MOTIVATION

Baryshnikov, *On Stokes sets* ('01)
Chapoton, *Stokes posets and serpent nests* ('16)
Garver–McConville, *Oriented flip graphs and non-crossing tree partitions* ('18)
Petersen–Pylyavskyy–Speyer, *A non-crossing standard monomial theory* ('10)
Santos–Stump-Welker, *Non-crossing sets and the Grassmann-assoc.* ('17)
McConville, *Lattice structures of grid Tamari orders* ('17)
TWO GENERALIZATIONS OF THE ASSOCIAHEDRON

dissection

subset of $\mathbb{Z}^2$
TWO GENERALIZATIONS OF THE ASSOCIAHEDRON

dissection
accordion

subset of $\mathbb{Z}^2$
monotone path
TWO GENERALIZATIONS OF THE ASSOCIAHEDRON

dissection
accordion
non-crossing complex

subset of $\mathbb{Z}^2$
monotone path
non-kissing complex
TWO GENERALIZATIONS OF THE ASSOCIAHEDRON

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subset of $\mathbb{Z}^2$
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simplicial associahedron = simplicial complex with
- vertices = internal diagonals of an \((n + 3)\)-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the \((n + 3)\)-gon
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![Diagram of simplicial associahedron with vertices and faces labeled.]

\[0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9\]

- diagonal
- crossing
- walk
- kissing
simplicial associahedron = simplicial complex with
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diagonal \(\leftrightarrow\) walk
crossing \(\leftrightarrow\) kissing
dissection \(\leftrightarrow\) non-kissing face
simplicial associahedron = simplicial complex with
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McConville, *Lattice structures of grid Tamari orders* ('17)
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McConville, *Lattice structures of grid Tamari orders* ('17)
Show that non-crossing and non-kissing complexes coincide

To this end, generalize both:

non-crossing complex to dissections of surfaces
non-kissing complex to gentle quivers

Palu–P.–Plamondon, *Non-kissing and non-crossing complexes for locally gentle algebras* ('18+)
Palu–P.–Plamondon, \textit{Non-kissing and non-crossing complexes for locally gentle algebras} (’18+)
$S =$ orientable surface with or without boundaries
$V$ and $V^*$ two families of marked points
$D$ and $D^*$ two dual dissections of $S$
$S = \text{orientable surface with or without boundaries}$

$V$ and $V^*$ two families of marked points

$D$ and $D^*$ two dual dissections of $S$

blossom vertices = white vertices, alternating with $V \cup V^*$ along the boundary of $S$
$S = \text{orientable surface with or without boundaries}$

$V$ and $V^*$ two families of marked points

$D$ and $D^*$ two dual dissections of $S$

blossom vertices = white vertices, alternating with $V \cup V^*$ along the boundary of $S$

$B$-curve = curve which at each endpoint either reaches a blossom point or infinitely circles around a puncture of $S$
**D-accordion** = $B$-curve $\alpha$ such that whenever $\alpha$ meets a face $f$ of $D$,
(i) it enters crossing an edge $a$ of $f$ and leaves crossing an edge $b$ of $f$
(ii) the two edges $a$ and $b$ of $f$ crossed by $\alpha$ are consecutive along the boundary of $f$,
(iii) $\alpha$, $a$ and $b$ bound a disk inside $f$ that does not contain $f^*$.

**D-accordion complex** = simplicial complex of pairwise non-crossing sets of $D$-accordions
$D^*$-slalom $= B$-curve $\alpha$ of $\bar{S}$ such that, whenever $\alpha$ crosses an edge $a^*$ of $D^*$ contained in two faces $f^*, g^*$ of $D^*$, the marked points $f$ and $g$ lie on opposite sides of $\alpha$ in the union of $f^*$ and $g^*$ glued along $a^*$.

