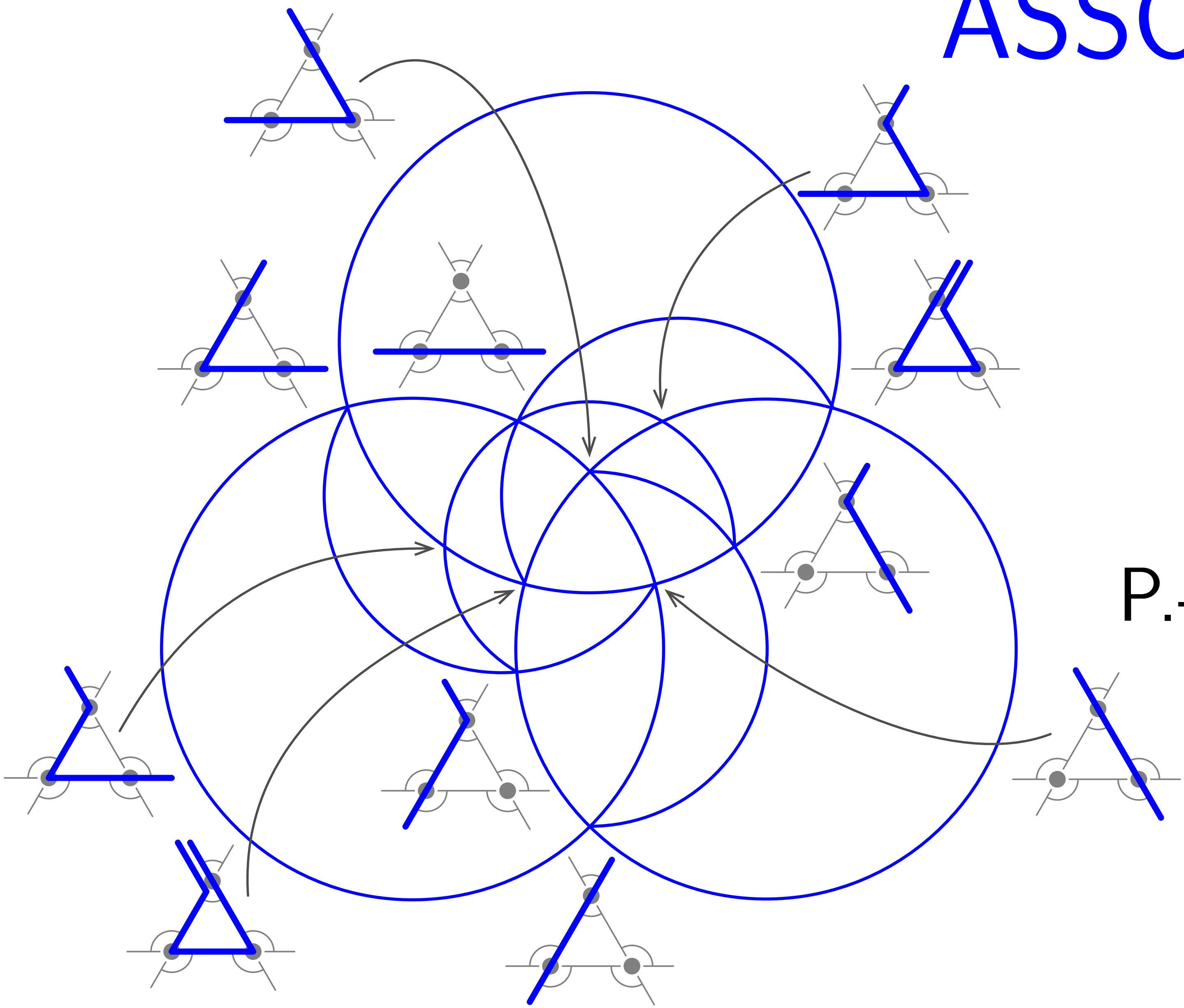
GENTLE ASSOCIAHEDRA



Y. PALU
Univ. Amiens

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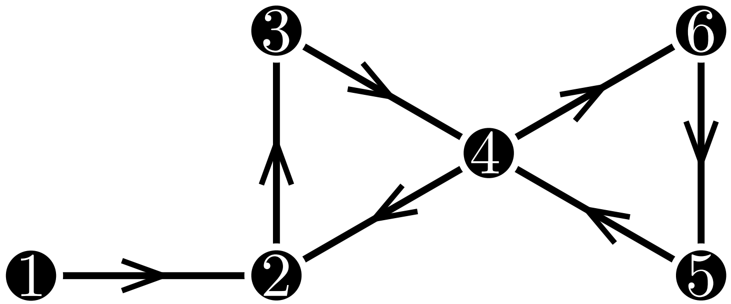
École Poytechnique

P.-G. PLAMONDON
Univ. Orsay

NON-KISSING COMPLEX

QUIVERS



quiver $=$ oriented graph
(loops and multiple edges allowed)

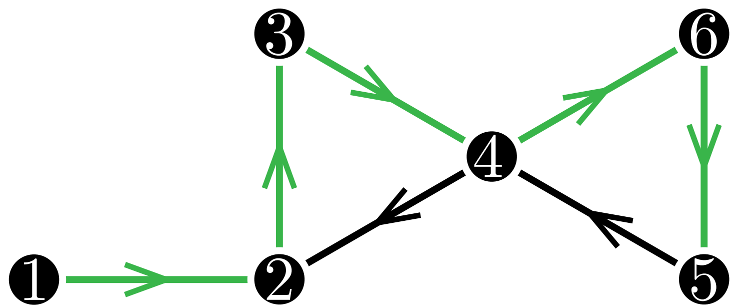
$Q = (Q_0, Q_1, s, t)$

$Q_0 =$ vertices

$Q_1 =$ edges

$s, t : Q_1 \to Q_0$ source and target maps

QUIVERS



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path $= \alpha_1 \dots \alpha_\ell$ with $\alpha_k \in Q_1$ and $t(\alpha_k) = s(\alpha_{k+1})$

path algebra $\mathbb{K}Q = \langle e_\pi \mid \pi \text{ path of } Q \rangle$ with concatenation product

$$e_{\alpha_1 \dots \alpha_\ell} \cdot e_{\beta_1 \dots \beta_k} = \begin{cases} e_{\alpha_1 \dots \alpha_\ell \beta_1 \dots \beta_k} & \text{if } t(\alpha_\ell) = s(\beta_1) \\ 0 & \text{otherwise} \end{cases}$$

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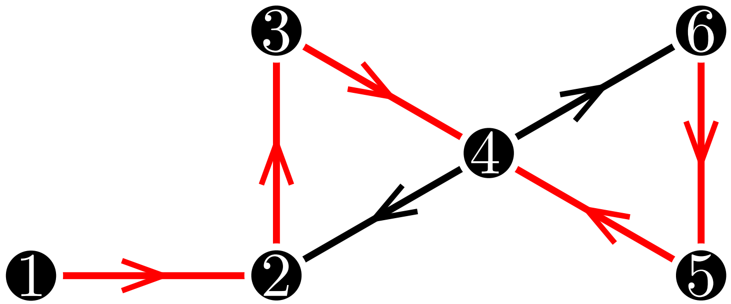

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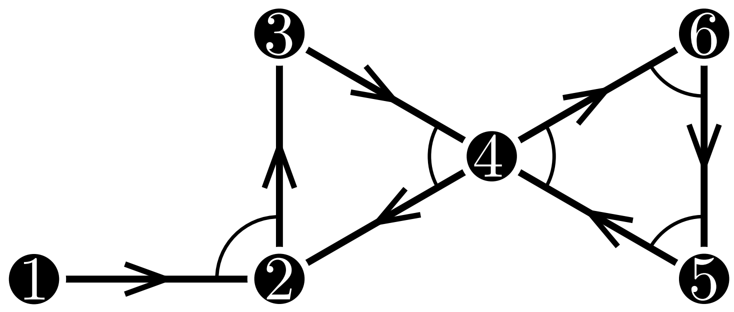

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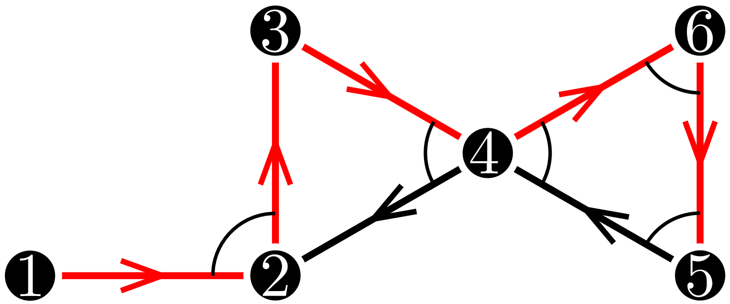
bound quiver = quiver with relations
$\bar{Q} = (Q, I)$ where I is an admissible ideal of $\mathbb{K}Q$.

Complicated way to say that we forbid certain paths

QUIVERS



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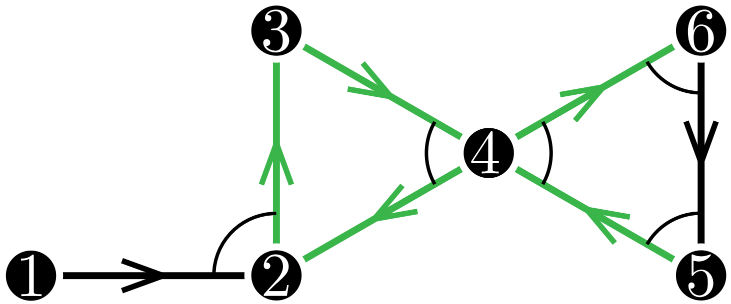

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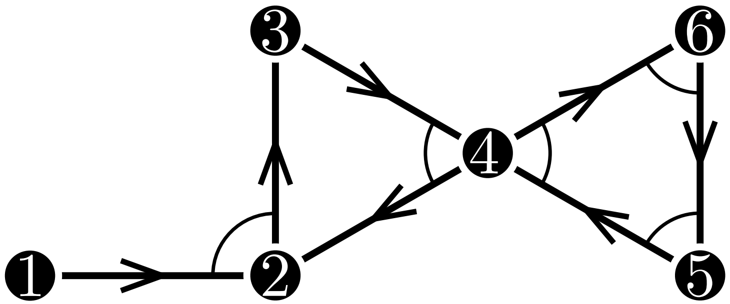
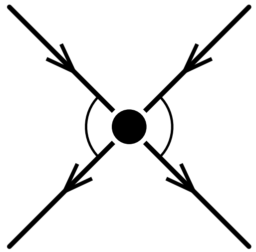


bound quiver $\bar{Q} = (Q, I)$

gentle quiver =

- forbidden paths all of length 2
- locally at each vertex, subgraph of



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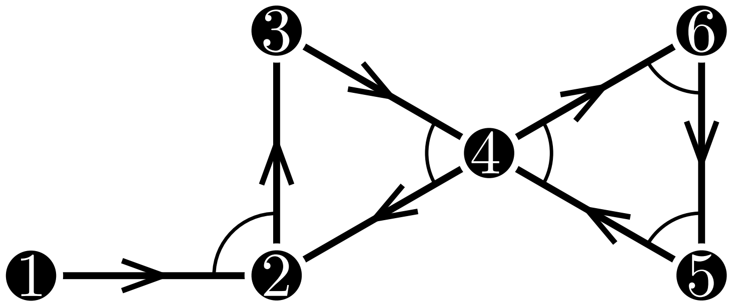

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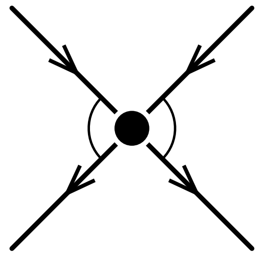
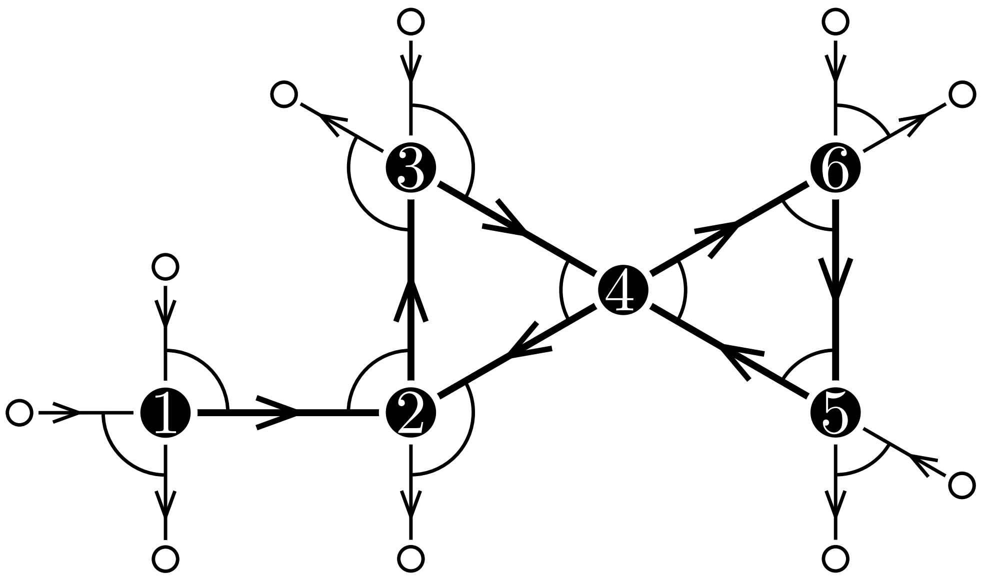
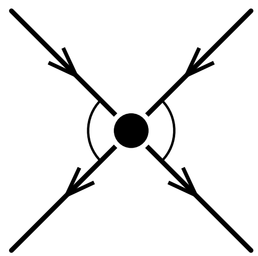
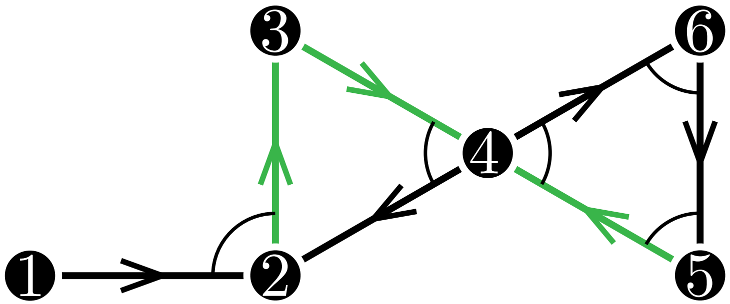




blossoming quiver $\bar{Q}^{❀} =$ add blossoms to complete each vertex to



STRINGS AND WALKS



string $\sigma = \alpha_1^{\varepsilon_1} \dots \alpha_\ell^{\varepsilon_\ell}$

with $\alpha_k \in Q_1$,

$\varepsilon_k \in \{-1, 1\}$

and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$

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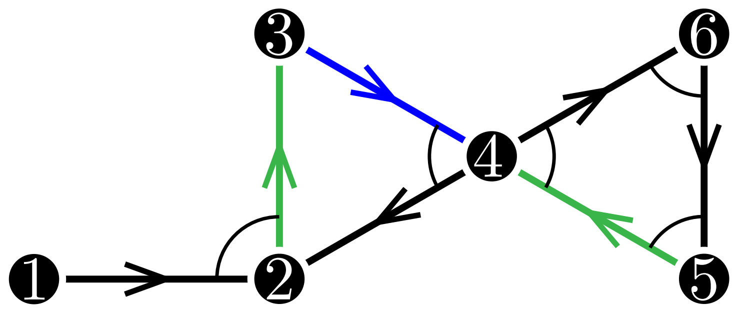

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substrings of $\sigma = \left\{ \alpha_i^{\varepsilon_i} \dots \alpha_j^{\varepsilon_j} \;\middle|\; 1 \le i \le j-1 \le k \right\}$

bottom substring of $\sigma =$ substring ρ of σ such that σ either ends
or has an outgoing arrow at each endpoint of ρ

$\Sigma_{\mathrm{bot}}(\sigma) = \{$ bottom substrings of σ $\}$

top substring of $\sigma =$ substring ρ of σ such that σ either ends
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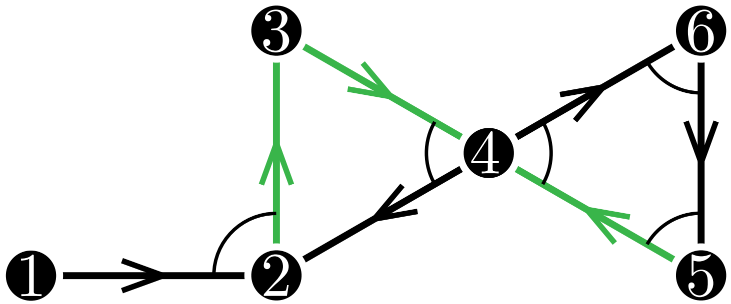
$\Sigma_{\mathrm{top}}(\sigma) = \{$ top substrings of σ $\}$

STRINGS AND WALKS



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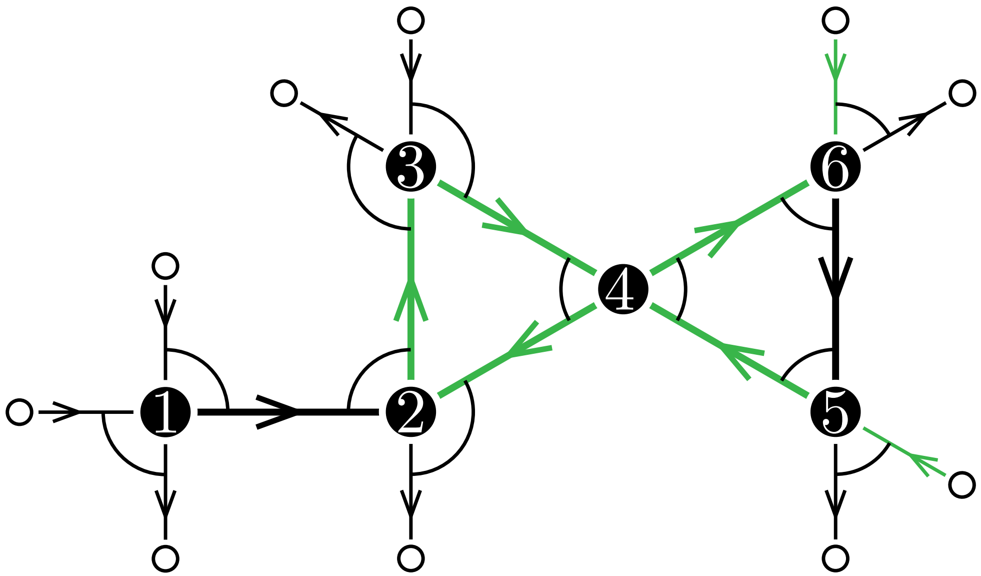
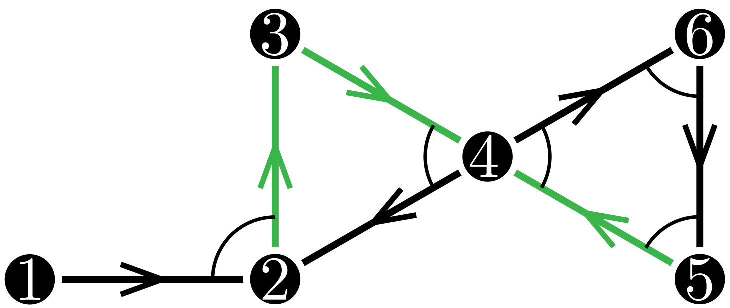


walk $\omega =$ maximal string in $Q^{❀}$
from blossoms to blossoms

STRINGS AND WALKS



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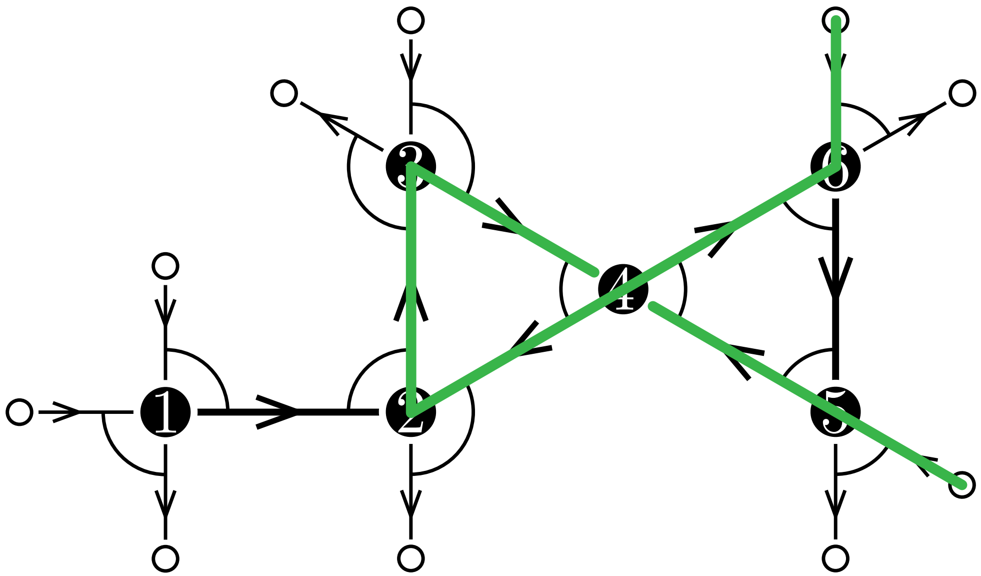

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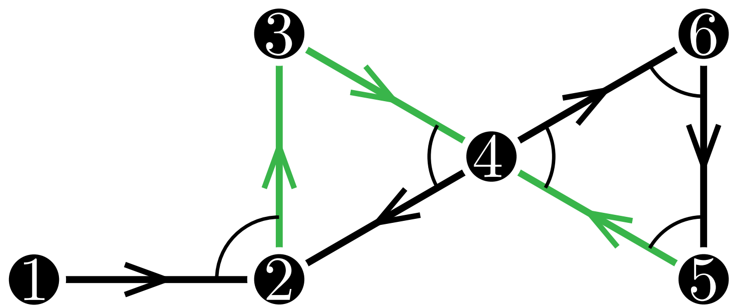

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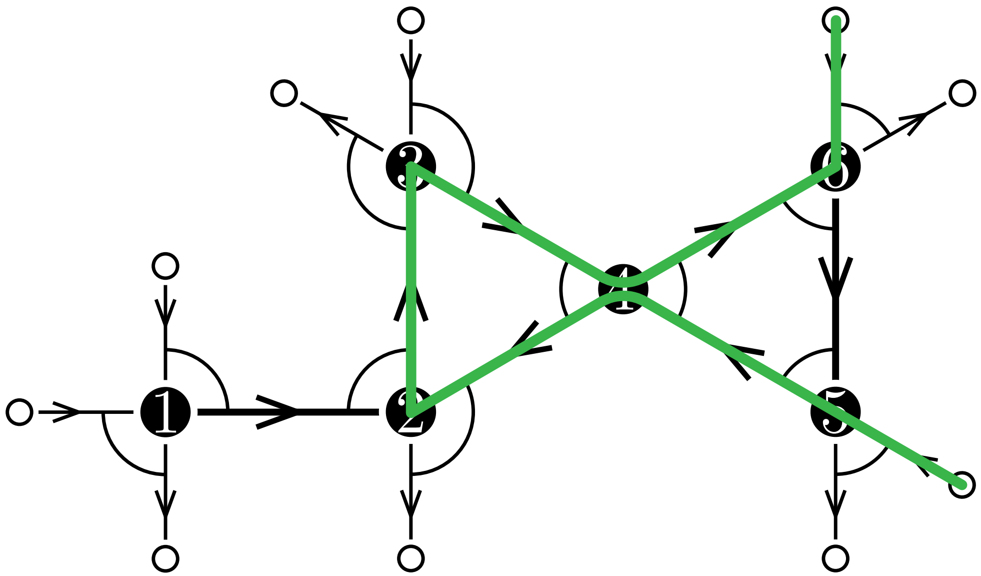

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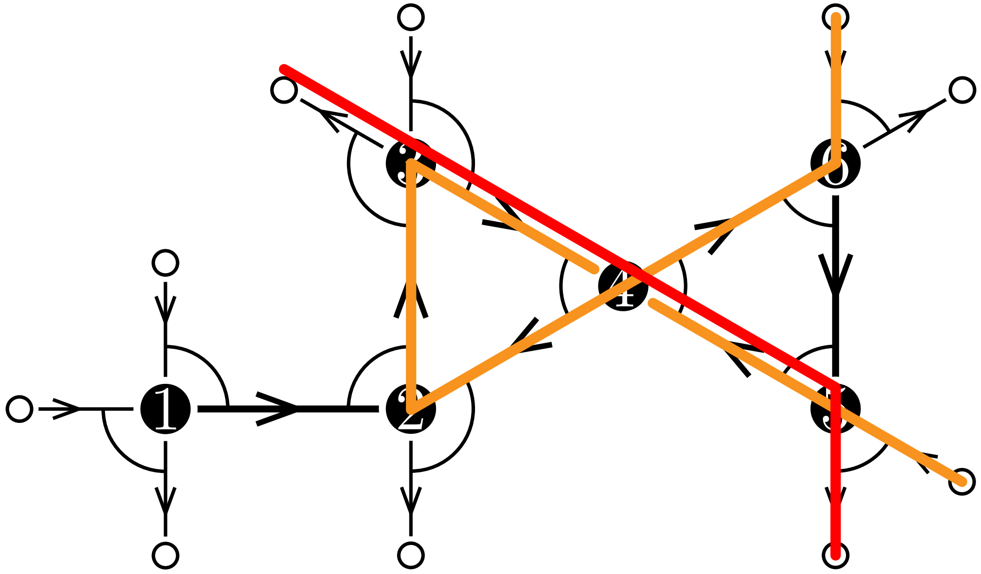
NON-KISSING COMPLEX



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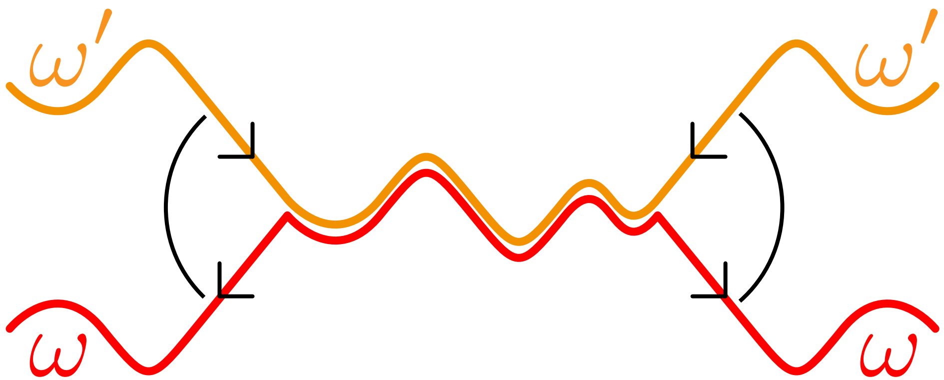

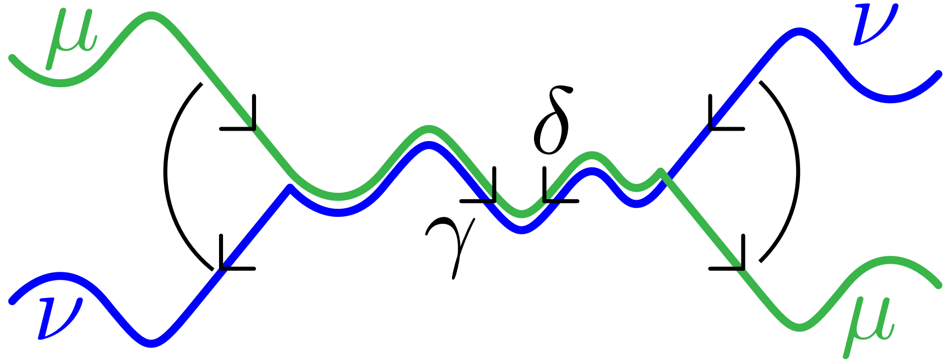

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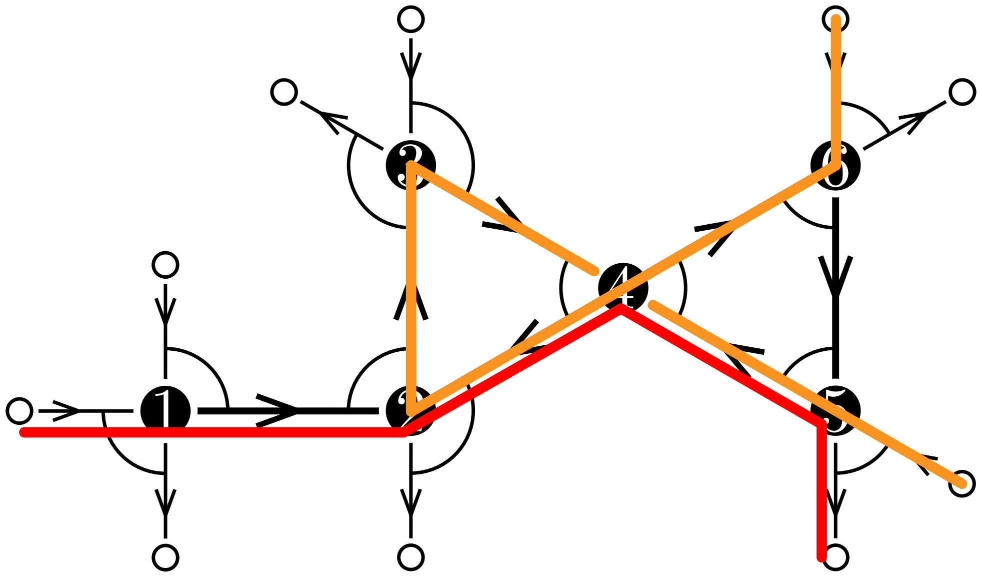

