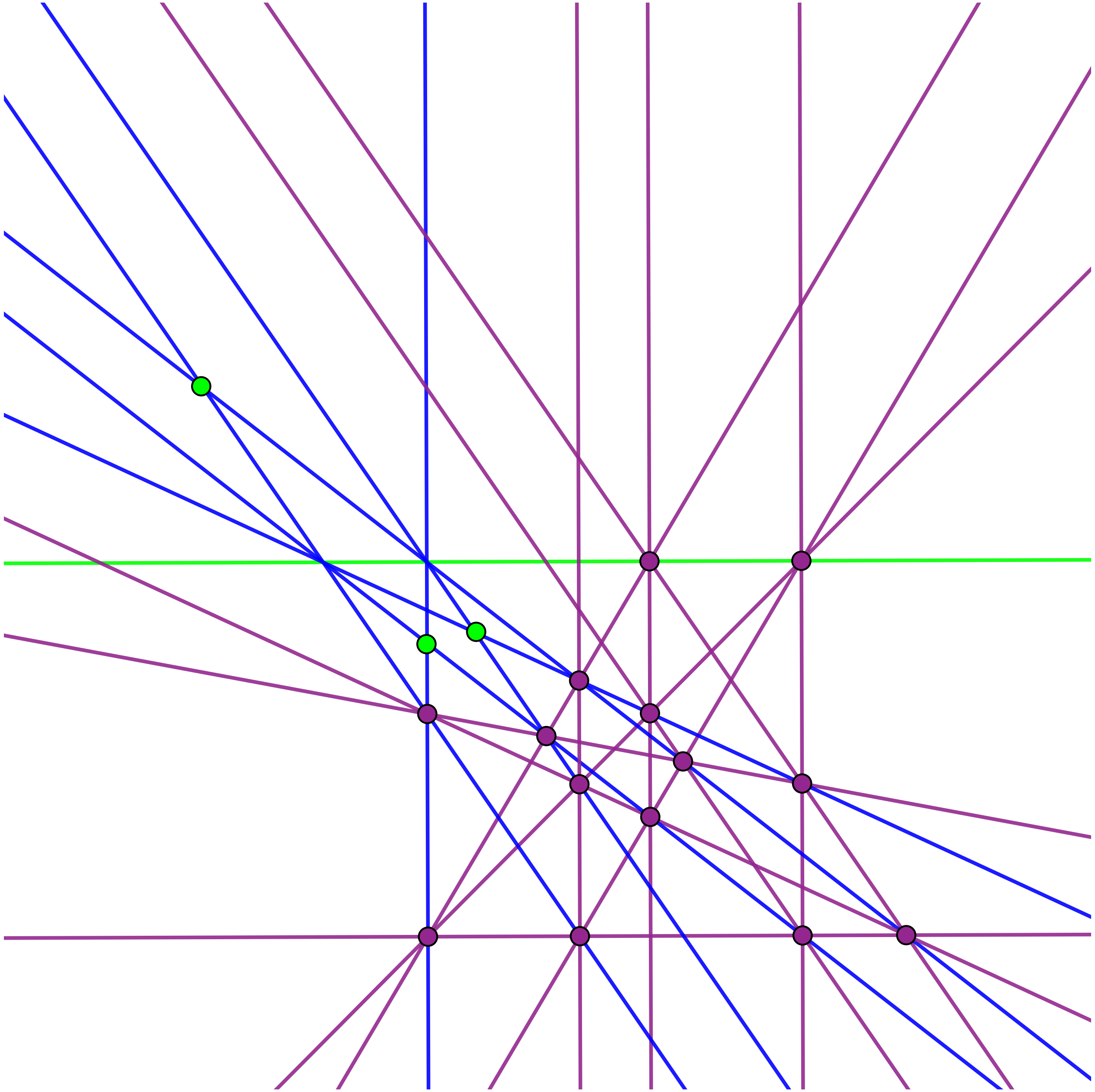