$D^*$-slalom complex $= \text{simplicial complex of pairwise non-crossing sets of } D^*$-slaloms
\textbf{D-ACCORDIONS} = \textbf{D*-SLALOMS}

$\text{(D, D*)-non-crossing complex} = \text{D-accordion complex} = \text{D*-slalom complex}$
NON-KISSING COMPLEX

GENTLE QUIVERS AND STRINGS

gentle quiver $\tilde{Q} =$

- **quiver** $Q$ = oriented graph $(Q_0, Q_1, s, t)$
- **relations** $I =$ forbid certain paths

where

- forbidden paths all of length 2
- locally at each vertex, subgraph of
GENTLE QUIVERS AND STRINGS

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where

- forbidden paths all of length 2
- locally at each vertex, subgraph of

\[
\text{string } \sigma = \alpha_1^{\varepsilon_1} \ldots \alpha_\ell^{\varepsilon_\ell} \text{ with } \alpha_k \in Q_1, \varepsilon_k \in \{-1, 1\}
\]

such that

- $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$
- contains no factor $\pi$ or $\pi^{-1}$ for any path $\pi \in I$
- contains no $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ for any arrow $\alpha \in Q_1$
blossoming quiver $Q^\ast =$
add blossoms to complete each vertex to

...
**BLOSSOMING QUIVERS AND WALKS**

**blossoming quiver** $\bar{Q}^\bullet = \text{add blossoms to complete each vertex to}

**walk** $\omega = \text{maximal string in } \bar{Q}^\bullet \text{ from blossoms to blossoms}
[reduced] non-kissing complex $\mathcal{NK}(\bar{Q}) =$

- vertices = [bending] walks in $\bar{Q}^*$
  (that are not self-kissing)
- faces = collections of pairwise non-kissing [bending] walks in $\bar{Q}^*$
NON-CROSSING VS NON-KISSING

Palu–P.–Plamondon, *Non-kissing and non-crossing complexes for locally gentle algebras* ('18+)
quiver $\overline{Q}_D$ of a dissection =

- vertices = edges of $D$ (boundary edges are blossom vertices)
- arrows = two consecutive edges around a face of $D$
- relations = three consecutive edges around a face of $D$
QUIVER OF A DISSECTION

quiver $\bar{Q}_D$ of a dissection =

- vertices = edges of $D$ (boundary edges are blossom vertices)
- arrows = two consecutive edges around a face of $D$
- relations = three consecutive edges around a face of $D$
surface $S_{\bar{Q}}$ of quiver $\bar{Q} = \text{surface obtained from the blossoming quiver } \bar{Q}^\ast$ as follows:

(i) for each arrow $\alpha \in \bar{Q}_1^\ast$, consider a lozenge

(ii) for any $\alpha, \beta \in \bar{Q}_1^\ast$ with $t(\alpha) = s(\beta)$, proceed to the following identifications:

• if $\alpha \beta \in I$, then glue $E_{nr}^t(\alpha)$ with $E_{sr}^s(\beta)$,
• if $\alpha \beta \notin I$, then glue $E_{nr}^t(\alpha)$ with $E_{sr}^s(\beta)$. 
PROP. The two previous constructions are inverse to each other and define bijections: pairs of dual dissections on a surface $\leftrightarrow$ gentle quivers

PROP. It defines isomorphisms between:
non-crossing complex of dissections $\leftrightarrow$ non-kissing complex of gentle quiver
non-kissing complex $\mathcal{NK}(\bar{Q}) =$

- vertices = walks in $\bar{Q}^*$ (that are not self-kissing)
- faces = collections of pairwise non-kissing walks in $\bar{Q}^*$

... generalizing the associahedron
DISTINGUISHED ARROWS AND FLIPS

McConville, *Lattice structures of grid Tamari orders* (’17)
DISTINGUISHED WALKS, ARROWS AND STRINGS

\[ F \text{ face of } \mathcal{N} \mathcal{K}(\bar{Q}) \]
$F$ face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$
$F$ face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{ \omega \in F \mid \alpha \in \omega \}$

$\omega \preceq_\alpha \omega'$ \text{ countercurrent order at } \alpha
Distinguished walks, arrows and strings

\( F \) face of \( \mathcal{NK}(\bar{Q}) \)

\( \alpha \in Q_1 \)

\( F_\alpha = \{ \omega \in F \mid \alpha \in \omega \} \)

\( \omega' \prec_\alpha \omega \) countercurrent order at \( \alpha \)

distinguished walk at \( \alpha \) in \( F = dw(\alpha, F) = \max_{\prec_\alpha} F_\alpha \)

distinguished arrows of \( \omega \) in \( F = da(\omega, F) = \{ \alpha \in Q_1 \mid \omega = dw(\alpha, F) \} \)
Distinguished walks, arrows and strings

\( F \) face of \( \mathcal{N}K(\bar{Q}) \)
\( \alpha \in Q_1 \)
\( F_\alpha = \{ \omega \in F \mid \alpha \in \omega \} \)