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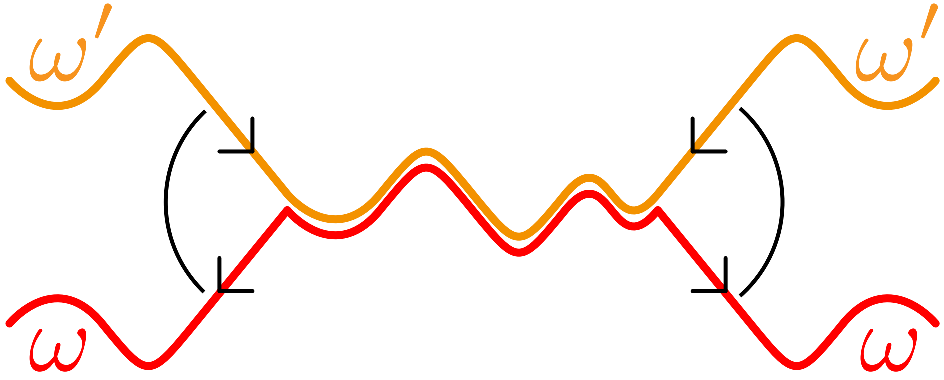

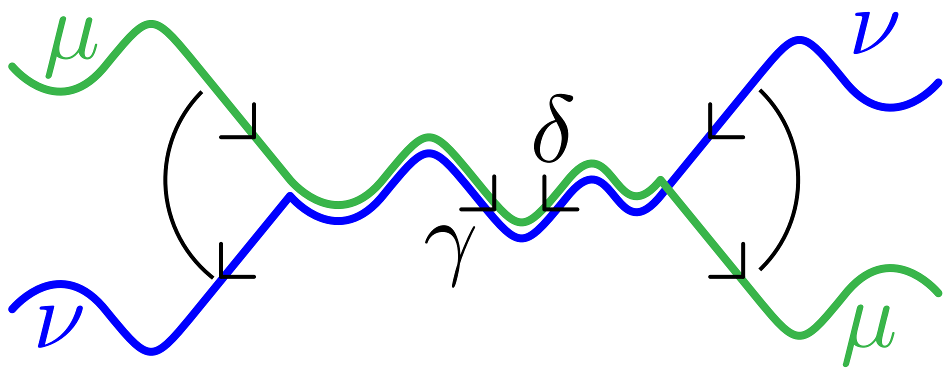

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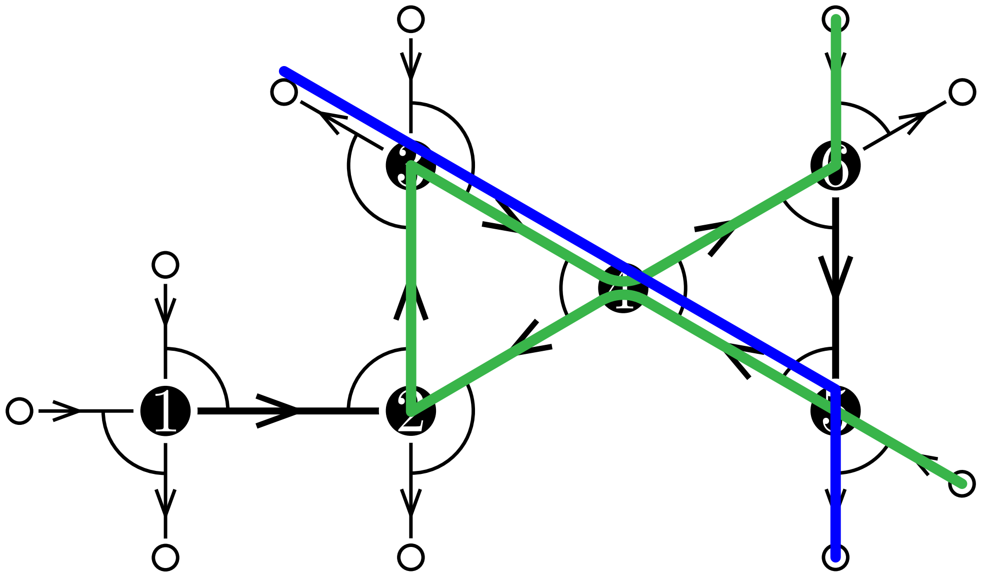

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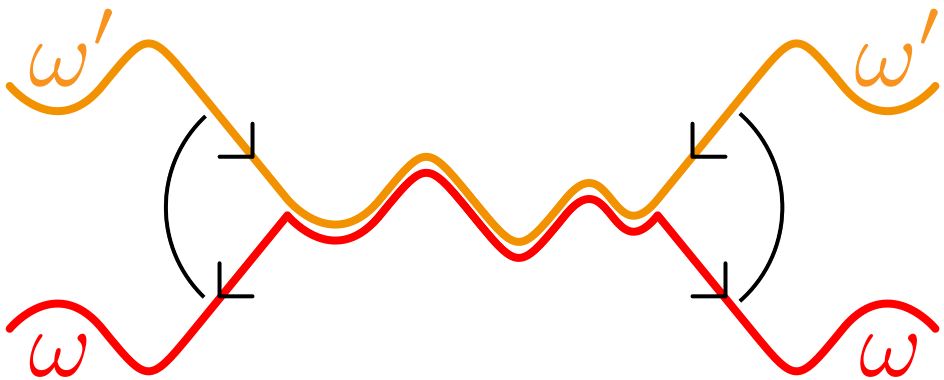

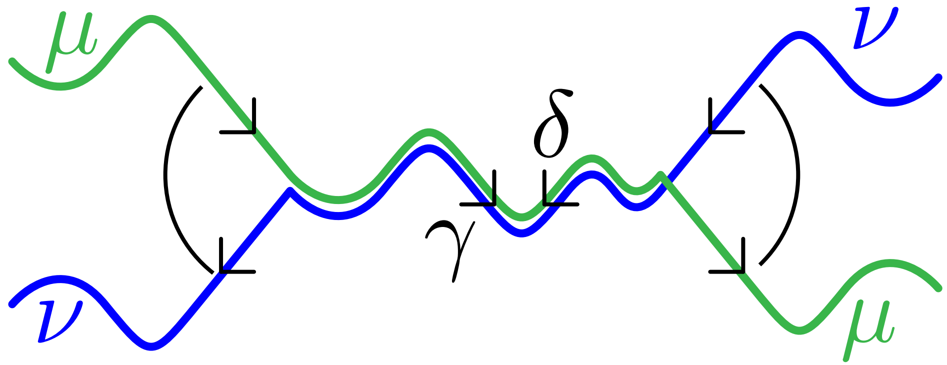

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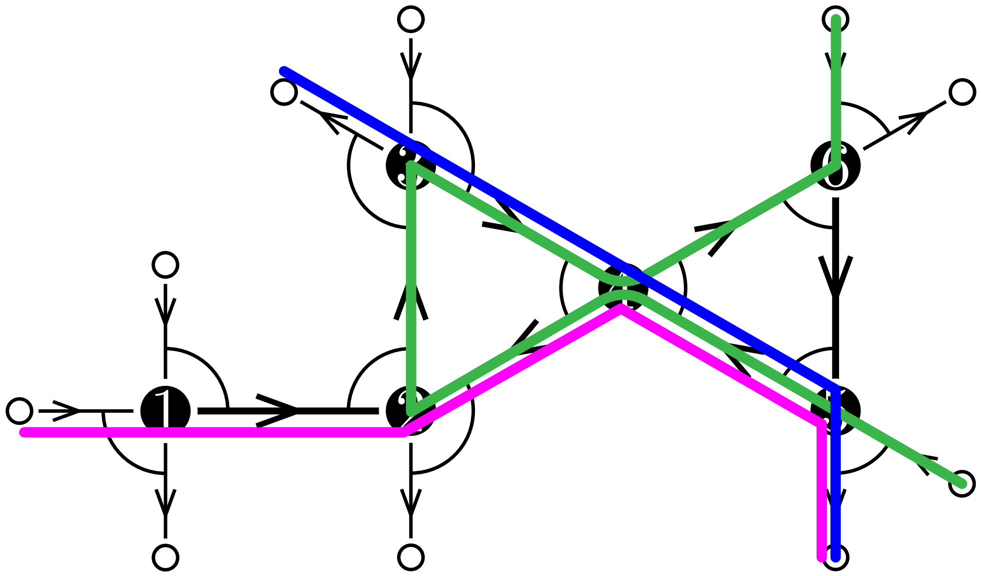

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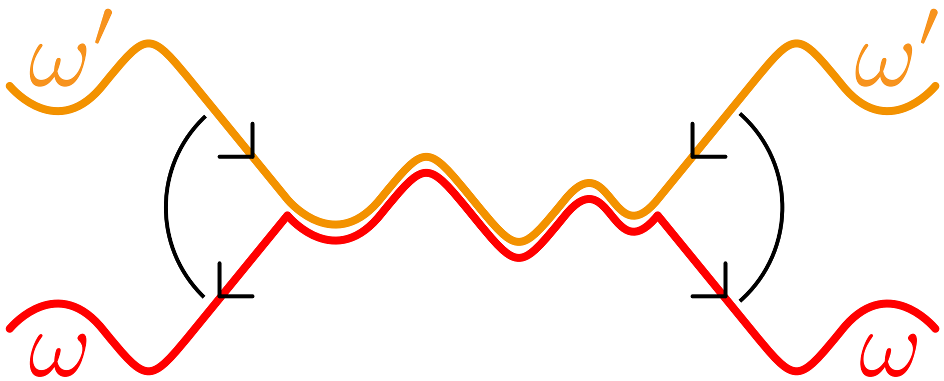

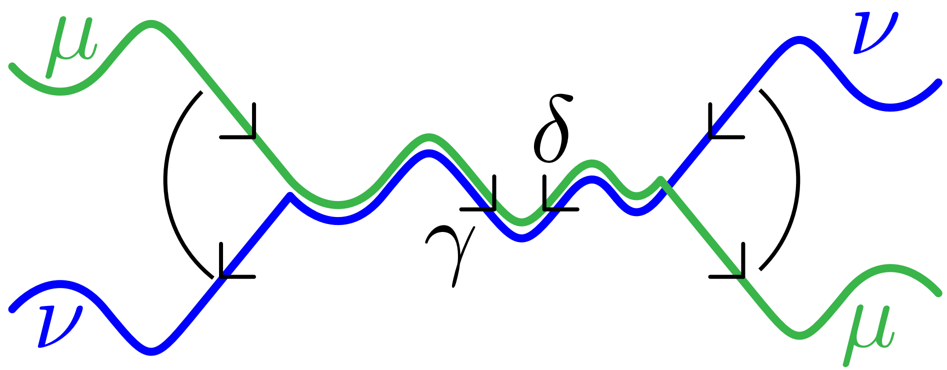

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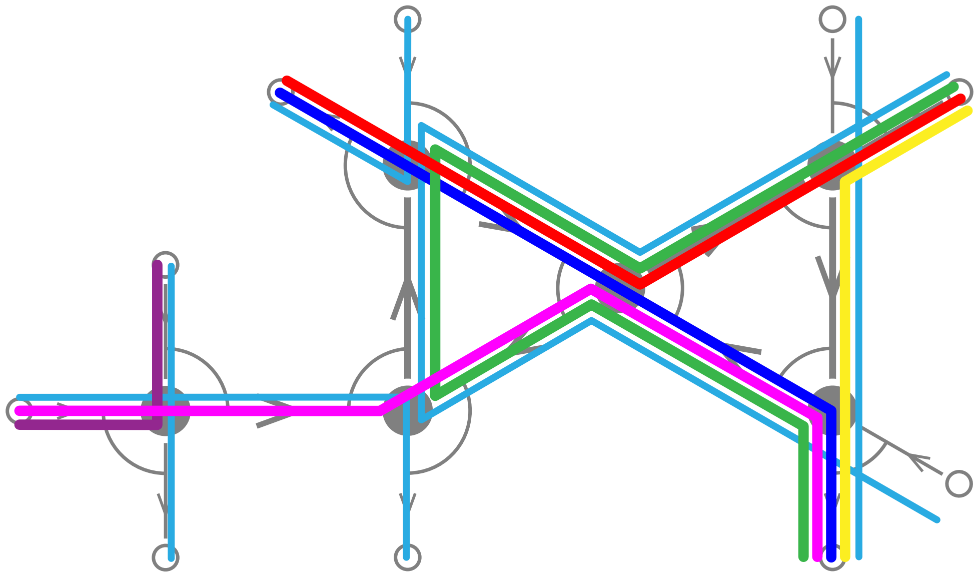

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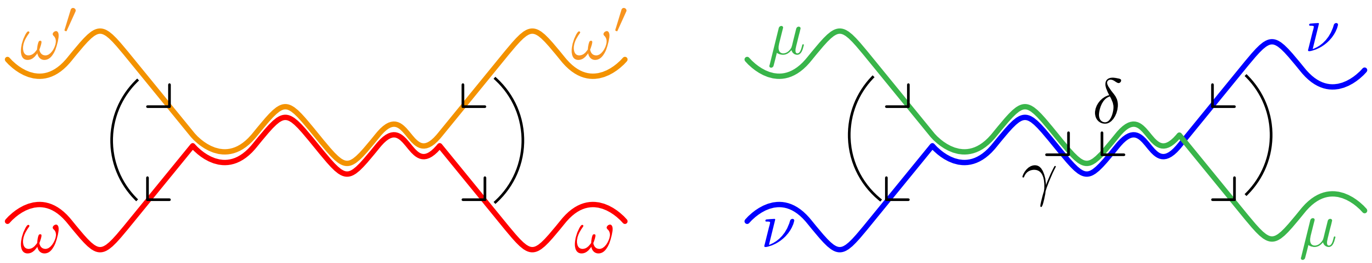
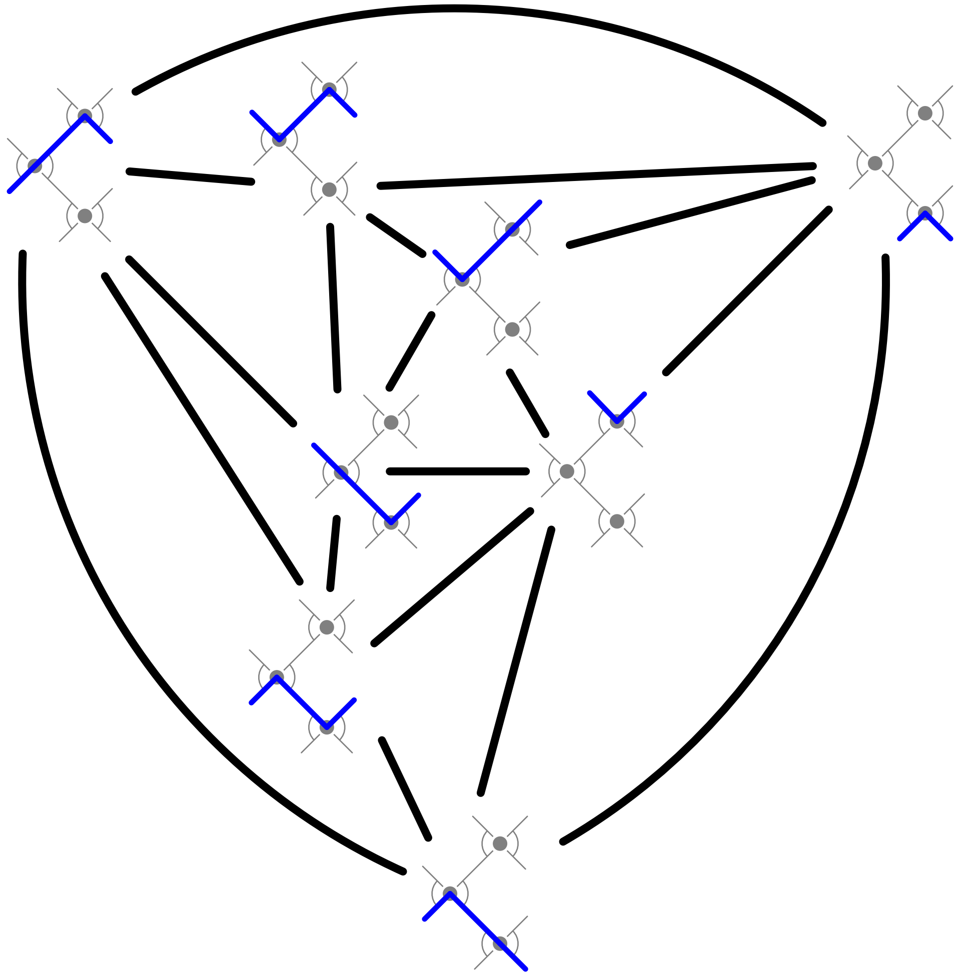
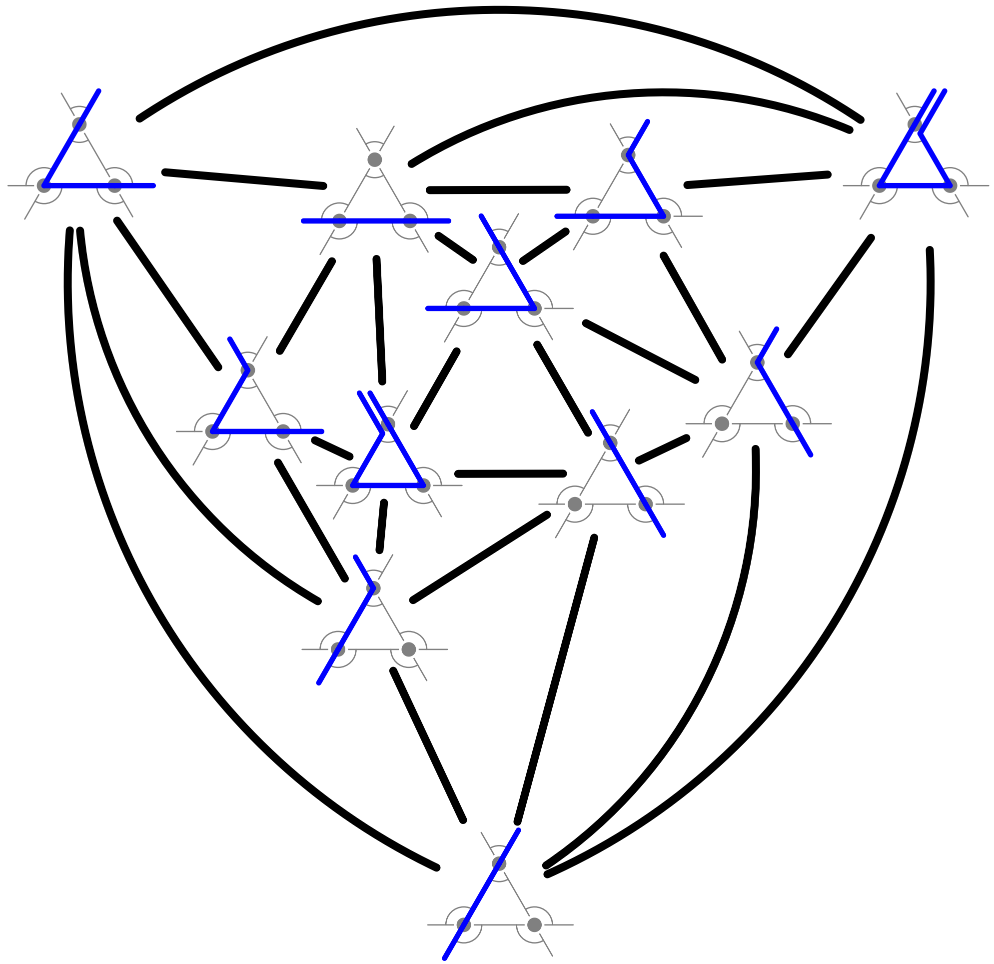


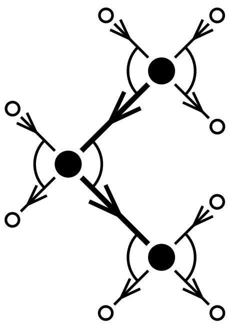
[reduced] non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ = simplicial complex with

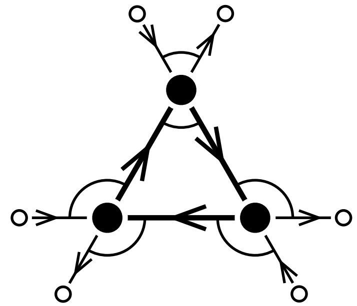
- vertices = [bended] walks of $\bar{Q}$ (that are not self-kissing)
- faces = collections of pairwise non-kissing [bended] walks of $\bar{Q}$

REDUCED NON-KISSING COMPLEX





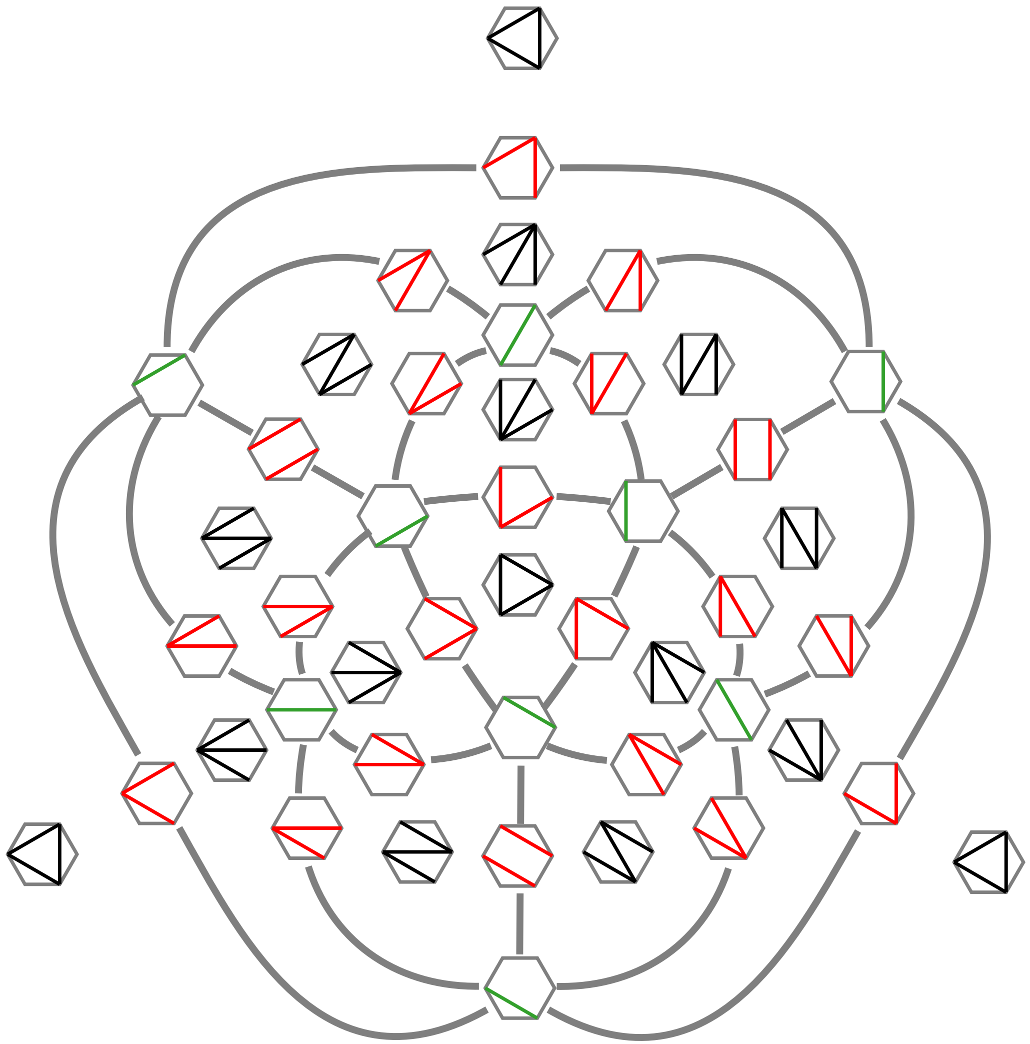




SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

[reduced] simplicial associahedron = simplicial complex with

- vertices = [internal] diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon



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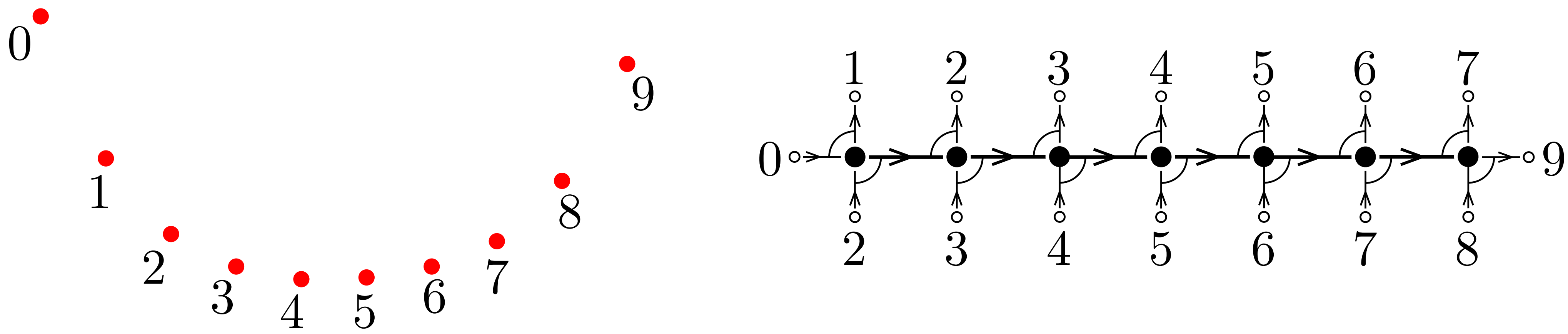

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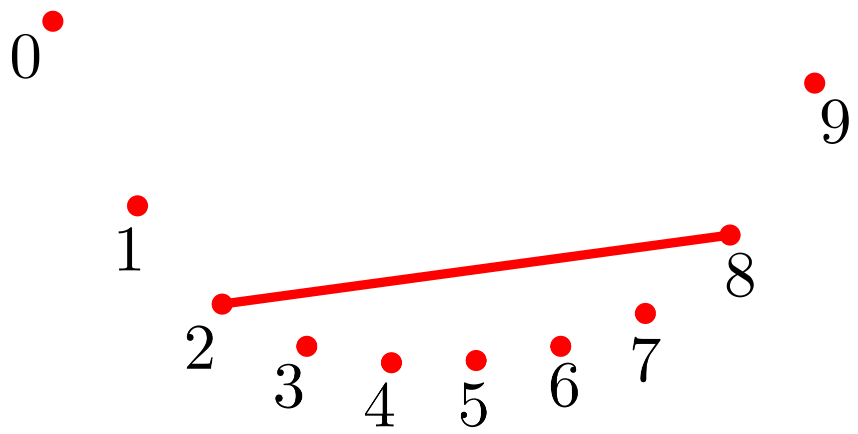

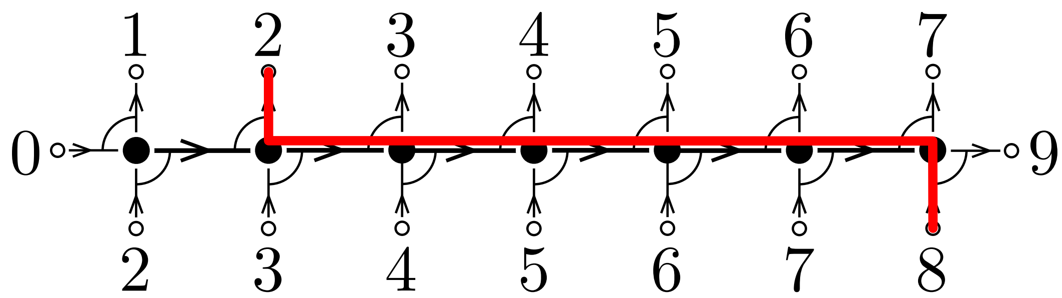

diagonal $\longleftrightarrow$ walk

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

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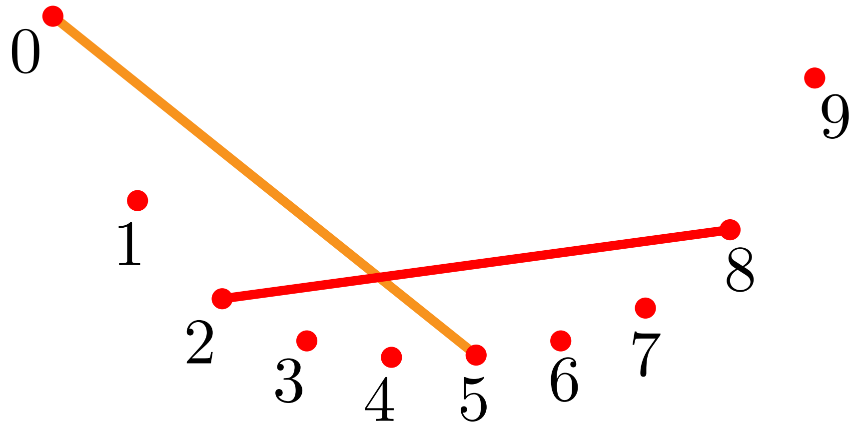

