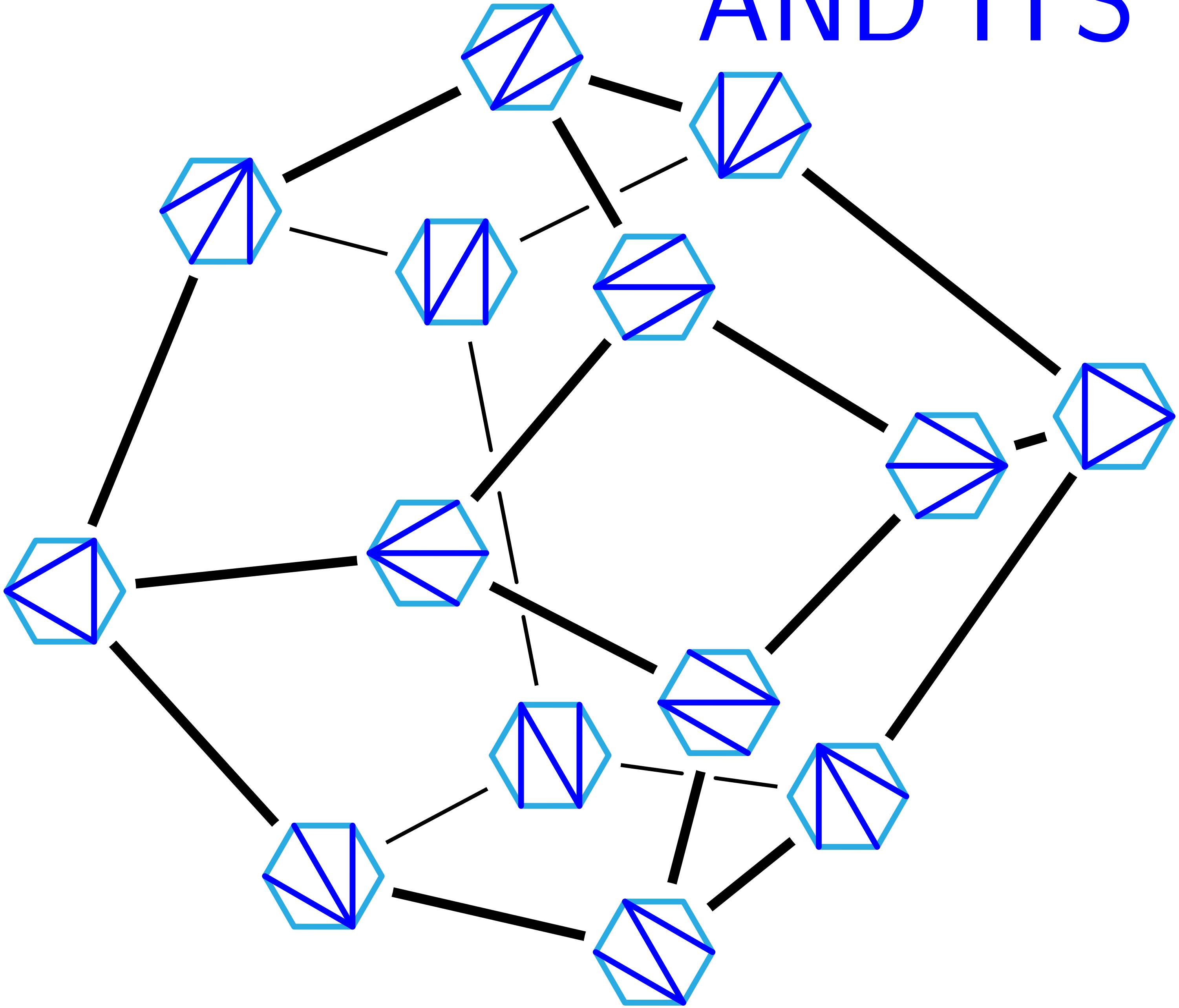
THE ASSOCIAHEDRON AND ITS FRIENDS

V. PILAUD

(CNRS & LIX)



Séminaire
Lotharingien
de Combinatoire
April 4 – 6, 2016

PROGRAM

I. THREE CONSTRUCTIONS OF THE ASSOCIAHEDRON

- Compatibility fan and cluster algebras
- Loday's associahedron and Hopf algebras
- Secondary polytope

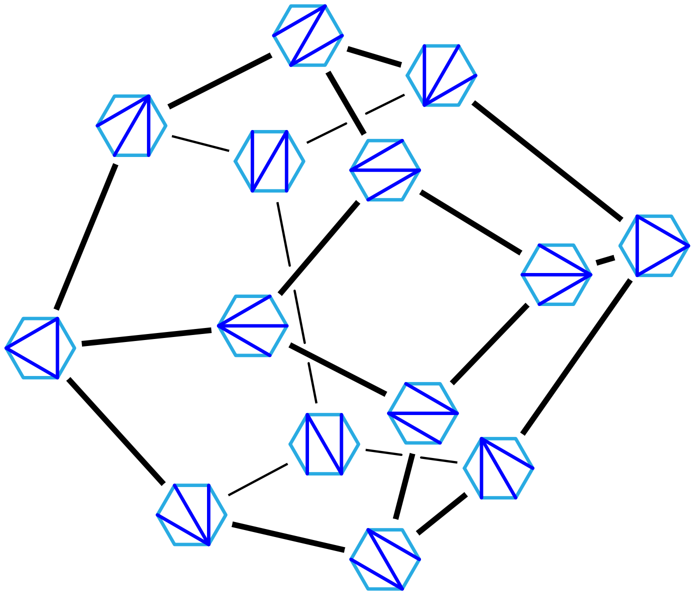
II. GRAPH ASSOCIAHEDRA

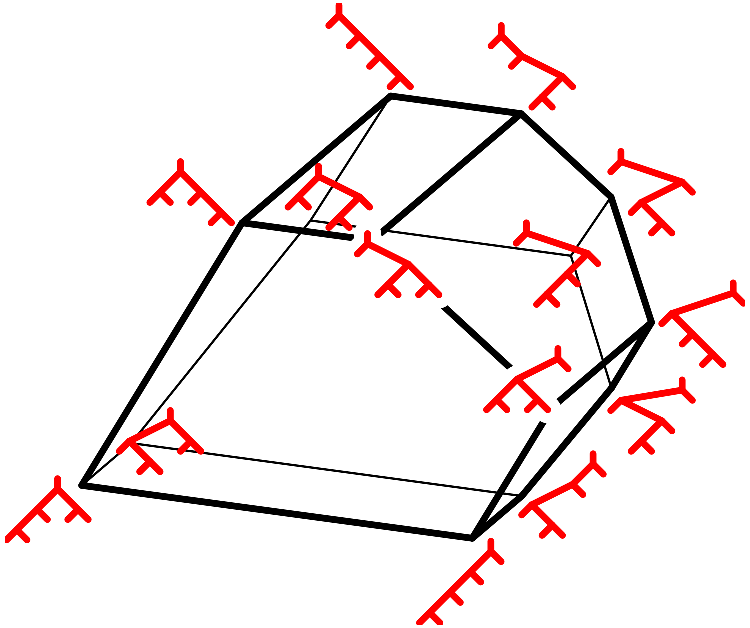
- Graphical nested complexes
- Compatibility fans for graphical nested complexes
- Signed tree associahedra

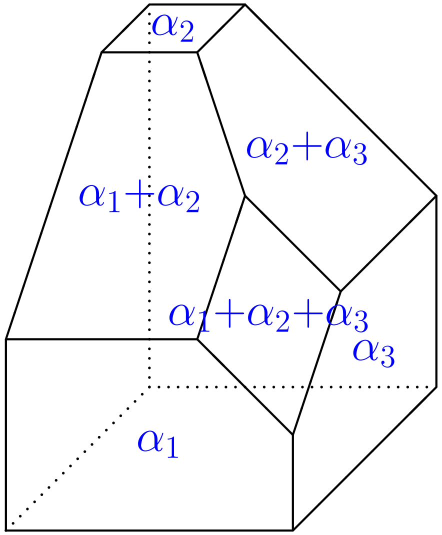
III. BRICK POLYTOPES AND THE TWIST ALGEBRA

- [Multi][pseudo]triangulations and sorting networks
- The brick polytope
- The twist algebra

I. THREE CONSTRUCTIONS OF THE ASSOCIAHEDRON







FANS & POLYTOPES

Ziegler, *Lectures on polytopes* ('95)
Matoušek, *Lectures on Discrete Geometry* ('02)

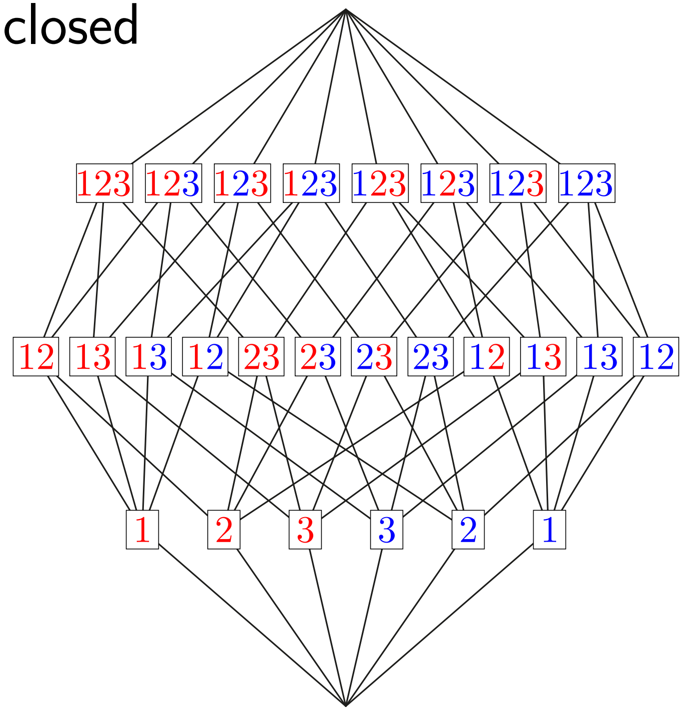
SIMPLICIAL COMPLEX

simplicial complex = collection of subsets of X downward closed

exm:

$$X = [n] \cup [n]$$

$$\Delta = \{I \subseteq X \mid \forall i \in [n],\ \{i, i\} \not\subseteq I\}$$

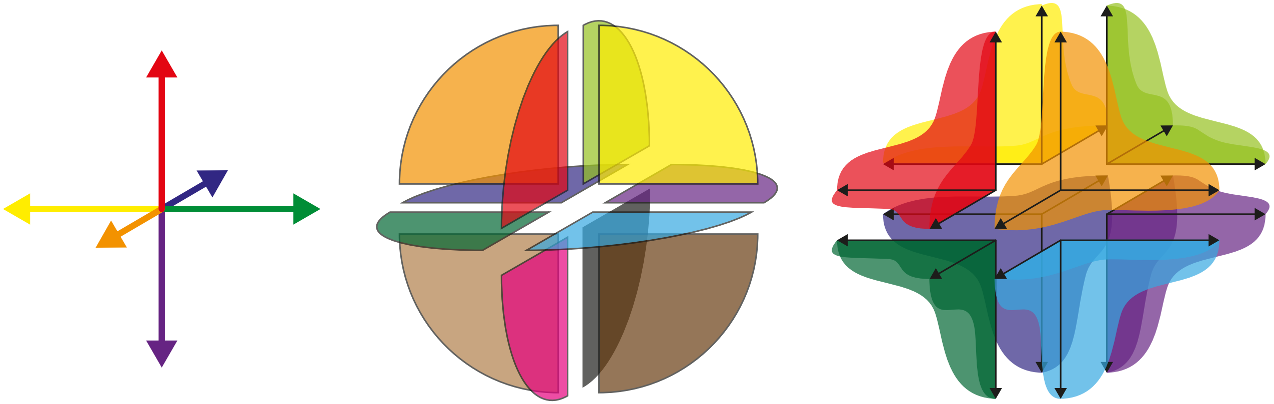


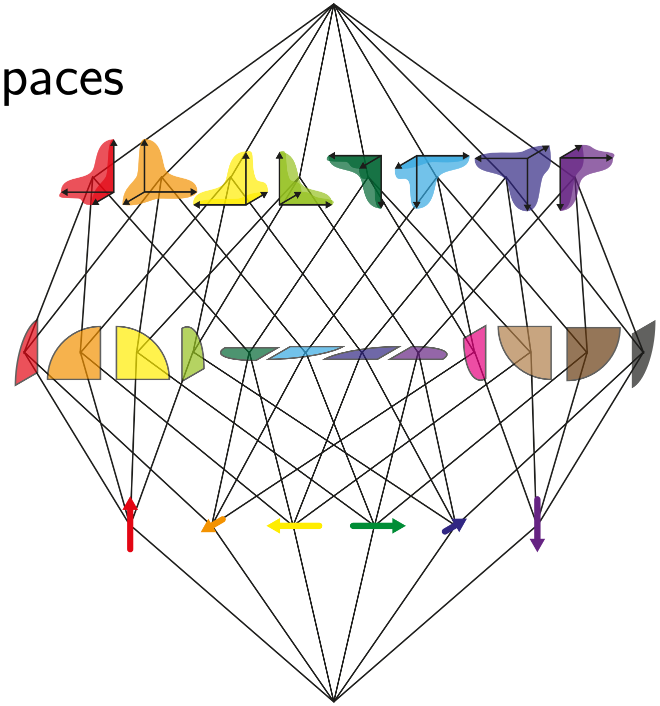
FANS

polyhedral cone $=$ positive span of a finite set of $\mathbb{R}^d$

$=$ intersection of finitely many linear half-spaces

fan $=$ collection of polyhedral cones closed by faces
and where any two cones intersect along a face





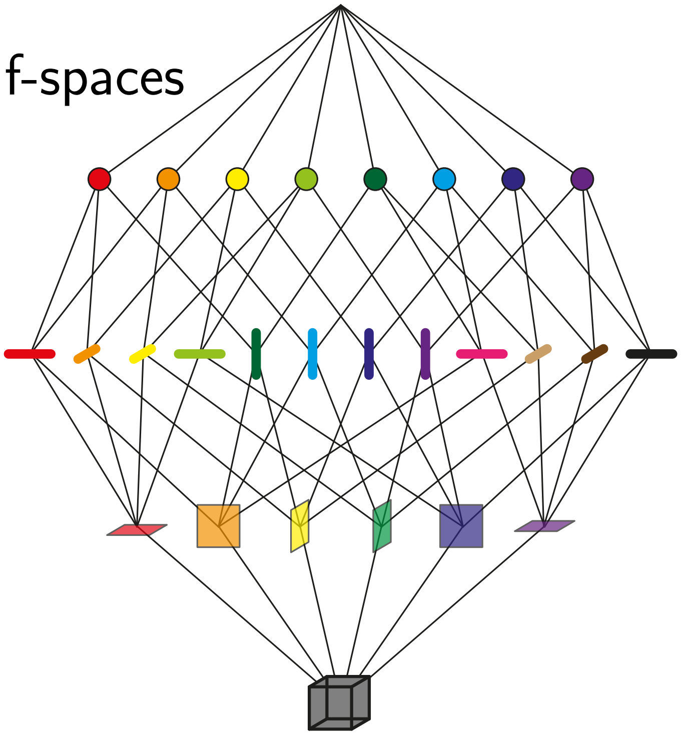
simplicial fan $=$ maximal cones generated by d rays

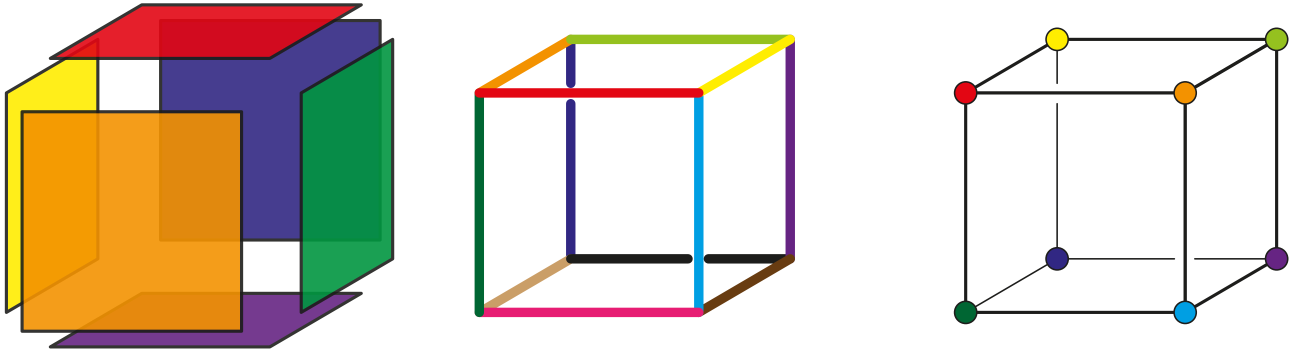
POLYTOPES

polytope $=$ convex hull of a finite set of $\mathbb{R}^d$
$=$ bounded intersection of finitely many affine half-spaces

face $=$ intersection with a supporting hyperplane

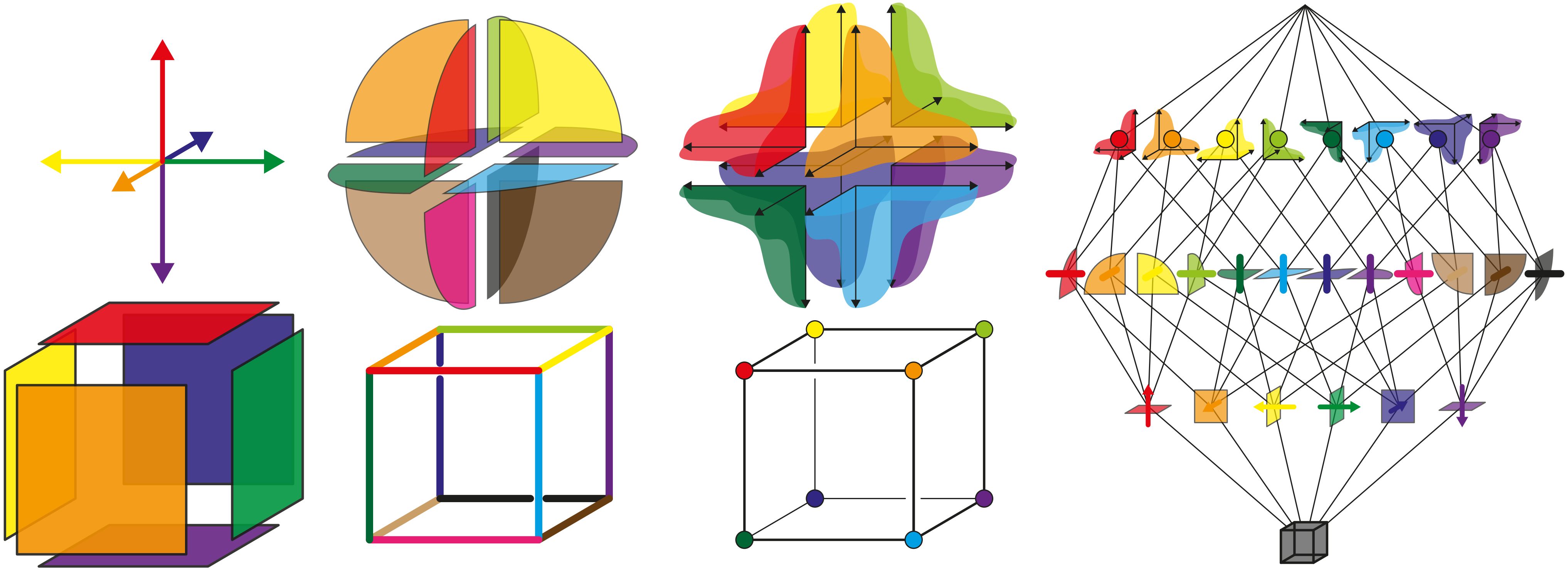
face lattice $=$ all the faces with their inclusion relations





simple polytope $=$ facets in general position $=$ each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