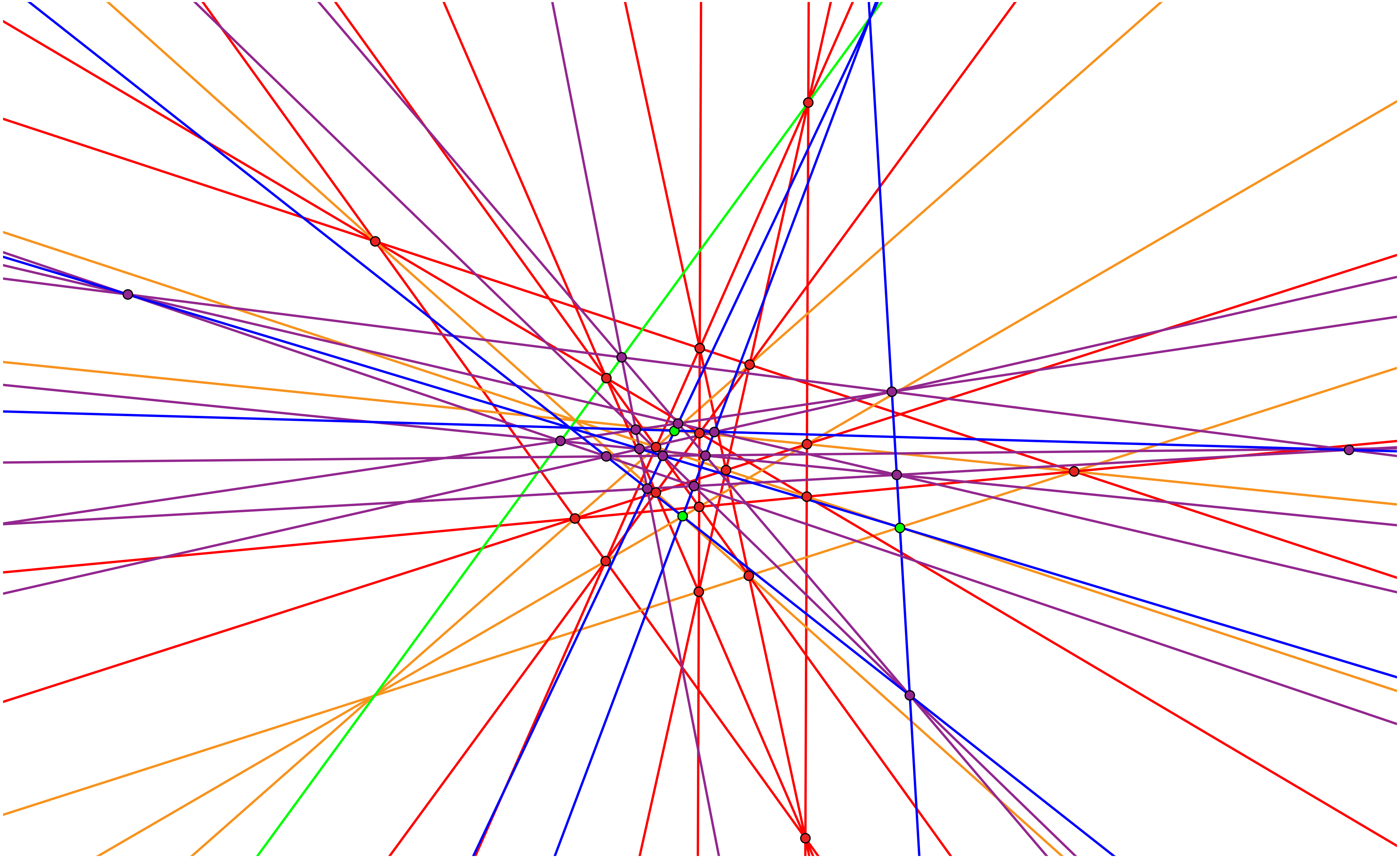