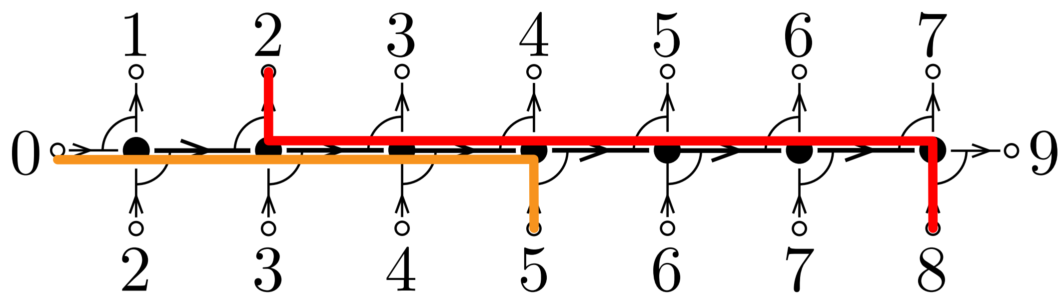

diagonal $\longleftrightarrow$ walk

crossing $\longleftrightarrow$ kissing

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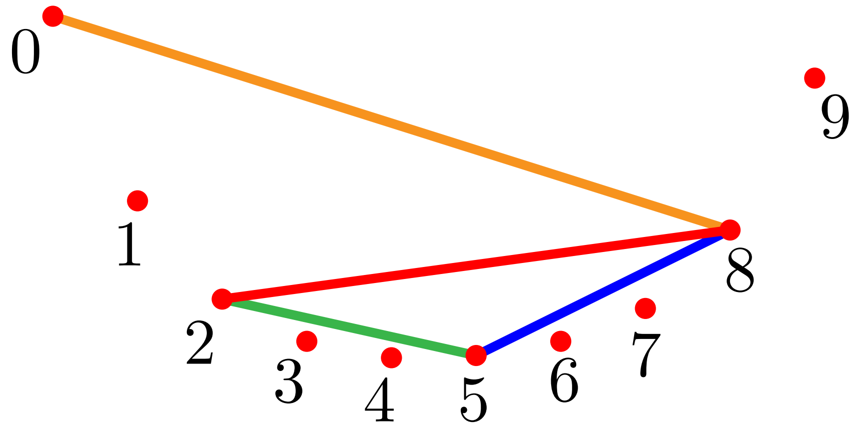

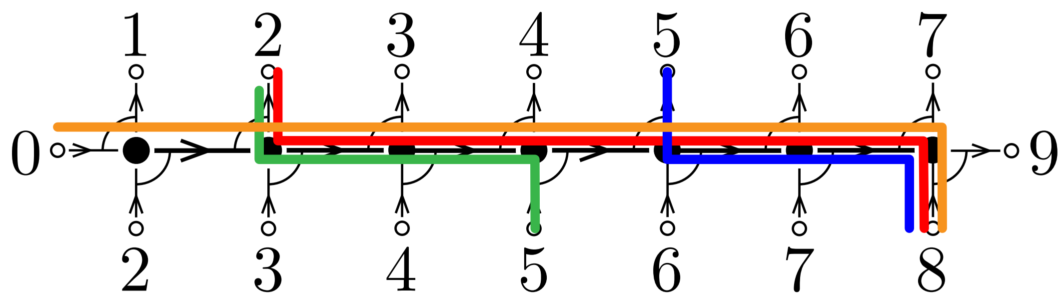

diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing
dissection	$\longleftrightarrow$	non-kissing face

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

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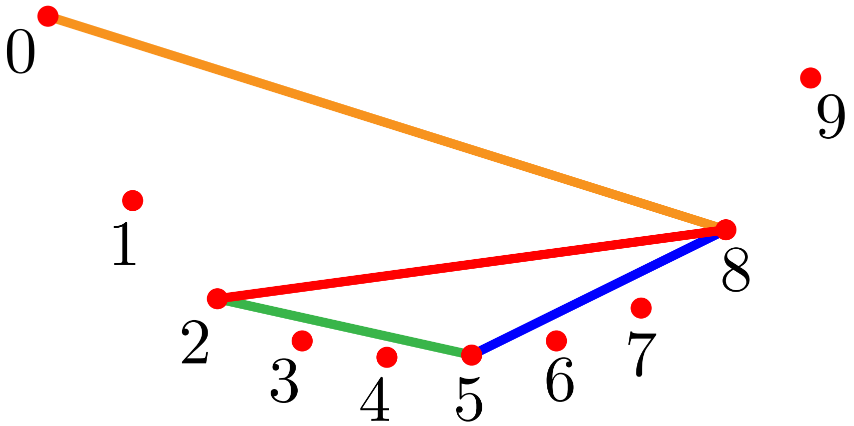
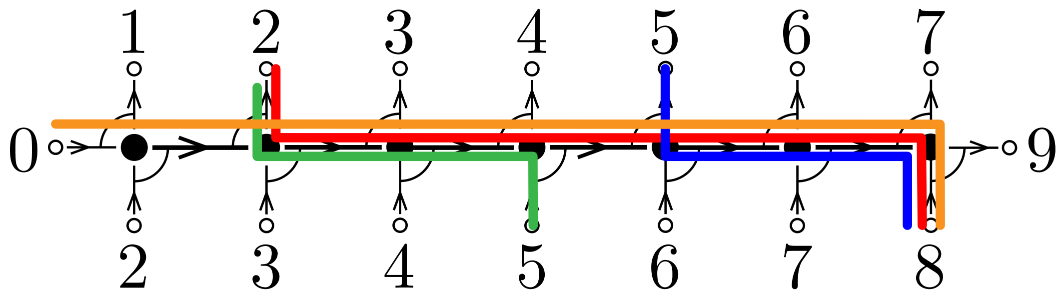
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon





diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing
dissection	$\longleftrightarrow$	non-kissing face
simplicial associahedron	$\longleftrightarrow$	non-kissing complex

McConville, Lattice structures of grid Tamari orders, 2017

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

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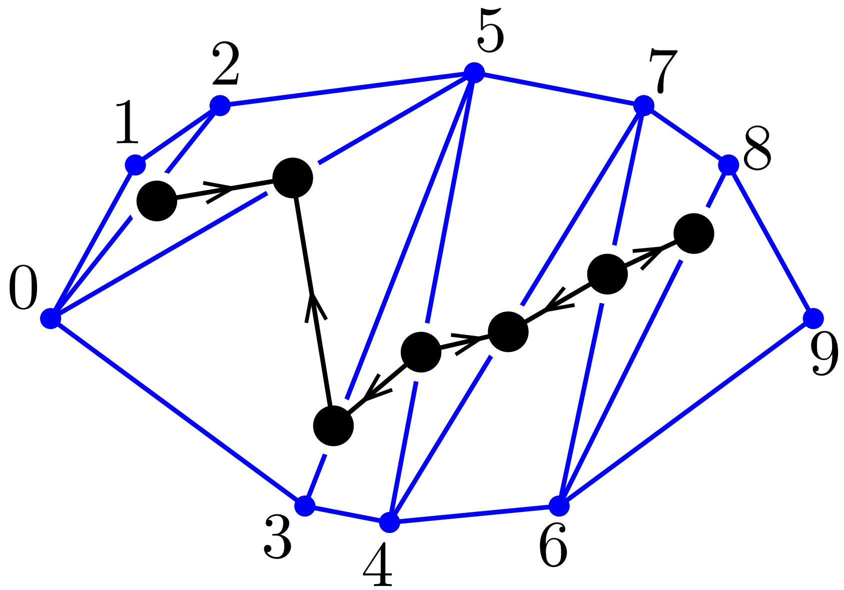
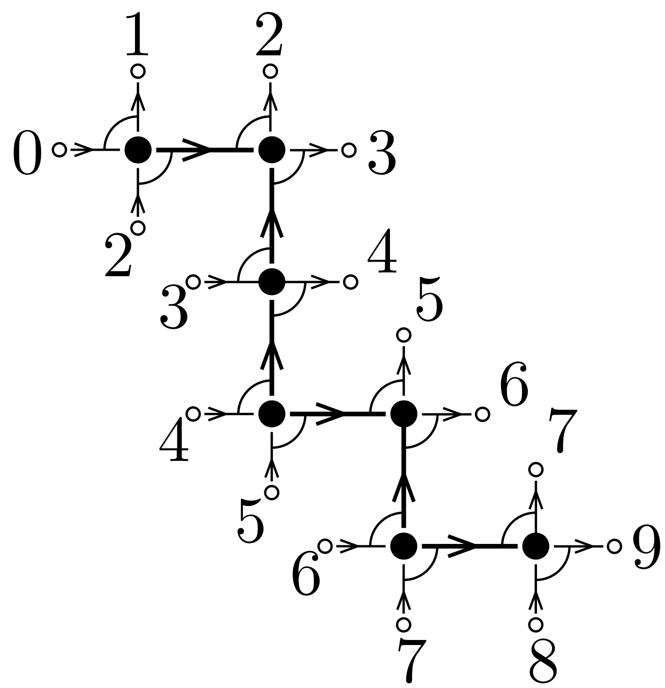
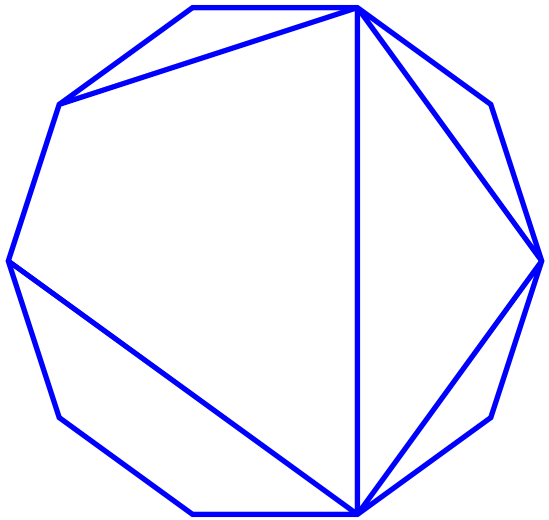


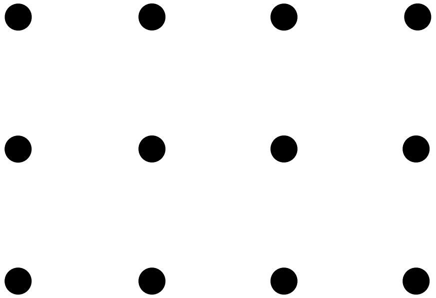


McConville, Lattice structures of grid Tamari orders, 2017

TWO FAMILIES OF NON-KISSING COMPLEXES

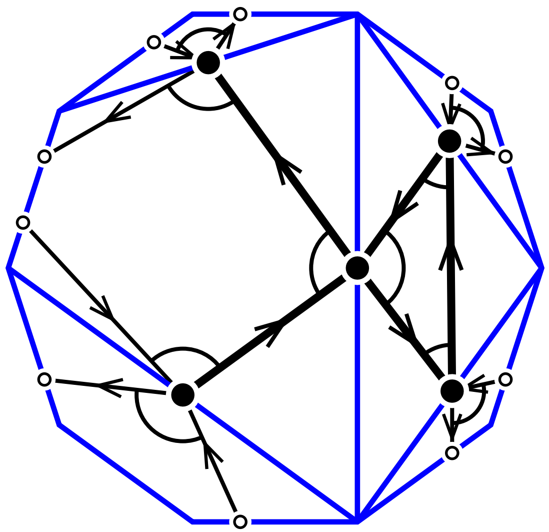


dissection

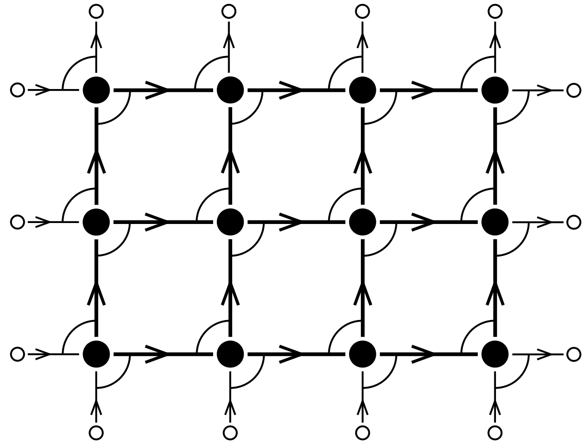


subset of $\mathbb{Z}^2$

TWO FAMILIES OF NON-KISSING COMPLEXES



dissection



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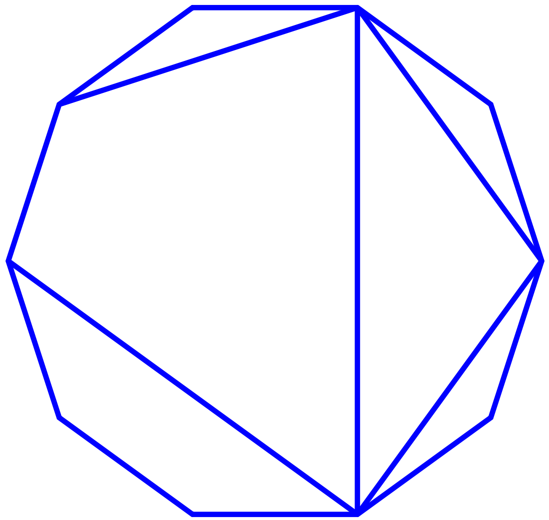
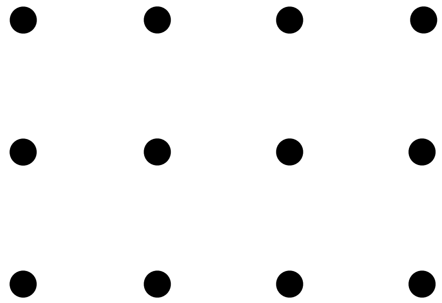


dissection



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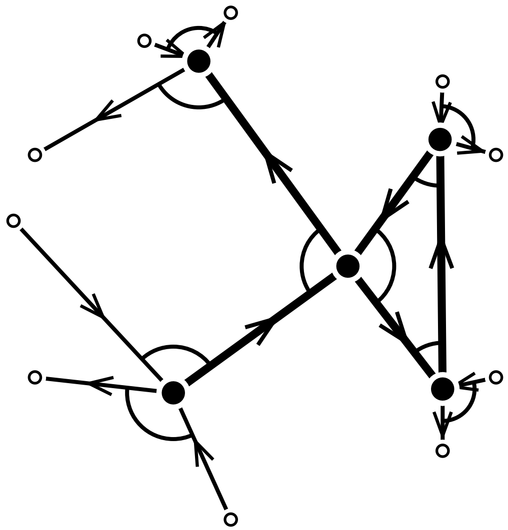
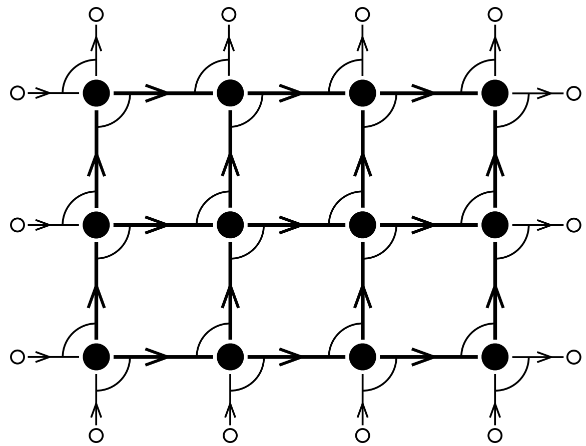
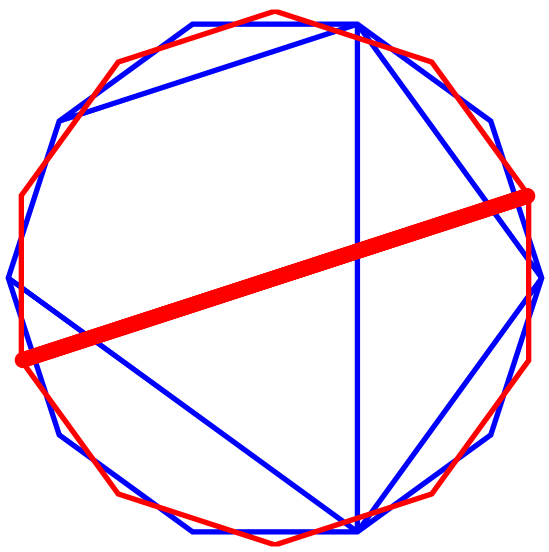


dissection quiver

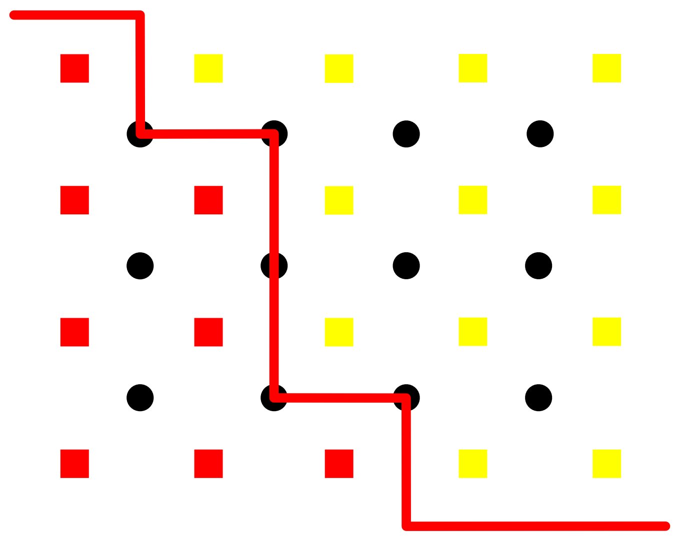


grid quiver

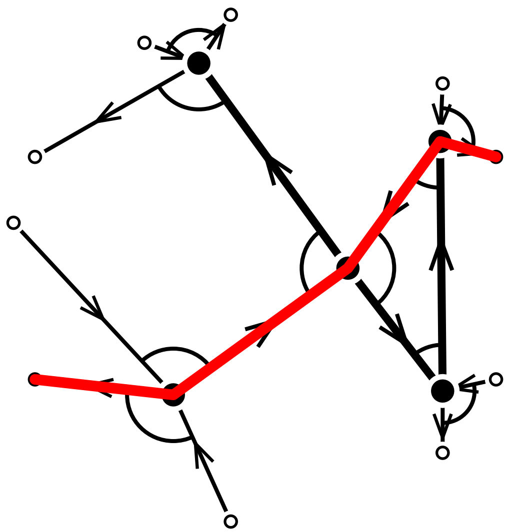
TWO FAMILIES OF NON-KISSING COMPLEXES



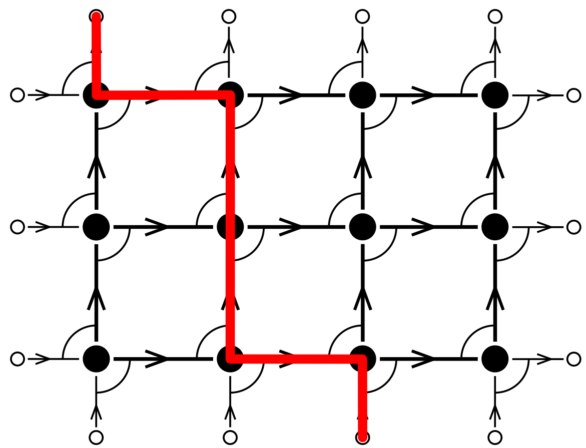
accordion



2457 subset of $[n+m]$

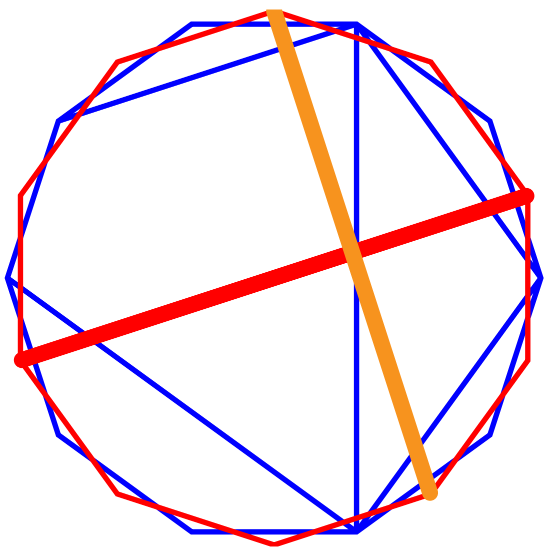


walk

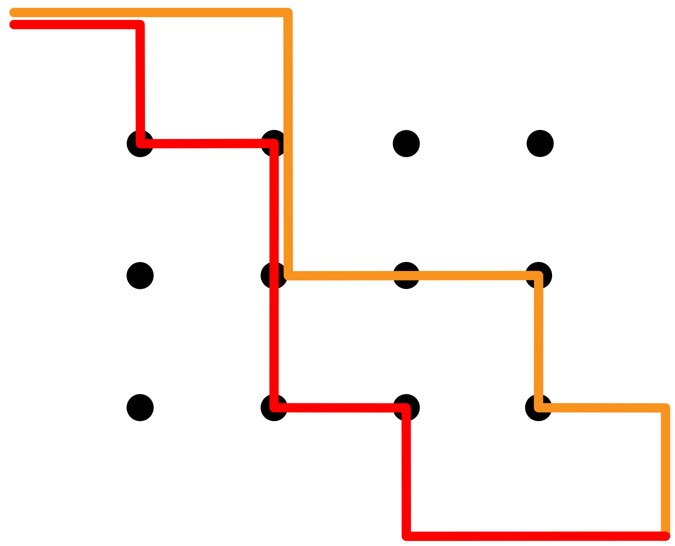


walk

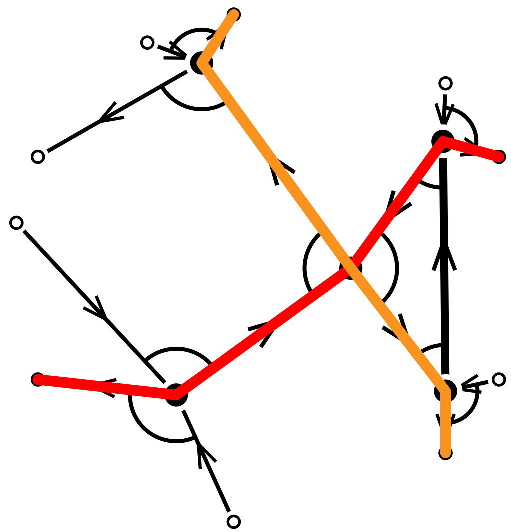
TWO FAMILIES OF NON-KISSING COMPLEXES



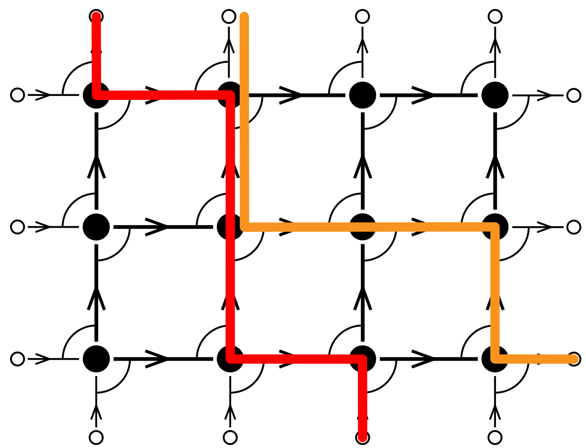
crossing accordions



crossing subsets of $[n+m]$



kissing walks



kissing walks

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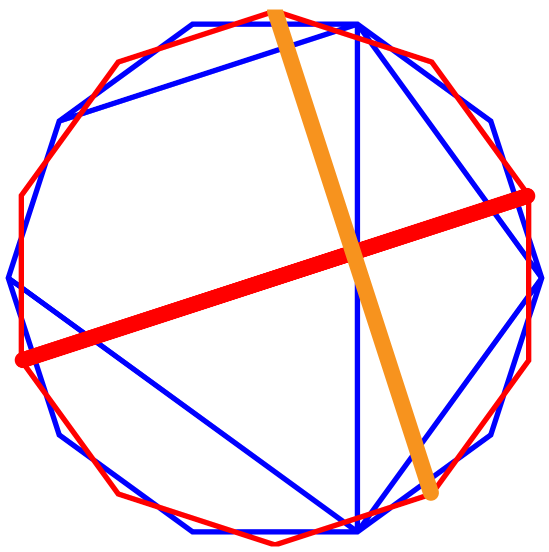
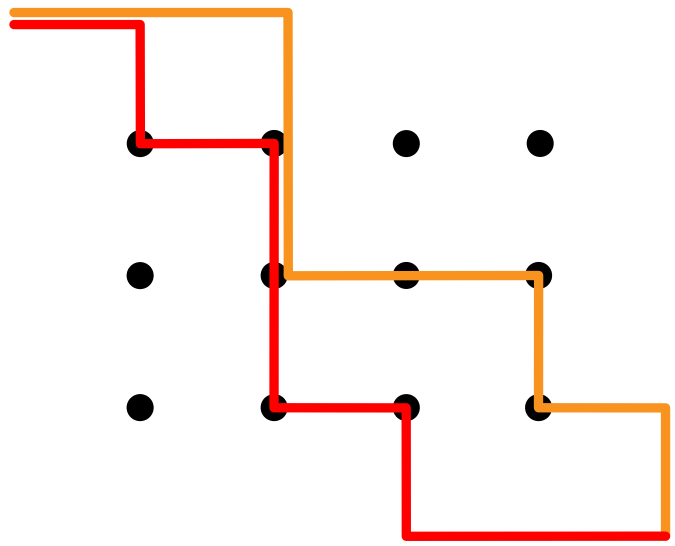


accordion complex

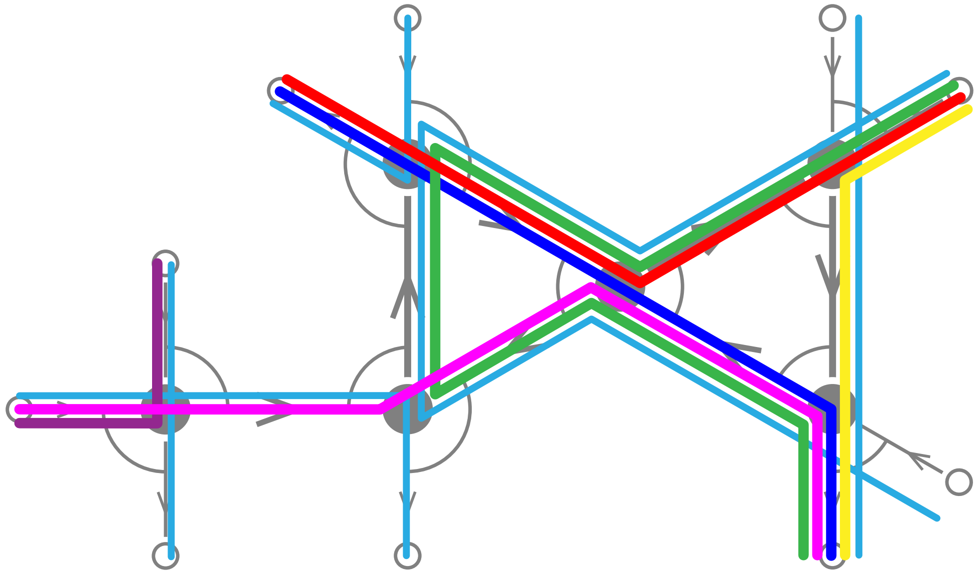


grid Tamari complex

Baryshnikov, On Stokes sets, 2001
Chapoton, Stokes posets and serpent nests, 2016
Garver – McConville, Oriented flip graphs and non-crossing tree partitions, 2017

Petersen – Pylyavskyy – Speyer, A non-crossing standard monomial theory, 2010
Santos – Stump – Welker, Non-crossing sets and the Grassmann-associahedron, 2017
McConville, Lattice structures of grid Tamari orders, 2017
Garver – McConville, Enumerative properties of grid-associahedra, 2017

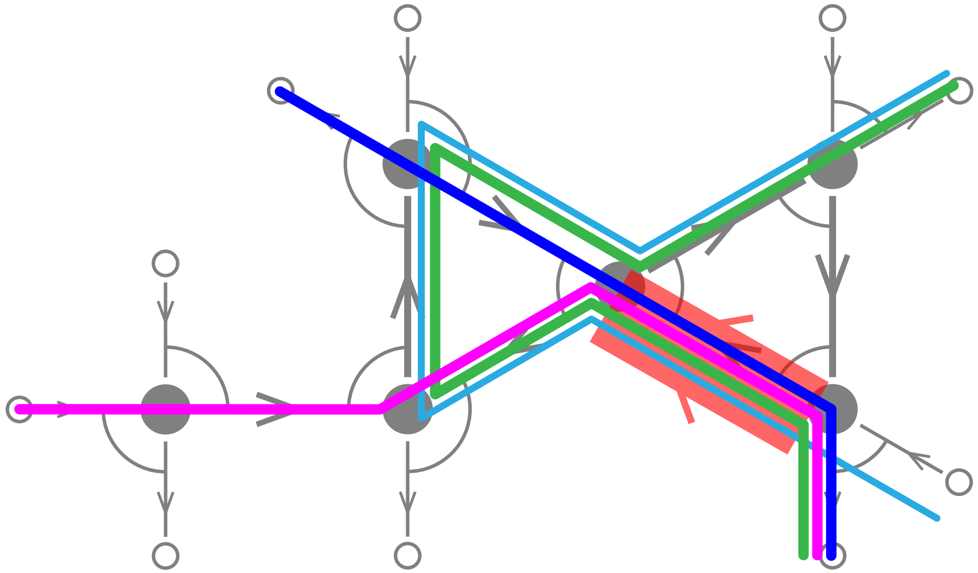
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$

$\alpha \in Q_1$

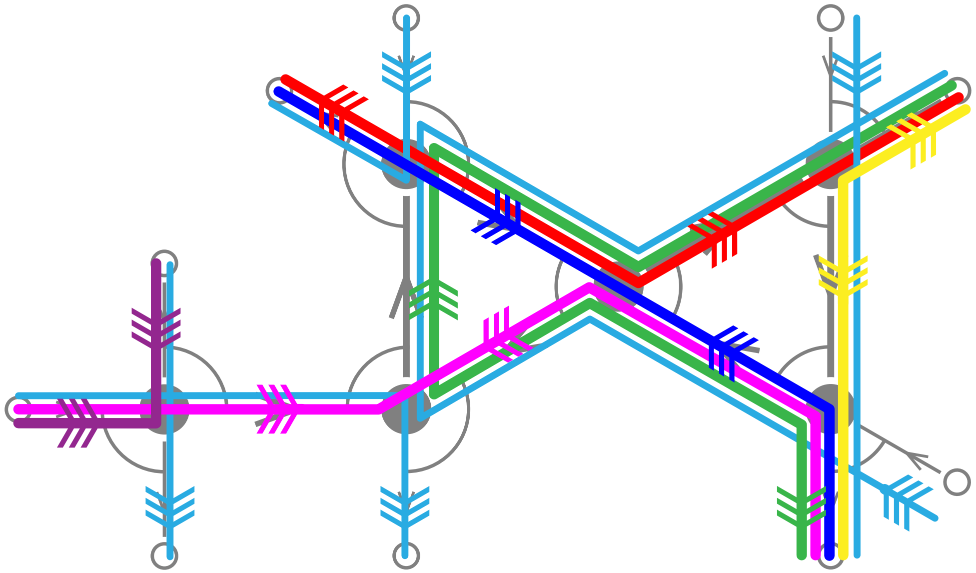
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$
$\alpha \in Q_1$