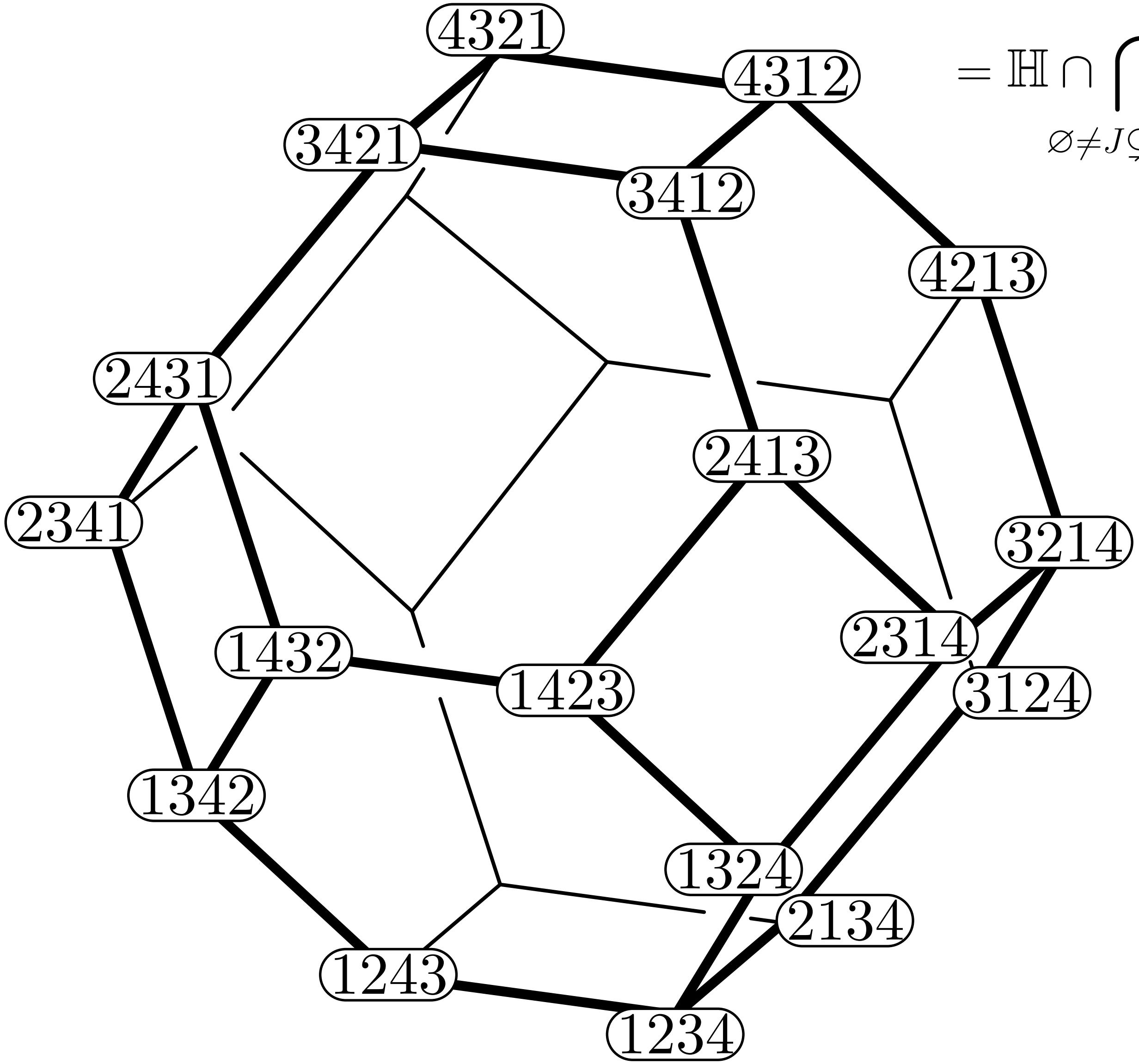
normal fan of P = $\{$ normal cone of F $\mid$ F face of P $\}$

simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

EXAMPLE: PERMUTAHEDRON

Hohlweg, *Permutahedra and associahedra* ('12)

PERMUTAHEDRON

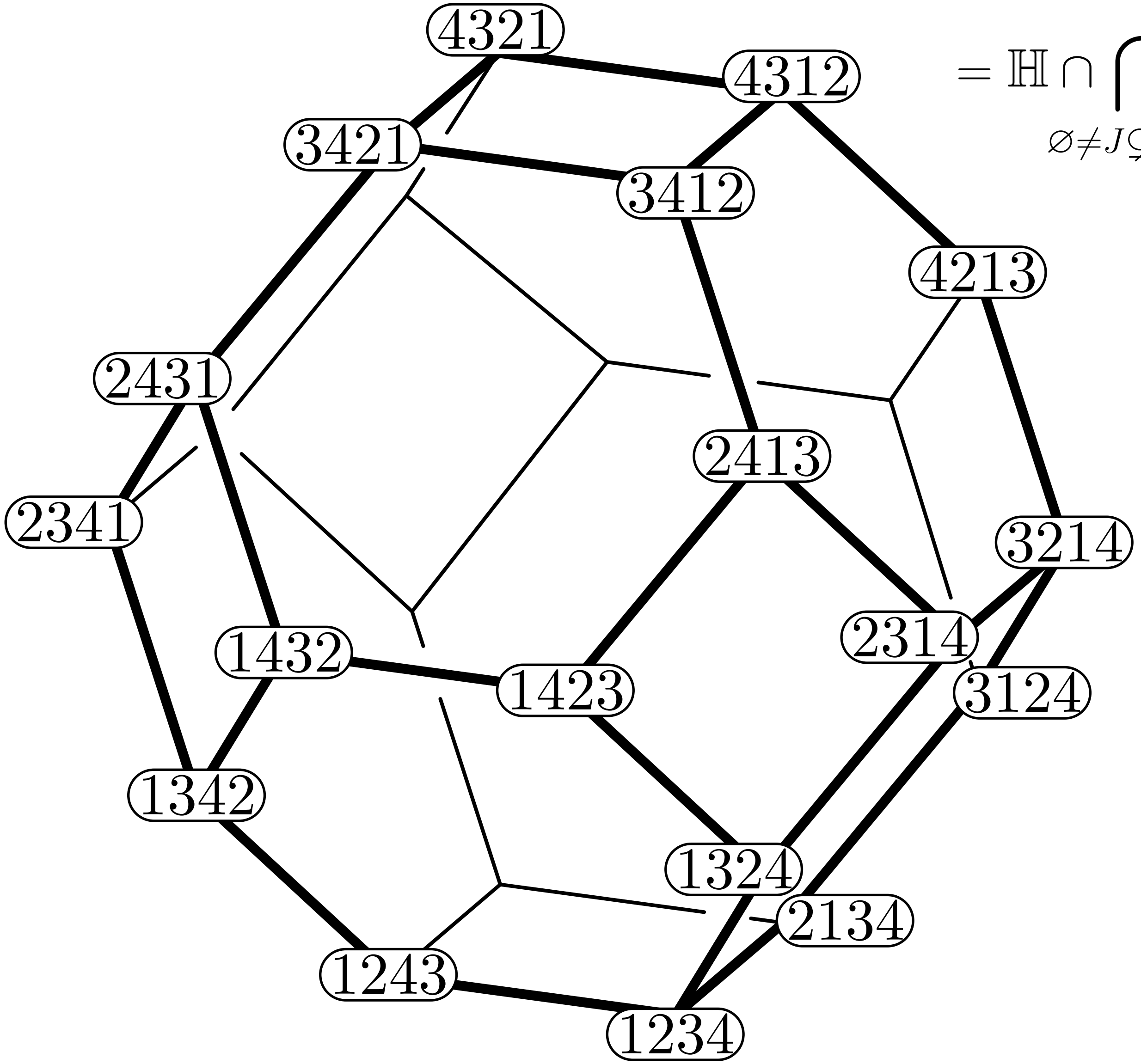


Permutohedron Perm(n)

$$= \operatorname{conv}\left\{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

PERMUTAHEDRON



Permutohedron $\mathsf{Perm}(n)$

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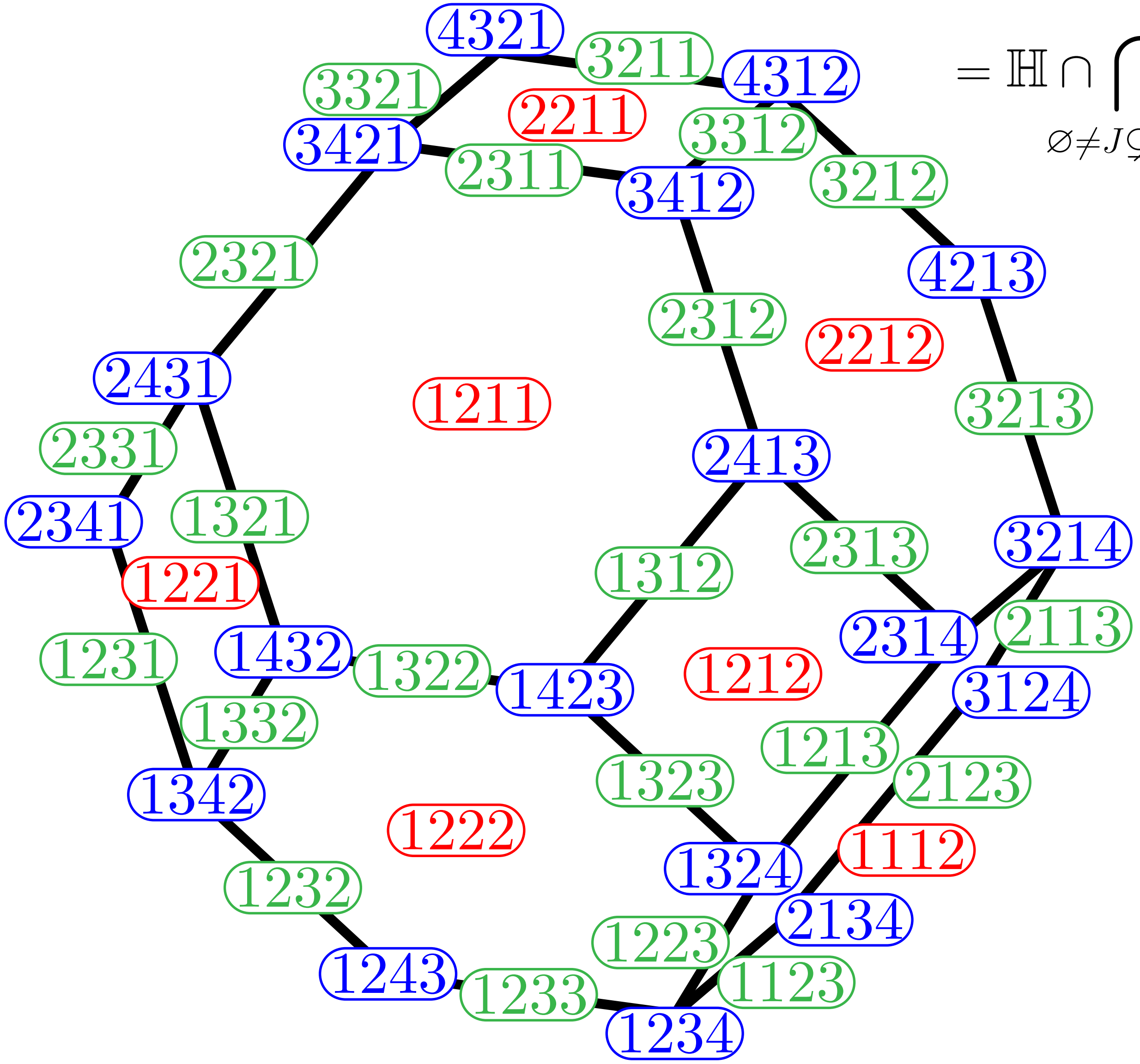
connections to

- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

PERMUTAHEDRON



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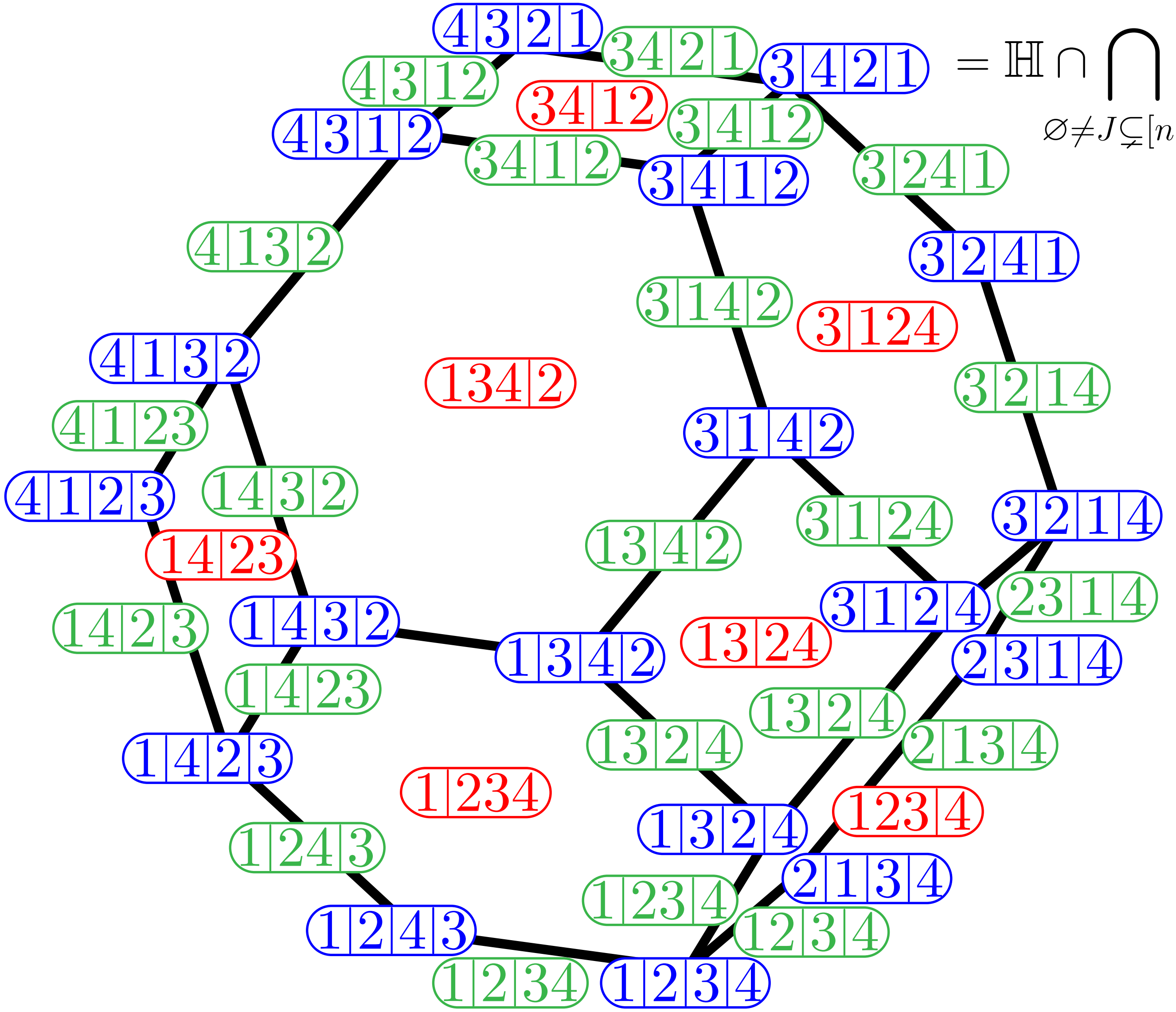
k-faces of Perm(n)