A FIRST (37_4)-CONFIGURATION



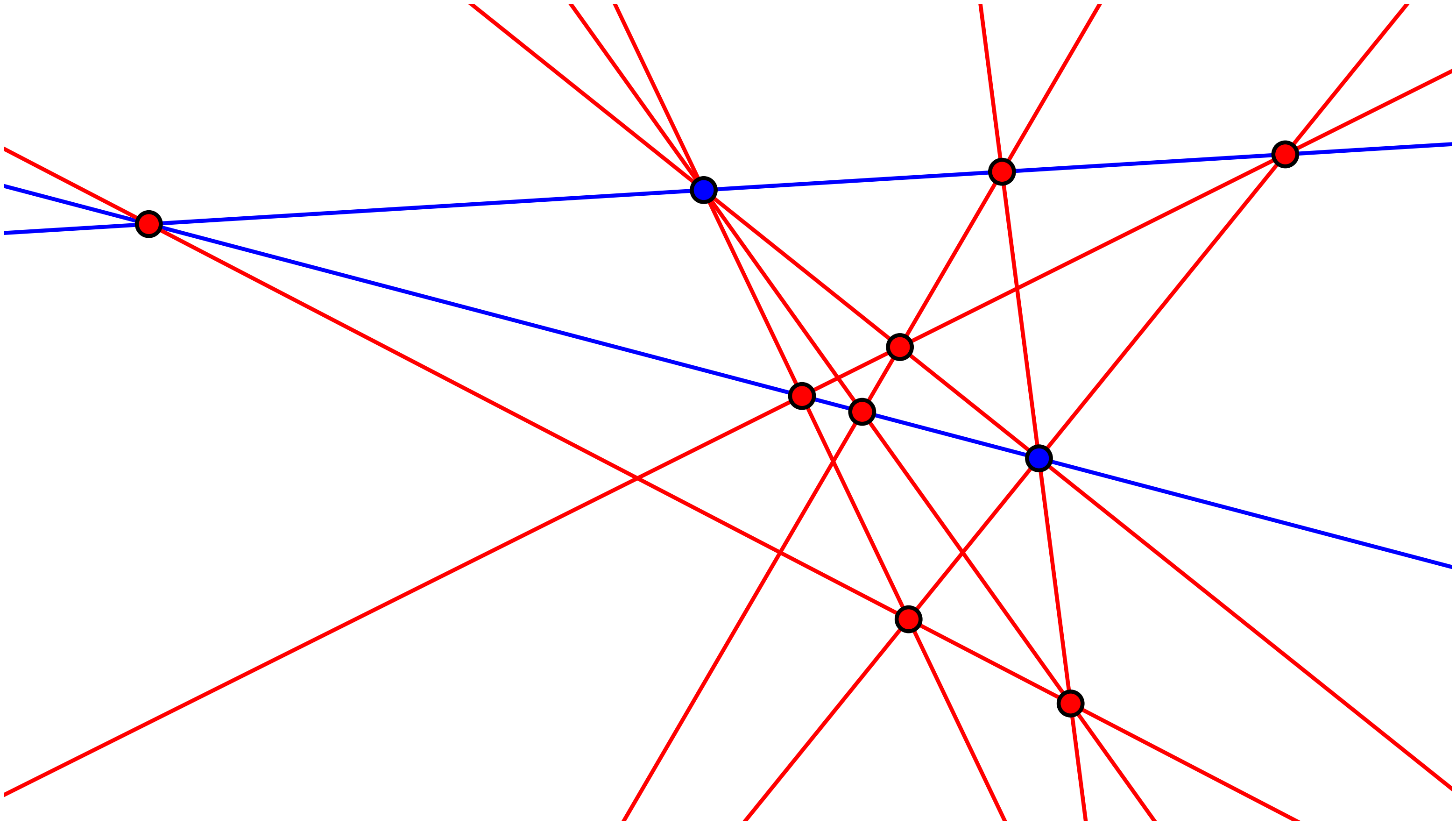
A FIRST (37_4)-CONFIGURATION



QUASI-CONFIGURATION

quasi-configuration = point-line configuration (P, L) where each point of P is contained in at least 3 lines of L and each line of L contains at least 3 points of P

$(n_{3|4})$-configurations = configuration (P, L) with n points and n lines, where each point of P is contained in 3 or 4 lines of L and each line of L contains 3 or 4 points of P



TOPOLOGICAL OBSTRUCTION

(P,L) a point-line configuration with p_i points of P contained in i lines of L
ℓ_j lines of L contained in j points of P

If (P,L) has a topological realization, then

$$0 \geq \sum_i i(i+1)p_i - 6\left(\sum_i p_i - 1\right) - \sum_j \ell_j \left(\sum_j \ell_j - 1\right)$$

Example 1. $p_4 = n$, $\ell_4 = n$ and $p_i = \ell_i = 0$ for all other values of i
inequality gives $0 \geq -n^2 + 15n + 6$ and thus $n \geq 16$