$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$
$\lambda \prec_\alpha \omega$ countercurrent order at α

DISTINGUISHED WALKS, ARROWS AND STRINGS



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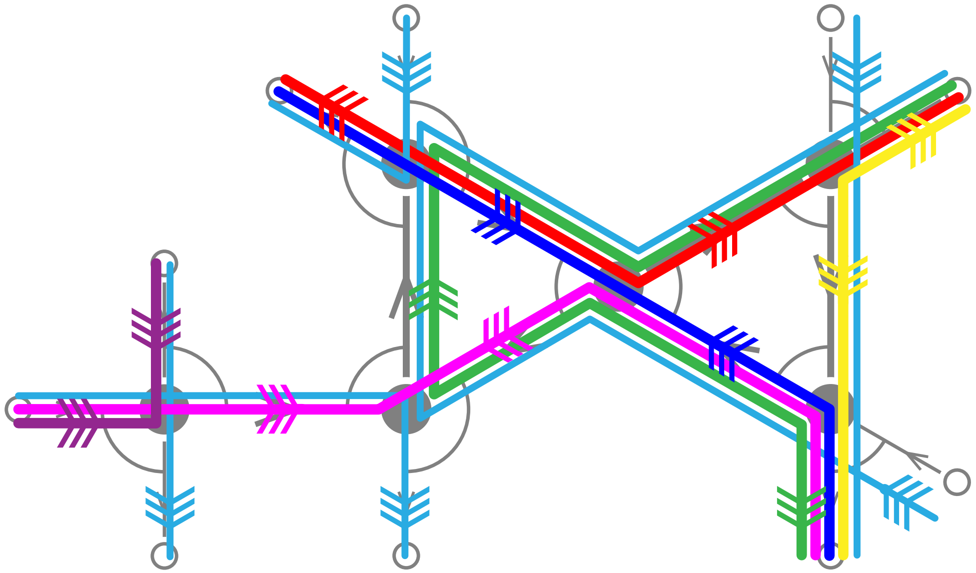
$\mathsf{dw}(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

$\mathsf{da}(\omega, F) = \{\alpha \in Q_1 \mid \omega = \mathsf{dw}(\alpha, F)\}$

DISTINGUISHED WALKS, ARROWS AND STRINGS



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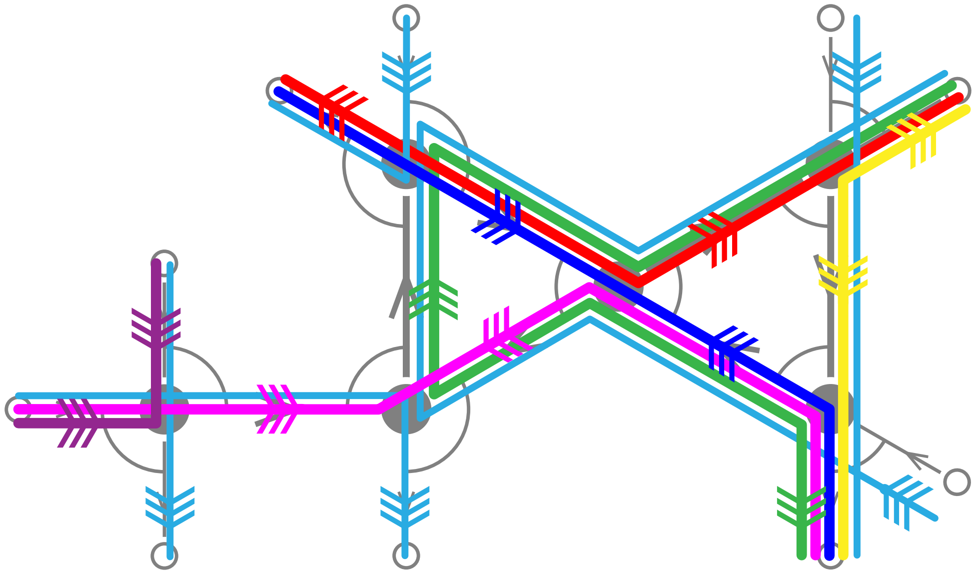
PROP. For any facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$,

- each bended walk of F contains 2 distinguished arrows in F pointing opposite,
- each straight walk of F contains 1 distinguished arrows in F pointing as the walk.

DISTINGUISHED WALKS, ARROWS AND STRINGS



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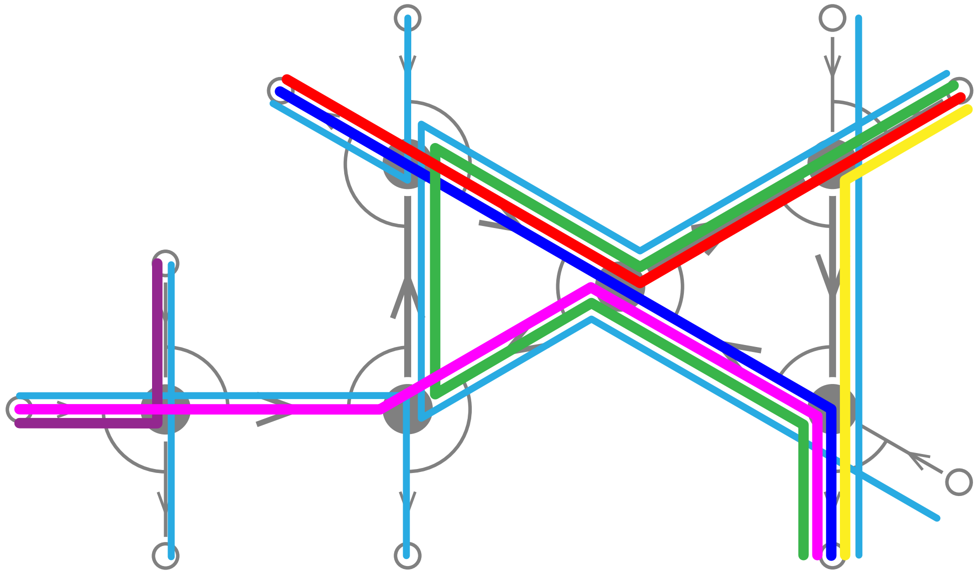
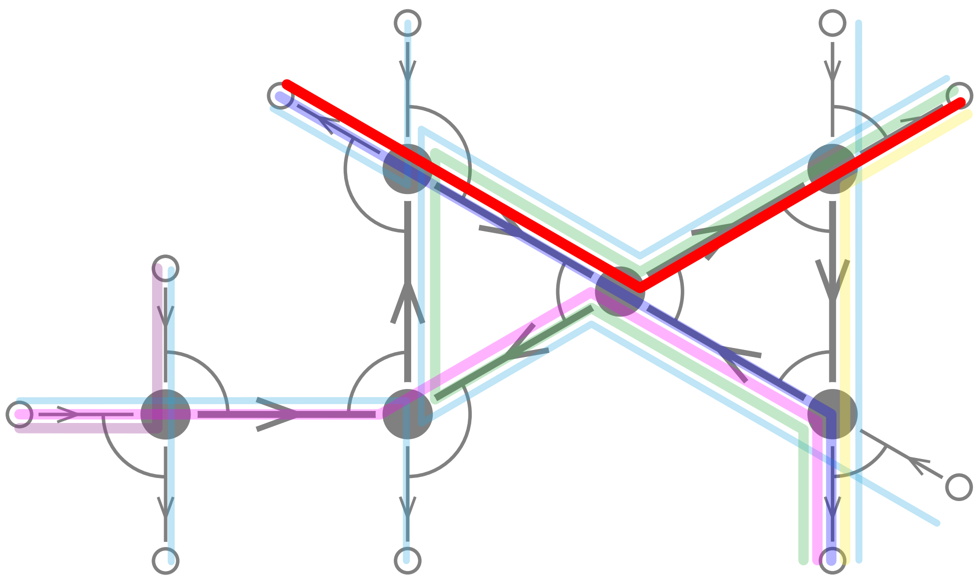
CORO. $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ is pure of dimension $|Q_0|$.

FLIPS

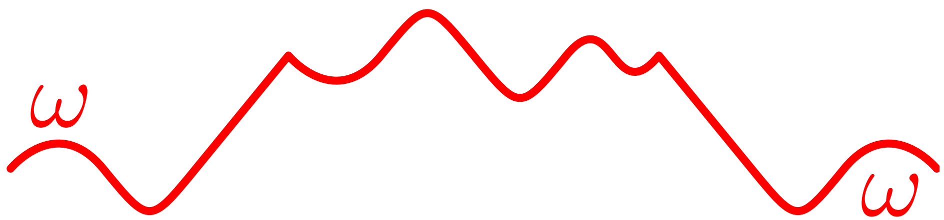


F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

FLIPS



F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)
$\omega \in F$ we want to "flip"



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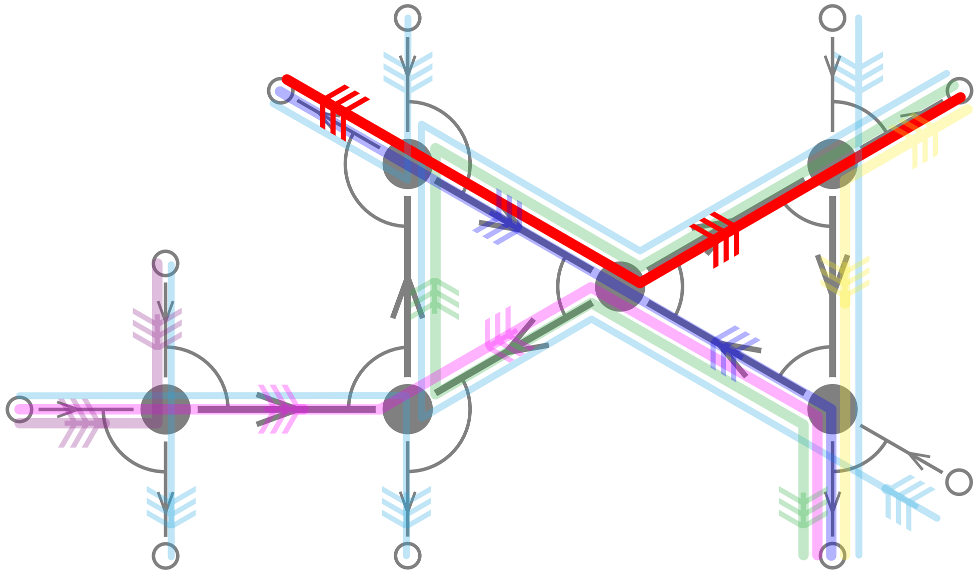

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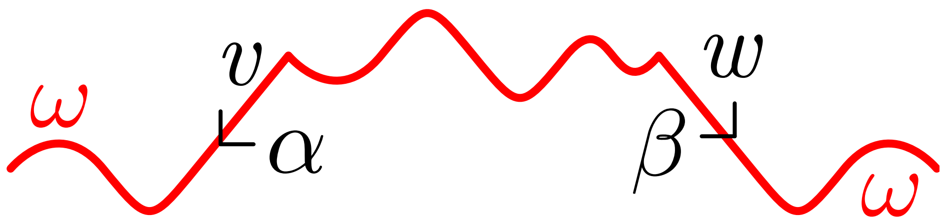

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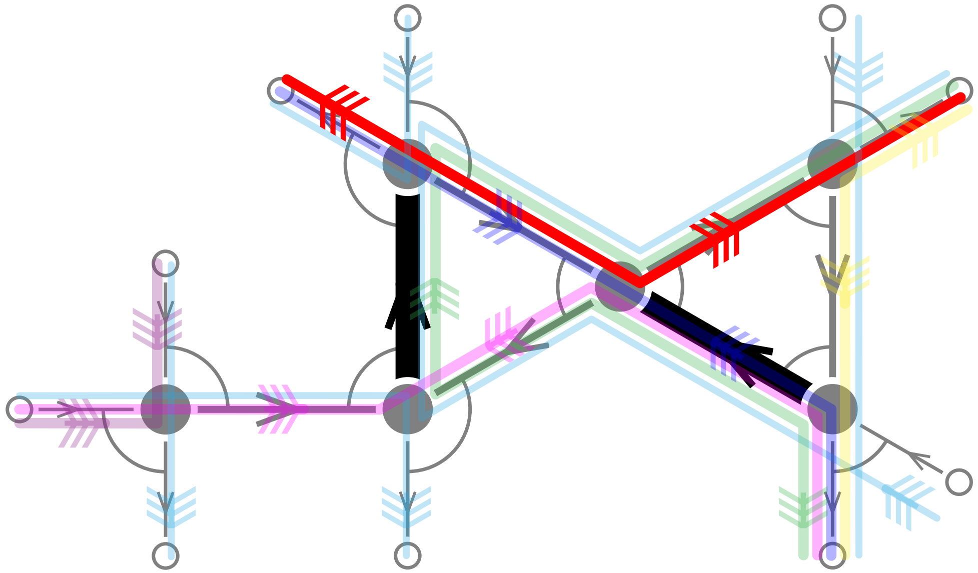

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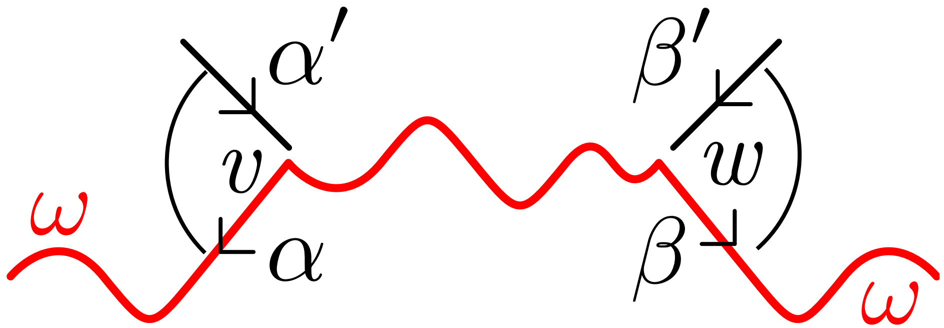
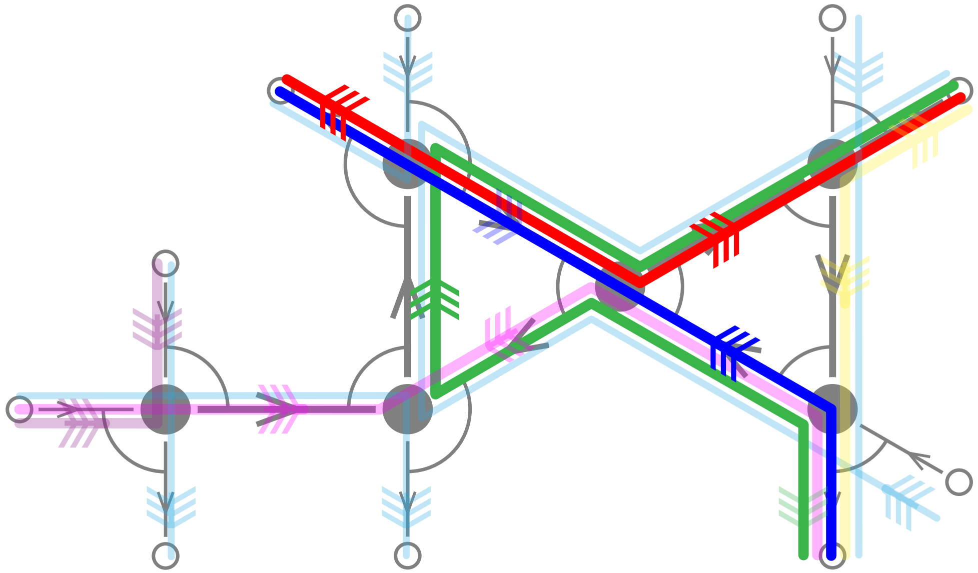
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$



FLIPS



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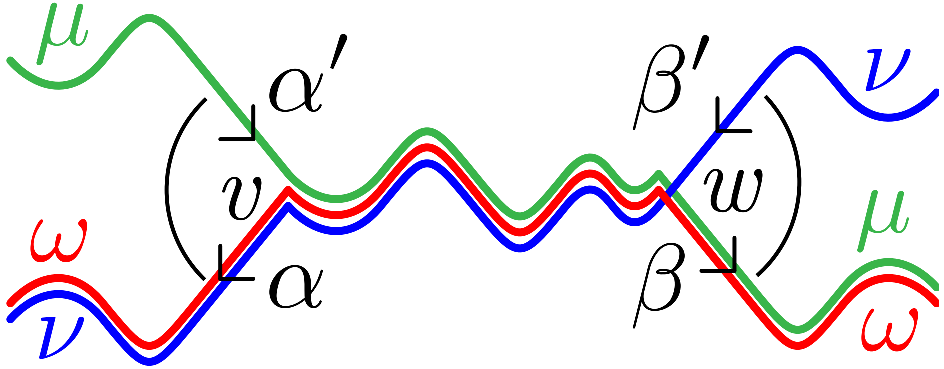
$\mu = \text{dw}(\alpha, F)$ and $\nu = \text{dw}(\beta, F)$



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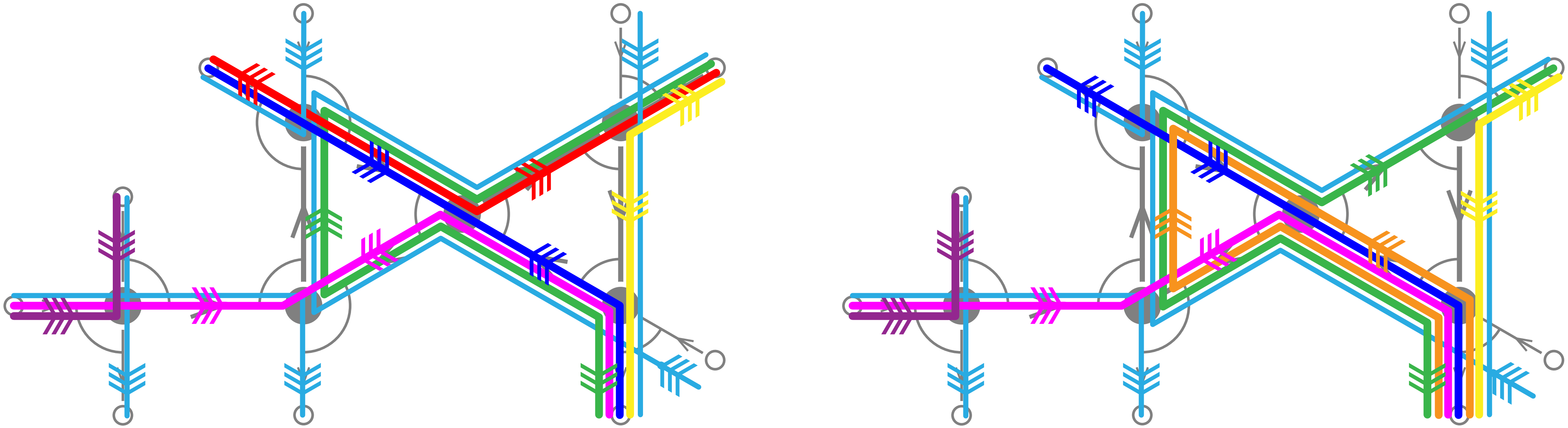

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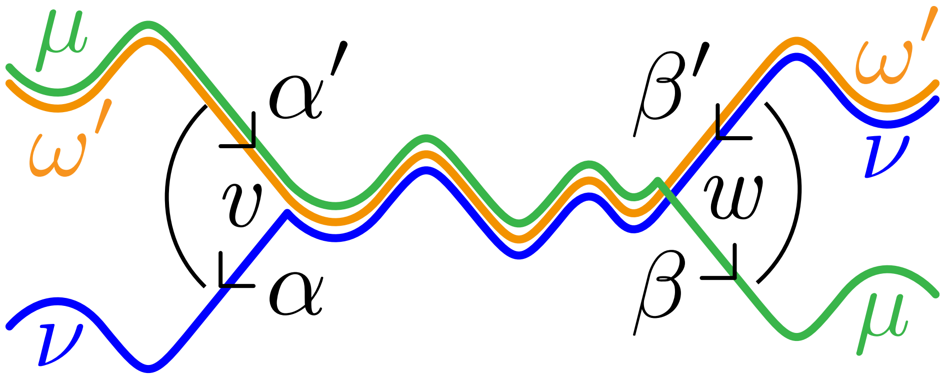
$\mu = \text{dw}(\alpha, F)$ and $\nu = \text{dw}(\beta, F)$

$\omega' = \mu[\cdot, v]\, \sigma\, \nu[w, \cdot]$



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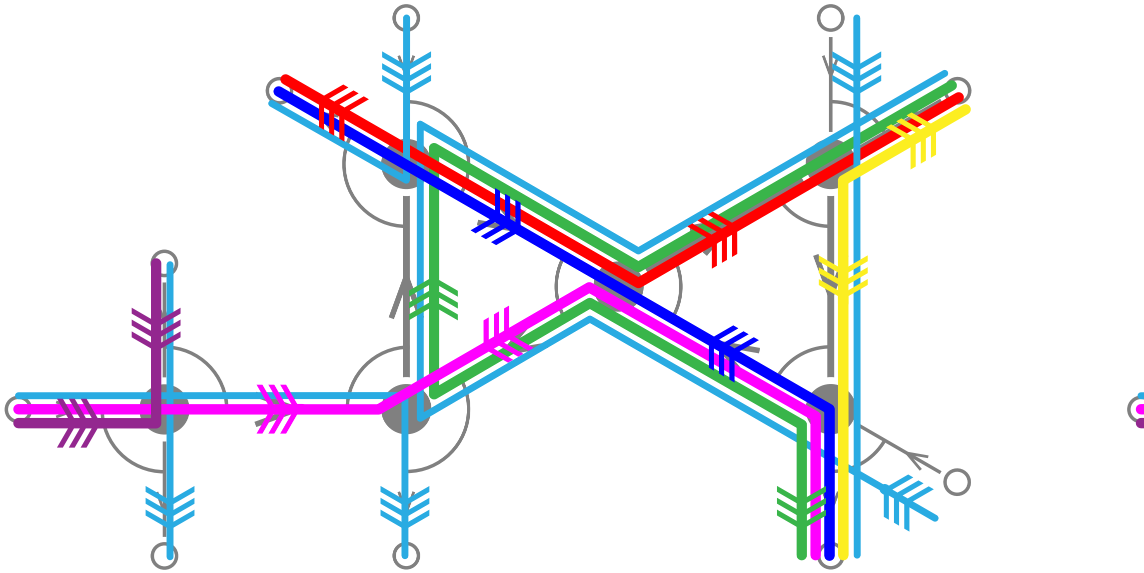
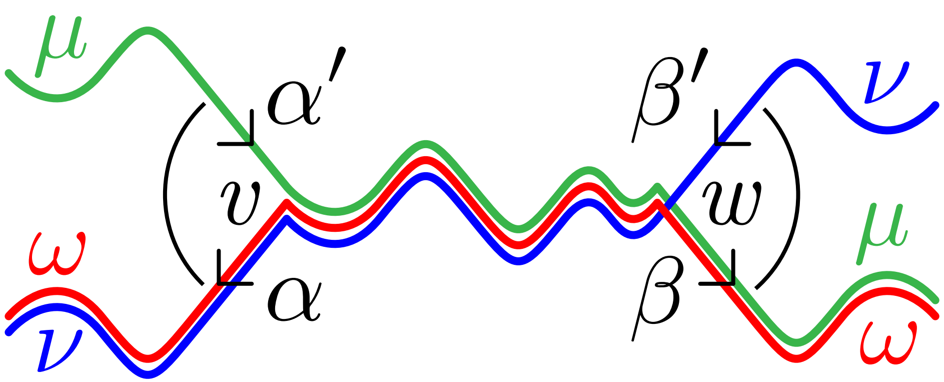
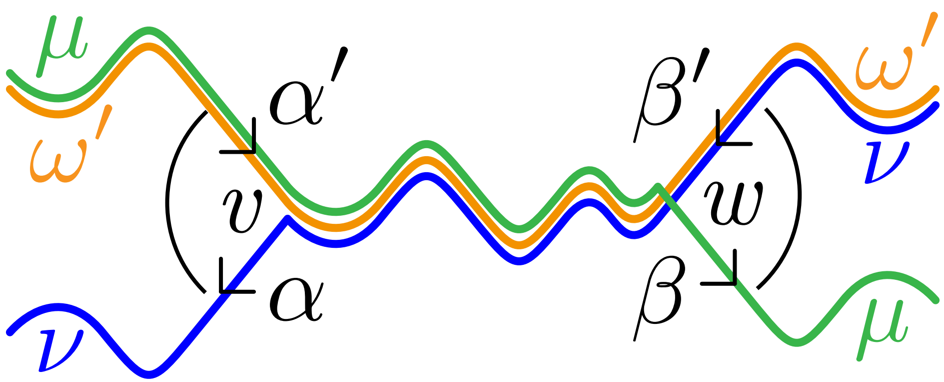
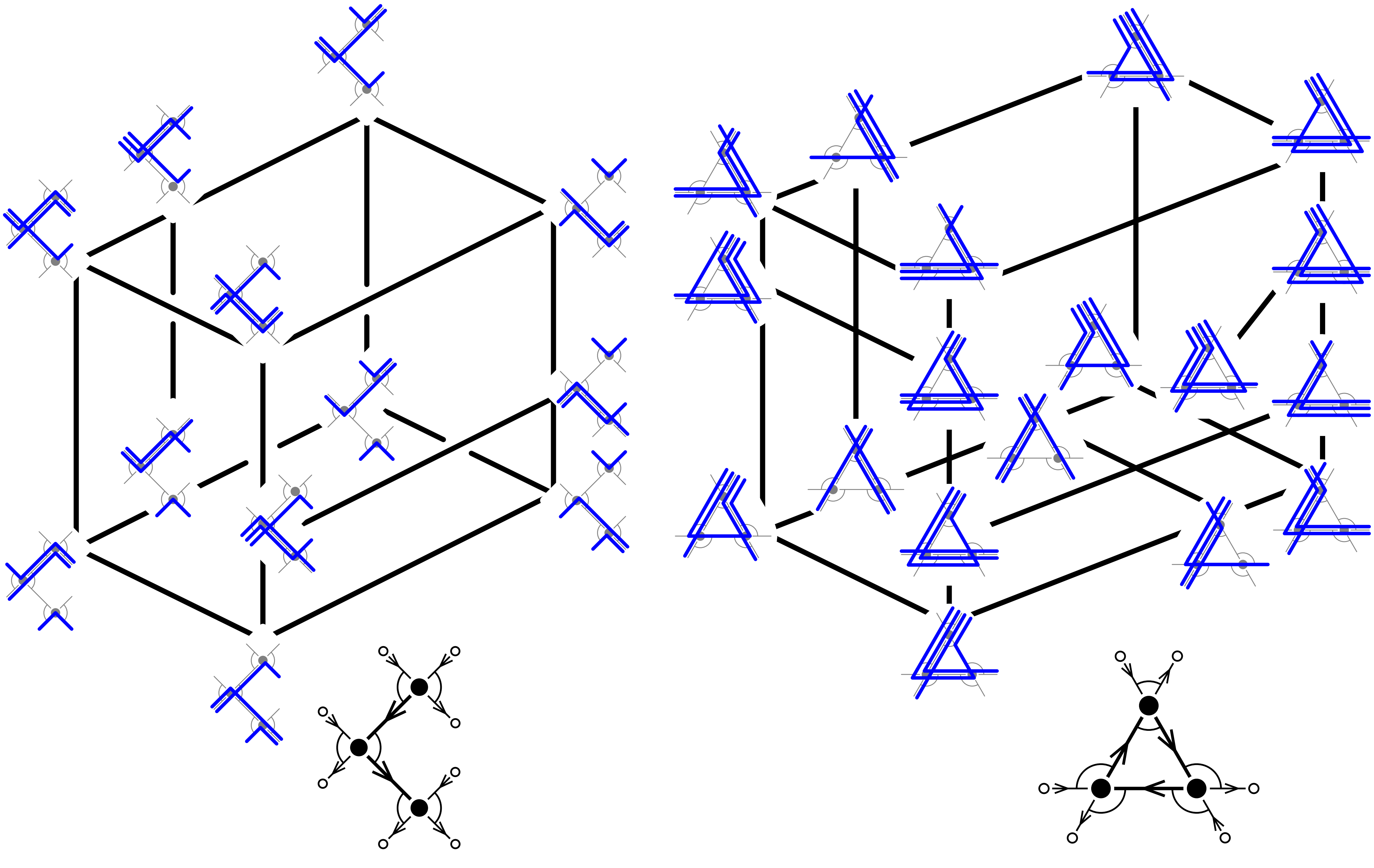


PROP. ω' kisses ω but no other walk of F. Moreover, ω' is the only such walk.