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

PERMUTAHEDRON



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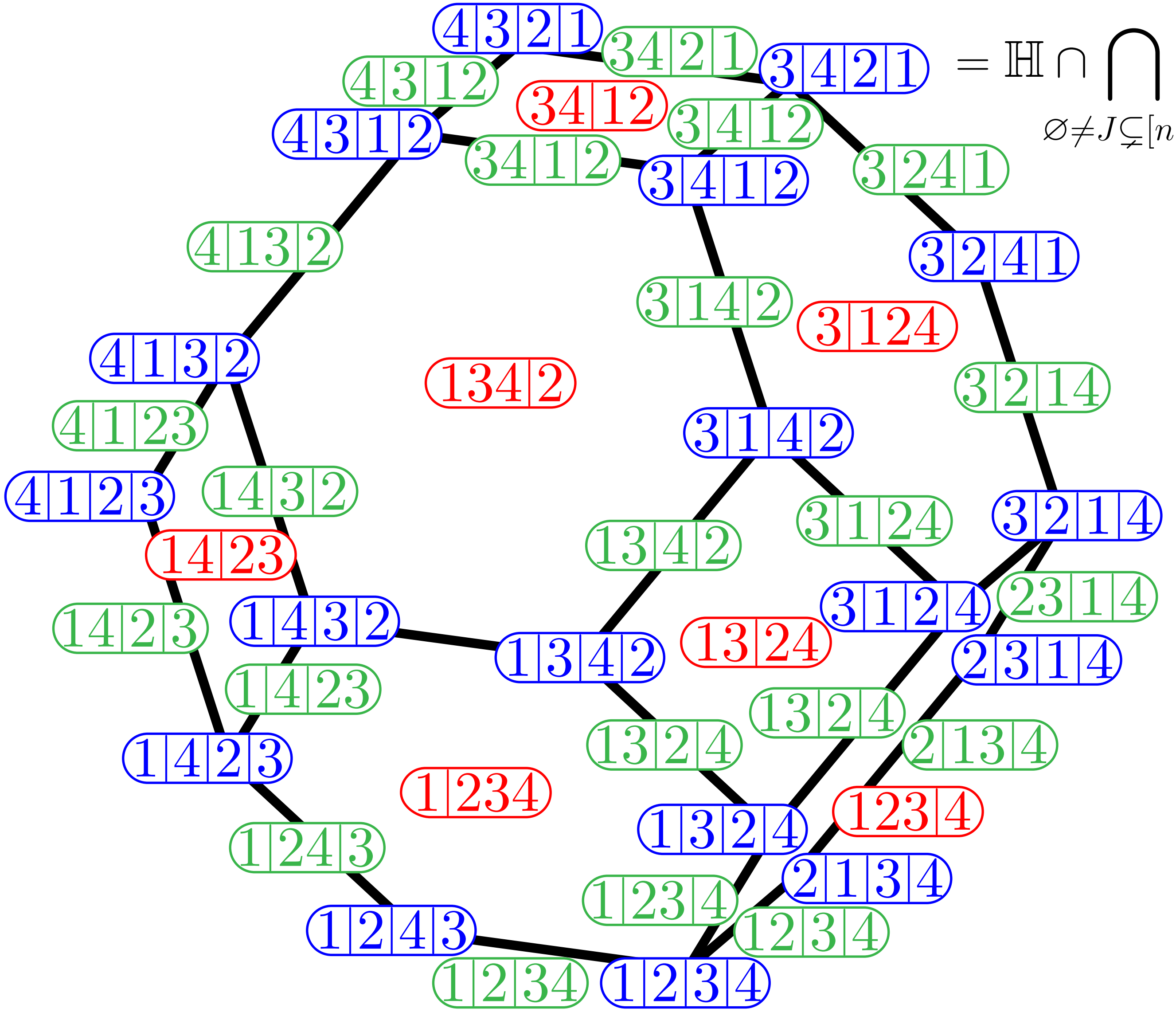
k-faces of Perm(n)

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

PERMUTAHEDRON



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connections to

- weak order
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k-faces of $\mathsf{Perm}(n)$

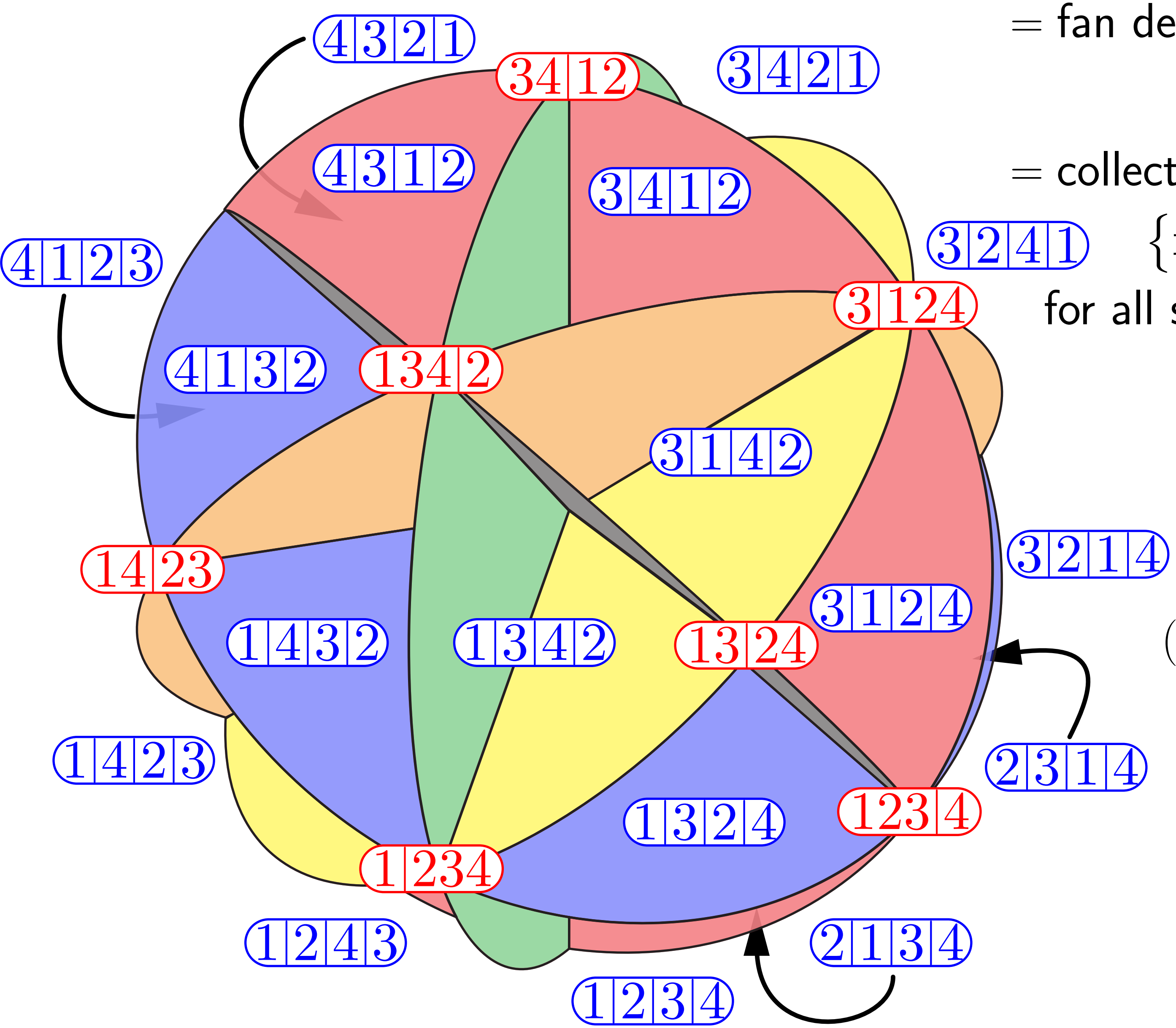
$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

$\equiv$ collections of $n-k$ nested subsets of $[n+1]$

COXETER ARRANGEMENT

Coxeter fan



= fan defined by the hyperplane arrangement

$$\left\{\mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; x_i = x_j\right\}_{1 \le i < j \le n+1}$$

= collection of all cones

$$\left\{\mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; x_i < x_j \text{ if } \pi(i) < \pi(j)\right\}$$

for all surjections $\pi : [n+1] \to [n+1-k]$

$(n-k)$-dimensional cones

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

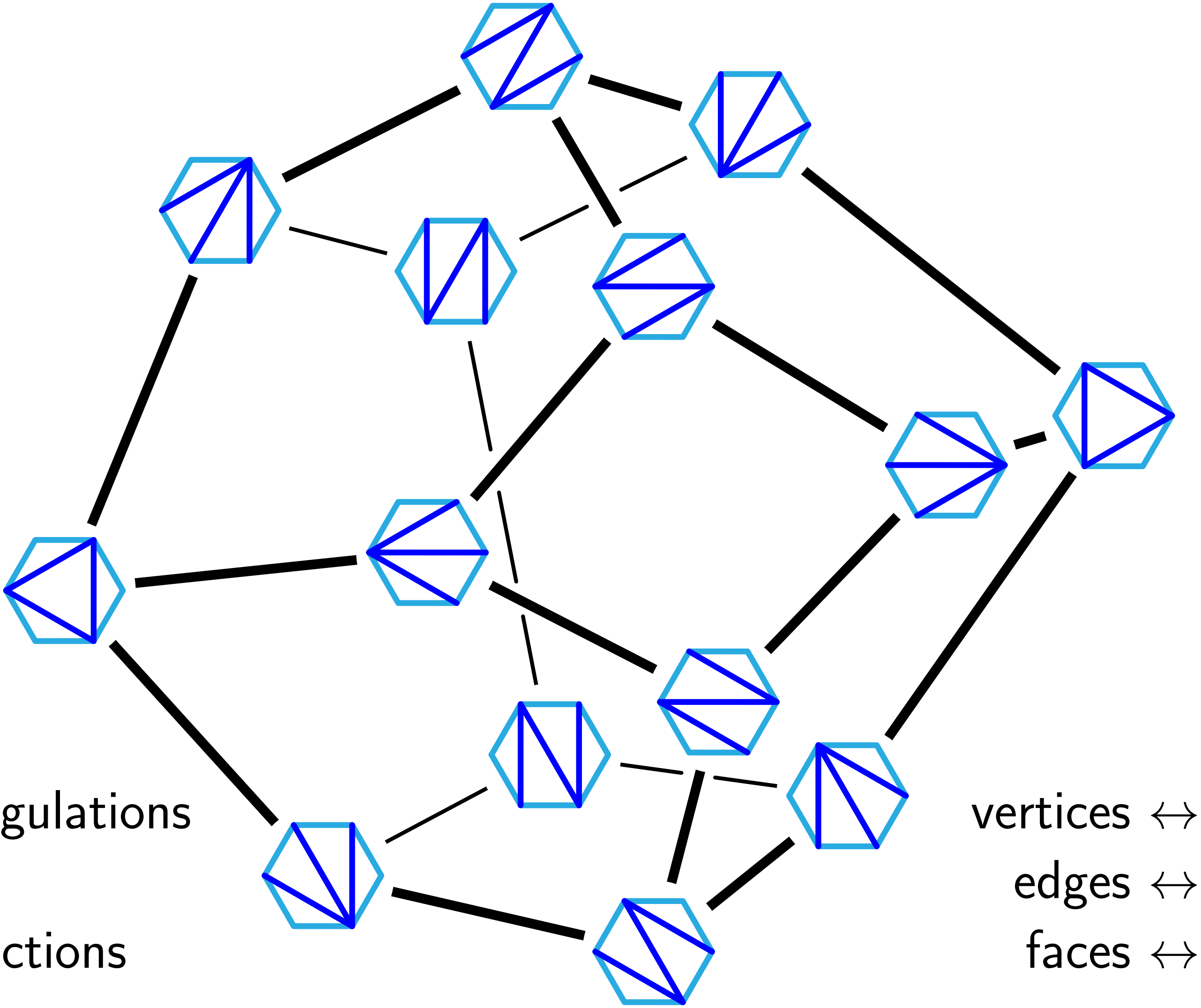
$\equiv$ collections of $n-k$ nested subsets of $[n+1]$

ASSOCIAHEDRA

Ceballos-Santos-Ziegler,
Many non-equivalent realizations of the associahedron ('11)

ASSOCIAHEDRON

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

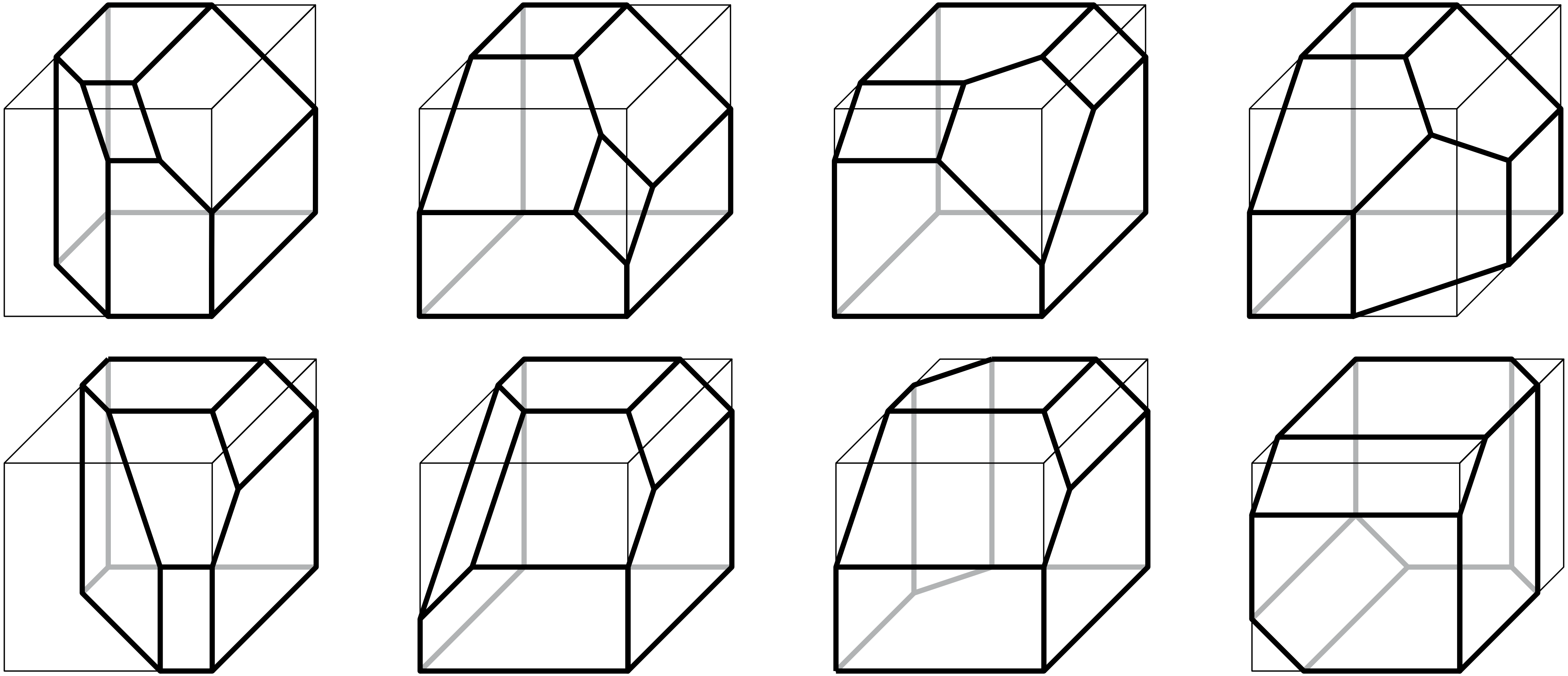


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion



(Pictures by Ceballos-Santos-Ziegler)

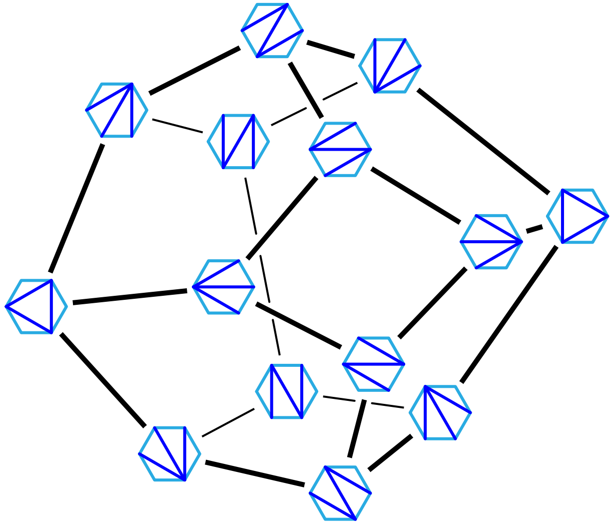
Tamari ('51) — Stasheff ('63) — Haimann ('84) — Lee ('89) —

... — Gel'fand-Kapranov-Zelevinski ('94) — ... — Chapoton-Fomin-Zelevinsky ('02) — ... — Loday ('04) — ...