\( \omega \prec_\alpha \omega' \) countercurrent order at \( \alpha \)

Distinguished walk at \( \alpha \) in \( F = dw(\alpha, F) = \max_{\prec_\alpha} F_\alpha \)
Distinguished arrows of \( \omega \) in \( F = da(\omega, F) = \{ \alpha \in Q_1 \mid \omega = dw(\alpha, F) \} \)

**PROP.** For any facet \( F \in \mathcal{N}K(\bar{Q}) \),
- each bending walk of \( F \) contains 2 distinguished arrows in \( F \) pointing opposite,
- each straight walk of \( F \) contains 1 distinguished arrows in \( F \) pointing as the walk.
Distinguished walks, arrows and strings

$F$ face of $\mathcal{NK}(\bar{Q})$

$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$

$\omega \prec_{\alpha} \omega'$ countercurrent order at $\alpha$

distinguished walk at $\alpha$ in $F = \text{dw}(\alpha, F) = \max_{\prec_{\alpha}} F_\alpha$

distinguished arrows of $\omega$ in $F = \text{da}(\omega, F) = \{\alpha \in Q_1 \mid \omega = \text{dw}(\alpha, F)\}$

**PROP.** For any facet $F \in \mathcal{NK}(\bar{Q})$,

- each bending walk of $F$ contains 2 distinguished arrows in $F$ pointing opposite,
- each straight walk of $F$ contains 1 distinguished arrows in $F$ pointing as the walk.

**CORO.** $\mathcal{NK}(\bar{Q})$ is pure of dimension $|Q_0|$. 
$F$ facet of $\mathcal{NK}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)
$F$ facet of $\mathcal{NK}(\hat{Q})$ (ie. maximal collection of pairwise non-kissing walks)
$\omega \in F$ we want to “flip”
$F$ facet of $\mathcal{NK}(Q)$ (ie. maximal collection of pairwise non-kissing walks) 
$\omega \in F$ we want to “flip” 
$\{\alpha, \beta\} = \text{da}(\omega, F)$
$F$ facet of $\mathcal{NK}(Q)$ (ie. maximal collection of pairwise non-kissing walks) 
$\omega \in F$ we want to "flip"
$\{\alpha, \beta\} = \text{da}(\omega, F)$
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$
F facet of $\mathcal{NK}(\mathcal{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to “flip”

$\{\alpha, \beta\} = d_a(\omega, F)$

$\alpha', \beta' \in Q_1$ such that $\alpha' \alpha \in I$ and $\beta' \beta \in I$

$\mu = d_w(\alpha', F)$ and $\nu = d_w(\beta', F)$

$\omega = \nu[\cdot, \nu] \sigma \mu[w, \cdot]$
$F$ facet of $N\mathcal{K}(Q)$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to "flip"

$\{\alpha, \beta\} = da(\omega, F)$

$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

$\mu = dw(\alpha', F)$ and $\nu = dw(\beta', F)$

$\omega = \nu[\cdot, v] \sigma \mu[w, \cdot]$

$\omega' = \mu[\cdot, v] \sigma \nu[w, \cdot]$
PROP. $\omega'$ kisses $\omega$ but no other walk of $F$. Moreover, $\omega'$ is the only such walk.
flip graph =
  • vertices = non-kissing facets
  • edges = flips
GENTLE ASSOCIAHEDRA

Manneville–P., *Geometric realizations of the accordion complex* ('17+)
Hohlweg–P.–Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
simplicial complex = collection of subsets of $X$ downward closed

exm:

$X = [n] \cup [n]$

$\Delta = \{ I \subseteq X | \forall i \in [n], \{ i, i \} \not\subseteq I \}$
**FANS**

**polyhedral cone** = positive span of a finite set of $\mathbb{R}^d$

= intersection of finitely many linear half-spaces

**fan** = collection of polyhedral cones closed by faces and where any two cones intersect along a face

**simplicial fan** = maximal cones generated by $d$ rays
**POLYTOPES**

polytope = convex hull of a finite set of \( \mathbb{R}^d \)

= bounded intersection of finitely many affine half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations

simple polytope = facets in general position = each vertex incident to \( d \) facets
$P$ polytope, $F$ face of $P$

normal cone of $F$ = positive span of the outer normal vectors of the facets containing $F$

normal fan of $P$ = \{ normal cone of $F$ \mid F$ face of $P$ \}
**G-VECTORS & C-VECTORS**