FLIP GRAPH



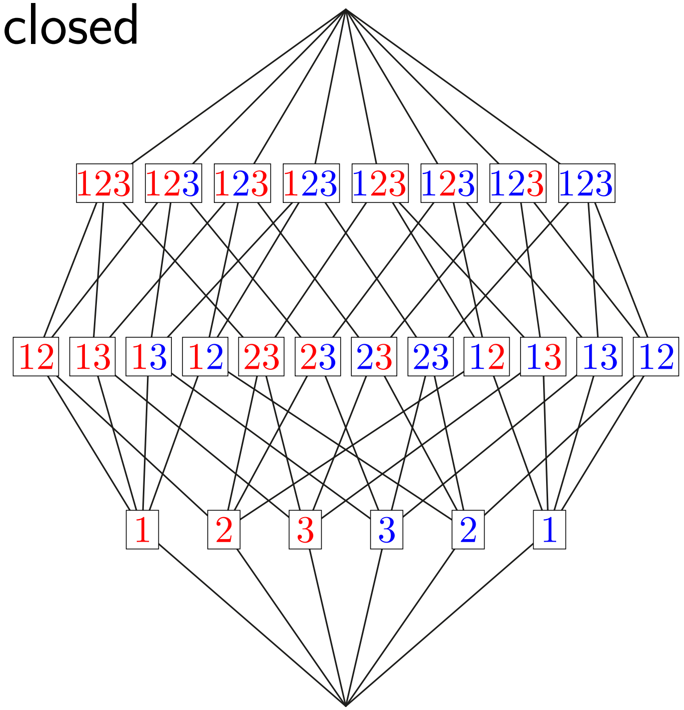
NON-KISSING ASSOCIAHEDRA

SIMPLICIAL COMPLEX

<u>simplicial complex</u> = collection of subsets of X downward closed

exm:

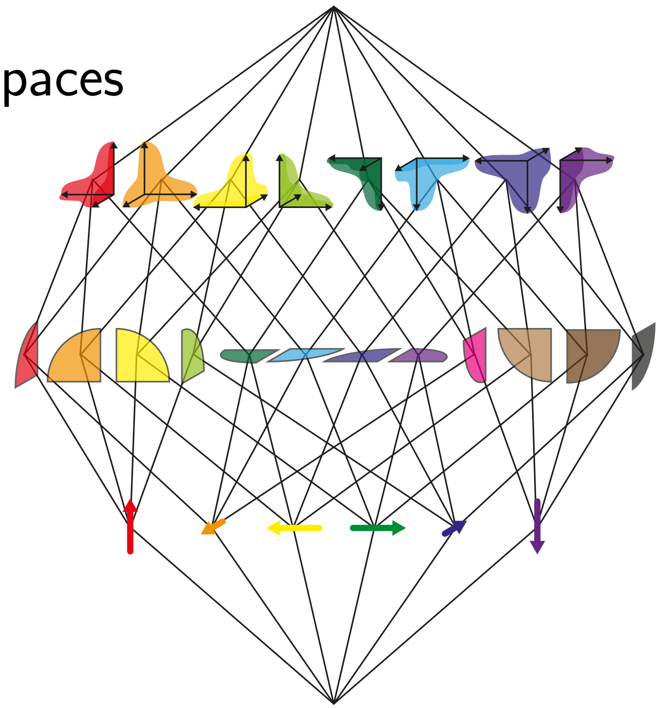
$$X = [n] \cup \underline{[n]}$$
$$\Delta = \{I \subseteq X \mid \forall i \in [n],\ \{i, \underline{i}\} \not\subseteq I\}$$

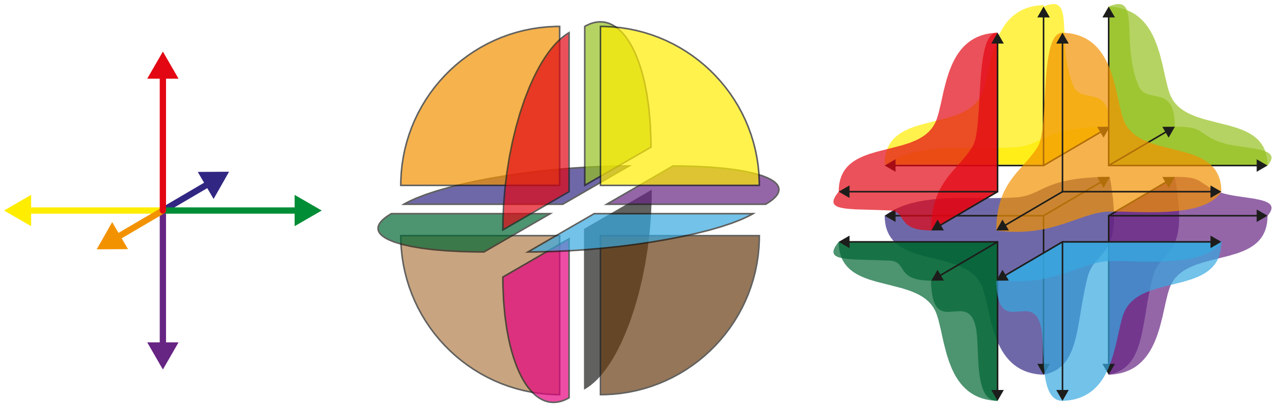


FANS

polyhedral cone = positive span of a finite set of $\mathbb{R}^d$
= intersection of finitely many linear half-spaces

fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face



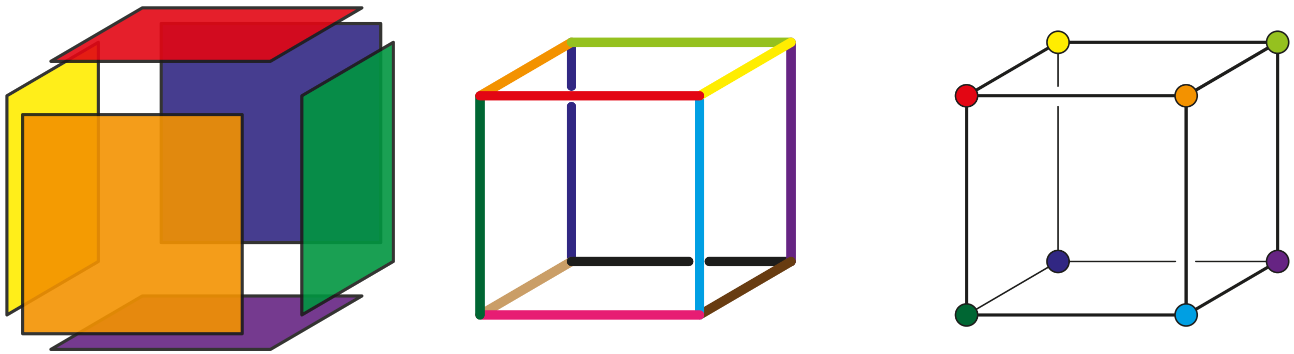


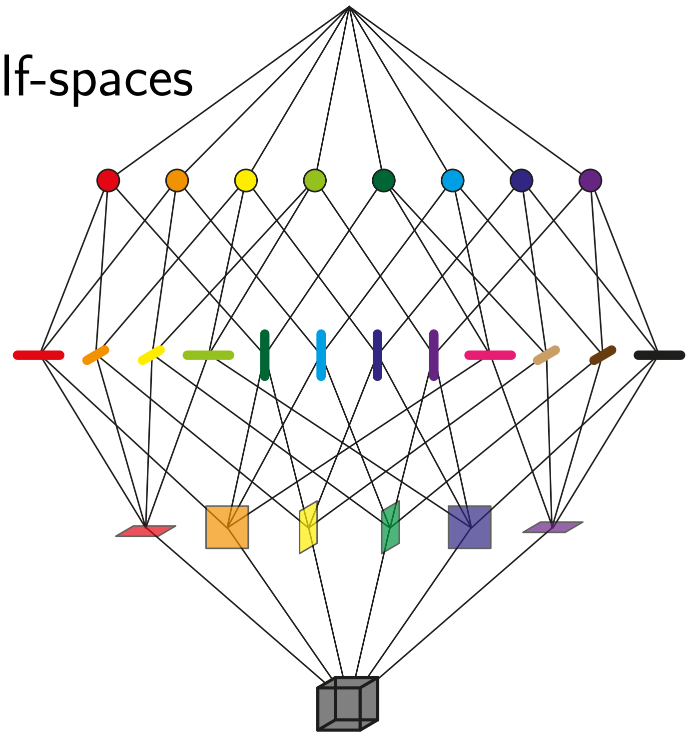
simplicial fan = maximal cones generated by d rays

POLYTOPES

polytope = convex hull of a finite set of $\mathbb{R}^d$
= bounded intersection of finitely many affine half-spaces

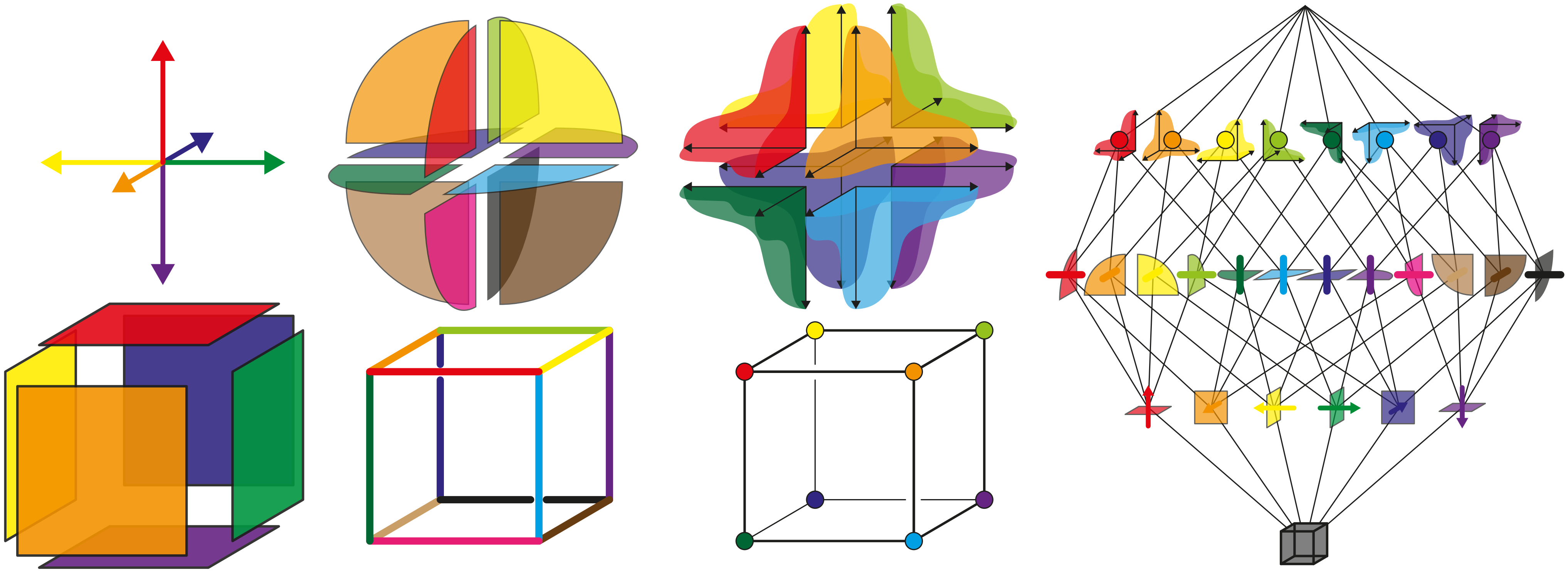
face = intersection with a supporting hyperplane
face lattice = all the faces with their inclusion relations





simple polytope = facets in general position = each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

normal fan of P = { normal cone of F | F face of P }

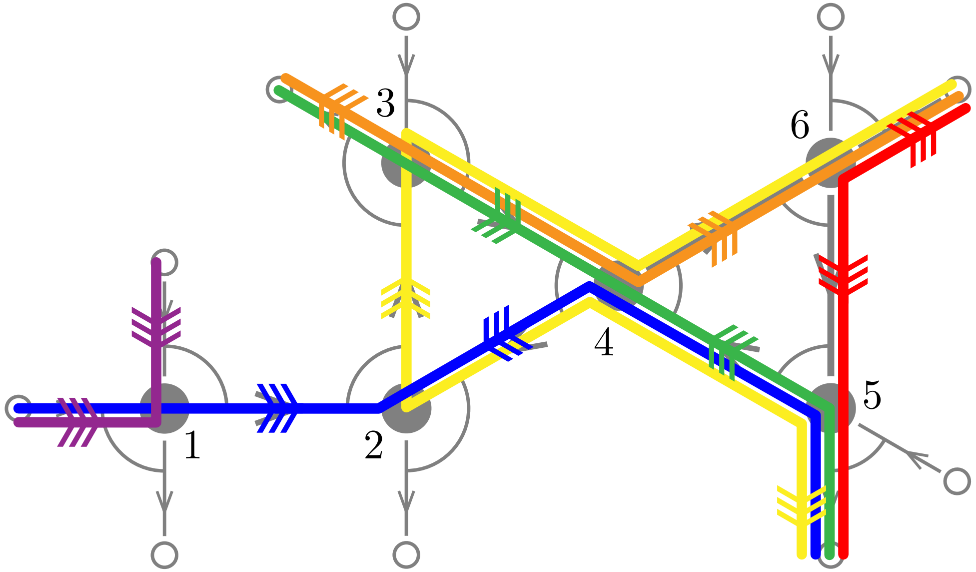
simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

G-VECTORS & C-VECTORS

multiplicity vector $\mathbf{m}_V$ of multiset $V = \{\{v_1, \dots, v_m\}\}$ of Q_0 $\;=\; \sum_{i \in [m]} \mathbf{e}_{v_i} \in \mathbb{R}^{Q_0}$

g-vector $\mathbf{g}(\omega)$ of a walk ω $\;=\; \mathbf{m}_{\mathsf{peaks}(\omega)} - \mathbf{m}_{\mathsf{deeps}(\omega)}$

c-vector $\mathbf{c}(\omega \in F)$ of a walk ω in a non-kissing facet F $\;=\; \varepsilon(\omega, F)\, \mathbf{m}_{\mathsf{ds}(\omega,F)}$



1	0	0	0	0	0	−1
2	0	0	0	0	−1	0
3	0	1	0	1	0	0
4	0	0	0	−1	0	0
5	0	0	1	0	1	0
6	1	0	0	0	0	0

$\mathbf{g}(F)$

1	0	0	0	0	0	−1
2	0	0	1	0	−1	0
3	0	1	0	0	0	0
4	0	1	1	−1	0	0
5	0	0	1	0	0	0
6	1	0	0	0	0	0

$\mathbf{c}(F)$

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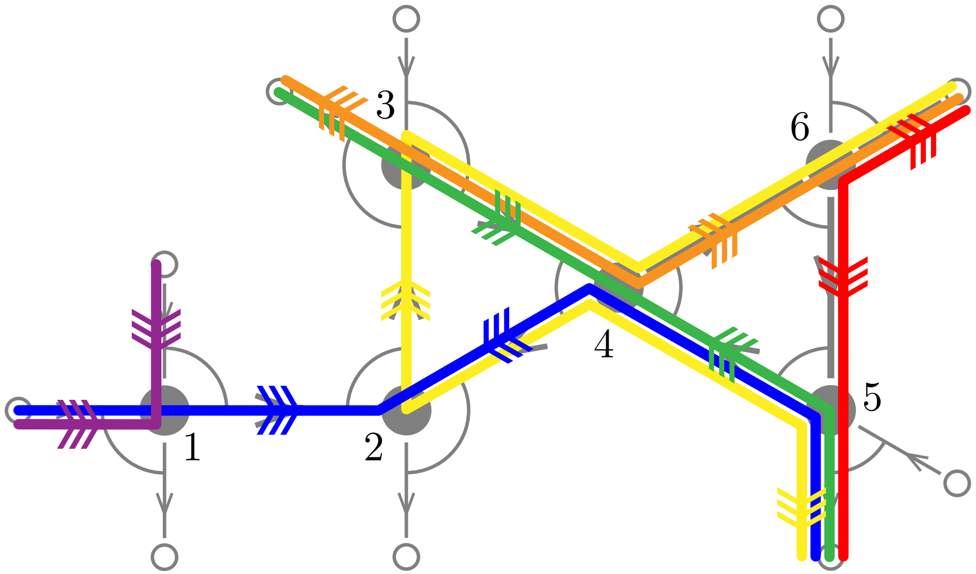


$$\mathbf{g}(F) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

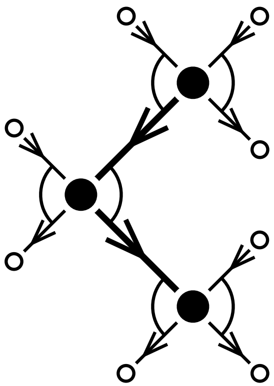
$$\mathbf{c}(F) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

PROP. For any non-kissing facet F, the sets of vectors

$$\mathbf{g}(F) := \{\mathbf{g}(\omega) \mid \omega \in F\} \qquad \text{and} \qquad \mathbf{c}(F) := \{\mathbf{c}(\omega \in F) \mid \omega \in F\}$$

form dual bases.

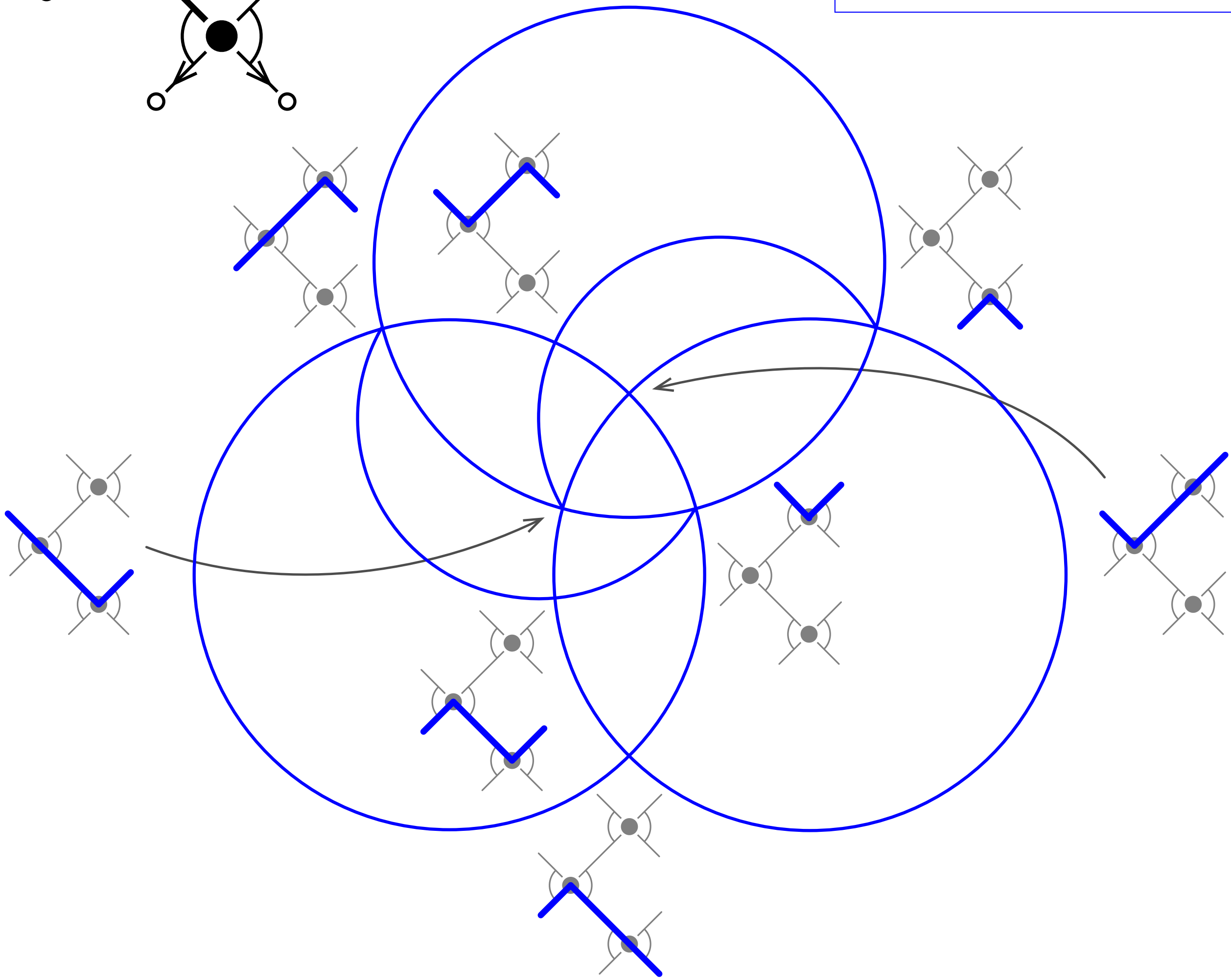
G-VECTOR FAN



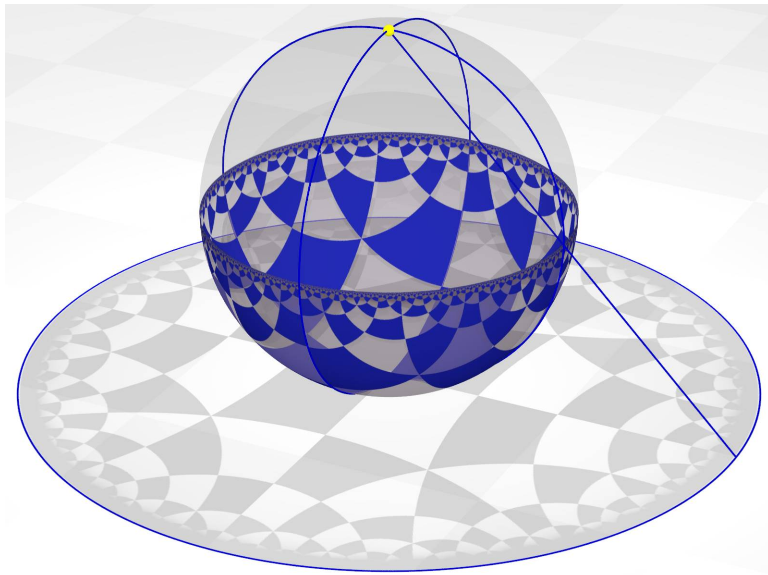
THM. For any gentle quiver $\bar{Q}$, the collection of cones

$$\mathcal{F}^{\mathbf{g}}(\bar{Q}) := \left\{ \mathbb{R}_{\geq 0}\mathbf{g}(F) \;\middle|\; F \in \mathcal{C}_{\mathrm{nk}}(\bar{Q}) \right\}$$

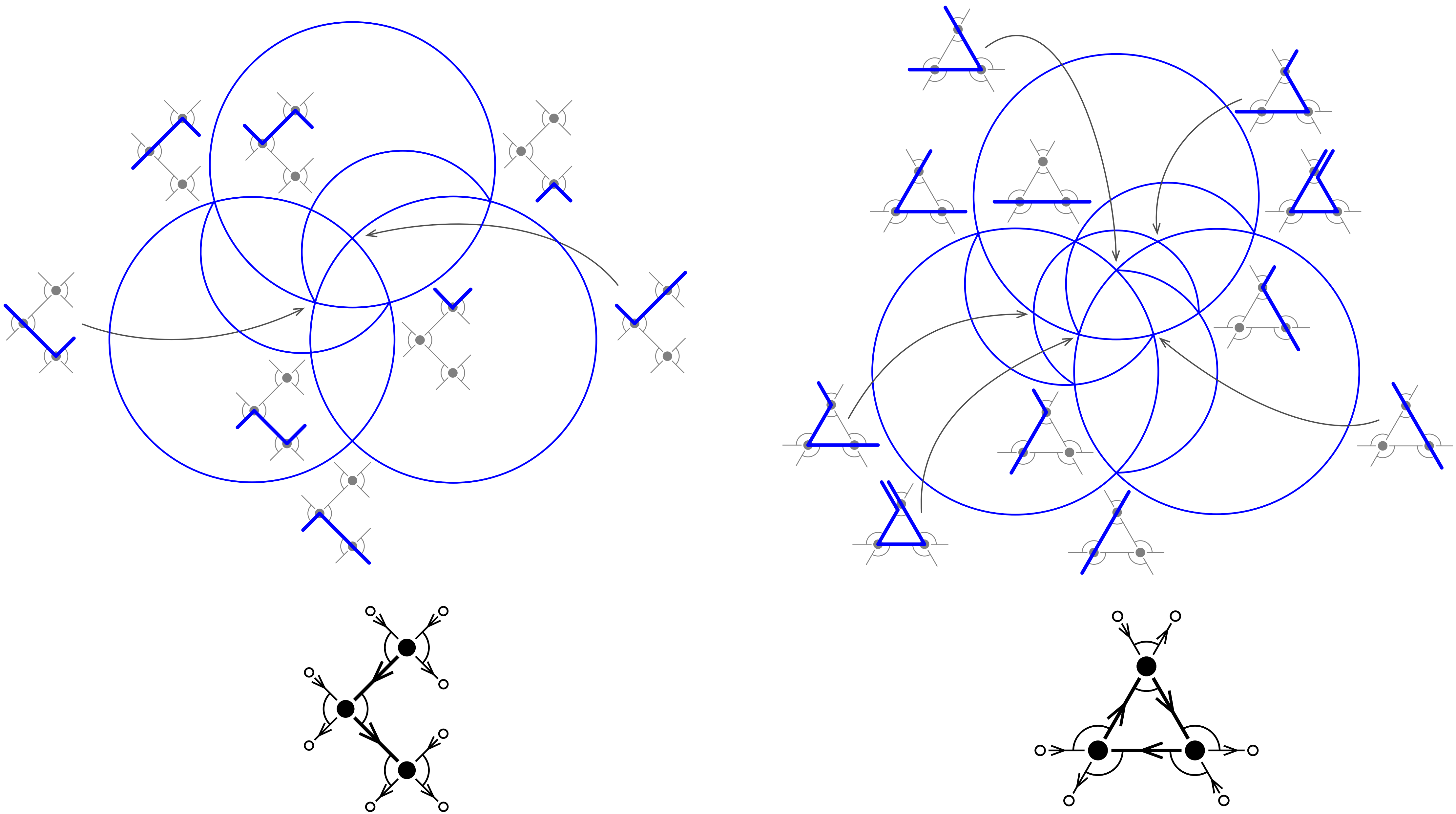
forms a compl. simpl. fan, called g-vector fan of $\bar{Q}$.



stereographic projection from $(1, 1, 1)$



G-VECTOR FAN



NON-KISSING ASSOCIAHEDRON

kissing number $\kappa(\omega, \omega') \;=\;$ number of times ω kisses ω'

kissing number $\mathsf{kn}(\omega) \;=\; \sum_{\omega'} \kappa(\omega, \omega') + \kappa(\omega', \omega)$

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{C}_{\rm nk}(\bar{Q})$, the two sets of $\mathbb{R}^{Q_0}$ given by

(i) the convex hull of the points

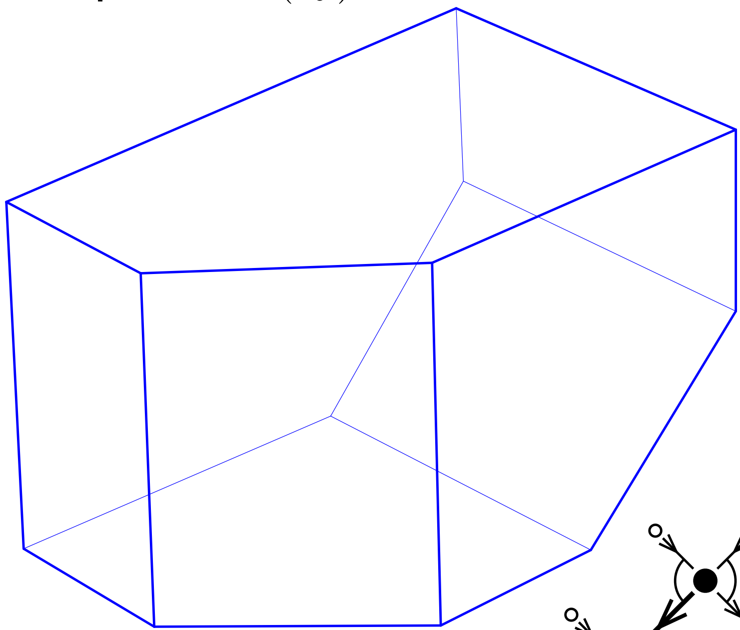
$$\mathbf{p}(F) := \sum_{\omega \in F} \mathsf{kn}(\omega)\, \mathbf{c}(\omega \in F),$$

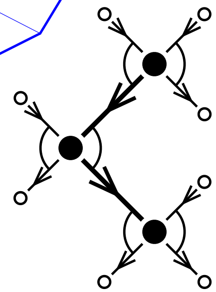
for all non-kissing facets $F \in \mathcal{C}_{\rm nk}(\bar{Q})$,

(ii) the intersection of the halfspaces

$$\mathbf{H}^{\geq}(\omega) := \left\{ \mathbf{x} \in \mathbb{R}^{Q_0} \;\middle|\; \langle\, \mathbf{g}(\omega) \mid \mathbf{x} \,\rangle \leq \mathsf{kn}(\omega) \right\}.$$

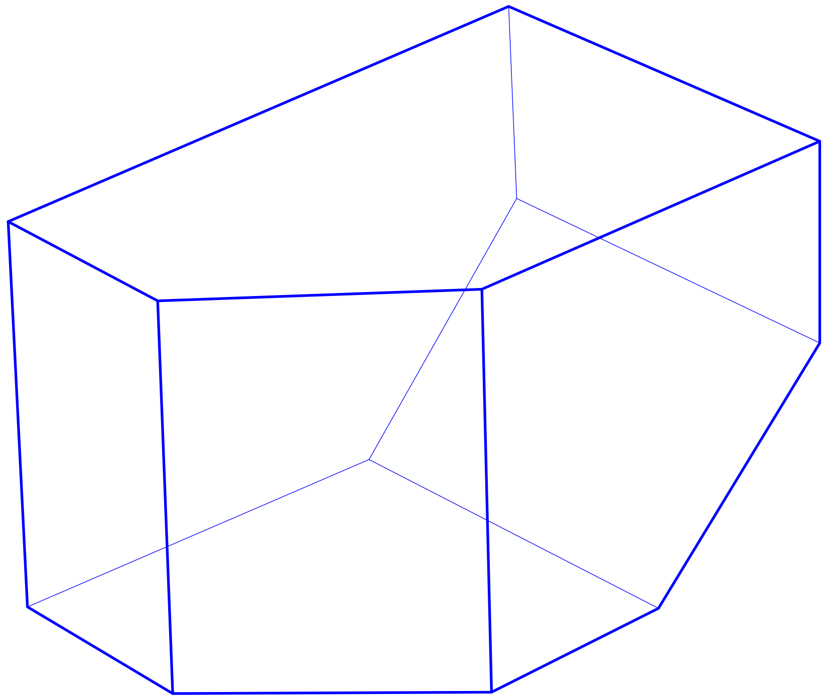
for all walks ω of $\bar{Q}$,

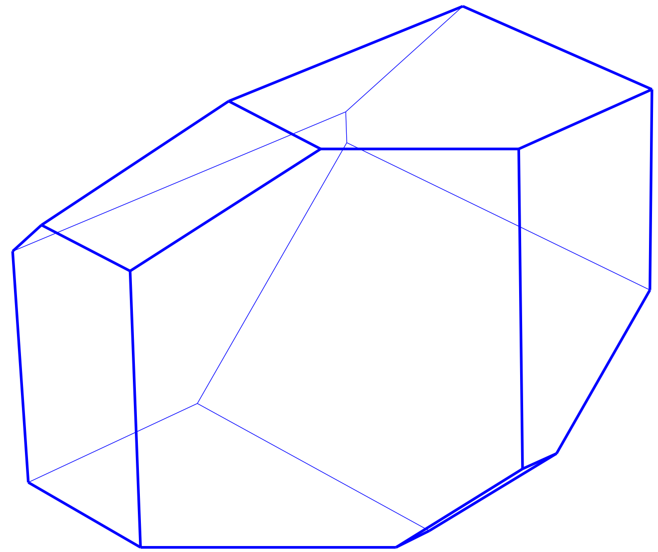


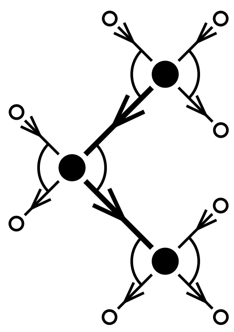


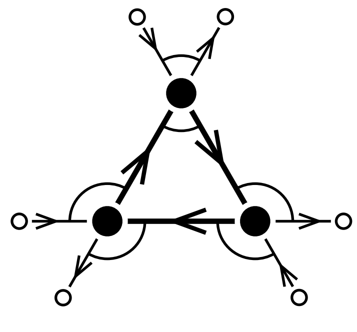
define the same polytope, whose normal fan is the g-vector fan $\mathcal{F}^{\rm g}$. We call it the $\bar{Q}$-associahedron and denote it by Asso.