— Ceballos-Santos-Ziegler ('11)

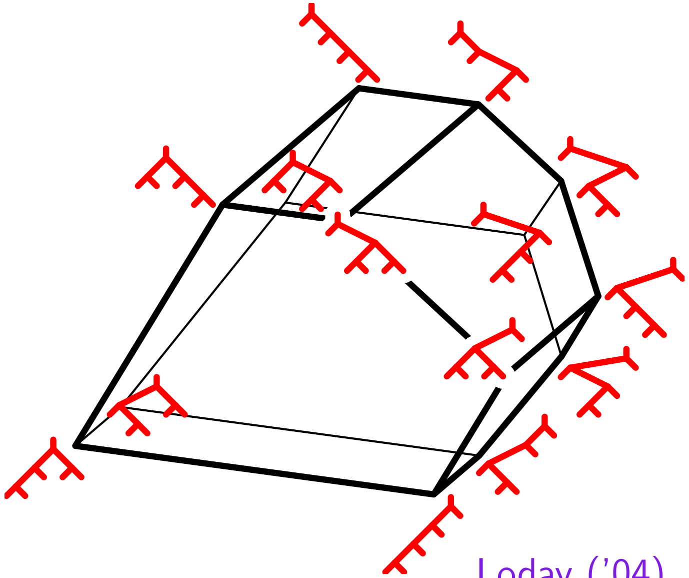
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



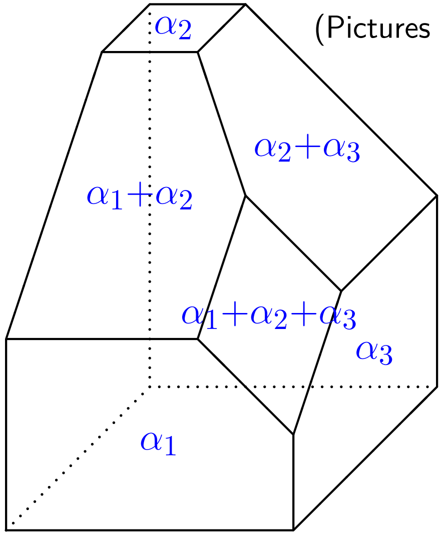
Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

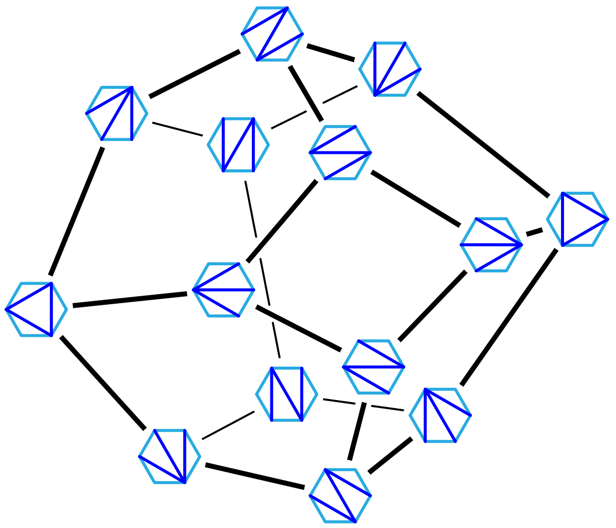


(Pictures by CFZ)

Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



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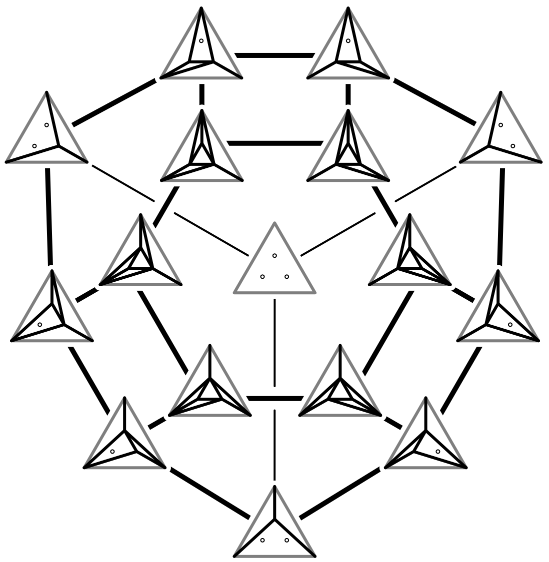

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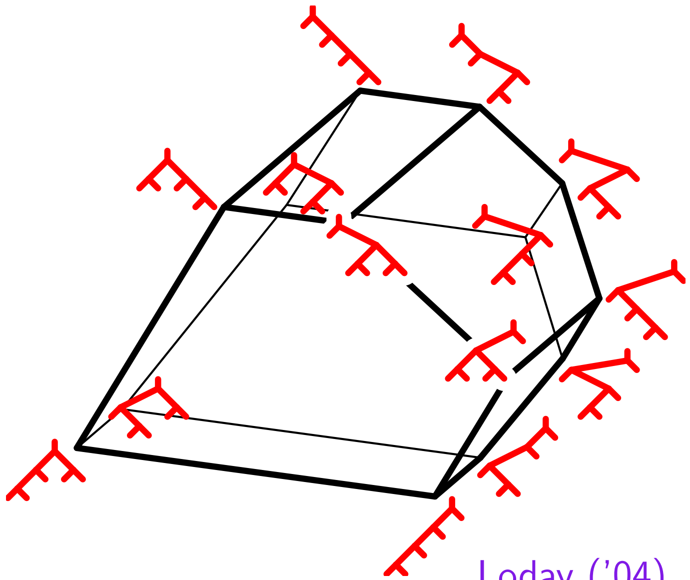
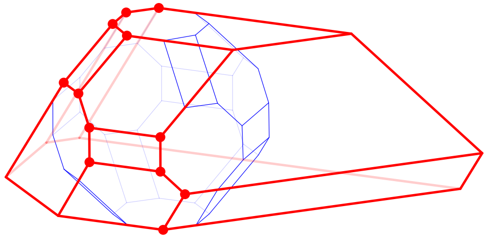
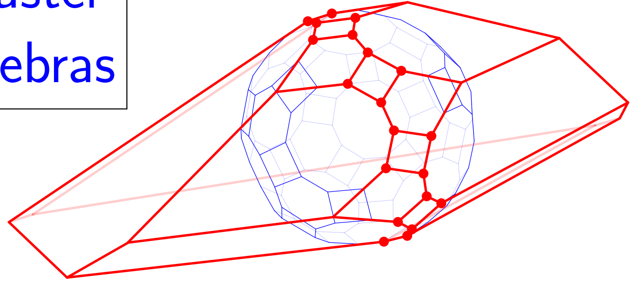


Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

Hopf algebra

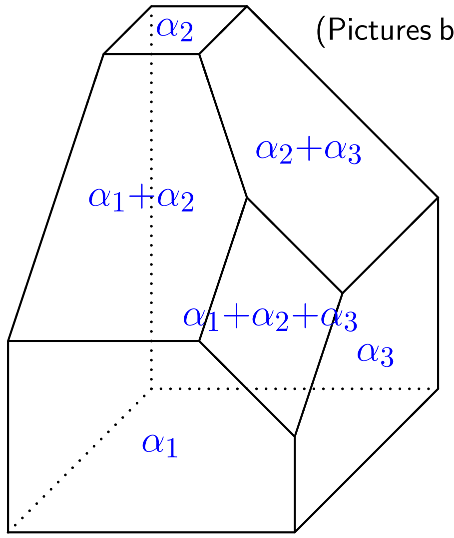


Cluster algebras



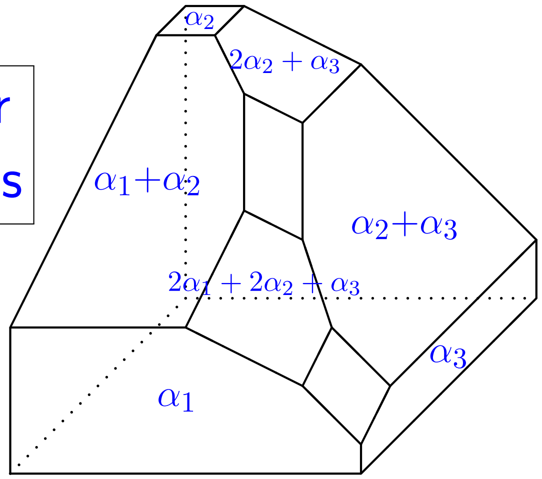
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras



COMPATIBILITY FAN

Chapoton-Fomin-Zelevinsky, *Polytopal realizations of generalized assoc.* ('02)
Ceballos-Santos-Ziegler, *Many non-equiv. realizations of the assoc.* ('11)
Manneville-P., *Compatibility fans for graphical nested complexes* ('15^+)

COMPATIBILITY FANS FOR ASSOCIAHEDRA

T° an initial triangulation
δ, δ' two internal diagonals

compatibility degree between δ and δ':

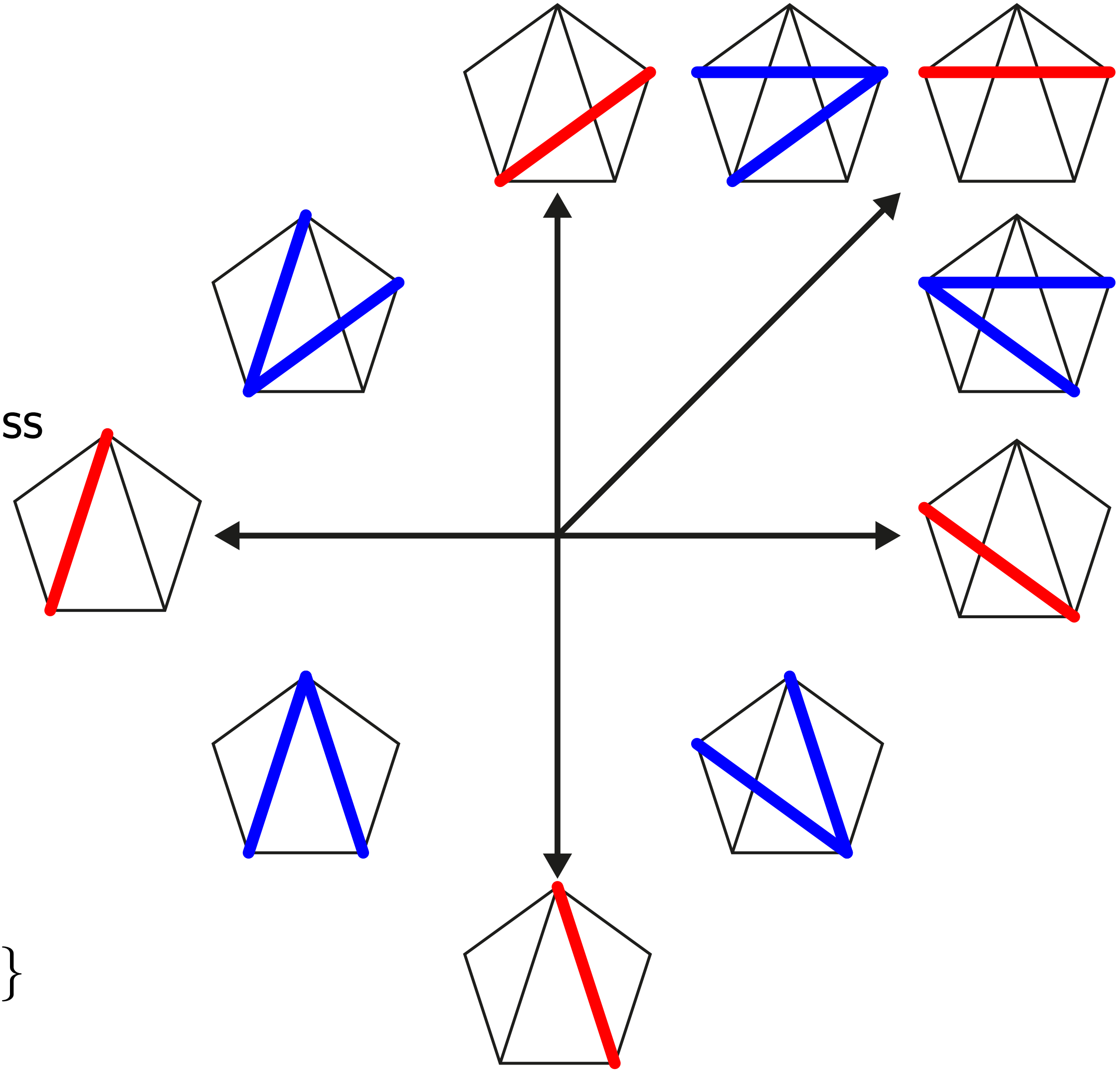
$$(\delta \,\|\, \delta') = \begin{cases} -1 & \text{if } \delta = \delta' \\ 0 & \text{if } \delta \text{ and } \delta' \text{ do not cross} \\ 1 & \text{if } \delta \text{ and } \delta' \text{ cross} \end{cases}$$

compatibility vector of δ wrt T°:

$$\mathbf{d}(\mathrm{T}^\circ, \delta) = \big[(\delta^\circ \,\|\, \delta) \big]_{\delta^\circ \in \mathrm{T}^\circ}$$

compatibility fan wrt T°:

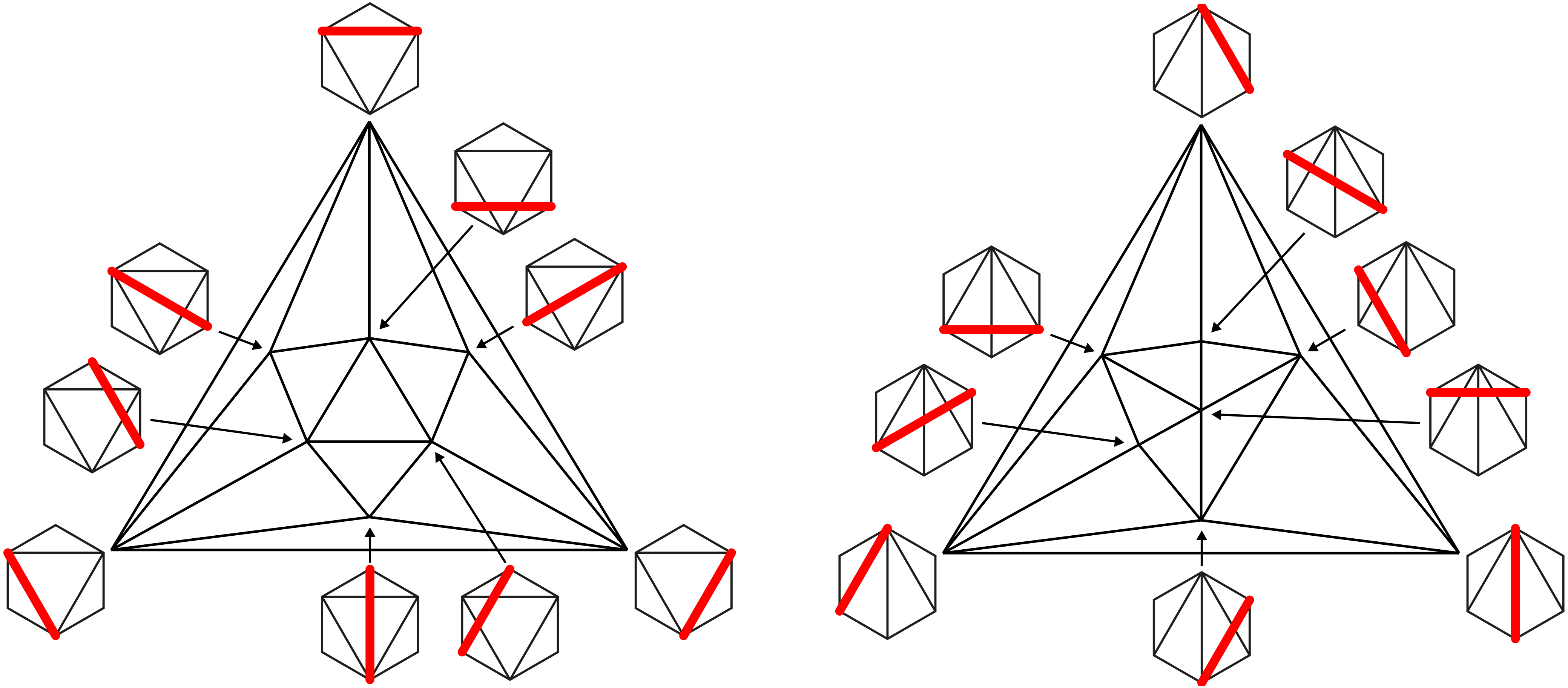
$$\mathcal{D}(\mathrm{T}^\circ) = \{ \mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D} \text{ dissection} \}$$



Fomin-Zelevinsky, *Y-Systems and generalized associahedra* ('03)
Fomin-Zelevinsky, *Cluster algebras II: Finite type classification* ('03)
Chapoton-Fomin-Zelevinsky, *Polytopal realizations of generalized associahedra* ('02)
Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11)

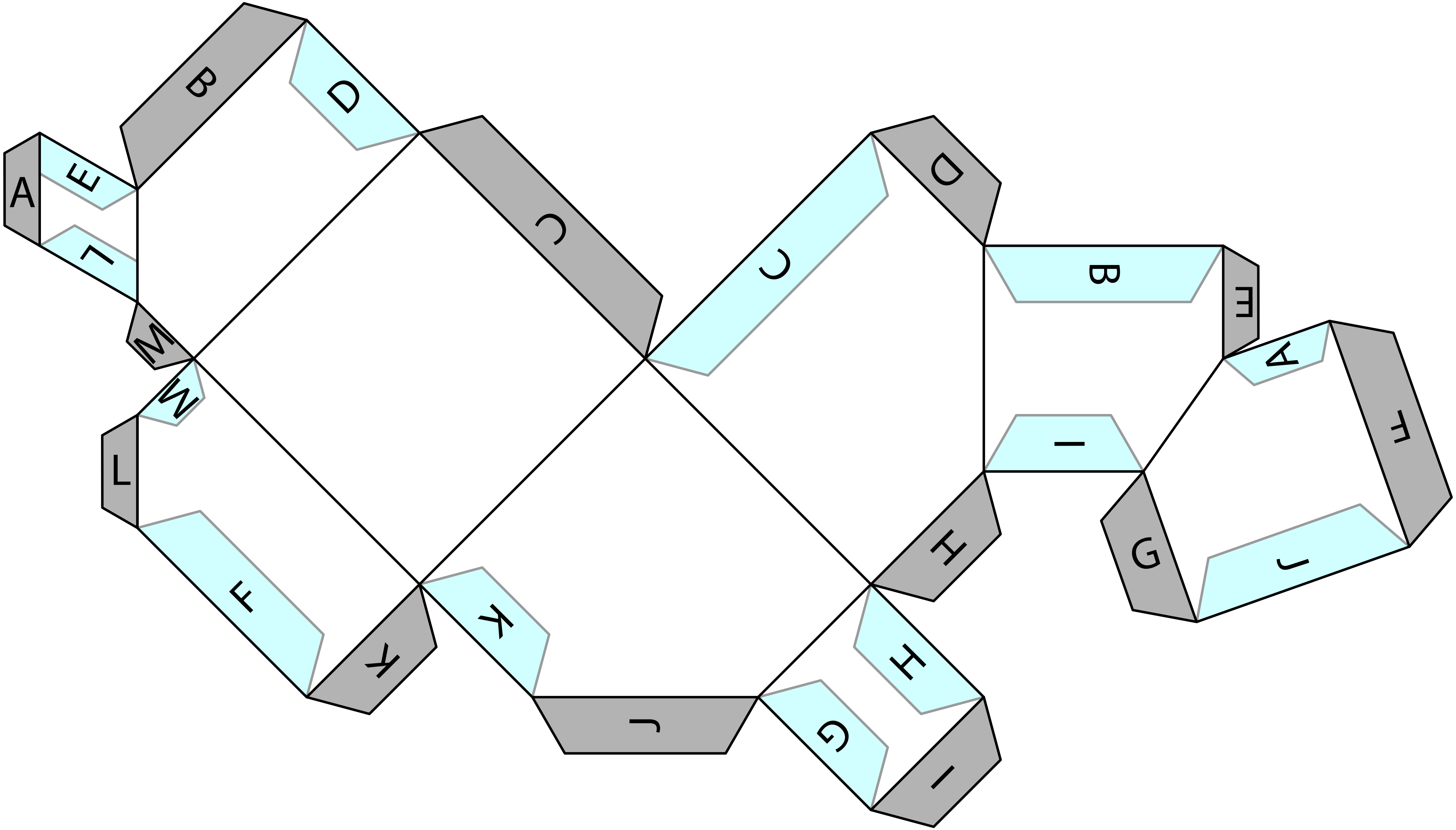
COMPATIBILITY FANS FOR ASSOCIAHEDRA

Different initial triangulations T° yield different realizations



THM. For any initial triangulation T°, the cones $\{\mathbb{R}_{\geq 0}\,\mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D}$ dissection$\}$ form a complete simplicial fan. Moreover, this fan is always polytopal.

Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11)



CHAPOTON-FOMIN-ZELEVINSKY’S ASSOCIAHEDRON

CLUSTER ALGEBRAS

Fomin-Zelevinsky, *Cluster Algebras I, II, III, IV* ('02 – '07)

CLUSTER ALGEBRAS

cluster algebra = commutative ring generated by distinguished cluster variables grouped into overlapping clusters

clusters computed by a mutation process :

cluster seed = algebraic data $\{x_1, \dots, x_n\}$, combinatorical data B (matrix or quiver)
cluster mutation = $\big(\{x_1, \dots, x_k, \dots, x_n\}, B\big) \xleftrightarrow{\mu_k} \big(\{x_1, \dots, x_k', \dots, x_n\}, \mu_k(B)\big)$

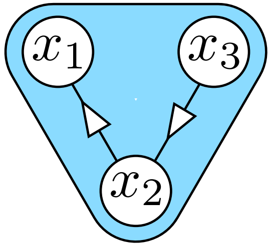
$$x_k \cdot x_k' = \prod_{\{i \mid b_{ik}>0\}} x_i^{b_{ik}} + \prod_{\{i \mid b_{ik}<0\}} x_i^{-b_{ik}}$$

$$\big(\mu_k(B)\big)_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i,j\} \\ b_{ij} + |b_{ik}| \cdot b_{kj} & \text{if } k \notin \{i,j\} \text{ and } b_{ik} \cdot b_{kj} > 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

cluster complex = simplicial complex w/ vertices = cluster variables & facets = clusters

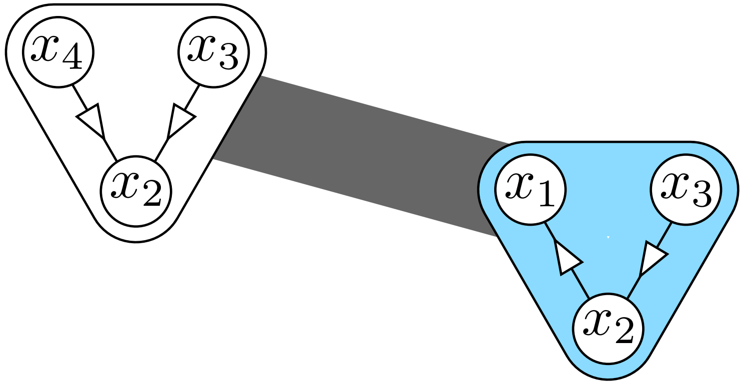
Fomin-Zelevinsky, *Cluster Algebras I, II, III, IV* ('02 – '07)

CLUSTER MUTATION



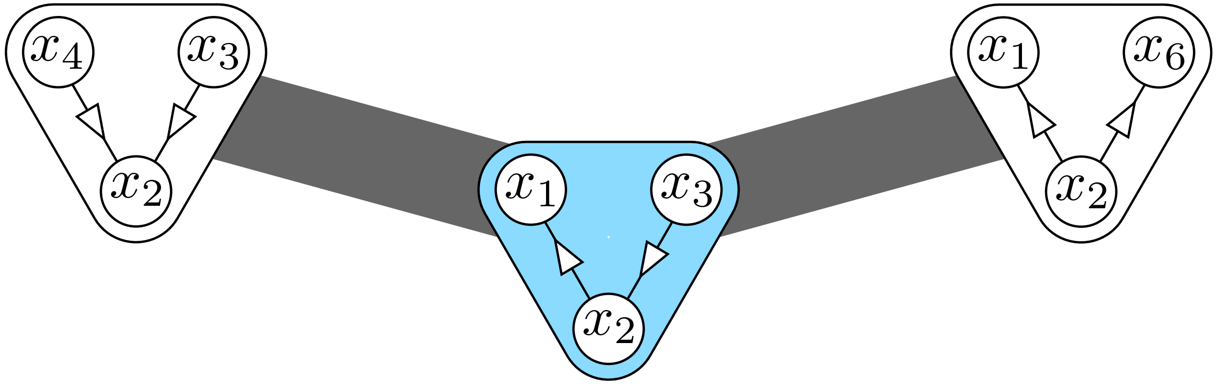
CLUSTER MUTATION

$$x_4 = \frac{1 + x_2}{x_1}$$



CLUSTER MUTATION

$$x_4 = \frac{1 + x_2}{x_1} \qquad x_6 = \frac{1 + x_2}{x_3}$$



CLUSTER MUTATION

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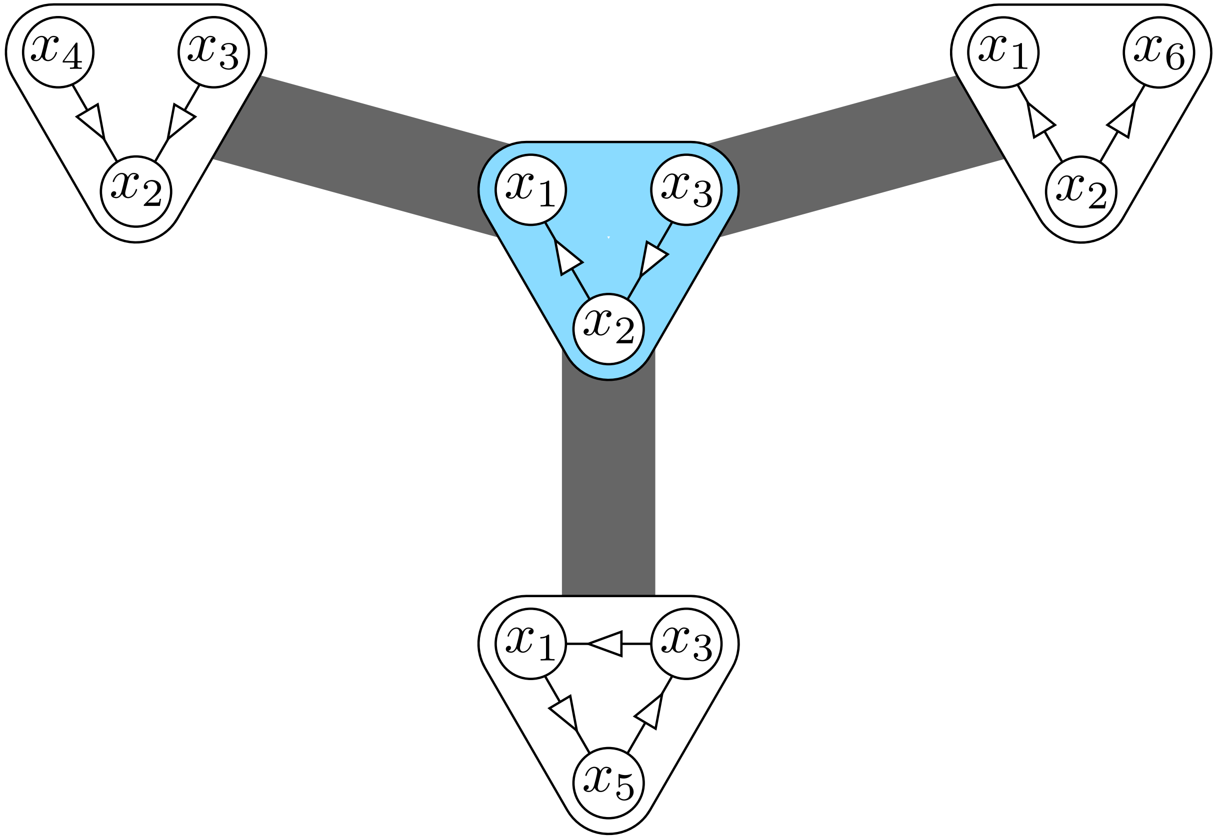
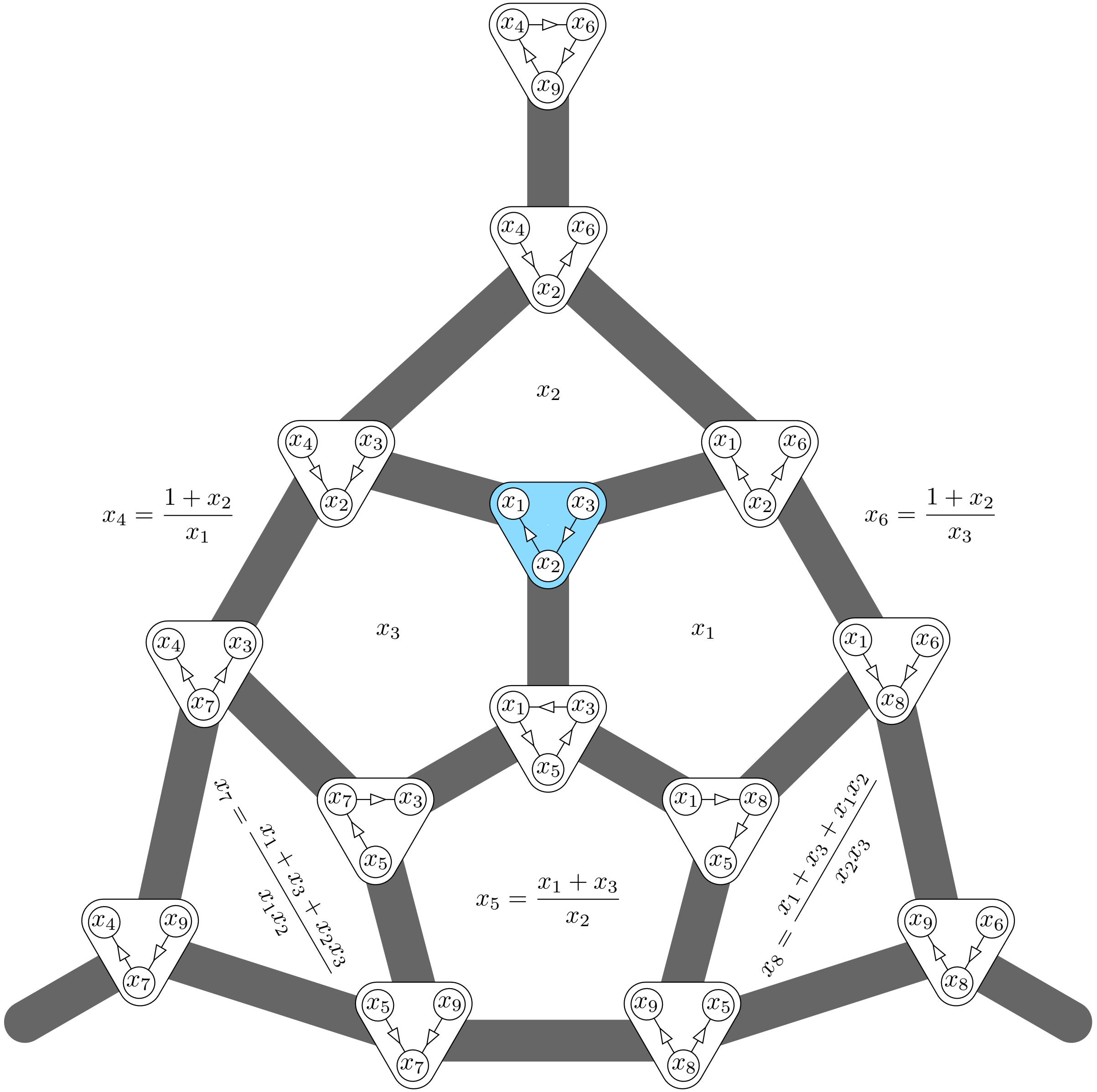


$$x_5 = \frac{x_1 + x_3}{x_2}$$

CLUSTER MUTATION GRAPH



$$x_4 = \frac{1+x_2}{x_1}$$

$$x_6 = \frac{1+x_2}{x_3}$$

$$x_7 = \frac{x_1+x_3+x_2x_3}{x_1x_2}$$

$$x_5 = \frac{x_1+x_3}{x_2}$$

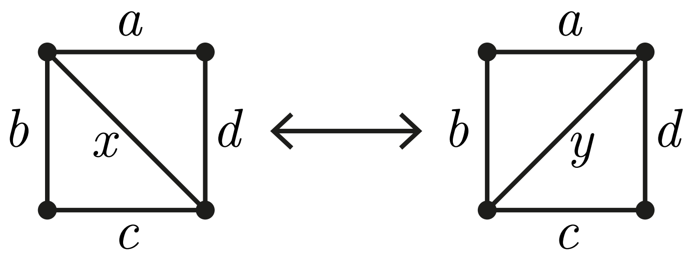
$$x_8 = \frac{x_1+x_3+x_1x_2}{x_2x_3}$$

$$x_9 = \frac{(x_1+x_3)(1+x_2)}{x_1x_2x_3}$$

CLUSTER ALGEBRA FROM TRIANGULATIONS

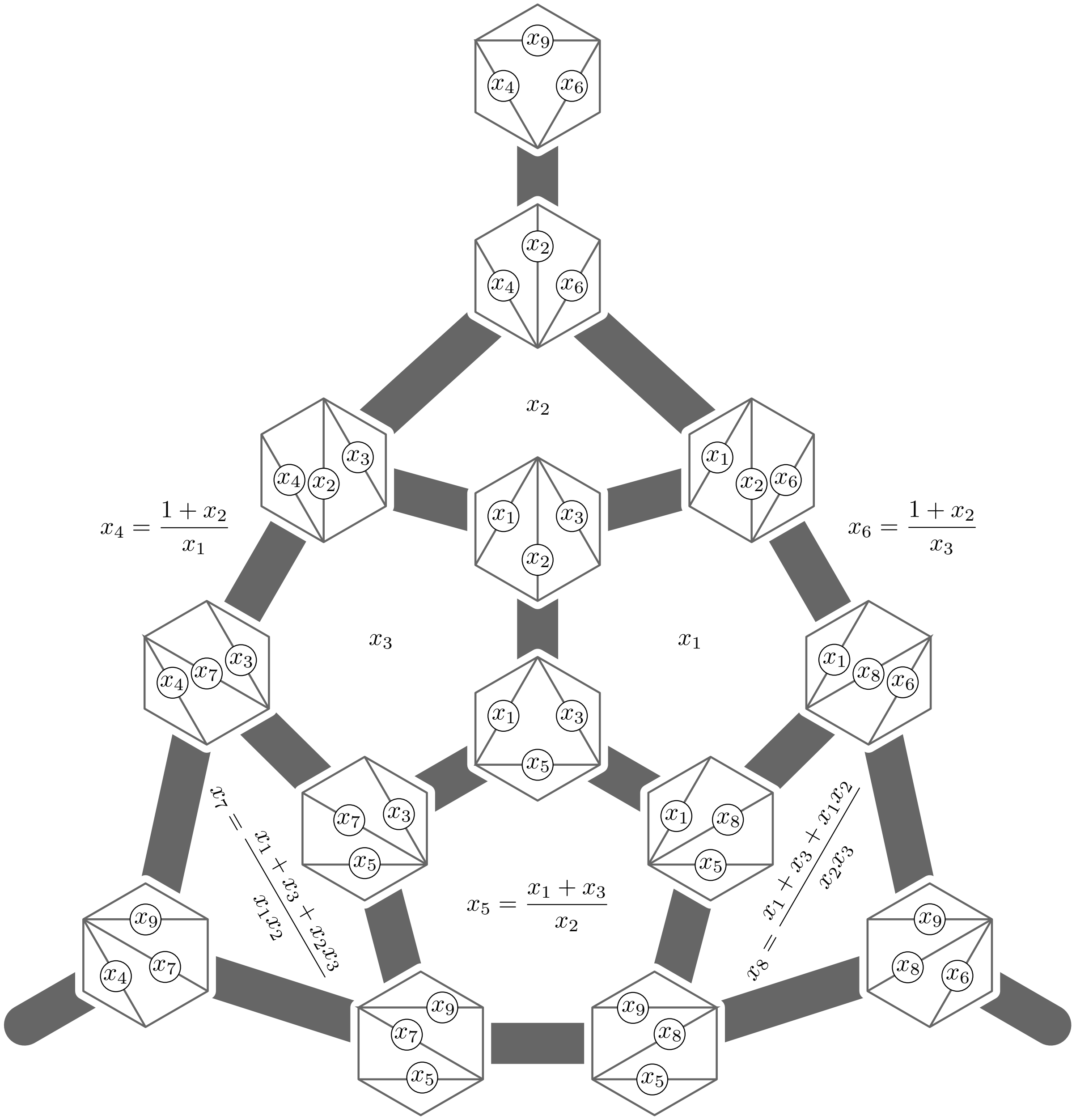
One constructs a cluster algebra from the triangulations of a polygon:

diagonals $\longleftrightarrow$ cluster variables
triangulations $\longleftrightarrow$ clusters
flip $\longleftrightarrow$ mutation



$\longleftrightarrow$ $xy = ac + bd$

CLUSTER MUTATION GRAPH



$$x_4 = \frac{1+x_2}{x_1}$$

$$x_6 = \frac{1+x_2}{x_3}$$

$$x_7 = \frac{x_1+x_3+x_2x_3}{x_1x_2}$$

$$x_5 = \frac{x_1+x_3}{x_2}$$

$$x_8 = \frac{x_1+x_3+x_1x_2}{x_2x_3}$$

$$x_9 = \frac{(x_1+x_3)(1+x_2)}{x_1x_2x_3}$$

CLUSTER ALGEBRAS

THM. (Laurent phenomenon)
All cluster variables are Laurent polynomials in the variables of the initial cluster seed.