Bokowski & Schewe. There are no realizable 15_4- and 16_4-configurations. 2005

TOPOLOGICAL OBSTRUCTION

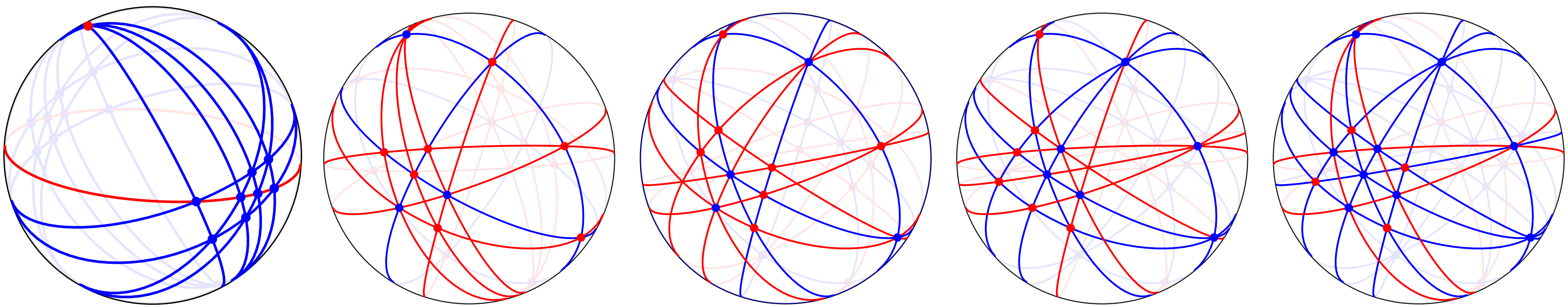
(P, L) a point-line configuration with p_i points of P contained in i lines of L
ℓ_j lines of L contained in j points of P

If (P, L) has a topological realization, then

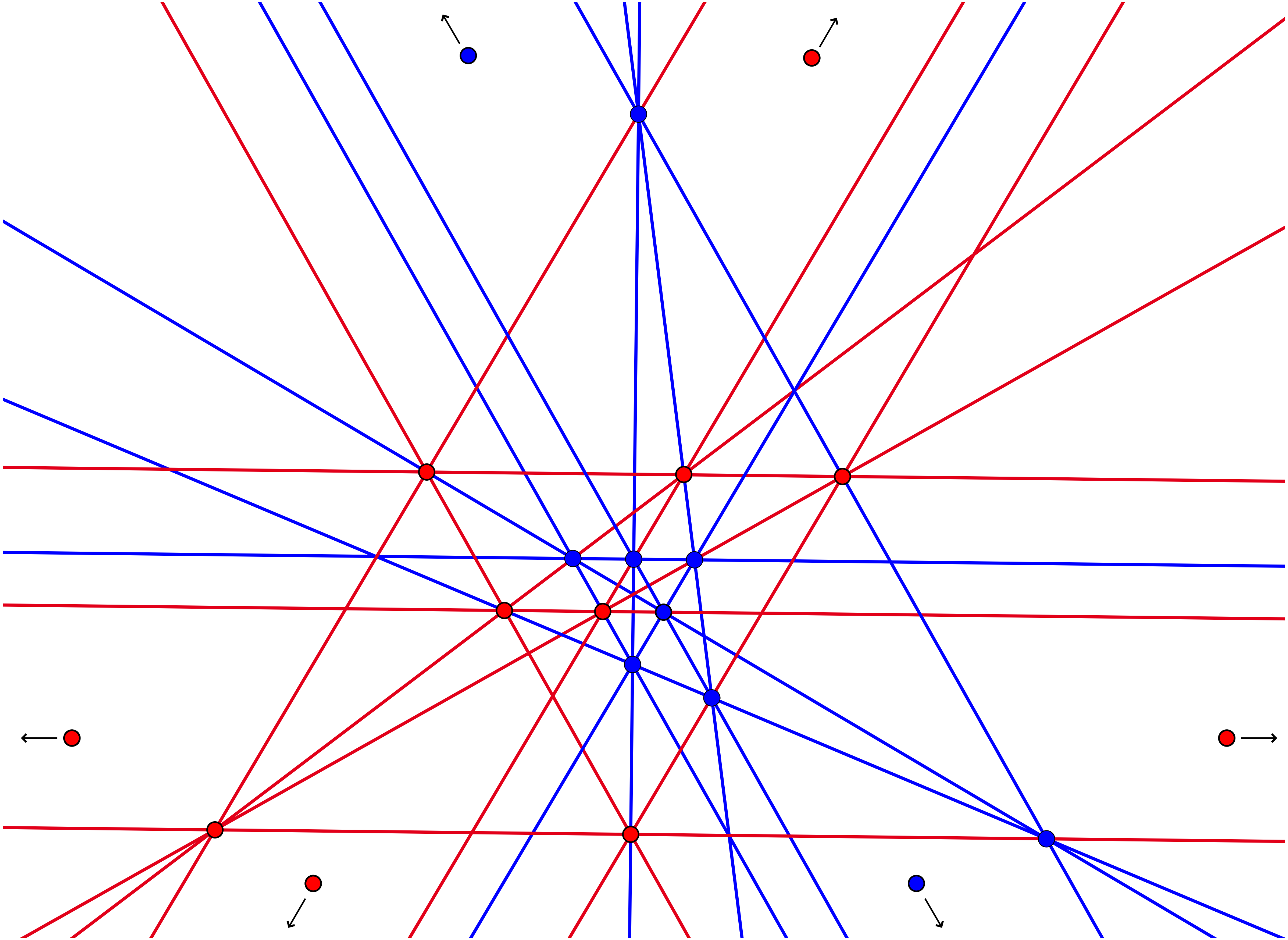
$$0 \geq \sum_i i(i+1)p_i - 6\left(\sum_i p_i - 1\right) - \sum_j \ell_j \left(\sum_j \ell_j - 1\right)$$