NON-KISSING ASSOCIAHEDRON

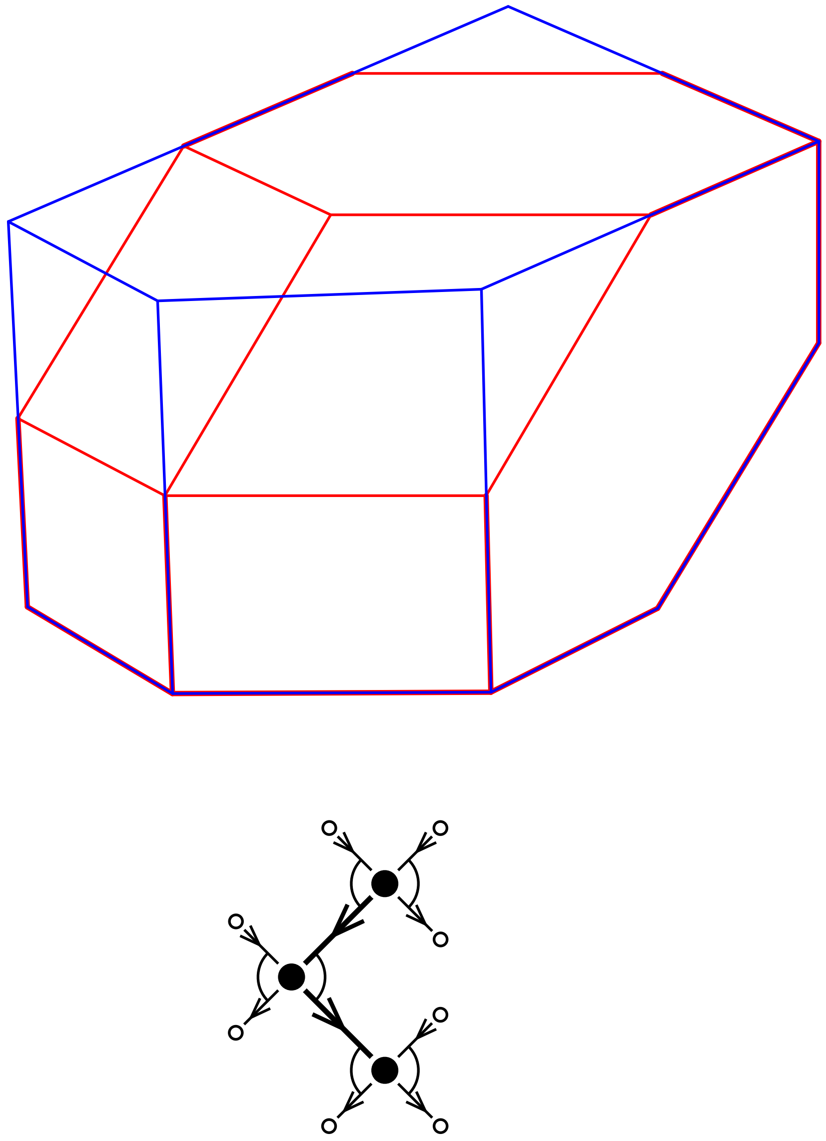


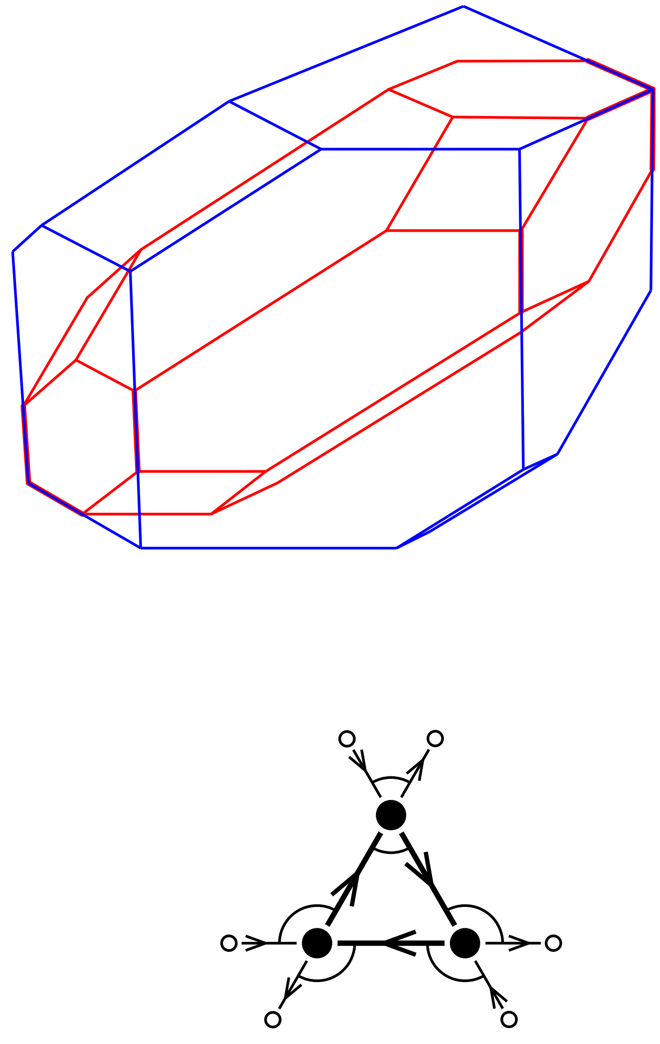






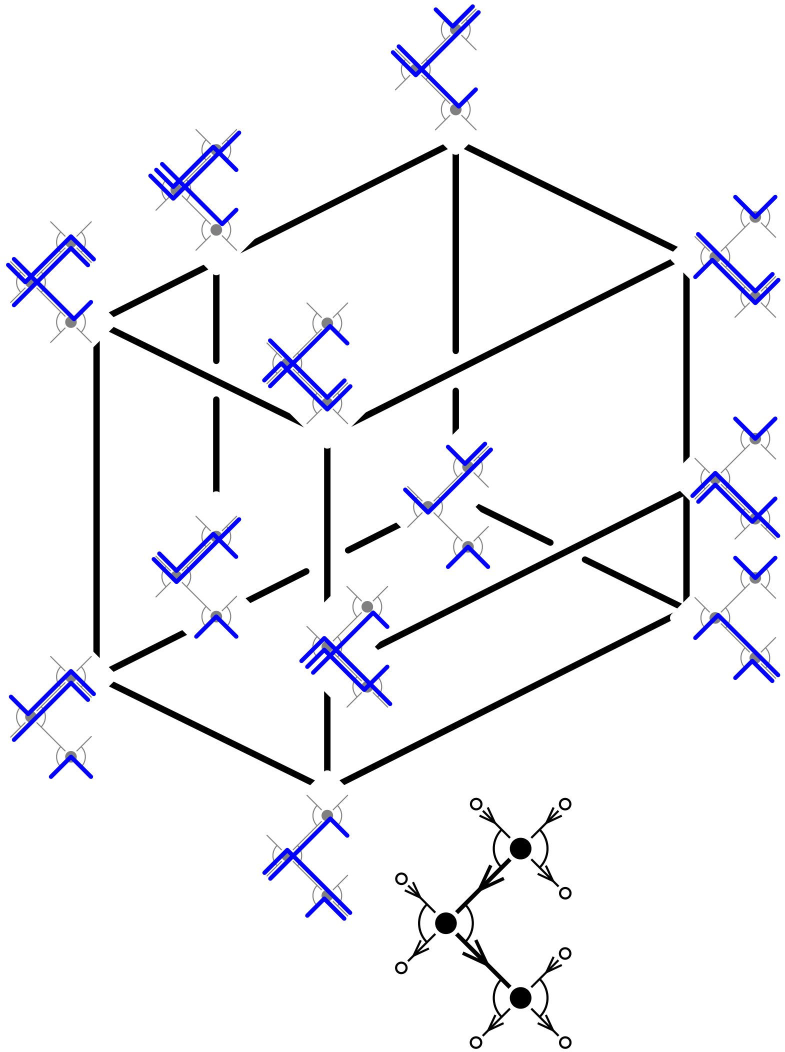
NON-KISSING ASSOCIAHEDRON VS ZONOTOPES

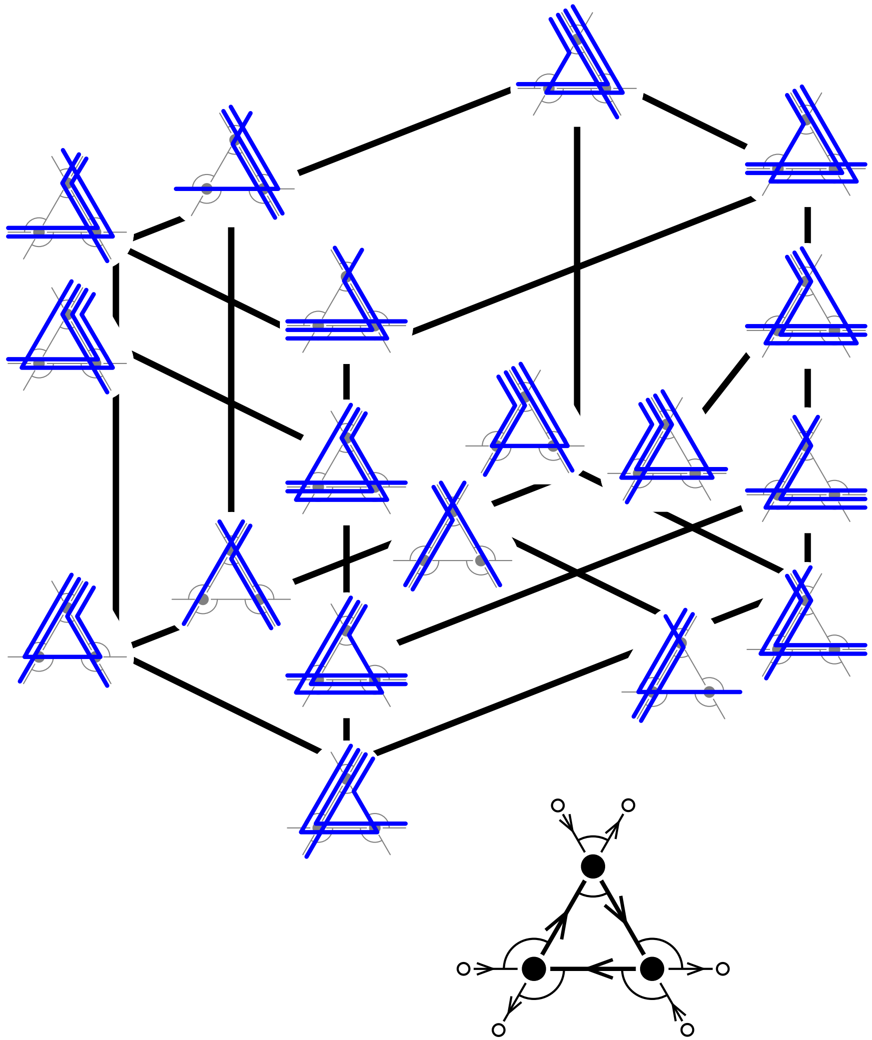




NON-KISSING LATTICE

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.





LATTICE QUOTIENTS

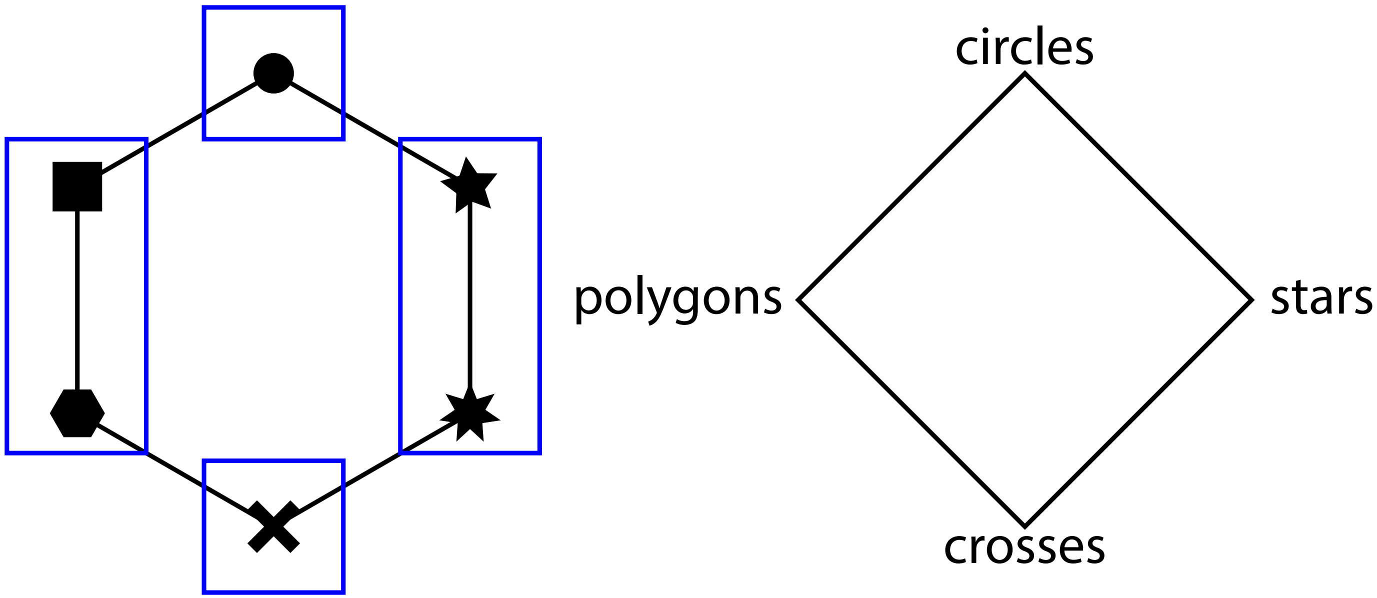
lattice $=$ poset $(L, \leq)$ with a meet $\wedge$ and a join $\vee$

lattice congruence $=$ equiv. rel. $\equiv$ on L which respects meets and joins

$$x \equiv x' \quad \text{and} \quad y \equiv y' \qquad \Longrightarrow \qquad x \wedge y \equiv x' \wedge y' \quad \text{and} \quad x \vee y \equiv x' \vee y'$$

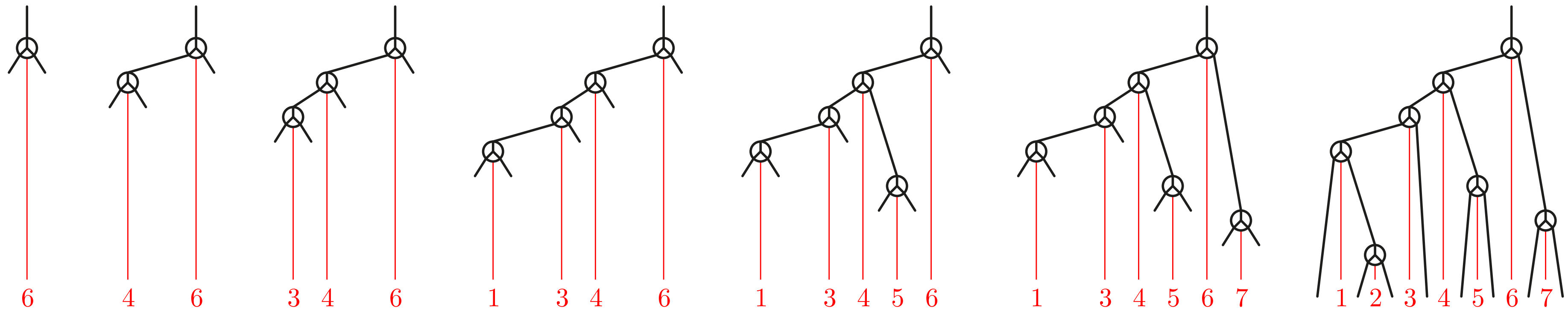
lattice quotient of $L/{\equiv}$ $=$ lattice on equiv. classes of L under $\equiv$ where

- $X \leq Y \iff \exists x \in X,\ y \in Y,\ \ x \leq y$
- $X \wedge Y$ $=$ equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ $=$ equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$



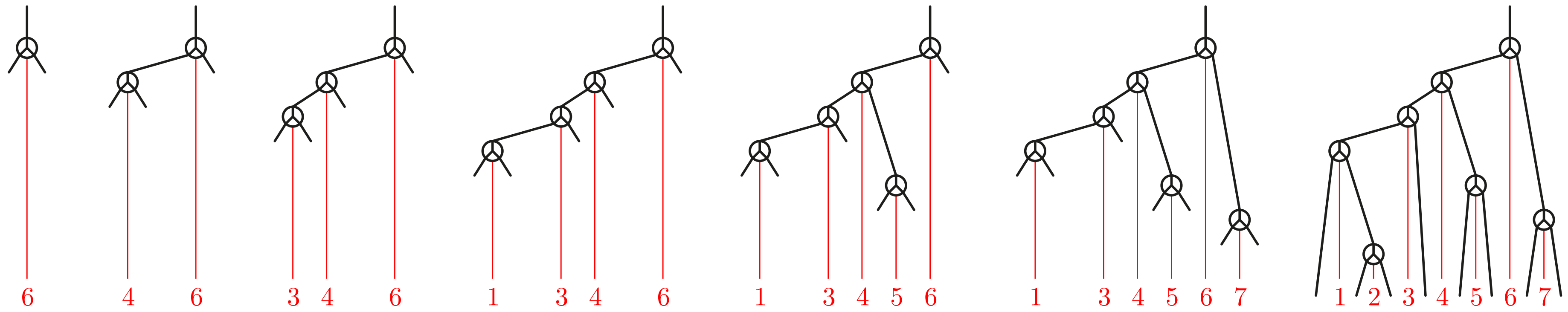
EXM: TAMARI LATTICE AS LATTICE QUOTIENT OF WEAK ORDER

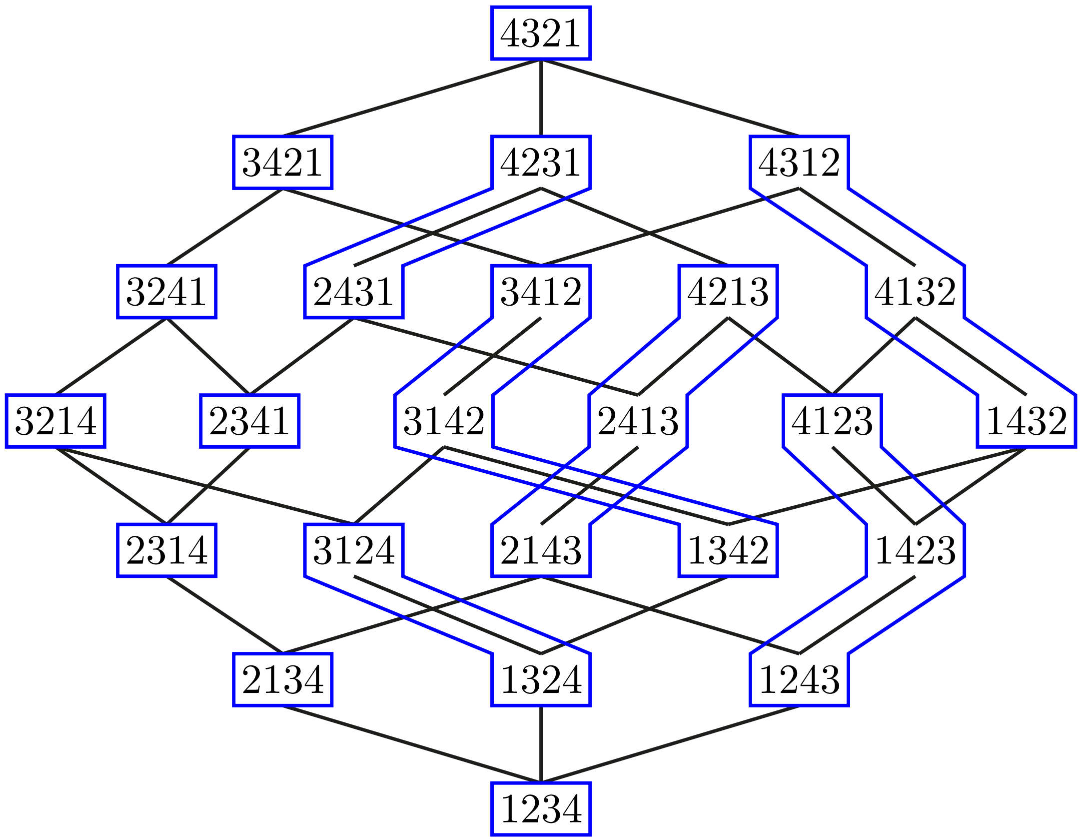
binary search tree insertion of 2751346

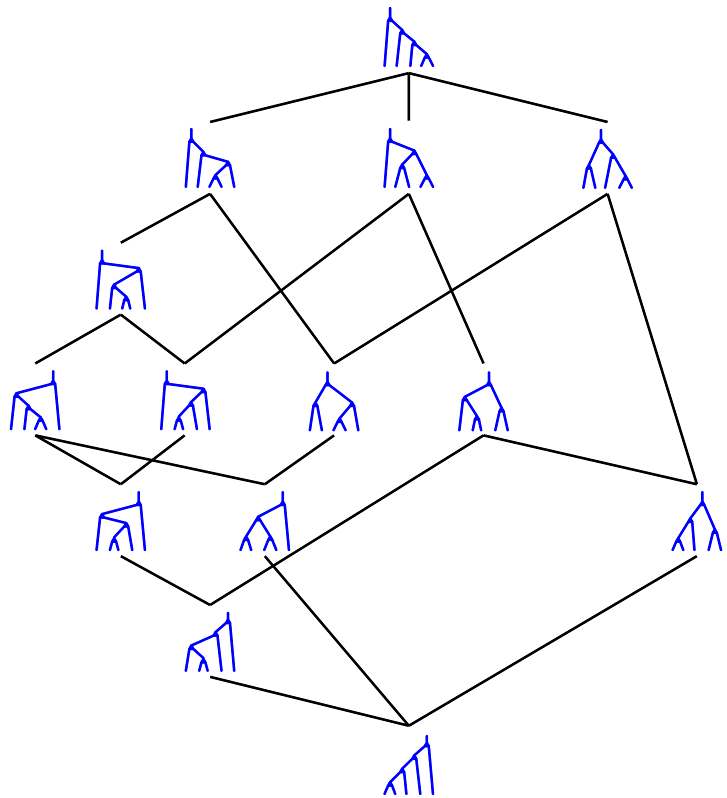


EXM: TAMARI LATTICE AS LATTICE QUOTIENT OF WEAK ORDER

binary search tree insertion of 2751346







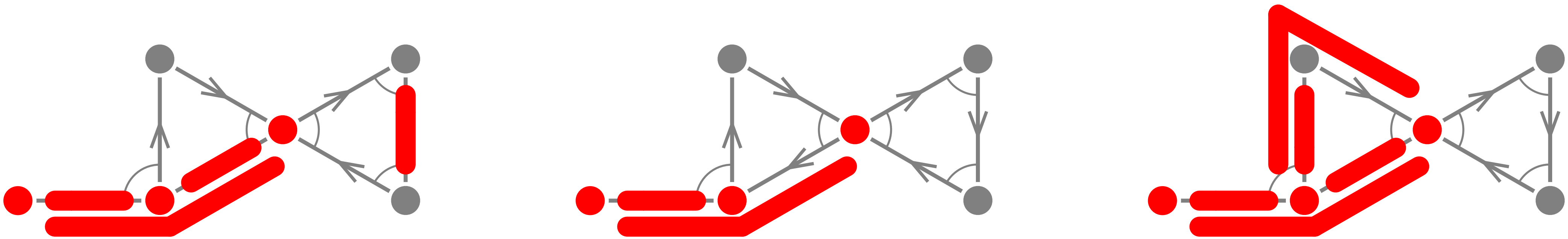
BICLOSED SETS OF SEGMENTS

σ, τ oriented strings

concatenation $\sigma \circ \tau = \{\sigma\alpha\tau \mid \alpha \in Q_1 \text{ and } \sigma\alpha\tau \text{ string of } \bar{Q}\}$

closure $S^{\mathrm{cl}} = \bigcup_{\substack{\ell \in \mathbb{N} \\ \sigma_1, \dots, \sigma_\ell \in S}} \sigma_1 \circ \dots \circ \sigma_\ell =$ all strings obtained by concatenation of some strings of S

closed $\iff S^{\mathrm{cl}} = S$ coclosed $\iff \bar{S}^{\mathrm{cl}} = \bar{S}$ biclosed $=$ closed and coclosed

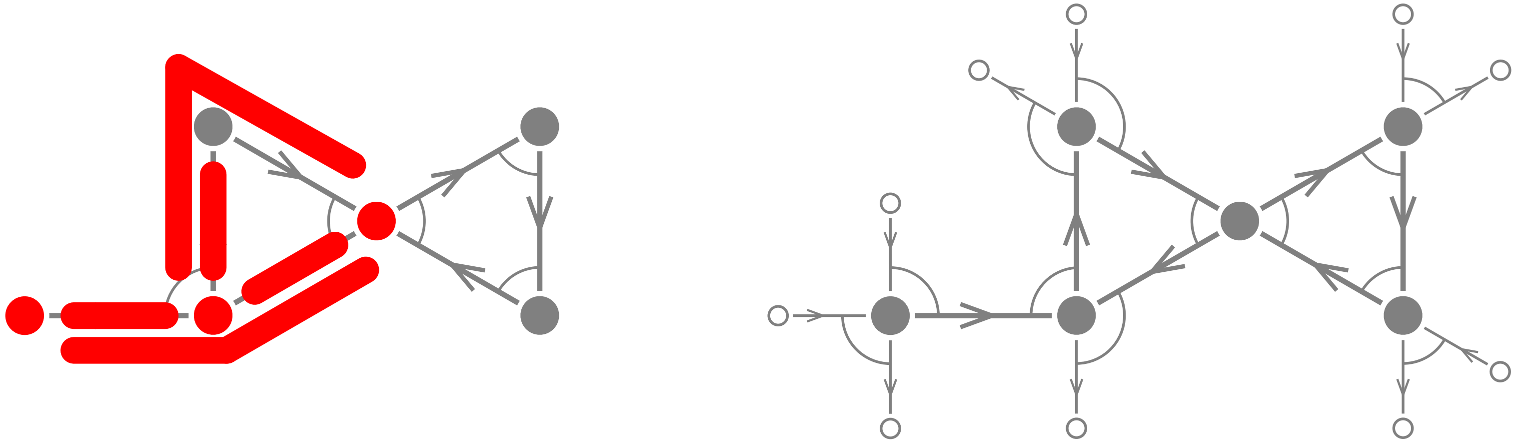


THM. For any gentle quiver $\bar{Q}$ such that $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ is finite, the inclusion poset on biclosed sets of strings of $\bar{Q}$ is a congruence-uniform lattice.