Multiplicity vector $m_V$ of multiset $V = \{v_1, \ldots, v_m\}$ of $Q_0 = \sum_{i \in [m]} e_{v_i} \in \mathbb{R}^{Q_0}$

g-vector $g(\omega)$ of a walk $\omega$ = $m_{\text{peaks}}(\omega) - m_{\text{deeps}}(\omega)$

c-vector $c(\omega \in F)$ of a walk $\omega$ in a non-kissing facet $F$ = $\varepsilon(\omega, F) m_{\text{ds}}(\omega, F)$

![Diagram with vectors and arrows]
multiplicity vector $m_V$ of multiset $V = \{v_1, \ldots, v_m\}$ of $Q_0 = \sum_{i \in [m]} e_{v_i} \in \mathbb{R}^{Q_0}$

g-vector $g(\omega)$ of a walk $\omega = m_{\text{peaks}(\omega)} - m_{\text{deeps}(\omega)}$

c-vector $c(\omega \in F)$ of a walk $\omega$ in a non-kissing facet $F = \varepsilon(\omega, F) m_{\text{ds}(\omega,F)}$

PROP. For any non-kissing facet $F$, the sets of vectors

$$g(F) := \{g(\omega) \mid \omega \in F\} \quad \text{and} \quad c(F) := \{c(\omega \in F) \mid \omega \in F\}$$

form dual bases.

Palu–P.–Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17+)
**THM.** For any gentle quiver $\tilde{Q}$, the collection of cones $\mathcal{F}^g(\tilde{Q}) := \{ \mathbb{R}_{\geq 0} g(F) \mid F \in \mathcal{NK}(\tilde{Q}) \}$ forms a compl. simpl. fan, called $g$-vector fan of $\tilde{Q}$.

s stereographic projection from $(1, 1, 1)$
kissing number \( \text{kn}(\omega) = \sum_{\omega'} \) number of times \( \omega \) and \( \omega' \) kiss

**THM.** For a gentle quiver \( \bar{Q} \) with finite non-kissing complex \( \mathcal{NK}(\bar{Q}) \), the two sets of \( \mathbb{R}^{Q_0} \) given by

(i) the convex hull of the points

\[
p(F) := \sum_{\omega \in F} \text{kn}(\omega) \cdot c(\omega \in F),
\]

for all non-kissing facets \( F \in \mathcal{NK}(\bar{Q}) \),

(ii) the intersection of the halfspaces

\[
\mathcal{H}^\geq(\omega) := \left\{ x \in \mathbb{R}^{Q_0} \mid \langle g(\omega) \mid x \rangle \leq \text{kn}(\omega) \right\}.
\]

for all walks \( \omega \) of \( \bar{Q} \), define the same polytope, whose normal fan is the g-vector fan \( \mathcal{F}^g \). We call it the \( \bar{Q} \)-associahedron and denote it by Asso.

Palu–P.–Plamondon, *Non-kissing complexes and \( \tau \)-tilting for gentle algebras* ('17+).
NON-KISSING LATTICE

McConville, *Lattice structures of grid Tamari orders* (’17)
**THM.** For a gentle quiver $\mathcal{Q}$ with finite non-kissing complex $\mathcal{NK}(\mathcal{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.

**LATTICE QUOTIENTS**

**lattice** = poset \((L, \leq)\) with a meet \(\land\) and a join \(\lor\)

**lattice congruence** = equiv. rel. \(\equiv\) on \(L\) which respects meets and joins

\[
x \equiv x' \quad \text{and} \quad y \equiv y' \quad \implies \quad x \land y \equiv x' \land y' \quad \text{and} \quad x \lor y \equiv x' \lor y'
\]

**lattice quotient of** \(L/\equiv\) = lattice on equiv. classes of \(L\) under \(\equiv\) where

- \(X \leq Y\) \iff \exists x \in X, y \in Y, x \leq y\)
- \(X \land Y\) = equiv. class of \(x \land y\) for any \(x \in X\) and \(y \in Y\)
- \(X \lor Y\) = equiv. class of \(x \lor y\) for any \(x \in X\) and \(y \in Y\)
EXM: TAMARI LATTICE AS LATTICE QUOTIENT OF WEAK ORDER

binary search tree insertion of 2751346
NON-KISSING LATTICE
**BICLOSED SETS OF STRINGS**