Example 2. the number of incidences of an $(n_{3|4})$-configuration is bounded by

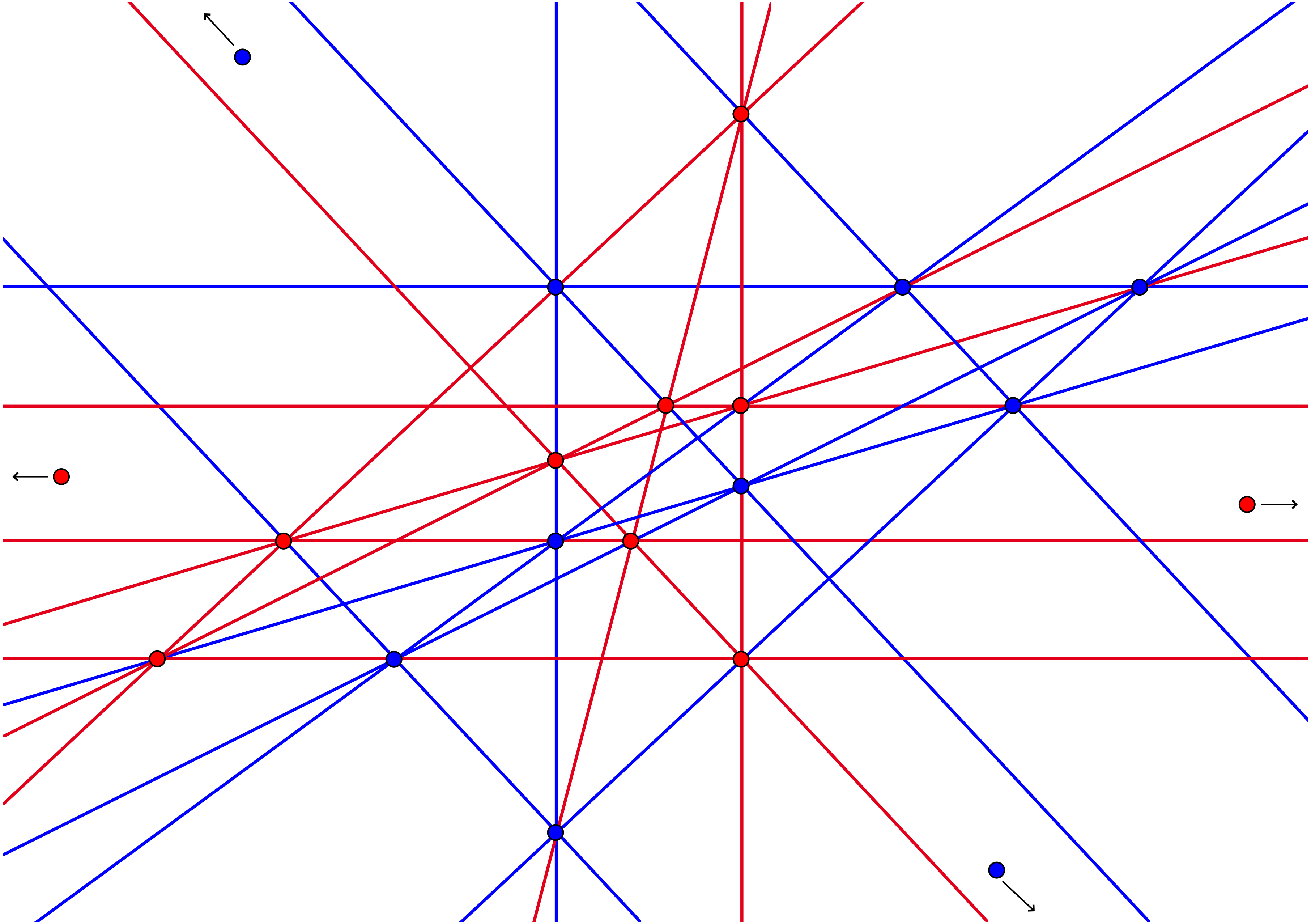
n	7	8	9	10	11	12	13	14	15	16
$\left\lfloor \min\left(4n, \frac{n^2+17n-6}{8}\right) \right\rfloor$	20	24	28	33	37	42	48	53	59	64