McConville, Lattice structures of grid Tamari orders, 2017
Garver – McConville, Oriented flip graphs and non-crossing tree partitions, 2017

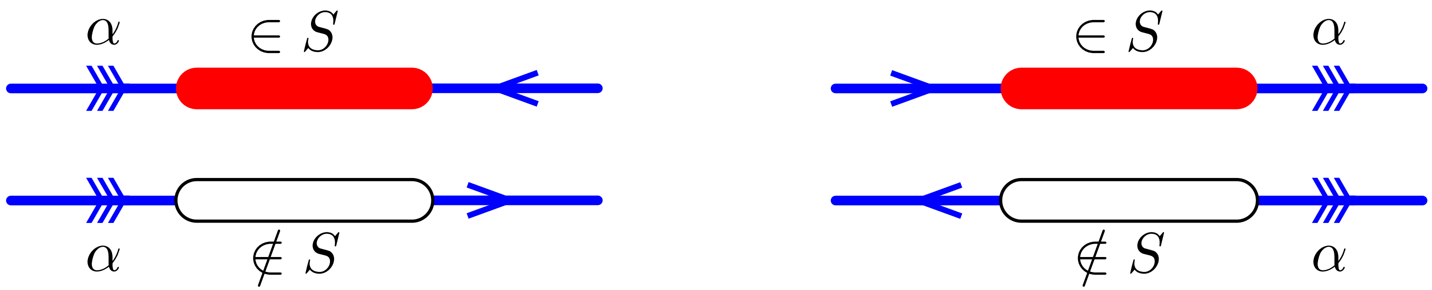
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville, Lattice structures of grid Tamari orders, 2017

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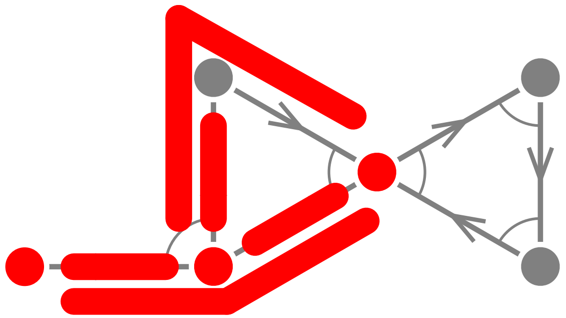

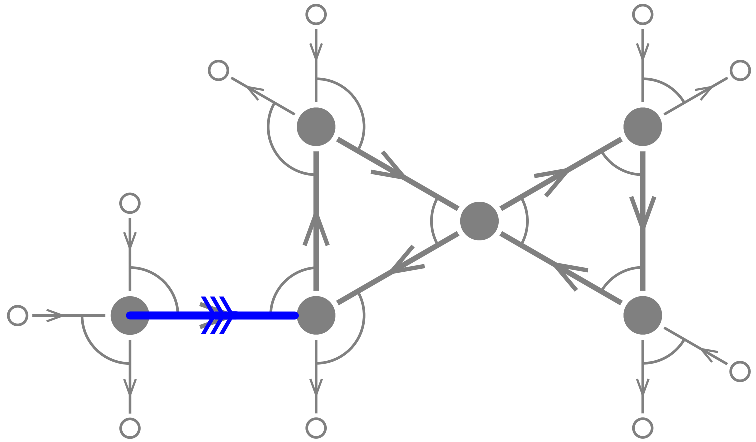

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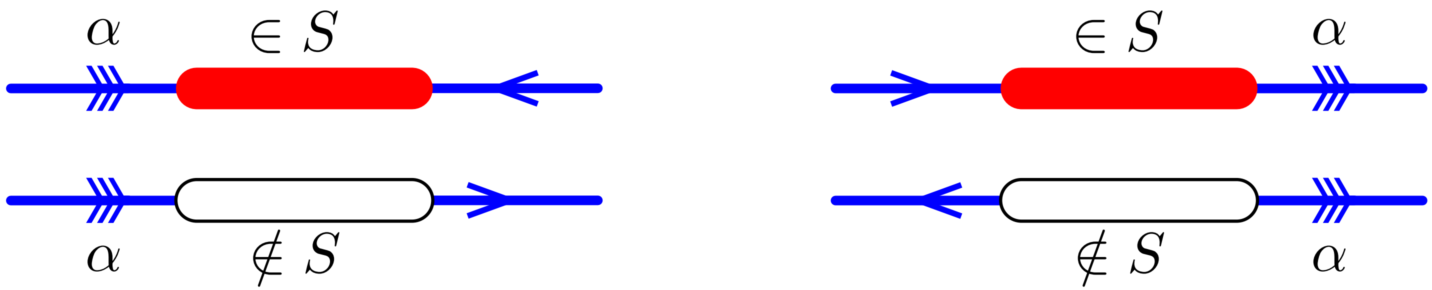

McConville, Lattice structures of grid Tamari orders, 2017

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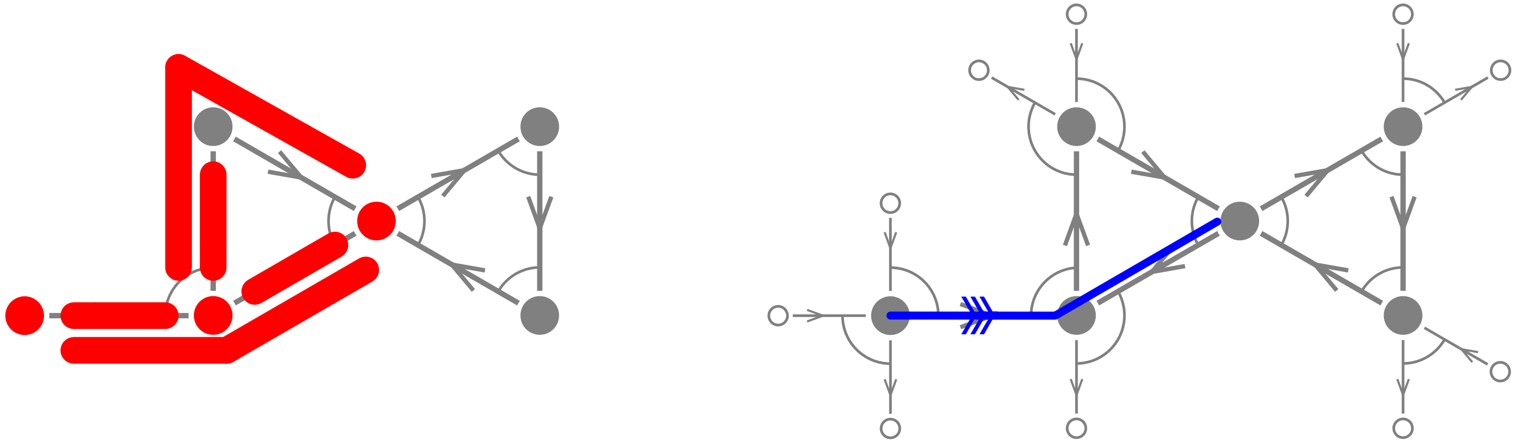

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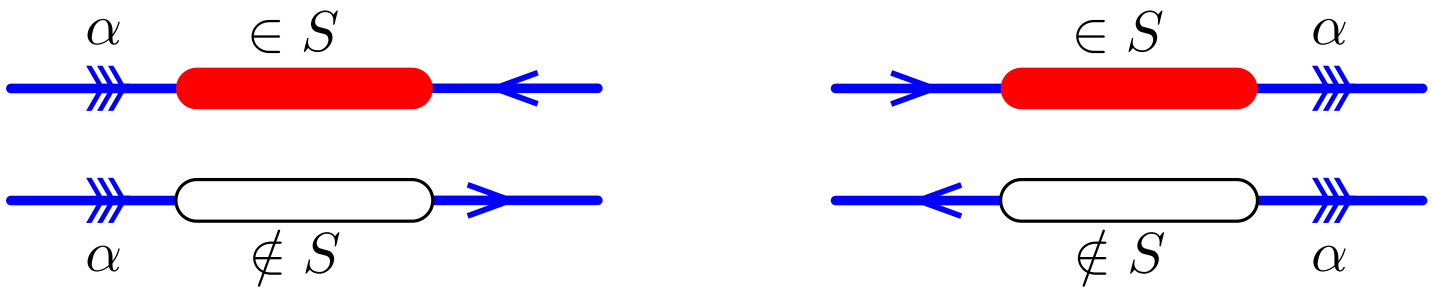

McConville, Lattice structures of grid Tamari orders, 2017

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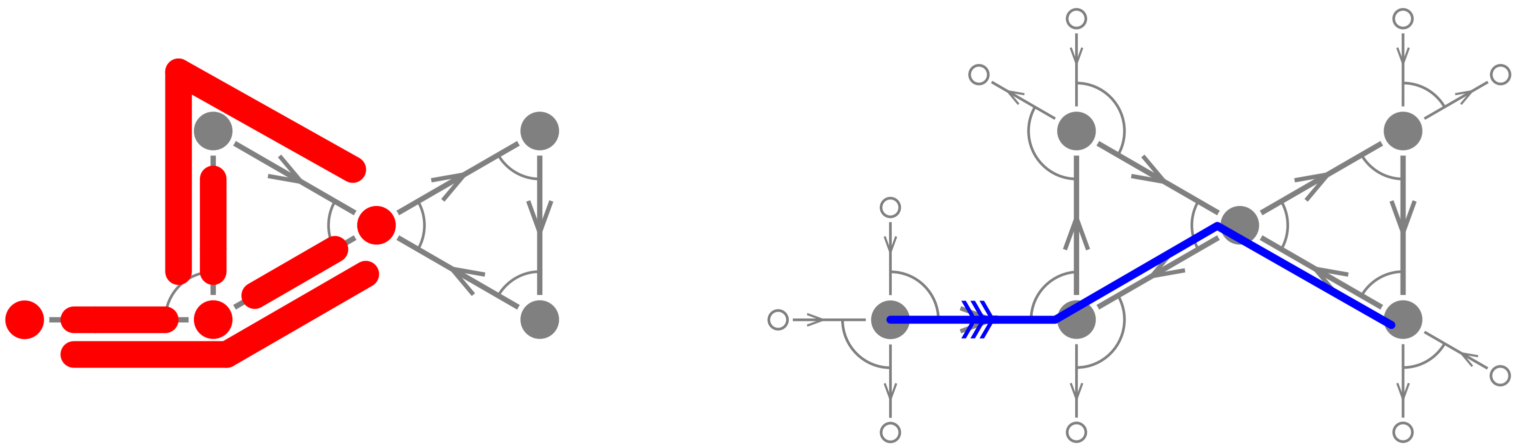

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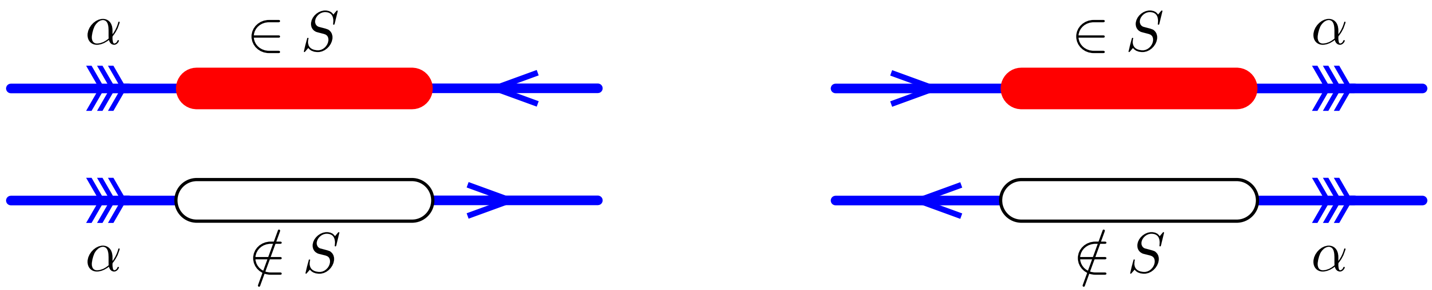
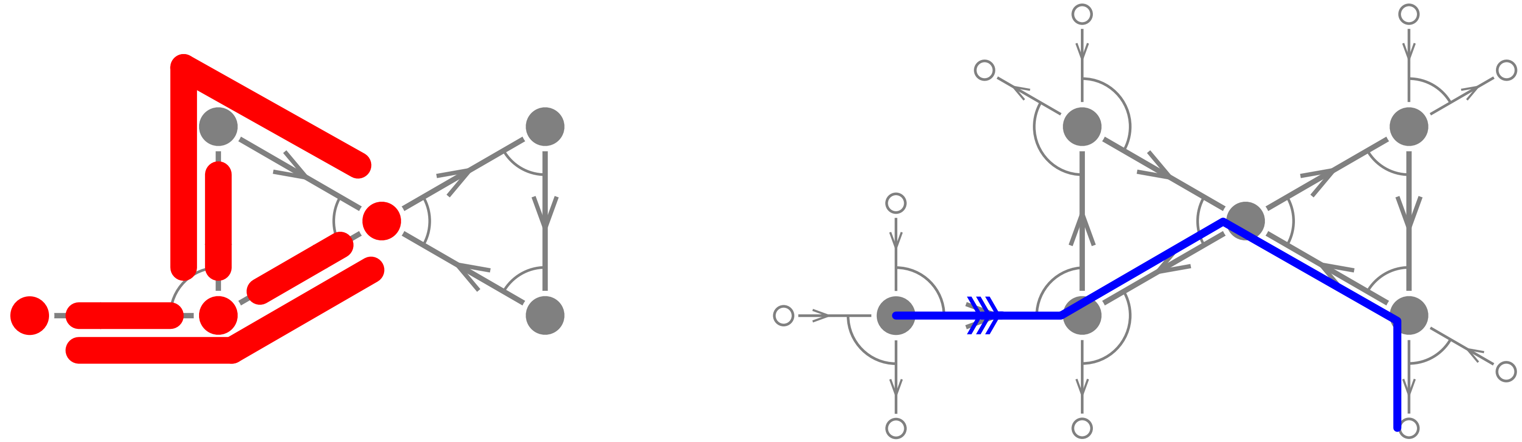


McConville, Lattice structures of grid Tamari orders, 2017

Surjection from biclosed sets of strings to non-kissing facets



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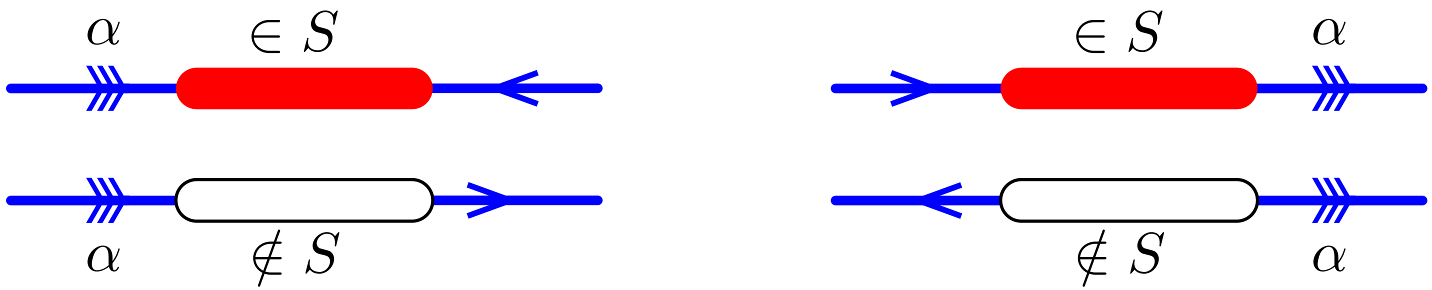

McConville, Lattice structures of grid Tamari orders, 2017

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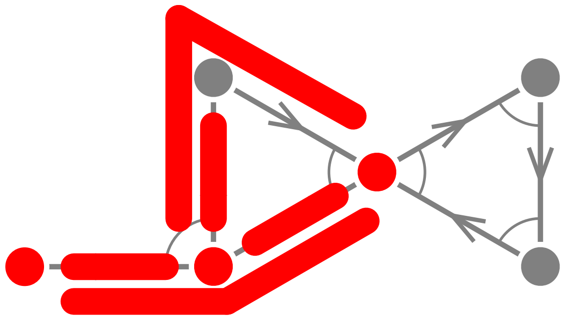

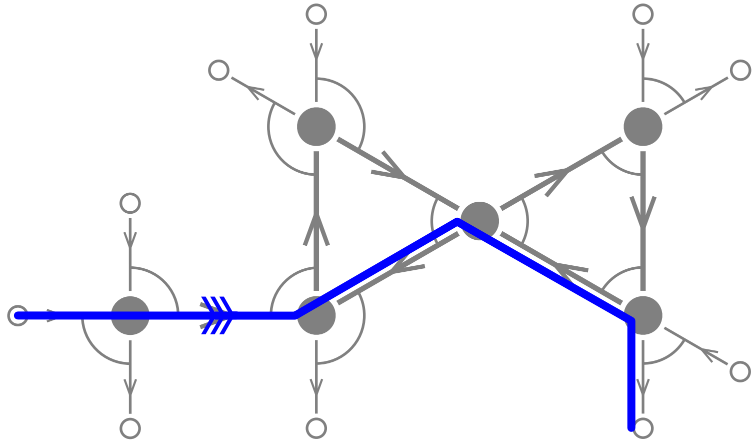

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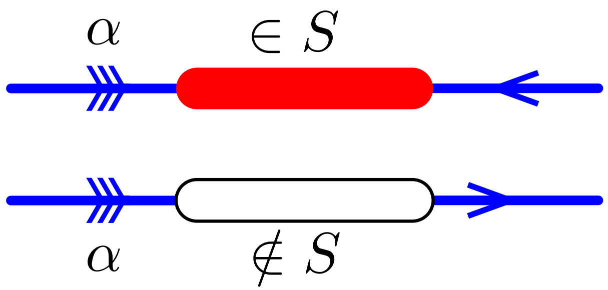
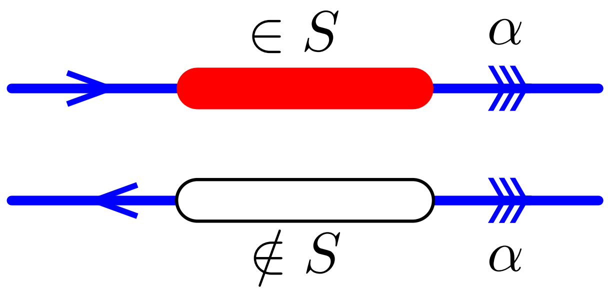
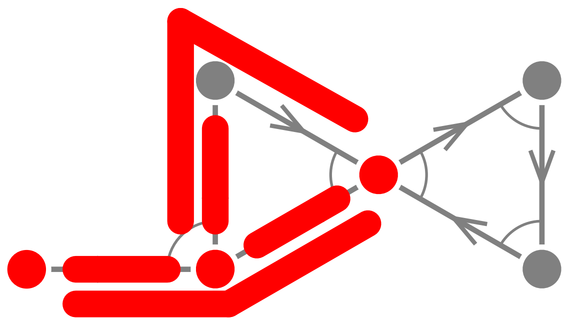
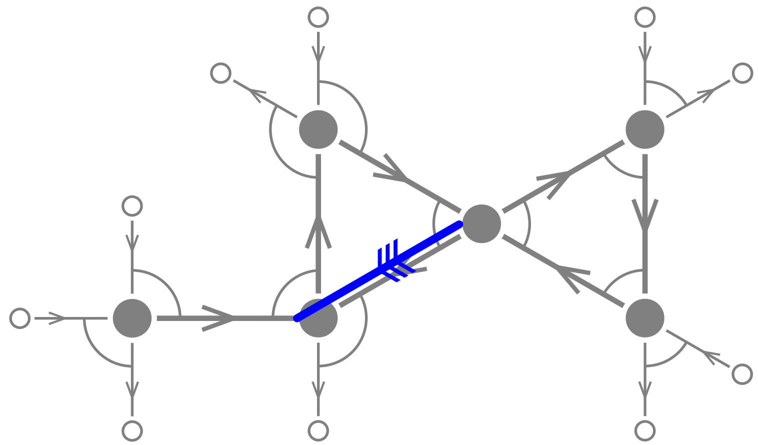




McConville, Lattice structures of grid Tamari orders, 2017

Surjection from biclosed sets of strings to non-kissing facets





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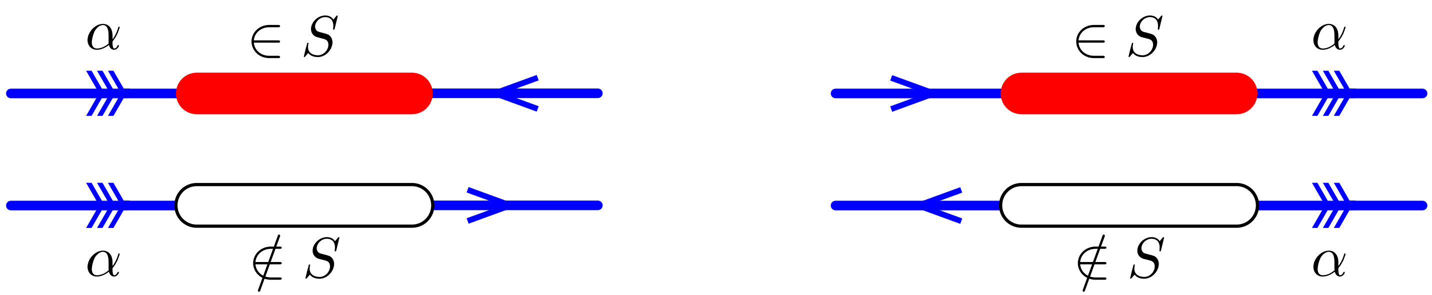

McConville, Lattice structures of grid Tamari orders, 2017

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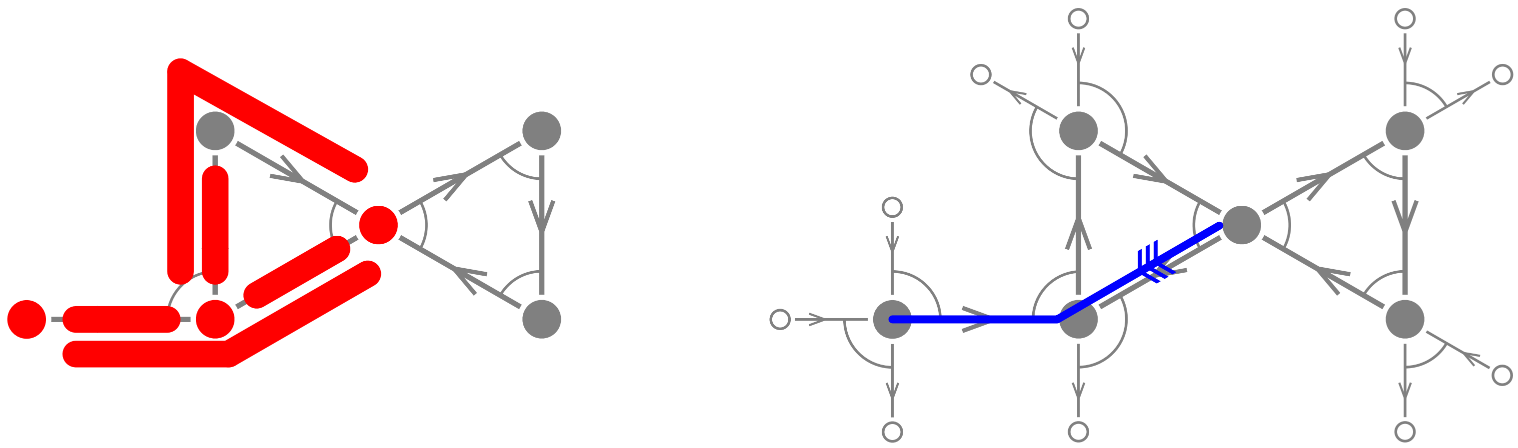

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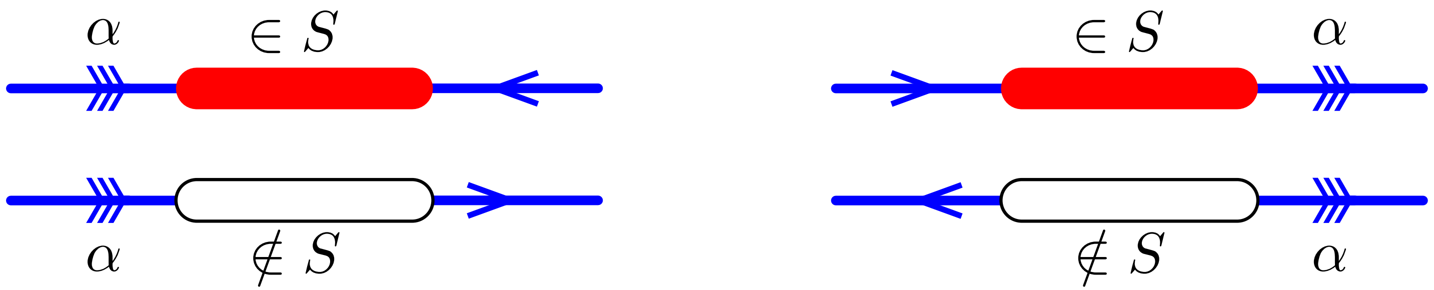

McConville, Lattice structures of grid Tamari orders, 2017

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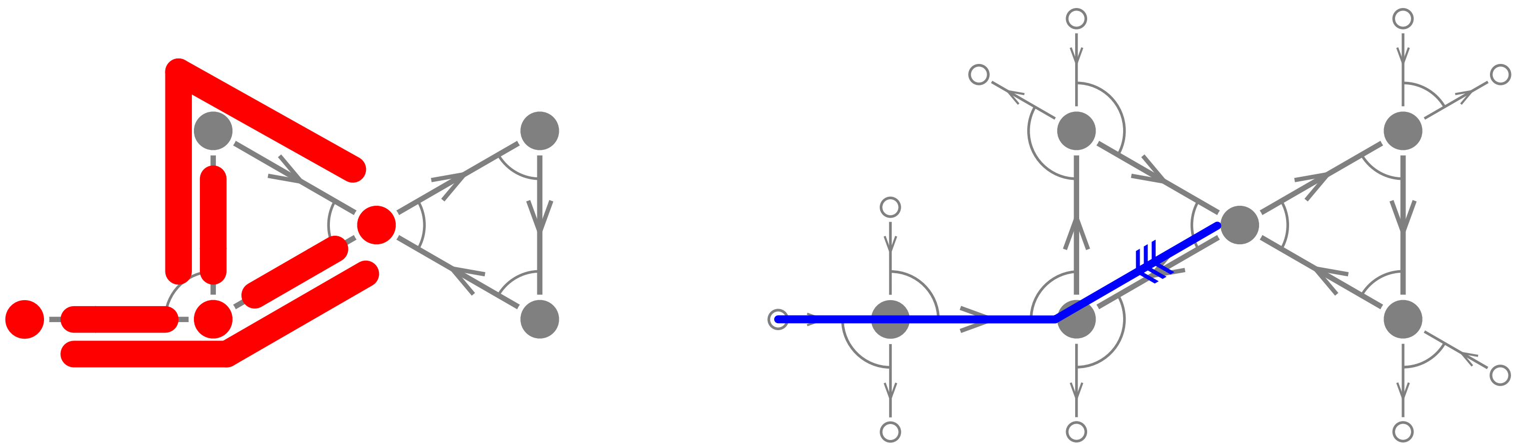

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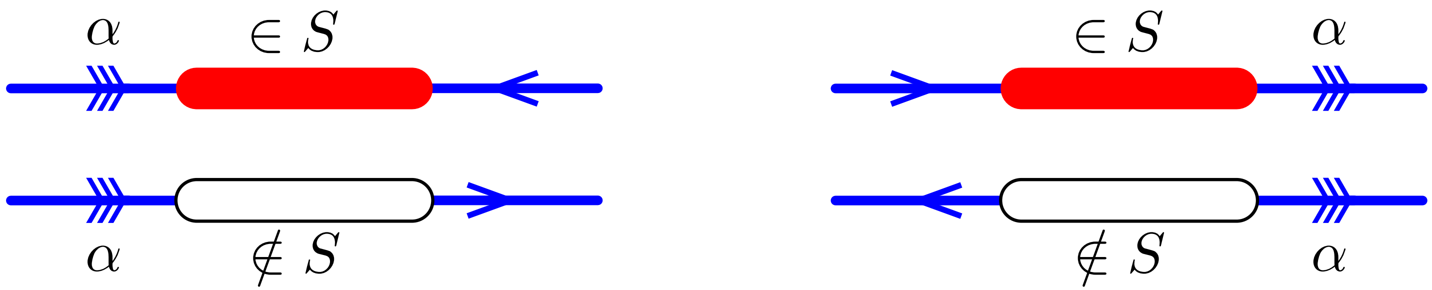

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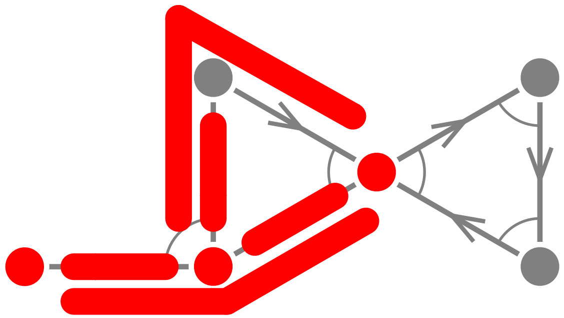
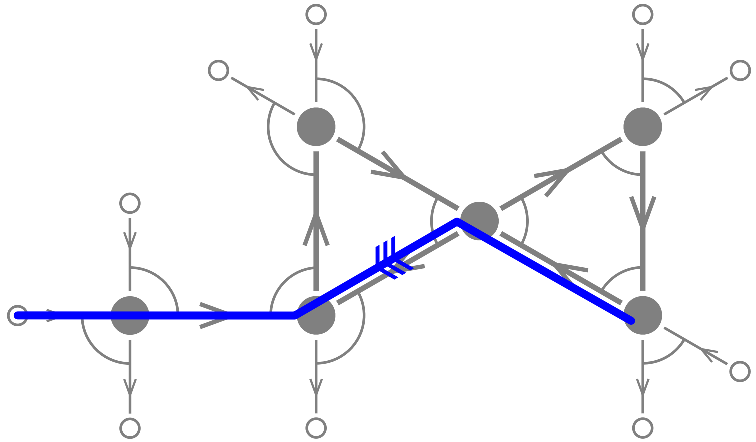
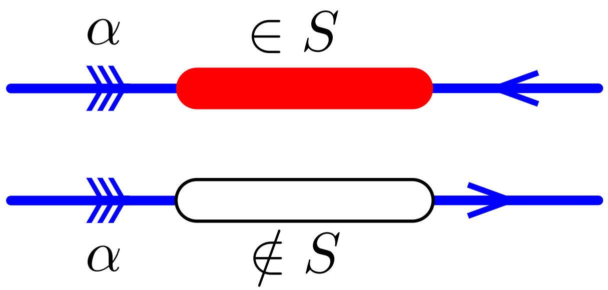
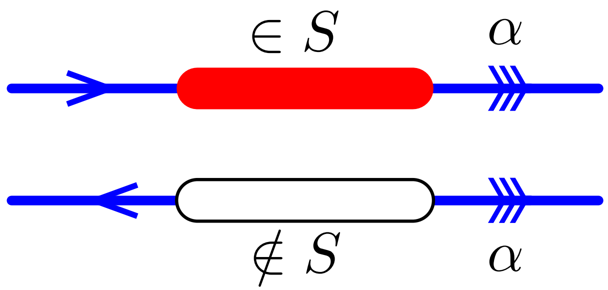




