THE QUEST FOR GEOMETRIC CONFIGURATIONS
METHODS, LIMITS & BY-PRODUCTS

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POINT-LINE CONFIGURATIONS
\((n_k)\)-configuration =

a set \(P\) of \(n\) points and a set \(L\) of \(n\) lines

with a point–line incidence relation st

each point is contained in \(k\) lines

and each line contains \(k\) points

Grünbaum. Configurations of points and lines. 2009
GEOMETRIC CONFIGURATIONS

\[ \mathbb{P} = \text{projective plane} \]
\[ = \text{space of vectorial lines of } \mathbb{R}^3 \]
\[ = \text{unit 2-sphere with antipodal points identified} \]
\[ = \text{unit disk with antipodal boundary points identified} \]

Geometric configuration = points and lines are ordinary points and lines in \( \mathbb{P} \)

Projective equivalence = equivalence under projective transformations
Pseudoline = non-separating simple closed curve in $\mathbb{P}$

Topological configuration = points are ordinary points in $\mathbb{P}$ and lines are pseudolines in $\mathbb{P}$

Topological equivalence = equivalence under homeomorphisms of $\mathbb{P}$
Combinatorial configuration = k-regular bipartite graph with girth at least 6 (Levi graph)

Combinatorial equivalence = automorphism of the Levi graph which sends points to points and lines to lines

Combinatorial duality = automorphism of the Levi graph which exchanges points and lines
Three different levels of configurations:

**Combinatorial configuration**
- just an abstract incidence structure
- combinatorial equivalence

**Topological configuration**
- ordinary points in $\mathbb{P}$
- pseudolines of $\mathbb{P}$
- topological equivalence
- mutation equivalence

**Geometric configuration**
- ordinary points in $\mathbb{P}$
- ordinary lines in $\mathbb{P}$
- projective equivalence
Two research directions on \((n_k)\)-configurations:

1. For a given \(k\), determine for which values of \(n\) do geometric, topological and combinatorial \((n_k)\)-configurations exist.

2. Enumerate and classify \((n_k)\)-configurations for given \(k\) and \(n\).
### EXISTENCE & ENUMERATION OF \((n_k)\)-CONFIGURATIONS

Two research directions on \((n_k)\)-configurations:

1. For a given \(k\), determine for which values of \(n\) do geometric, topological and combinatorial \((n_k)\)-configurations exist
2. Enumerate and classify \((n_k)\)-configurations for given \(k\) and \(n\)

### EXISTENCE OF \((n_k)\)-CONFIGURATIONS

<table>
<thead>
<tr>
<th>(k)</th>
<th>Combinatorial conf.</th>
<th>Topological conf.</th>
<th>Geometric configurations</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>exist iff (n \geq 7)</td>
<td>exist iff (n \geq 9)</td>
<td>exist iff (n \geq 9)</td>
</tr>
<tr>
<td>4</td>
<td>exist iff (n \geq 13)</td>
<td>exist iff (n \geq 17)</td>
<td>exist iff (n \geq 18) with the possible exceptions of (n = 19, 22, 23, 26, 37, 43)</td>
</tr>
</tbody>
</table>


Bokowski & Schewe. On the finite set of missing geometric configurations \((n_4)\). 2011
EXISTENCE & ENUMERATION OF ($n_k$)-CONFIGURATIONS

Two research directions on ($n_k$)-configurations:

1. For a given $k$, determine for which values of $n$ do geometric, topological and combinatorial ($n_k$)-configurations exist

2. Enumerate and classify ($n_k$)-configurations for given $k$ and $n$

Combinatorial ($13_4$)-conf.  Topological ($17_4$)-conf.  Geometric ($18_4$)-conf.

Bokowski, Grünbaum & Schewe  Bokowski & Schewe
EXISTENCE & ENUMERATION OF \( (n_k) \)-CONFIGURATIONS

Two research directions on \( (n_k) \)-configurations:

1. For a given \( k \), determine for which values of \( n \) do geometric, topological and combinatorial \( (n_k) \)-configurations exist.