Fomin-Zelevinsky, *Cluster algebras I: Fundations* ('02)

CLUSTER ALGEBRAS

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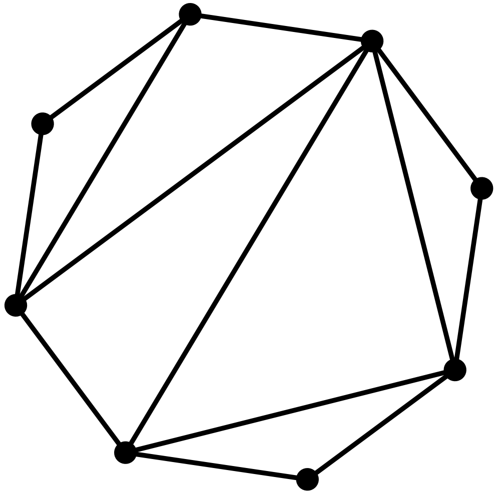
Fomin-Zelevinsky, *Cluster algebras I: Fundations* ('02)

Exm: Conway-Coxeter friezes



	...		0		0		0		0		0		0		0		0		0		...
...		1		1		1		1		1		1		1		1		1		...	
	...		2		1		3		3		1		3		1		4		2		...
...		7		1		2		8		2		2		2		3		7		...	
	...		3		1		5		5		3		1		5		5		3		...
...		2		2		2		3		7		1		2		8		2		...	
	...		1		3		1		4		2		1		3		3		1		...
...		1		1		1		1		1		1		1		1		1		...	
	...		0		0		0		0		0		0		0		0		0		...

CLUSTER ALGEBRAS

THM. (Laurent phenomenon)
All cluster variables are Laurent polynomials in the variables of the initial cluster seed.

Fomin-Zelevinsky, *Cluster algebras I: Fundations* ('02)

THM. (Classification)
Finite type cluster algebras are classified by the Cartan-Killing classification for crystallographic root systems.

Fomin-Zelevinsky, *Cluster algebras II: Finite type classification* ('03)

for a root system Φ, and an acyclic initial cluster $X = \{x_1, \dots, x_n\}$, there is a bijection

$$\begin{array}{ccc} \text{cluster variables of } \mathcal{A}_\Phi & \xleftrightarrow{\theta_X} & \Phi_{\ge -1} = \Phi^+ \cup -\Delta \\ y = \dfrac{F(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} & \xleftrightarrow{\theta_X} & \beta = d_1\alpha_1 + \dots + d_n\alpha_n \\ \text{cluster of } \mathcal{A}_\Phi & \xleftrightarrow{\theta_X} & X\text{-cluster in } \Phi_{\ge -1} \\ \text{cluster complex of } \mathcal{A}_\Phi & \xleftrightarrow{\theta_X} & X\text{-cluster complex in } \Phi_{\ge -1} \end{array}$$

see a short introduction to finite Coxeter groups

CLUSTER/DENOMINATOR/COMPATIBILITY FAN

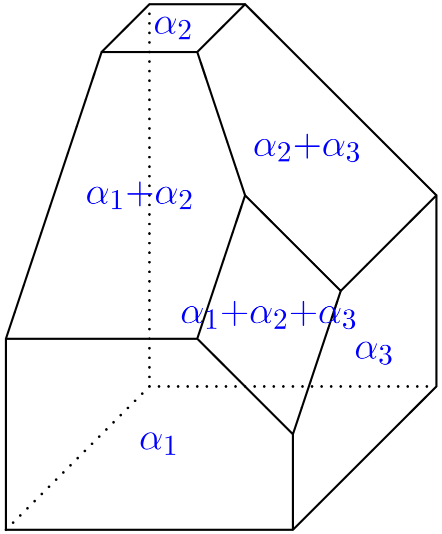
THM. Φ crystallographic root system, X acyclic initial cluster of $\mathcal{A}_\Phi$,
θ_X : cluster variables of $\mathcal{A}_\Phi \longmapsto$ almost postive roots $\Phi_{\geq -1} = \Phi^+ \cup -\Delta$

The collection of cones

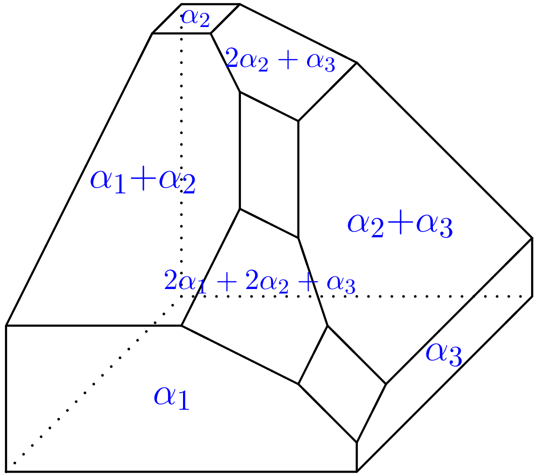
$$\{\mathbb{R}_{\geq 0} \cdot \theta(Y) \mid Y \text{ cluster of } \mathcal{A}_\Phi\}$$

is a complete simplicial fan, called cluster fan, denominator fan, or compatibility fan. Moreover, this fan is always polytopal.

Chapoton-Fomin-Zelevinsky, *Polytopal realizations of generalized associahedra* ('02)



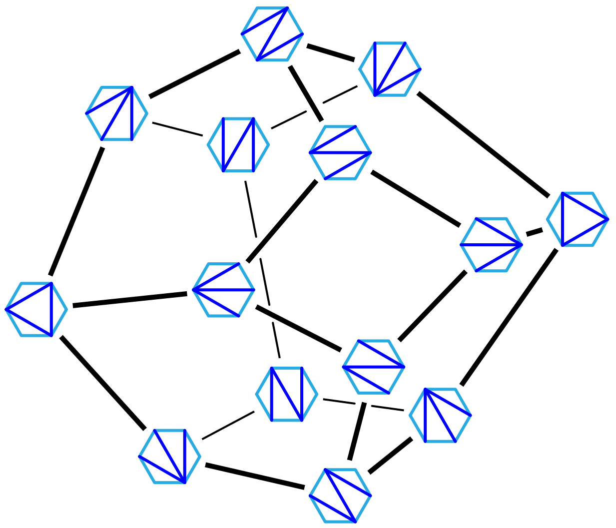
Type A



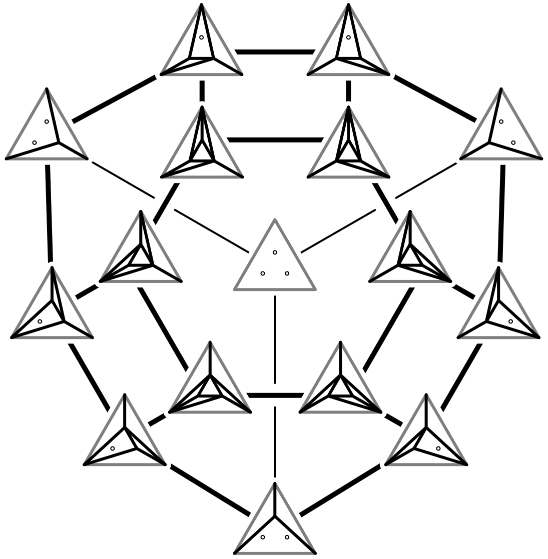
Type B

THREE FAMILIES OF REALIZATIONS

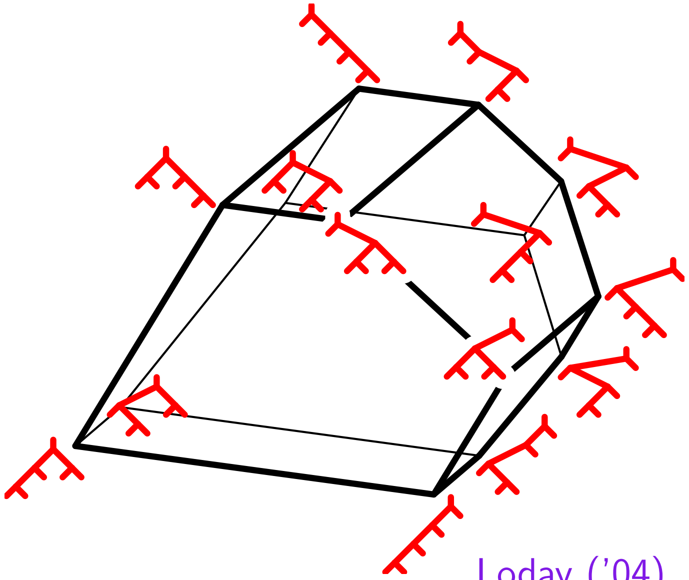
SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)



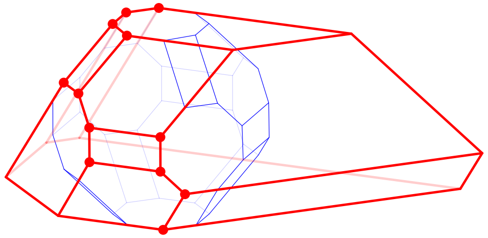
LODAY'S ASSOCIAHEDRON

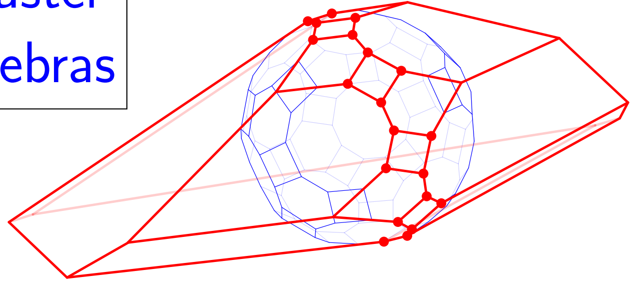


Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

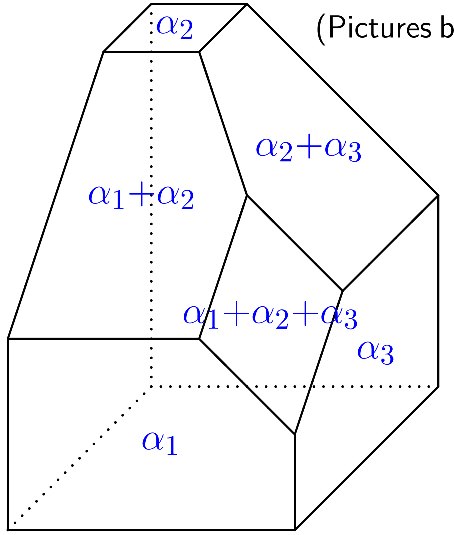
Hopf algebra

Cluster algebras





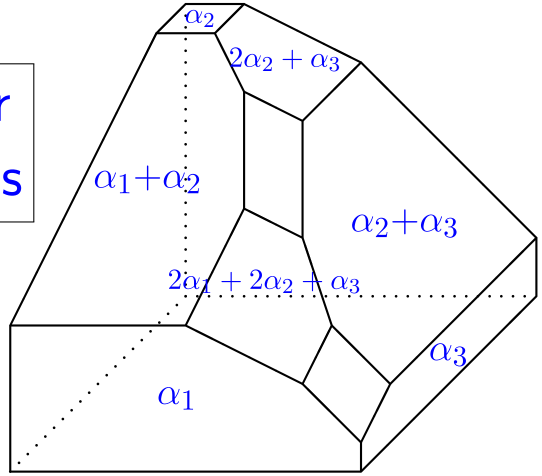
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON



(Pictures by CFZ)

Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras



LODAY'S ASSOCIAHEDRON

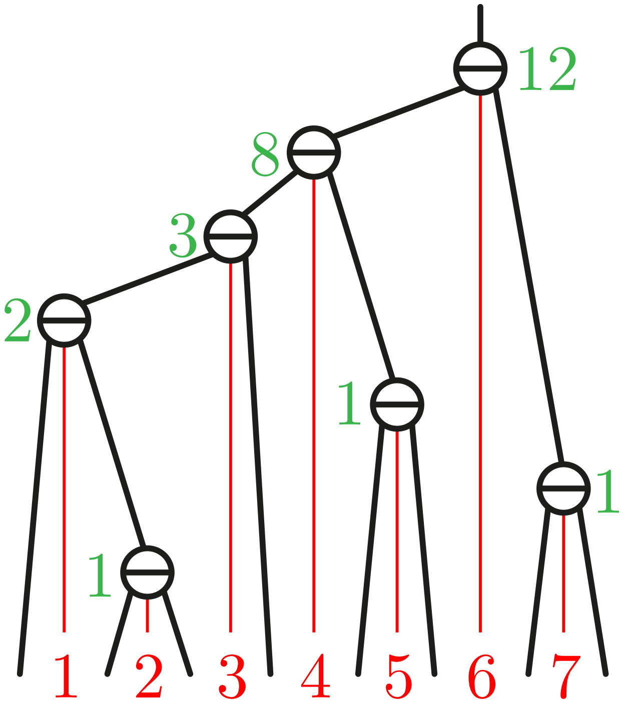
Loday, *Realization of the Stasheff polytope* ('04)

LODAY'S ASSOCIAHEDRON

$$\mathsf{Asso}(n) := \operatorname{conv}\left\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\right\} = \mathbb{H} \cap \bigcap_{1\le i\le j\le n+1} \mathbf{H}^{\geq}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i)\cdot r(\mathrm{T},i)\big]_{i\in[n+1]} \qquad \mathbf{H}^{\geq}(i,j) := \left\{\mathbf{x}\in\mathbb{R}^{n+1} \;\middle|\; \sum_{i\le k\le j} x_i \ge \binom{j-i+2}{2}\right\}$$

Loday, *Realization of the Stasheff polytope* ('04)