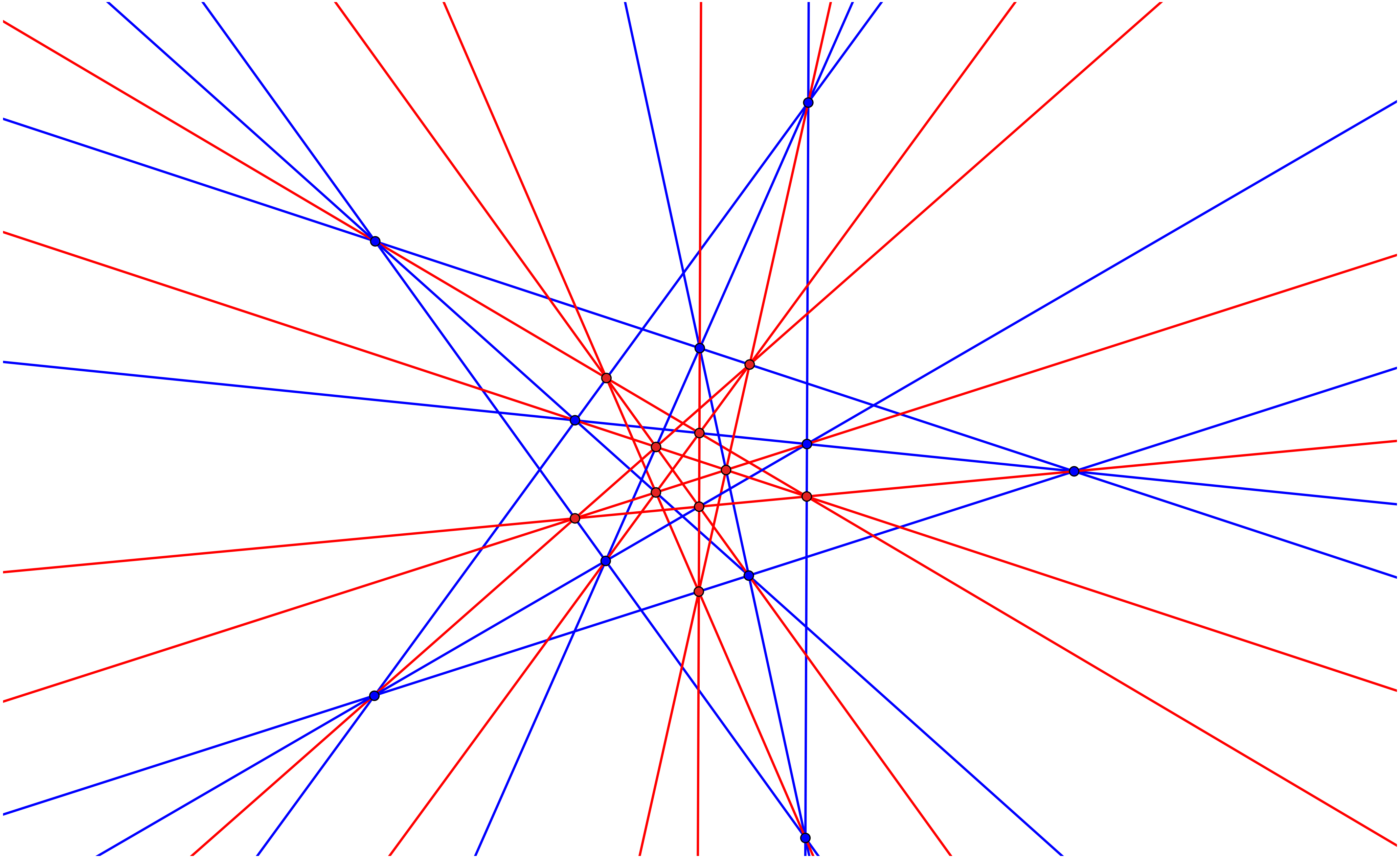


SPLITTING CONFIGURATIONS



SPLITTING CONFIGURATIONS





MANY RESEARCH DIRECTIONS

Enumerate and classify small quasi-configurations

For example, what are the optimal $(14_{3|4})$-, $(15_{3|4})$- and $(16_{3|4})$-configurations?