2. **Enumerate** and classify \( (n_k) \)-configurations for given \( k \) and \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \text{comb}_3(n) )</th>
<th>( \text{topo}_3(n) )</th>
<th>( \text{geom}_3(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \leq 6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>3</td>
<td>3</td>
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<td>10</td>
<td>10</td>
<td>10</td>
<td>9</td>
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<tr>
<td>11</td>
<td>31</td>
<td>31</td>
<td>31</td>
</tr>
<tr>
<td>12</td>
<td>229</td>
<td>229</td>
<td>229</td>
</tr>
<tr>
<td>13</td>
<td>2,036</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>19</td>
<td>7,640,941,062</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \text{comb}_4(n) )</th>
<th>( \text{topo}_4(n) )</th>
<th>( \text{geom}_4(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \leq 12 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>19</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>1,972</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>18</td>
<td>971,191</td>
<td>16</td>
<td>?</td>
</tr>
<tr>
<td>19</td>
<td>269,224,652</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
CONTRIBUTION

APPROACH

1. Generate all topological \((n_k)\)-configurations (up to combinatorial equivalence), without enumerating first combinatorial \((n_k)\)-configurations

2. Study their geometric realizations

RESULTS

1. Confirm and complete former results on \((18_4)\)-configurations
   In particular, discover a new geometric \((18_4)\)-configuration

2. Enumeration of the 4028 topological \((19_4)\)-configurations, 222 of which are self-dual

3. First examples of topological \((19_4)\)-configurations with a non-trivial symmetry group

4. There is no geometric \((19_4)\)-configuration (to be confirmed!)

5. Study sub-configurations and quasi-configurations
   In particular, obtain the first \((37_4)\)- and \((43_4)\)-configurations
TOPOLOGICAL CONFIGURATIONS
**Sweeping algorithm** to generate all topological \((n_k)\)-configurations for fixed \(k\) and \(n\)

- No need to enumerate all combinatorial \((n_k)\)-configurations
- Focus on mutation equivalence classes of topological configurations
- Requires to reduce the output up to combinatorial equivalence (multiscale invariant technique)
Sweeping algorithm to generate all topological \((n_k)\)-configurations for fixed \(k\) and \(n\).
**MUTATION EQUIVALENCE**

**Mutation** = local transformation where only one pseudoline moves, sweeping a single vertex of the remaining arrangement

**Admissible mutation** = a mutation where all perturbed crossings are not in \( P \)

**Mutation equivalent configurations** = configurations in the same connected component of the admissible mutations

We enumerate at least one representative in each mutation equivalence class
We enumerate at least one representative in each mutation equivalence class. It enables us to assume that sweep events are of two kinds:
CLIQUE AND COCLIQUE DISTRIBUTIONS

(P, L) a combinatorial point-line configuration

\textit{j-clique} of (P, L) = set of j points of P pairwise related by lines of L
For \( p \in P \), define \( \gamma(p) = \# \{ \text{j-clique of (P, L) containing } p \} \) \( j \geq 3 \)
clique distribution of (P, L) = \( \gamma(P) = \{ \gamma(p) \mid p \in P \} \)

\textit{j-coclique} of (P, L) = set of j lines of L pairwise crossing at points of P
For \( ` \in L \), define \( \delta(`) = \# \{ \text{j-coclique of (P, L) containing } ` \} \) \( j \geq 3 \)
coclique distribution of (P, L) = \( \delta(L) = \{ \delta(`) \mid ` \in L \} \)
Clique and coclique distributions are \textit{combinatorial invariants}

Two different use:

1. either \textit{separate isomorphism classes} of combinatorial configurations
   (two configurations with different invariants cannot be combinatorially equivalent)

2. or \textit{guess combinatorial isomorphisms}
   (any isomorphism between two configurations respects the combinatorial invariants)
DERIVATION OF INVARIANTS

\( \gamma : P \to X \) such that \( \gamma(P) = \{ \gamma(p) \mid p \in P \} \) are combinatorial invariants of \((P, L)\).

\( \delta : L \to Y \) such that \( \delta(L) = \{ \delta(`) \mid ` \in L \} \) are combinatorial invariants of \((P, L)\).

\( \gamma \) derivative of \(\gamma\) is the function \(\gamma' : L \to X^k\) defined by \(\gamma'(`) = \{ \gamma(p) \mid p \in P, p \in ` \}\).

\( \delta \) derivative of \(\delta\) is the function \(\delta' : P \to Y^k\) defined by \(\delta'(p) = \{ \delta(`) \mid ` \in L, p \in ` \}\).