LODAY'S ASSOCIAHEDRON

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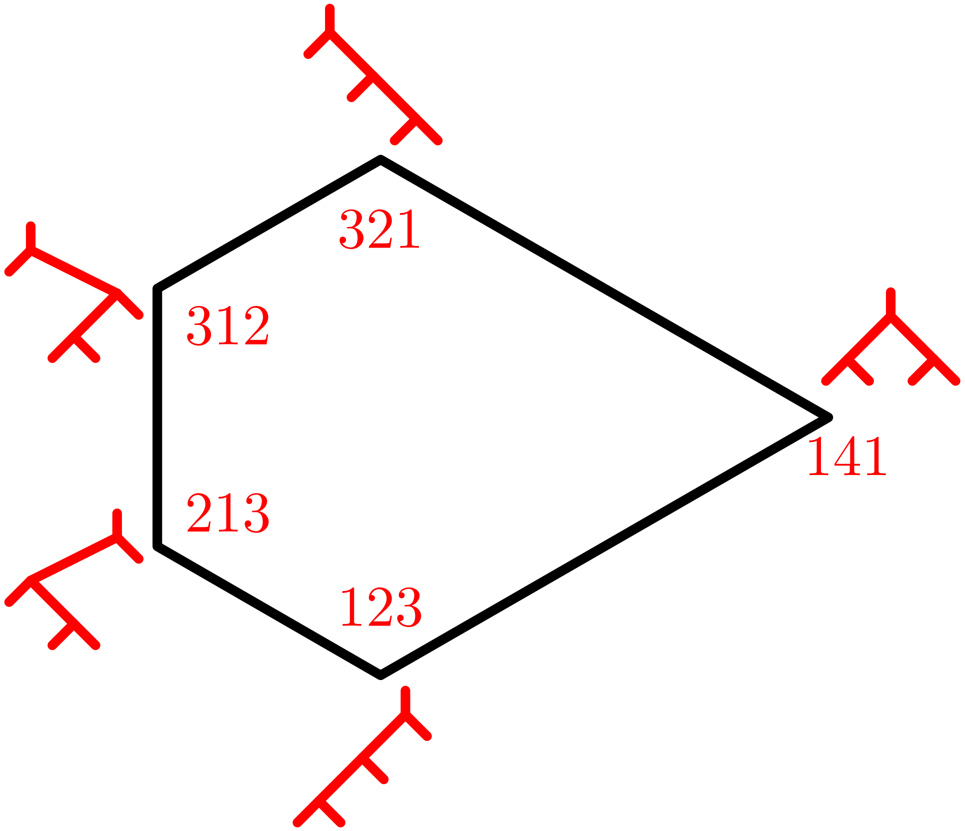
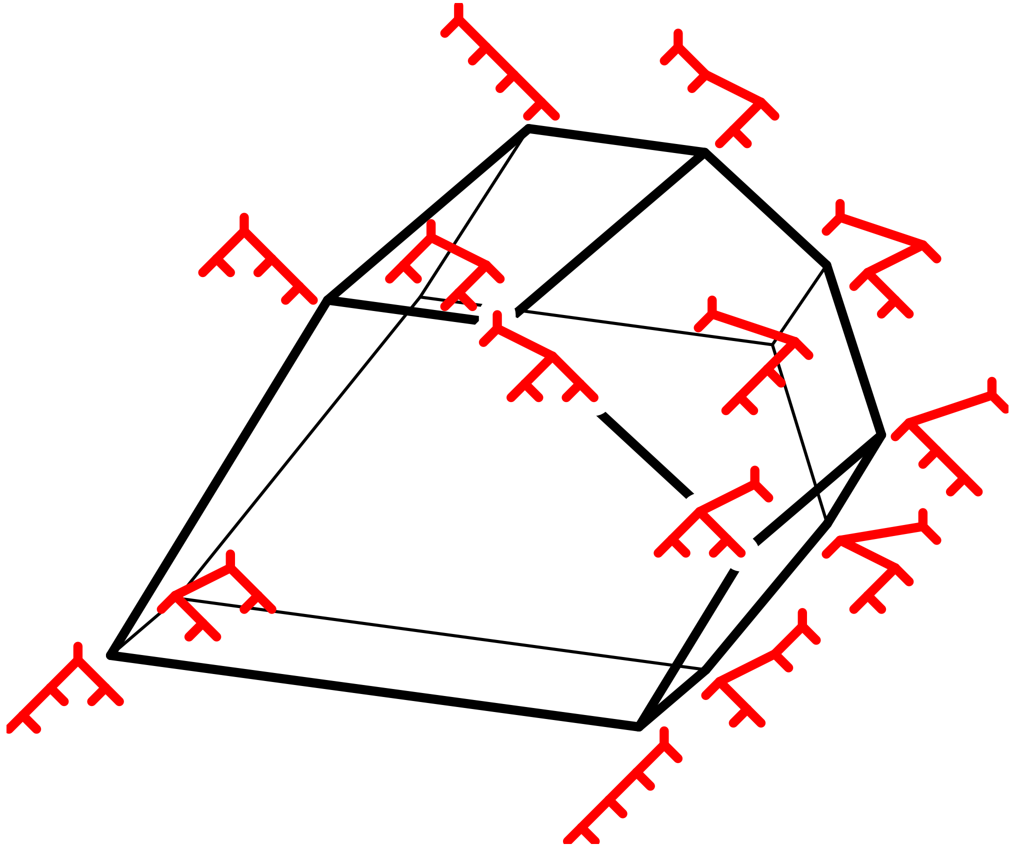
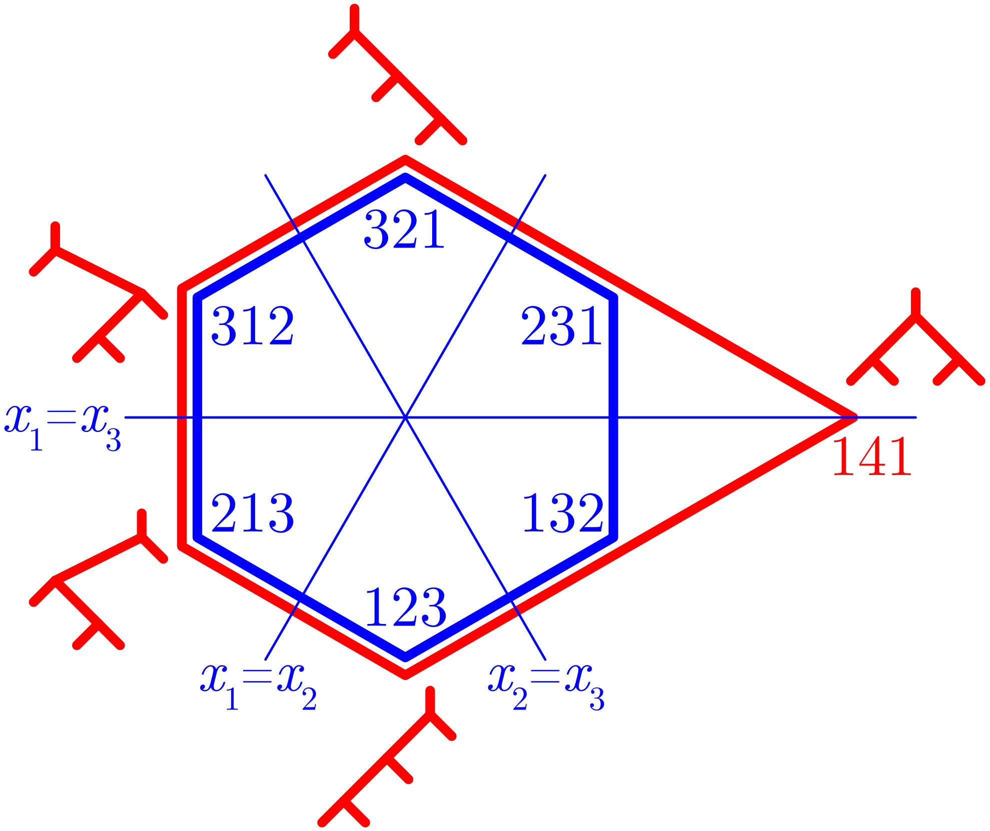


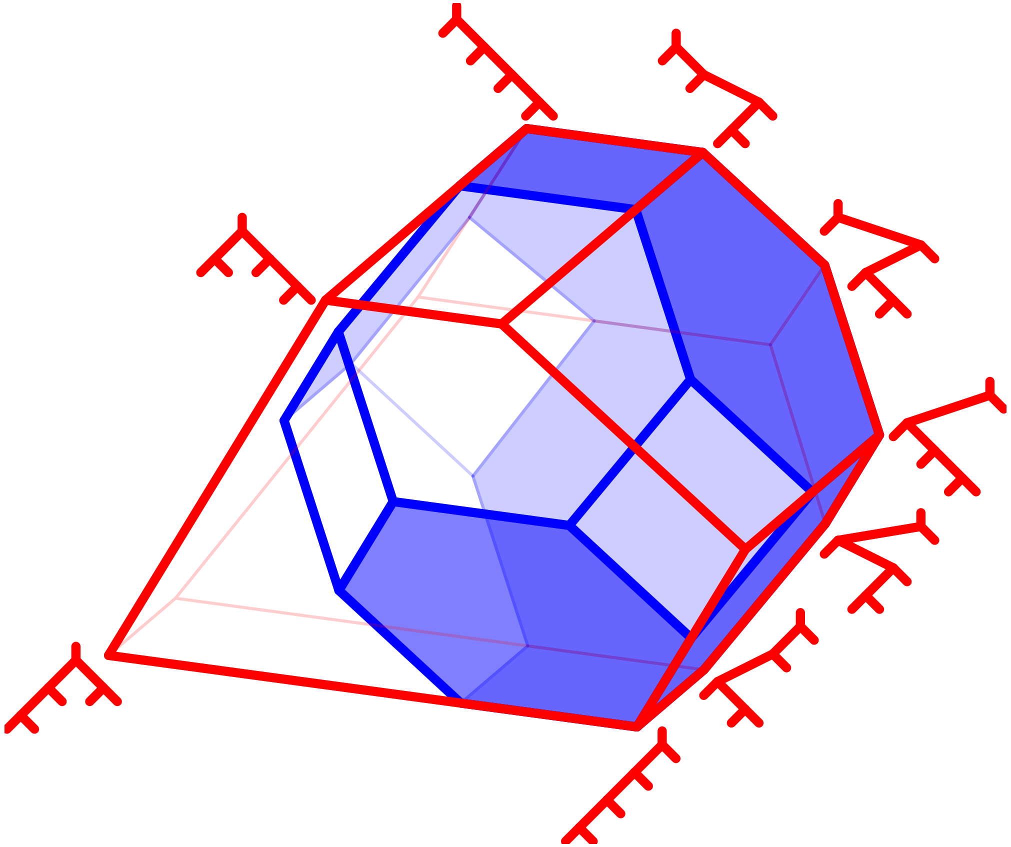


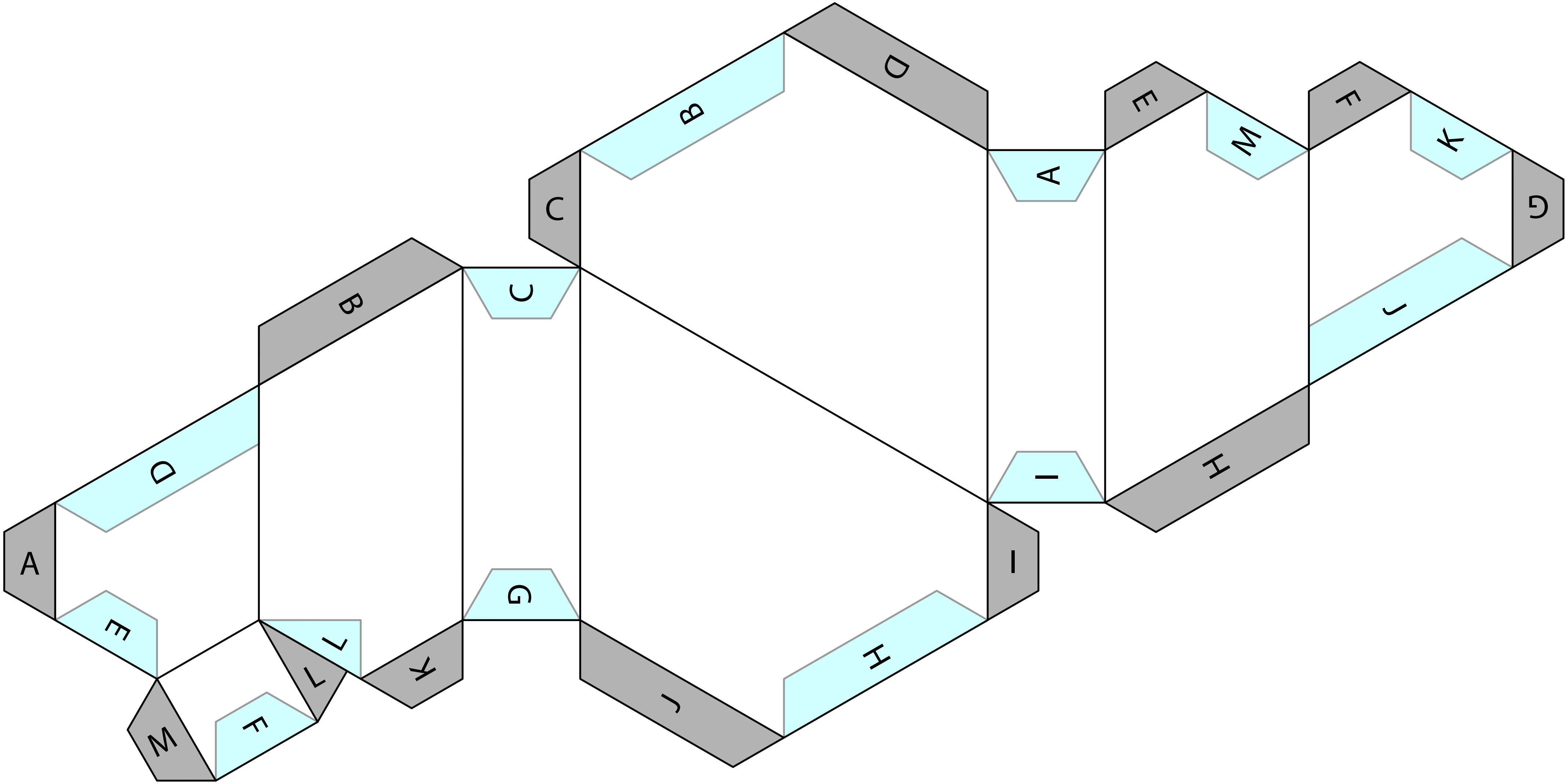
LODAY'S ASSOCIAHEDRON

Asso(n) obtained by deleting inequalities in facet description of the permutahedron

$$\begin{aligned}\text{normal cone of } \mathbf{L}(\mathrm{T}) \text{ in Asso}(n) &= \{\mathbf{x} \in \mathbb{H} \mid x_i < x_j \text{ for all } i \to j \text{ in } \mathrm{T}\} \\ &= \bigcup_{\sigma^{-1} \in \mathcal{L}(\mathrm{T})} \text{normal cone of } \sigma \text{ in Perm}(n)\end{aligned}$$



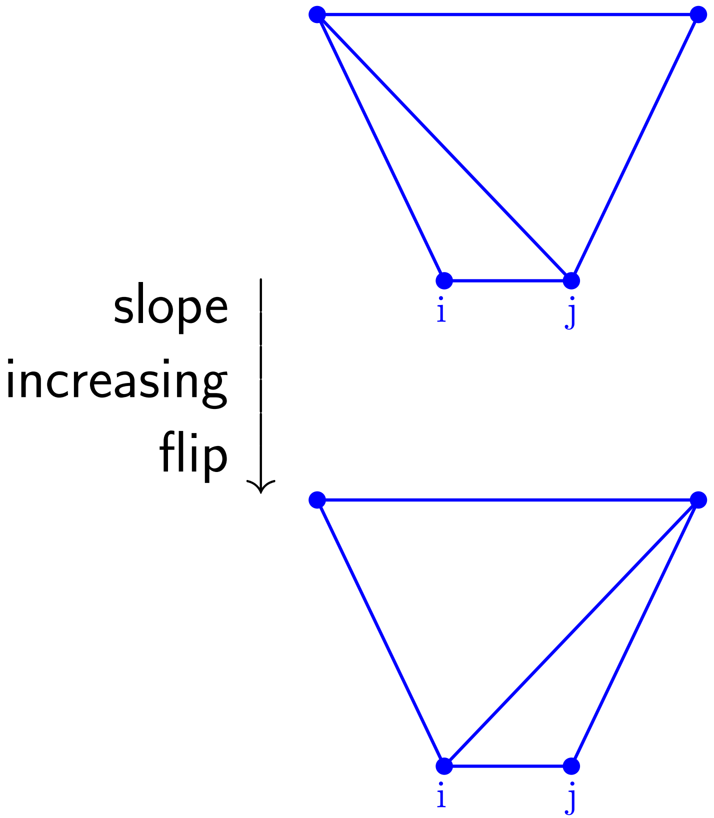




LODAY’S ASSOCIAHEDRON

TAMARI LATTICE

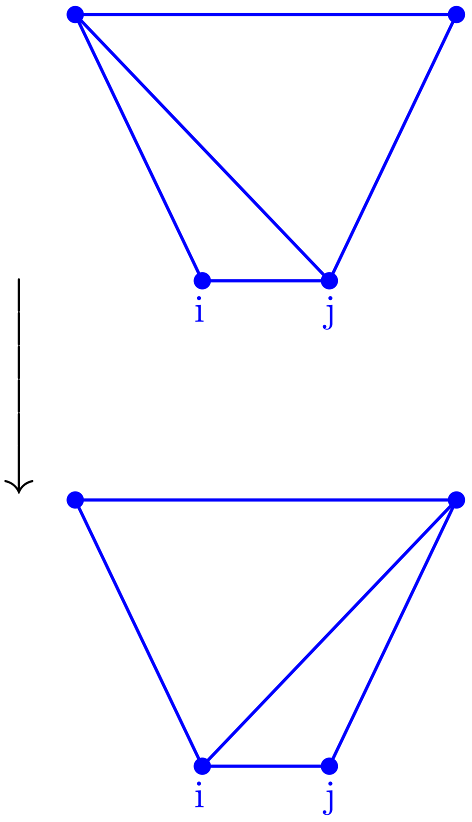
Tamari lattice = slope increasing flips on triangulations



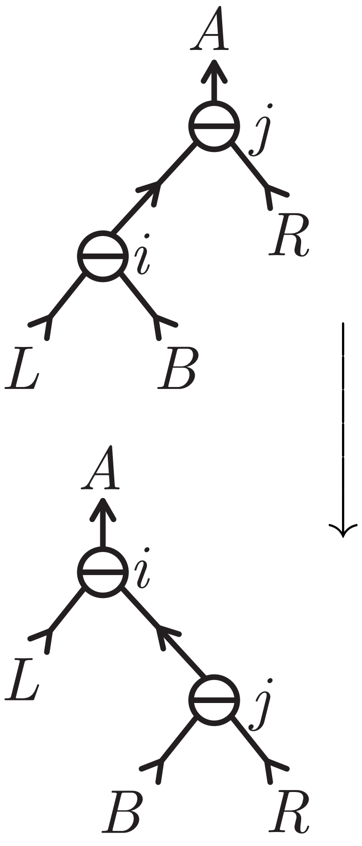
Tamari Festschrift ('12)