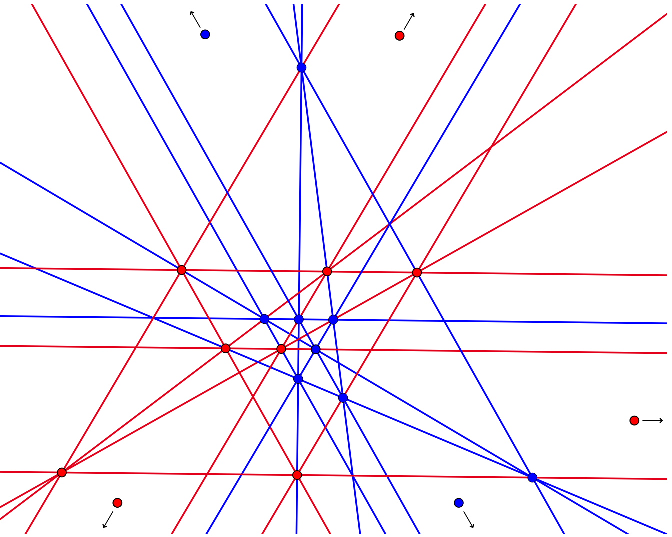
Create large configurations from small quasi-configurations

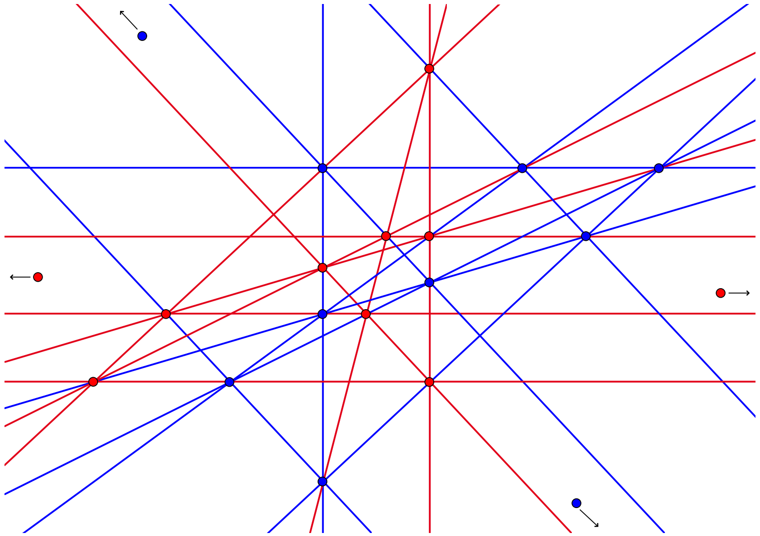
For example, can we create (22_4)-, (23_4)-, or (26_4)-configurations from $(11_{3|4})$-, $(12_{3|4})$-, and $(13_{3|4})$-configurations?