Then \(\delta(P)\) and \(\gamma'(L)\) are still combinatorial invariants of \((P, L)\).

They refine the initial invariants \(\gamma(P)\) and \(\delta(L)\).
MULTISCALE TECHNIQUE

$C$ a set of combinatorial configurations to be reduced up to combinatorial equivalence

$$\gamma : P \to X \quad \text{such that} \quad \gamma(P) = \{\gamma(p) \mid p \in P\}$$

$$\delta : L \to Y \quad \delta(L) = \{\delta(`) \mid ` \in L\}$$

are combinatorial invariants of $(P, L)$

Separate the configurations of $C$ into different classes according to $(\gamma(P), \delta(L))$

Compute the derivative invariants $\delta'(P)$ and $\gamma'(L)$

In each class, we have three possibilities:

- $\delta'(P)$ and $\gamma'(L)$ are not constant
  $\implies$ refine into subclasses according to $(\delta'(P), \gamma'(L))$ and reiterate the refinement

- $\delta'(P)$ and $\gamma'(L)$ constant, but provide more information about possible isomorphisms
  $\implies$ reiterate the refinement

- Otherwise, $\delta'(P)$ and $\gamma'(L)$, as well as their further derivatives, provide precisely the same information about possible isomorphisms
  $\implies$ start a brute-force search for possible isomorphisms
Confirmation: 16 topological $(18_4)$-configurations up to combinatorial equivalence

About 1 hour for the enumeration process (compared to several months of CPU time with previous methods)

New result: 4028 topological $(19_4)$-configurations up to combinatorial equivalence, 222 of which are self-dual

The automorphism groups of the Levi graphs of these $(19_4)$-configurations are:

<table>
<thead>
<tr>
<th>group $G$</th>
<th>1</th>
<th>$\mathbb{Z}_2$</th>
<th>$\mathbb{Z}_2 \times \mathbb{Z}_2$</th>
<th>$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$</th>
<th>$D_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td># of configurations $(P, L)$ with $\text{Aut}(\mathcal{L}G(P, L)) \cong G$</td>
<td>3726</td>
<td>283</td>
<td>14</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
Symmetry group $\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$:

- horizontal reflection
- vertical reflection
- self-polarity $(a,A)(b,B)\ldots(s,S)$
GEOMETRIC CONFIGURATIONS
CONSTRUCTION SEQUENCES

**INPUT:** A combinatorial configuration \((P, L)\)

**OUTPUT:** A system of polynomial equalities and inequalities with a solution iff \((P, L)\) is geometrically realizable

Choose a projective base \(\{p, q, r, s\}\) in \((P, L)\) (meaning 4 points, no 3 on a line)

Initialize the set of already constructed points \(\Pi \leftarrow \{u_p, u_q, u_r, u_s\}\) and lines \(\Lambda \leftarrow \emptyset\)

the set of equalities \(\mathbb{E} \leftarrow \emptyset\) and inequalities \(\mathbb{I} \leftarrow \emptyset\)

Repeat

- for each non constructed line \(\` \in L \setminus \Lambda\),
  - if we have already constructed at least two points \(p, q\) contained in \(\`\), then
    \[
    \Lambda \leftarrow \Lambda \cup \{u \cdot = u_p \wedge u_q\} \quad \mathbb{E} \leftarrow \mathbb{E} \cup \{u_r \cdot u \cdot = 0 \mid r \in \`\} \quad \mathbb{I} \leftarrow \mathbb{I} \cup \{u_r \cdot u \cdot \neq 0 \mid r \notin \`\}
    \]
  - if no new line can be added this way, then choose one arbitrary non constructed line \(\` \in L \setminus \Lambda\), and set
    \[
    \Lambda \leftarrow \Lambda \cup \{u \cdot = [x, y, z]\} \quad \mathbb{E} \leftarrow \mathbb{E} \cup \{u_r \cdot u \cdot = 0 \mid r \in \`\} \quad \mathbb{I} \leftarrow \mathbb{I} \cup \{u_r \cdot u \cdot \neq 0 \mid r \notin \`\}
    \]
- dualize to go to the next step

until all points and lines are constructed
GEOMETRIC (184)-CONFIGURATIONS

Bokowski & Schewe
coordinates in $\mathbb{Q}[1 + \sqrt{5}]$

NEW!!
coordinates in $\mathbb{Q}\left[\sqrt[3]{108 + 12\sqrt{93}}\right]$
Inspiration for a new general construction?
GEOMETRIC $(19_4)$-CONFIGURATIONS

There is no geometric $(19_4)$-configuration.