σ, τ oriented strings

**concatenation** \( \sigma \circ \tau = \{ \sigma \alpha \tau \mid \alpha \in Q_1 \text{ and } \sigma \alpha \tau \text{ string of } \bar{Q} \} \)

**closure** \( S^{cl} = \bigcup_{\ell \in \mathbb{N}} \sigma_1 \circ \cdots \circ \sigma_{\ell} = \text{all strings obtained by concatenation of some strings of } S \)

closed \( \iff S^{cl} = S \)

coclosed \( \iff \bar{S}^{cl} = \bar{S} \)

biclosed = closed and coclosed

---

**THM.** For any gentle quiver \( \bar{Q} \) such that \( \mathcal{NK}(\bar{Q}) \) is finite, the inclusion poset on biclosed sets of strings of \( \bar{Q} \) is a congruence-uniform lattice.

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McConville, *Lattice structures of grid Tamari orders* ('17)

Garver–McConville, *Oriented flip graphs and non-crossing tree partitions* ('17+)

Palu–P.–Plamondon, *Non-kissing complexes and \( \tau \)-tilting for gentle algebras* ('17+)
Surjection from biclosed sets of strings to non-kissing facets

\( S \) biclosed, \( \alpha \in Q_1 \)

\( \omega(\alpha, S) = \text{walk constructed with the local rules:} \)

\[
\begin{align*}
\alpha & \in S \\
\alpha & \notin S
\end{align*}
\]
Surjection from biclosed sets of strings to non-kissing facets

$S$ biclosed, $\alpha \in Q_1$

$\omega(\alpha, S) =$ walk constructed with the local rules:

McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

\[ \omega(\alpha, S) = \text{walk constructed with the local rules:} \]

- If \( \alpha \in S \):
  - \( \alpha \) moves to the right.
  - \( \alpha \) moves to the left.

- If \( \alpha \notin S \):
  - \( \alpha \) moves to the right.

McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

\( S \) biclosed, \( \alpha \in Q_1 \)

\( \omega(\alpha, S) = \) walk constructed with the local rules:

1. \( \alpha \in S \)
2. \( \alpha \notin S \)

McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

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\( \alpha \notin S \)
Surjection from biclosed sets of strings to non-kissing facets

\[ S \text{ biclosed}, \ \alpha \in Q_1 \]
\[ \omega(\alpha, S) = \text{walk constructed with the local rules:} \]

- \( \alpha \in S \)
- \( \alpha \notin S \)

McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

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McConville, *Lattice structures of grid Tamari orders* (’17)
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McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

\( S \) biclosed, \( \alpha \in Q_1 \)

\( \omega(\alpha, S) = \text{walk constructed with the local rules:} \)

\[
\begin{align*}
\alpha \in S & \quad \alpha \\
\alpha \notin S & \quad \alpha
\end{align*}
\]

McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

PROP. \( \eta(S) := \{ \omega(\alpha, S) \mid \alpha \in Q_1 \} \) is a non-kissing facet.

McConville, *Lattice structures of grid Tamari orders* ('17)
inversion set of 2751346
Surjection from biclosed sets of strings to non-kissing facets

**PROP.** \(\eta(S) := \{\omega(\alpha, S) \mid \alpha \in Q_1\}\) is a non-kissing facet.

**THM.** The map \(\eta\) defines a lattice morphism from biclosed sets to non-kissing facets.

McConville, *Lattice structures of grid Tamari orders* ('17)
NON-KISSING LATTICE
THM. For a gentle quiver $\widetilde{Q}$ with finite non-kissing complex $\mathcal{NK}(\tilde{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.

Palu–P.–Plamondon, Non-kissing complexes and $\tau$-tilting for gentle algebras (’17+)

Much more nice combinatorics:
• join-irreducible elements of $\mathcal{L}_{nk}(\tilde{Q})$ are in bijection with distinguishable strings
• canonical join complex of $\mathcal{L}_{nk}(\tilde{Q})$ is a generalization of non-crossing partitions
non-kissing complex $\mathcal{N}\mathcal{K}(\bar{Q}) =$
- vertices = walks in $\bar{Q}^*$ (that are not self-kissing)
- faces = collections of pairwise non-kissing walks in $\bar{Q}^*$

... generalizing the associahedron

Flip graph

Associahedron

Tamari lattice