S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S)$ = walk constructed with the local rules:





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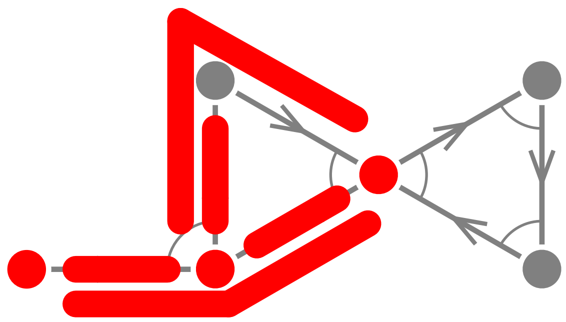
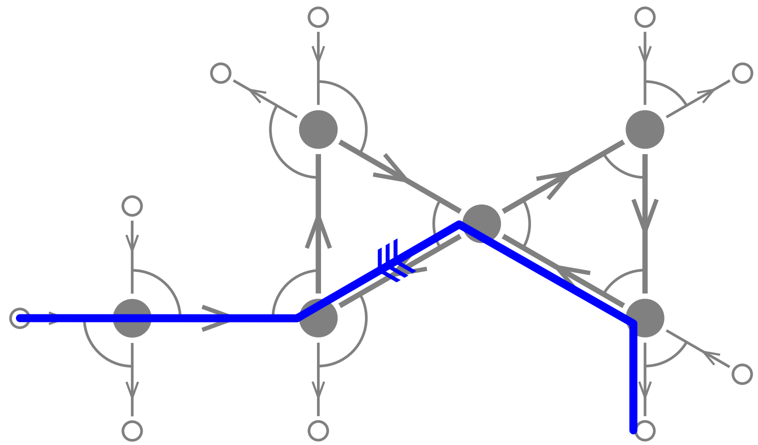
Surjection from biclosed sets of strings to non-kissing facets





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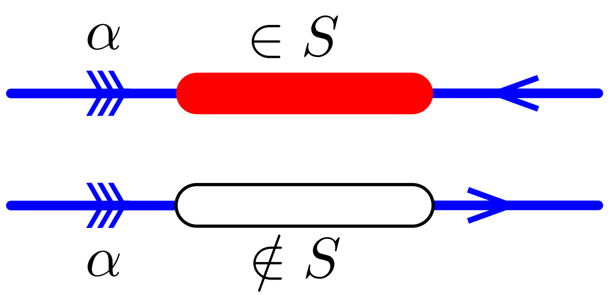

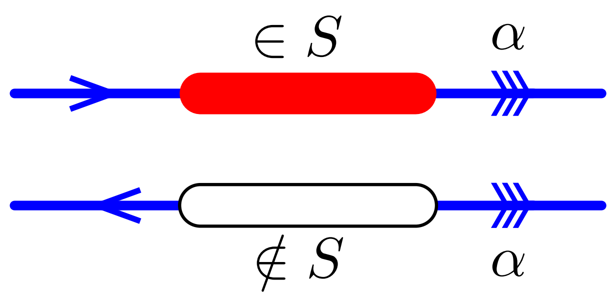

McConville, Lattice structures of grid Tamari orders, 2017

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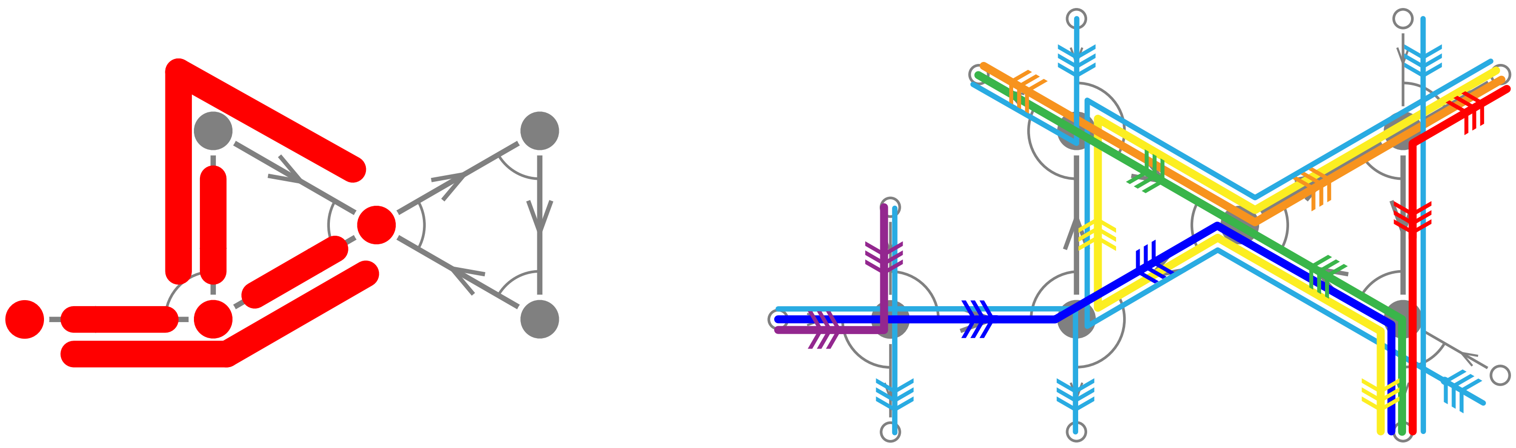
Surjection from biclosed sets of strings to non-kissing facets

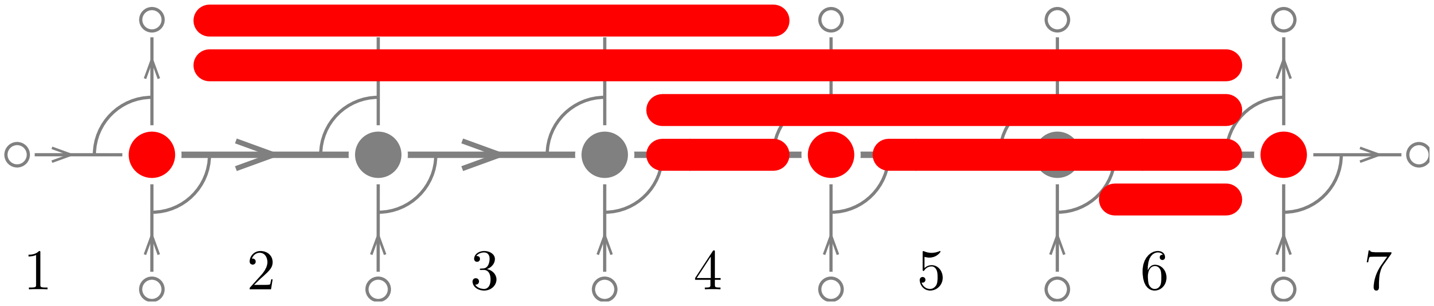


PROP. $\eta(S) \coloneqq \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

McConville, Lattice structures of grid Tamari orders, 2017

EXM: BINARY SEARCH TREE INSERTION AGAIN

inversion set of 2751346



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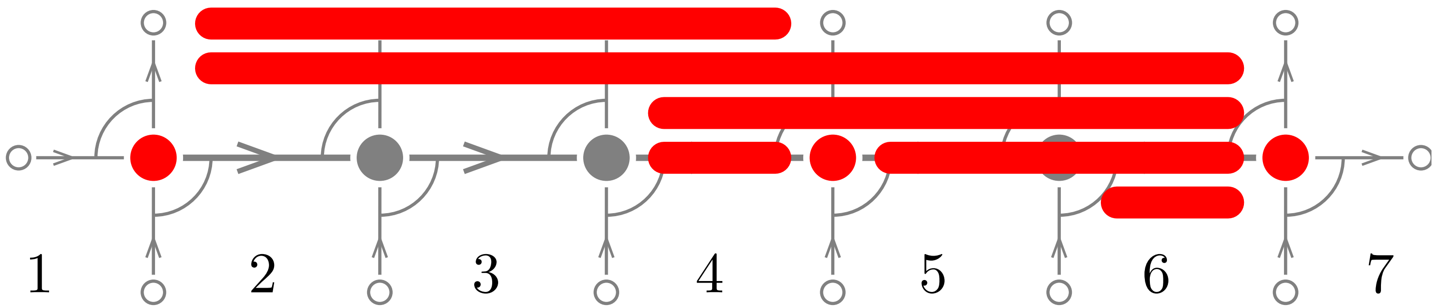
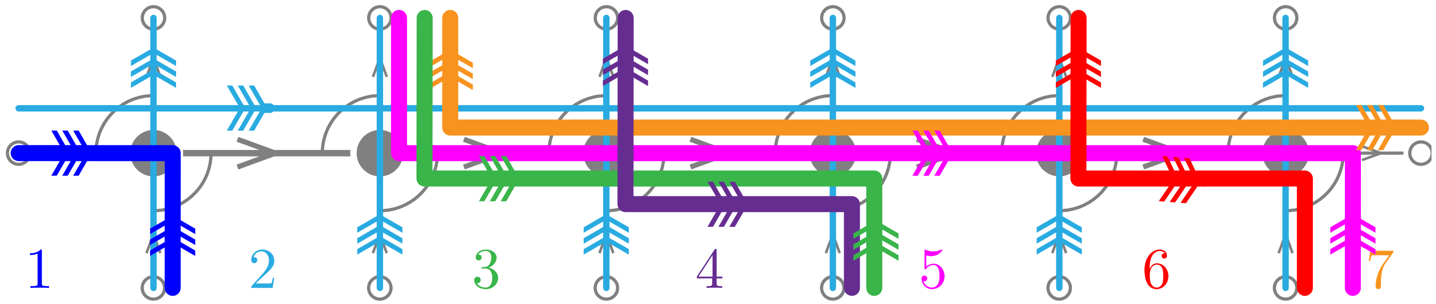
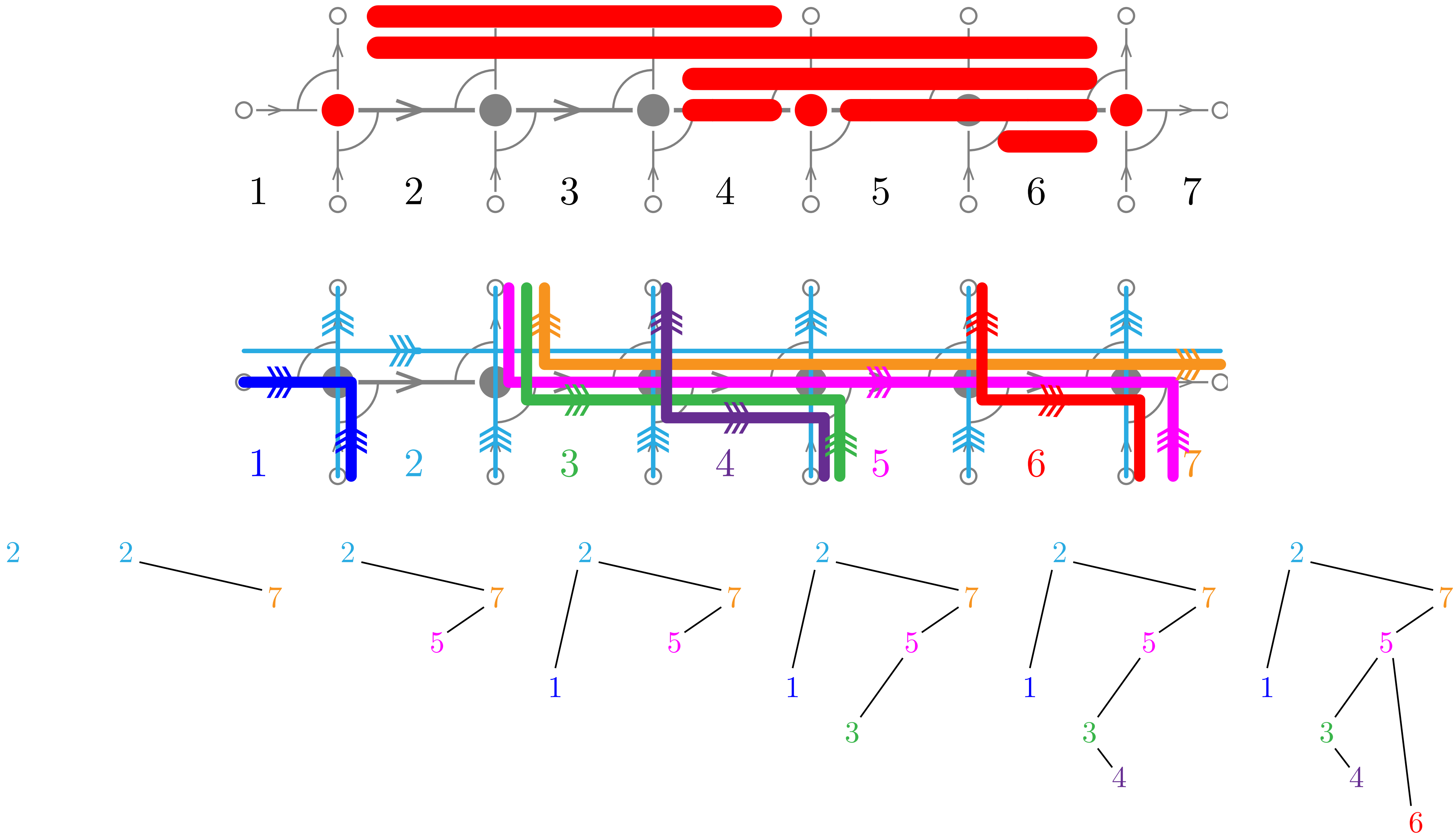
inversion set of 2751346





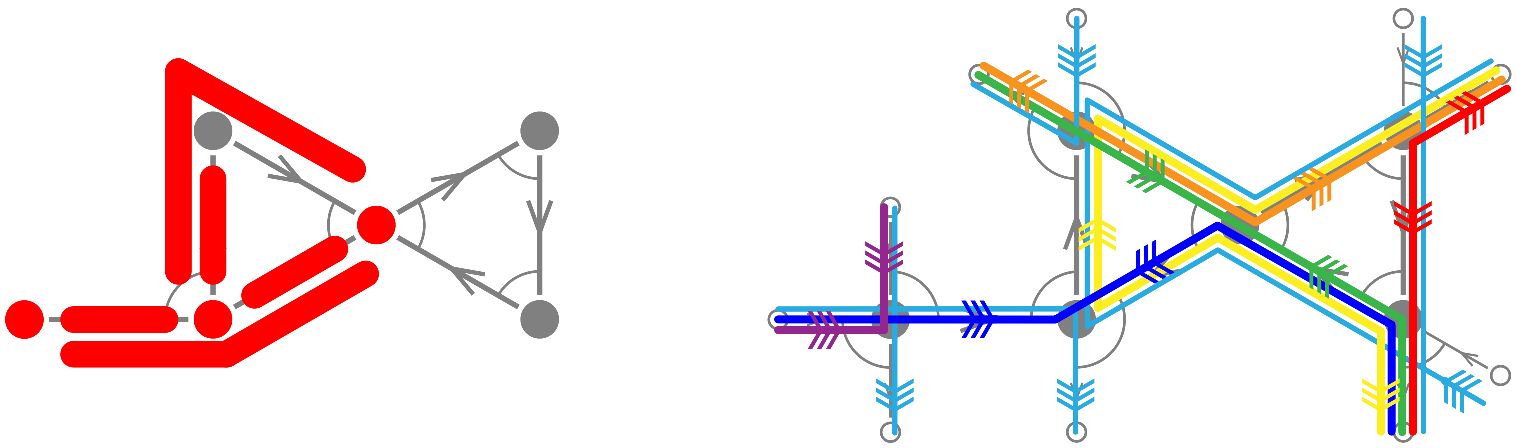
EXM: BINARY SEARCH TREE INSERTION AGAIN

inversion set of 2751346



NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets

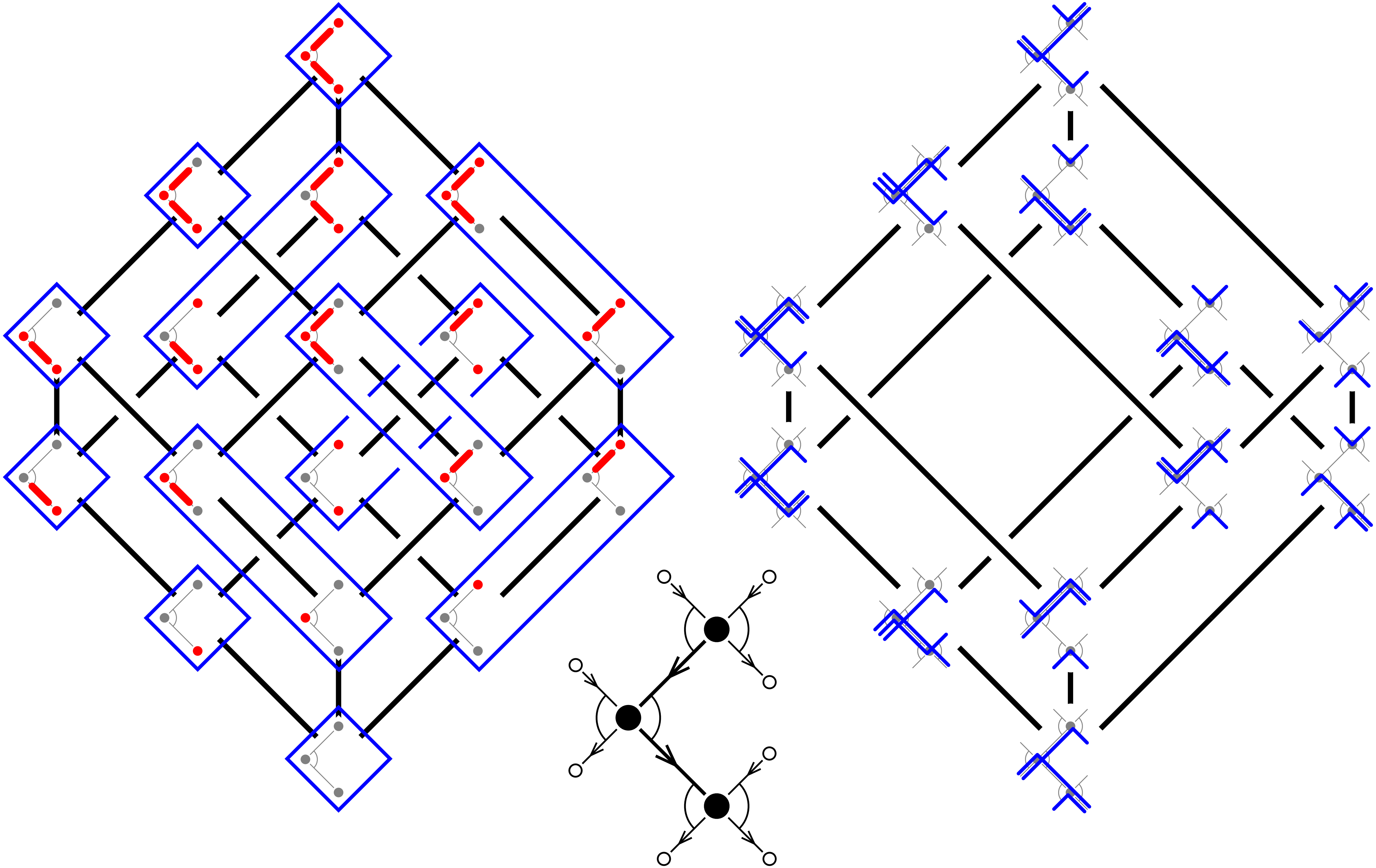


PROP. $\eta(S) := \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

THM. The map η defines a lattice morphism from biclosed sets to non-kissing facets.

McConville, Lattice structures of grid Tamari orders, 2017

NON-KISSING LATTICE

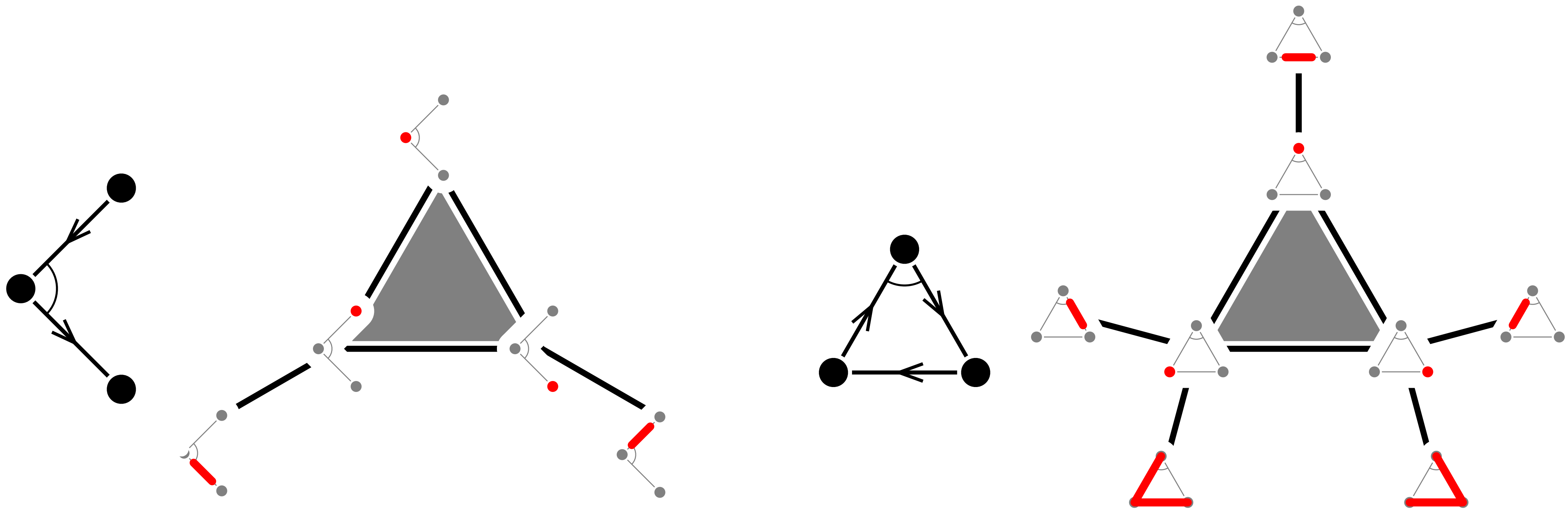


NON-KISSING LATTICE

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.

Much more nice combinatorics:

- join-irreducible elements of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ are in bijection with distinguishable strings
- canonical join complex of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ is a generalization of non-crossing partitions



SUMMARY

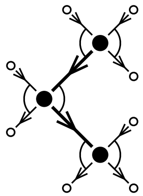
$\bar{Q}$ gentle quiver

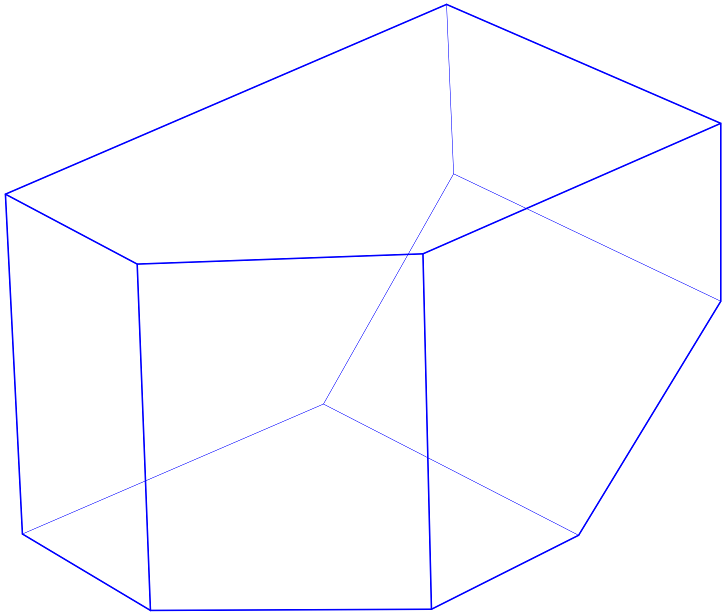
[reduced] non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ = simplicial complex with

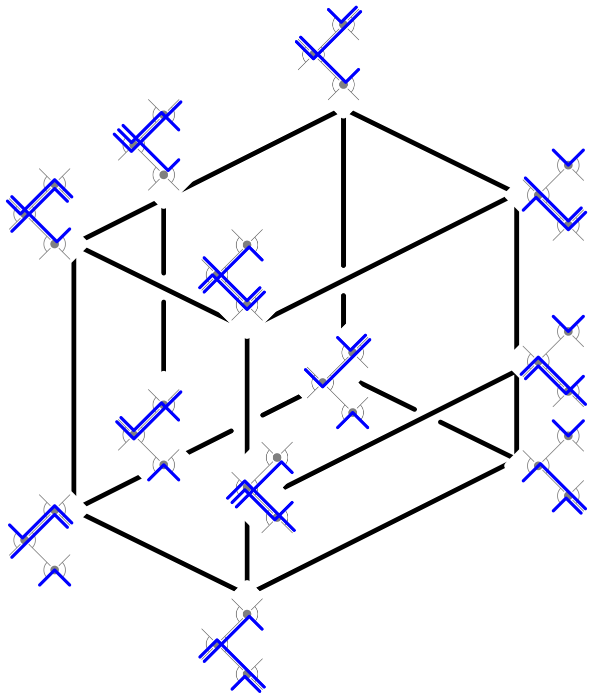
- vertices = [bended] walks of $\bar{Q}$
- faces = collections of pairwise non-kissing [bended] walks of $\bar{Q}$

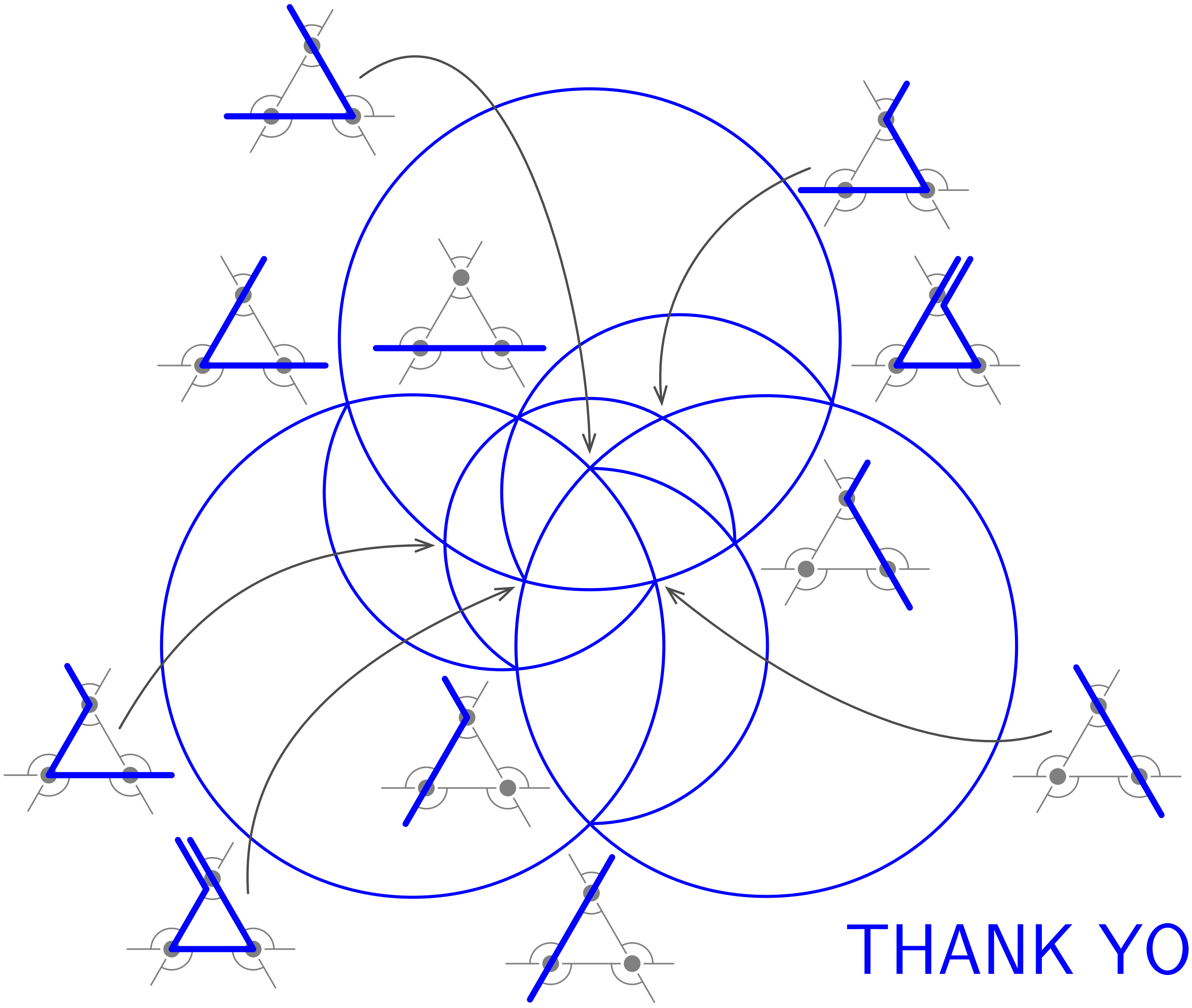
THM. For a gentle quiver $\bar{Q}$ with finite $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$, the non-kissing flip graph is

- the 1-skeleton of a polytope,
- the Hasse diagram of a congruence-uniform lattice.









THANK YOU