Based on the following steps:

- Enumeration of 119 879 topological $(19_4)$-configurations. (Java)
- Reduction to 4 028 combinatorial equivalence classes. (Haskell)
- 222 configurations are self-dual. For the other pairs, keep only one representative. Obtain 2 125 configurations with non-isomorphic Levi graphs. (Haskell)
- Only 512 configurations do not contradict Pappus’ Theorem. (Haskell)
- For each configuration, compute an optimal construction sequence and derive a corresponding instance of the Existencial Theory of the Real. (Haskell)
- Check that this instance has no solution. (Maple)

To be confirmed: relies on Maple to solve 512 systems of equalities and inequalities on at most 2 variables with maximum degree 24.
SUBCONFIGURATIONS & QUASI-CONFIGURATIONS
MOTIVATION

We can use smaller point-line configurations to

1. prove that a given large configuration is not geometrically realizable
   (example: configurations containing a non-pappus subconfiguration)

2. construct large configurations from small pieces
   (example: Jürgen’s recent \((37_4)\)- and \((43_4)\)-configurations)
A FIRST \((43_4)\)-CONFIGURATION
A FIRST (434)-CONFIGURATION
A FIRST $\left(43_4\right)$-CONFIGURATION
A FIRST $\binom{37}{4}$-CONFIGURATION
A FIRST $\binom{37}{4}$-CONFIGURATION
A FIRST $\langle 37_4 \rangle$-CONFIGURATION
A FIRST (37₄)-CONFIGURATION
quasi-configuration = point-line configuration \((P, L)\) where each point of \(P\) is contained in at least 3 lines of \(L\) and each line of \(L\) contains at least 3 points of \(P\)

\((n_{3|4})\)-configurations = configuration \((P, L)\) with \(n\) points and \(n\) lines, where each point of \(P\) is contained in 3 or 4 lines of \(L\) and each line of \(L\) contains 3 or 4 points of \(P\)
(P, L) a point-line configuration with \( p_i \) points of P contained in \( i \) lines of L
\( \ell_j \) lines of L contained in \( j \) points of P

If \((P, L)\) has a topological realization, then

\[
0 \geq \sum_i i(i + 1)p_i - 6 \left( \sum_i p_i - 1 \right) - \sum_j \ell_j \left( \sum_j \ell_j - 1 \right)
\]

Example 1. \( p_4 = n, \ell_4 = n \) and \( p_i = \ell_i = 0 \) for all other values of \( i \)
inequality gives \( 0 \geq -n^2 + 15n + 6 \) and thus \( n \geq 16 \)

Bokowski & Schewe. There are no realizable 15_4- and 16_4-configurations. 2005
(P, L) a point-line configuration with \( p_i \) points of P contained in i lines of L
and \( \ell_j \) lines of L contained in j points of P

If (P, L) has a topological realization, then

\[
0 \geq \sum_i i(i + 1)p_i - 6\left(\sum_i p_i - 1\right) - \sum_j \ell_j \left(\sum_j \ell_j - 1\right)
\]

Example 2. the number of incidences of an \((n_{3/4})\)-configuration is bounded by

<table>
<thead>
<tr>
<th>n</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\min\left(4n, \frac{n^2 + 17n - 6}{8}\right))</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>33</td>
<td>37</td>
<td>42</td>
<td>48</td>
<td>53</td>
<td>59</td>
<td>64</td>
</tr>
</tbody>
</table>
SPLITTING CONFIGURATIONS
SPLITTING CONFIGURATIONS
MANY RESEARCH DIRECTIONS

Enumerate and classify small quasi-configurations
For example, what are the optimal \((14_{3|4})\)-, \((15_{3|4})\)- and \((16_{3|4})\)-configurations?

Create large configurations from small quasi-configurations
For example, can we create \((22_{4})\)-, \((23_{4})\)-, or \((26_{4})\)-configurations from \((11_{3|4})\)-, \((12_{3|4})\)-, and \((13_{3|4})\)-configurations?

Study splittings of configurations
Are there arbitrary large unsplittable \((n_{4})\)-configurations?
What is the smallest unsplittable configuration?