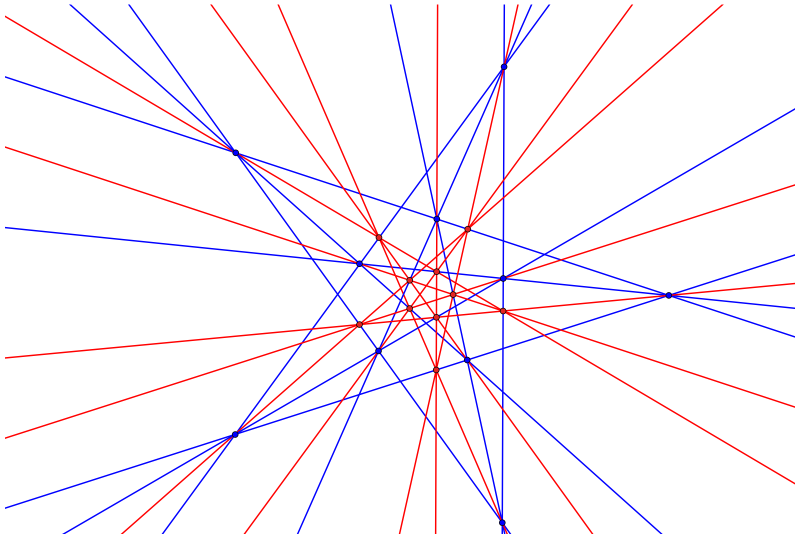
Study splittings of configurations

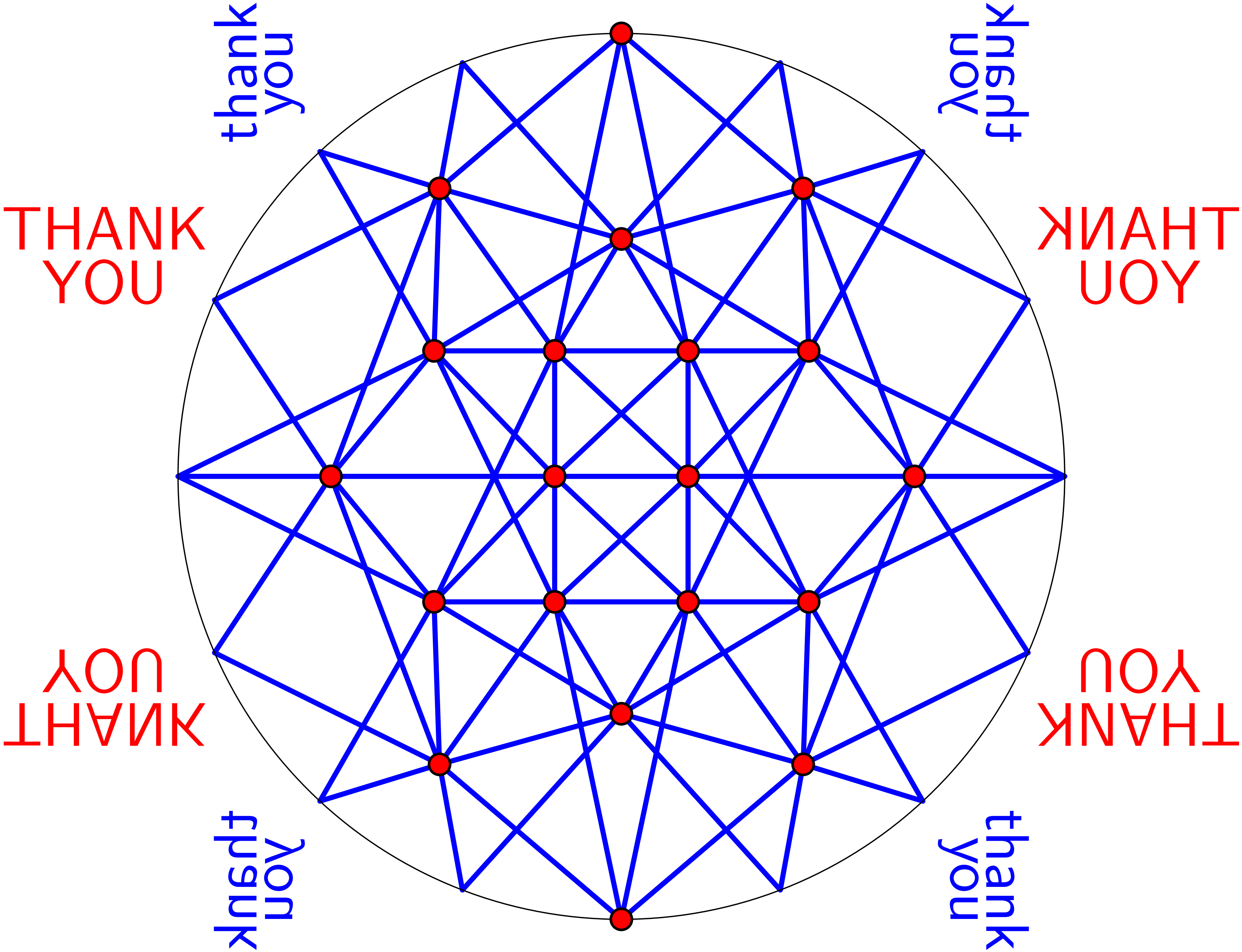
Are there arbitrary large unsplittable (n_4)-configurations?

What is the smallest unsplittable configuration?








THANK YOU
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