TAMARI LATTICE

Tamari lattice = slope increasing flips on triangulations
= right rotations on binary trees



slope increasing flip

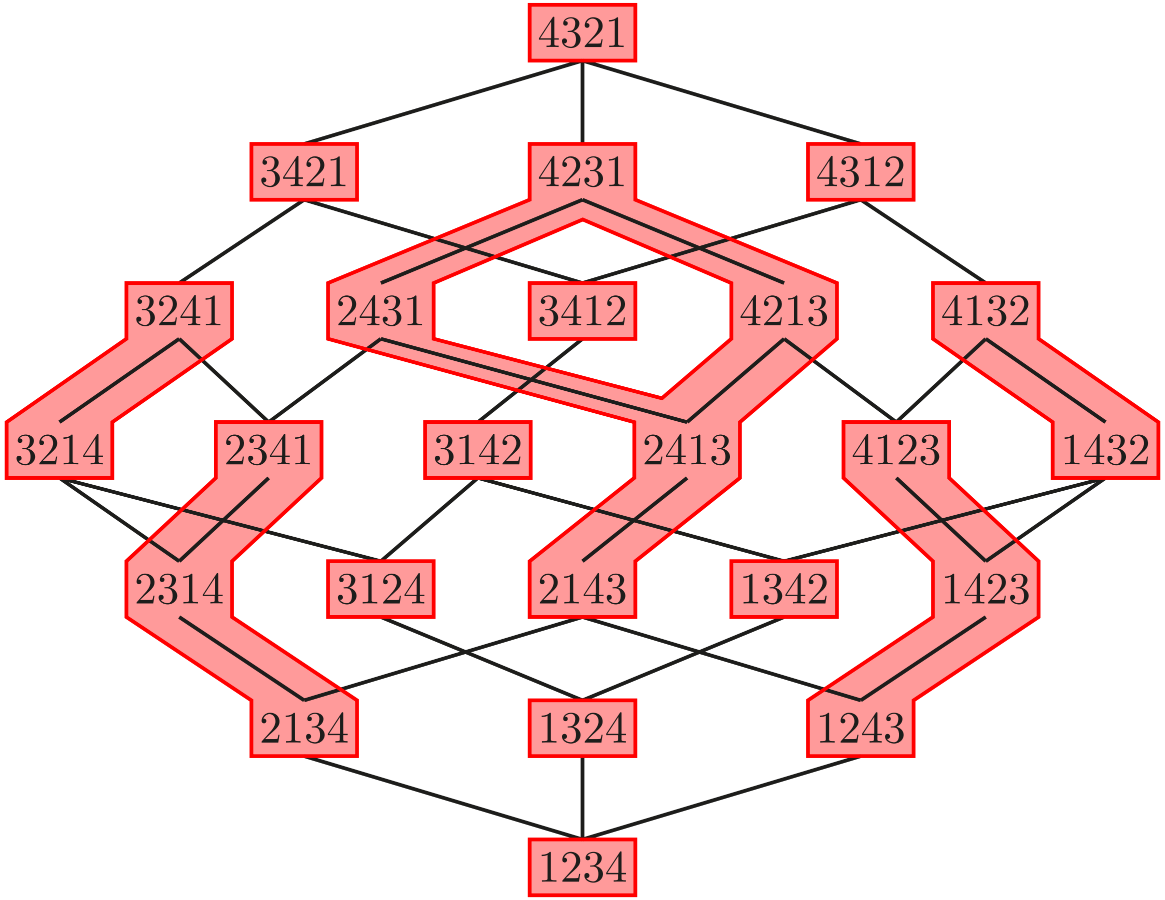


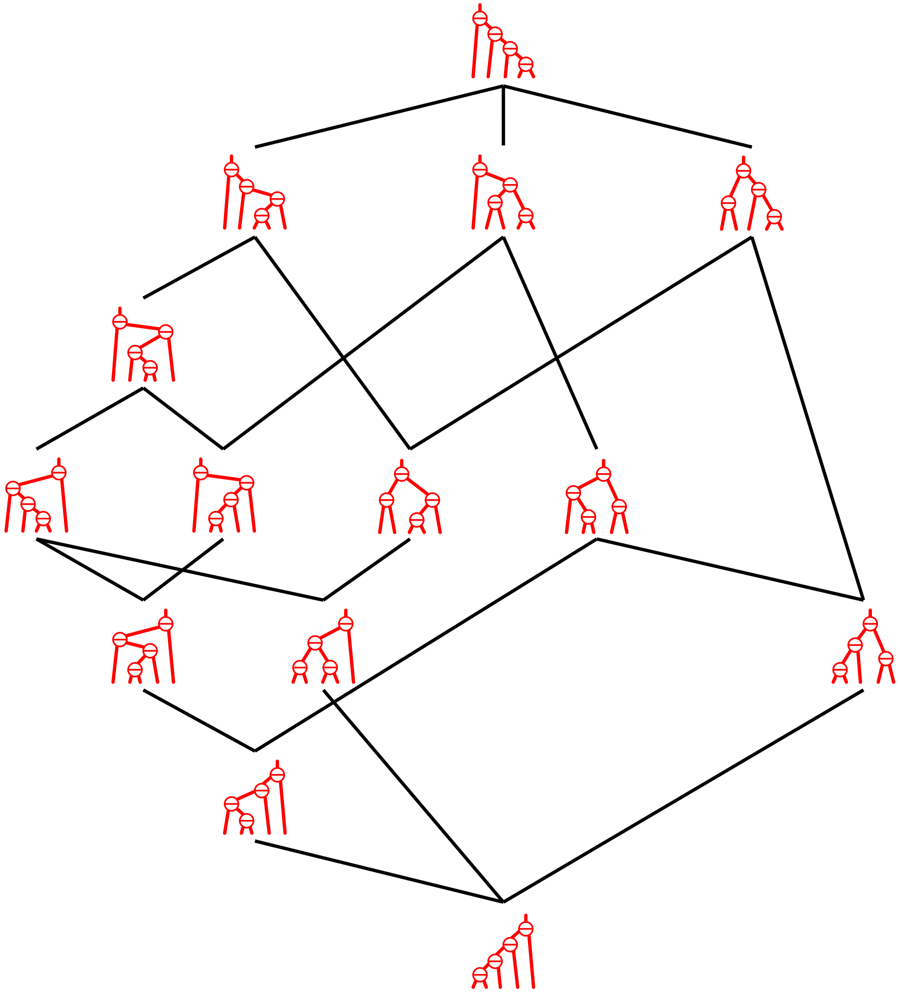
right rotation

Tamari Festschrift ('12)

TAMARI LATTICE

Tamari lattice = slope increasing flips on triangulations
= right rotations on binary trees
= lattice quotient of the weak order by the sylvester congruence





Tamari Festschrift ('12)

TAMARI LATTICE

Tamari lattice = slope increasing flips on triangulations
= right rotations on binary trees
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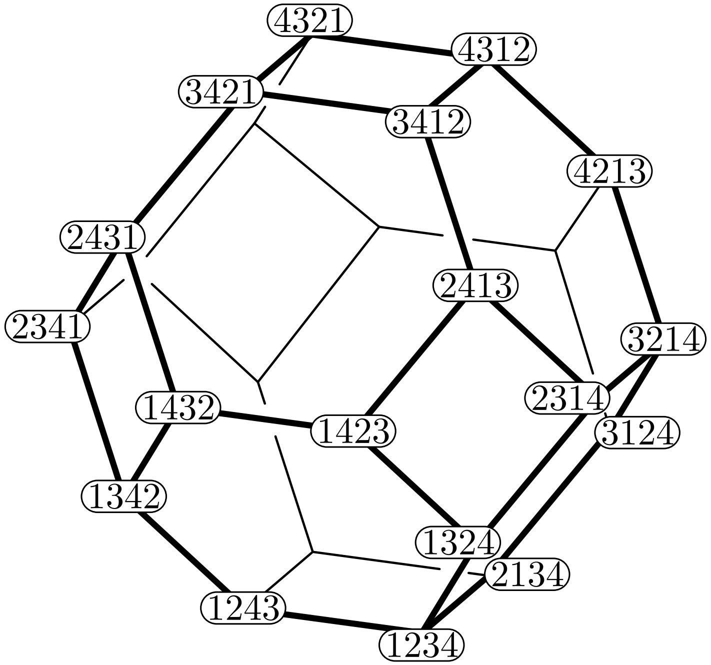
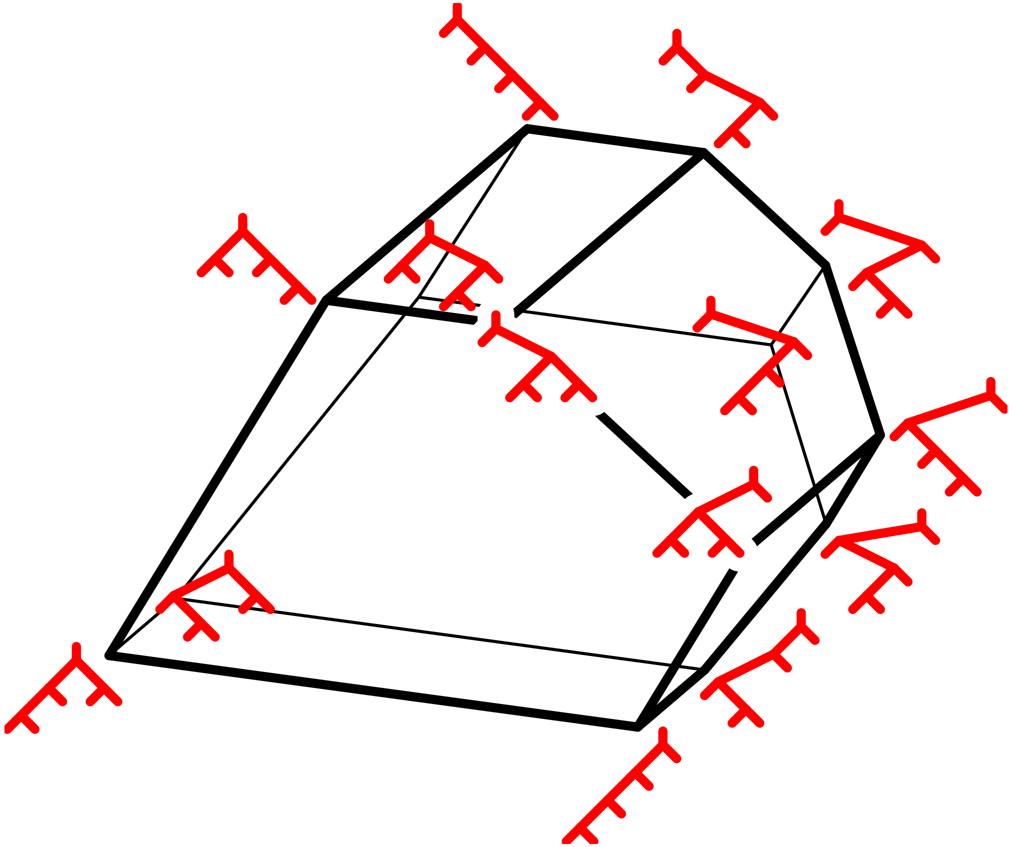
= orientation of the graph of the associahedron in direction $e \to w_\circ$





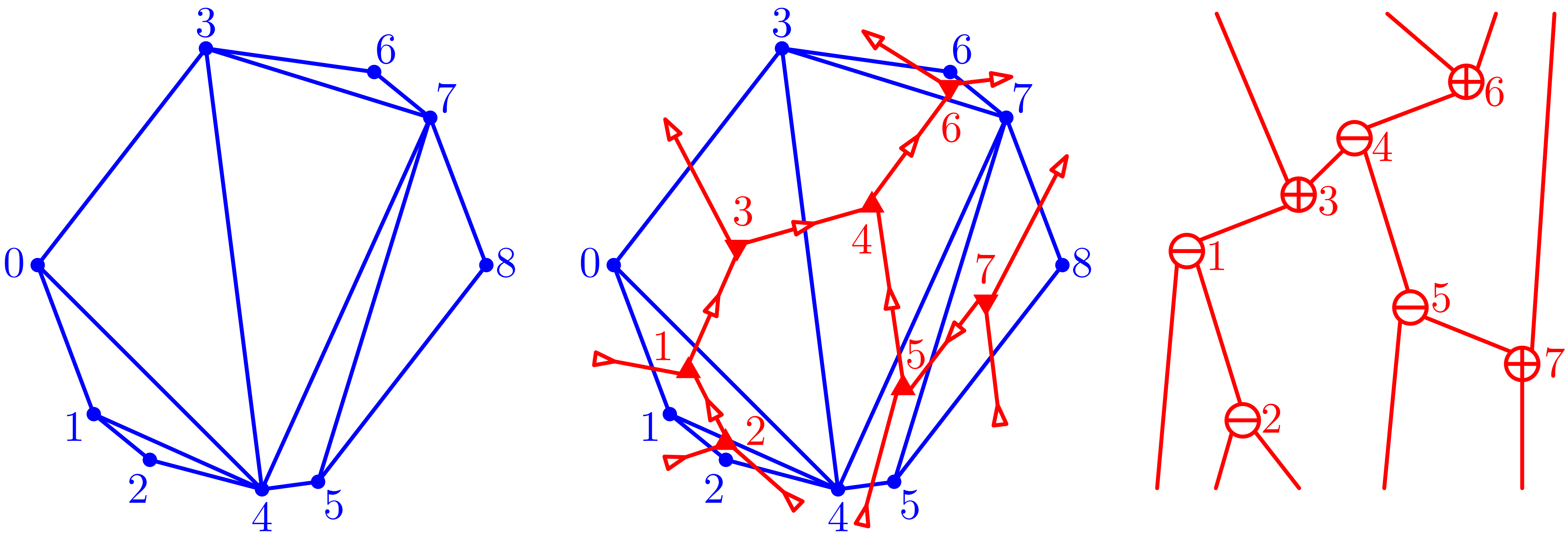
Tamari Festschrift ('12)

HOHLWEG-LANGE'S ASSOCIAHEDRON

Loday, *Realization of the Stasheff polytope* ('04)
Reading, *Cambrian lattices* ('06)
Reading-Speyer, *Cambrian fans* ('09)
Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)
Lange-P., *Using spines to revisit a construction of the associahedron* ('15)

CAMBRIAN TREES AND TRIANGULATIONS

Cambrian trees are dual to triangulations of polygons



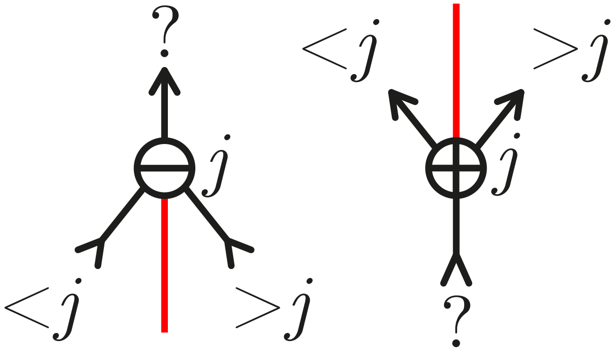
vertices above or below $[0, 9]$	$\longleftrightarrow$	signature
triangle $i < j < k$	$\longleftrightarrow$	node j

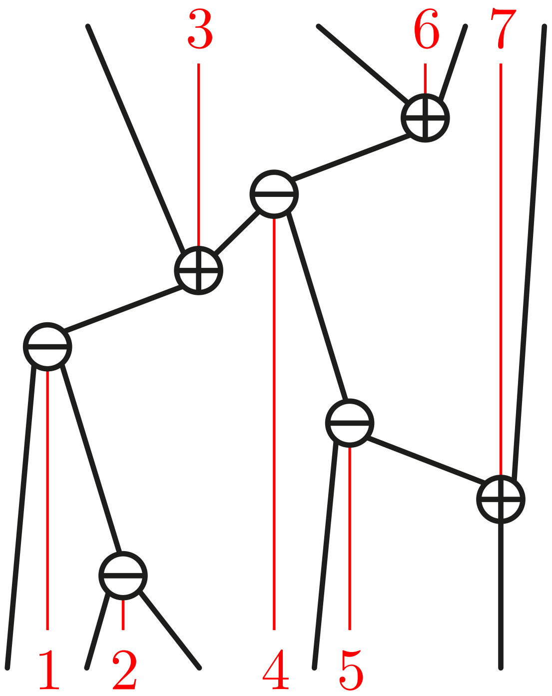
For any signature ε, there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ ε-Cambrian trees

Lange-P., *Using spines to revisit a construction of the associahedron* ('15)

CAMBRIAN TREES

Cambrian tree =
directed and labeled tree such that





Lange-P., *Using spines to revisit a construction of the associahedron* ('15)

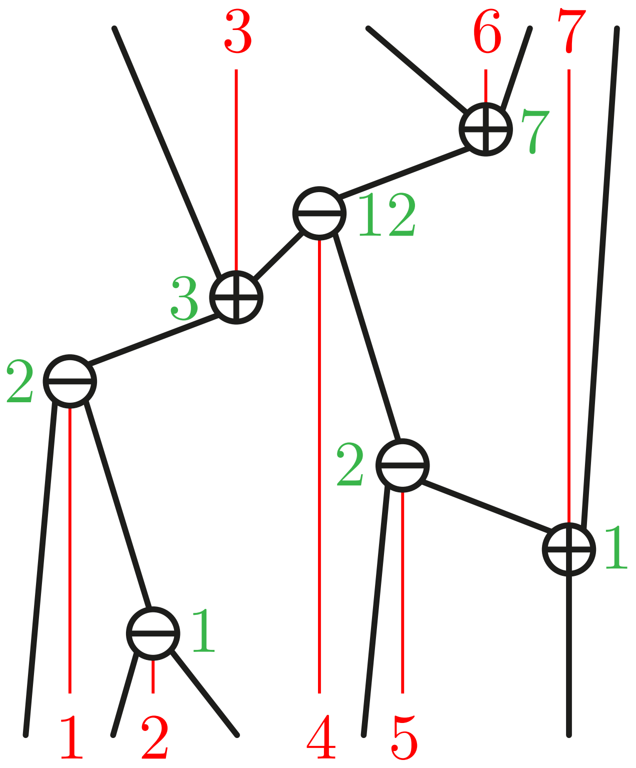
HOHLWEG-LANGE'S ASSOCIAHEDRA

for any $\varepsilon \in \pm^{n+1}$, $\mathrm{Asso}(\varepsilon) := \mathrm{conv}\left\{ \mathbf{HL}(\mathrm{T}) \mid \mathrm{T}\ \varepsilon\text{-Cambrian tree} \right\}$

$$\text{with } \mathbf{HL}(\mathrm{T})_j := \begin{cases} \ell(\mathrm{T}, j) \cdot r(\mathrm{T}, j) & \text{if } \varepsilon(j) = - \\ n + 2 - \ell(\mathrm{T}, j) \cdot r(\mathrm{T}, j) & \text{if } \varepsilon(j) = + \end{cases}$$

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

Lange-P., *Using spines to revisit a construction of the associahedron* ('15)



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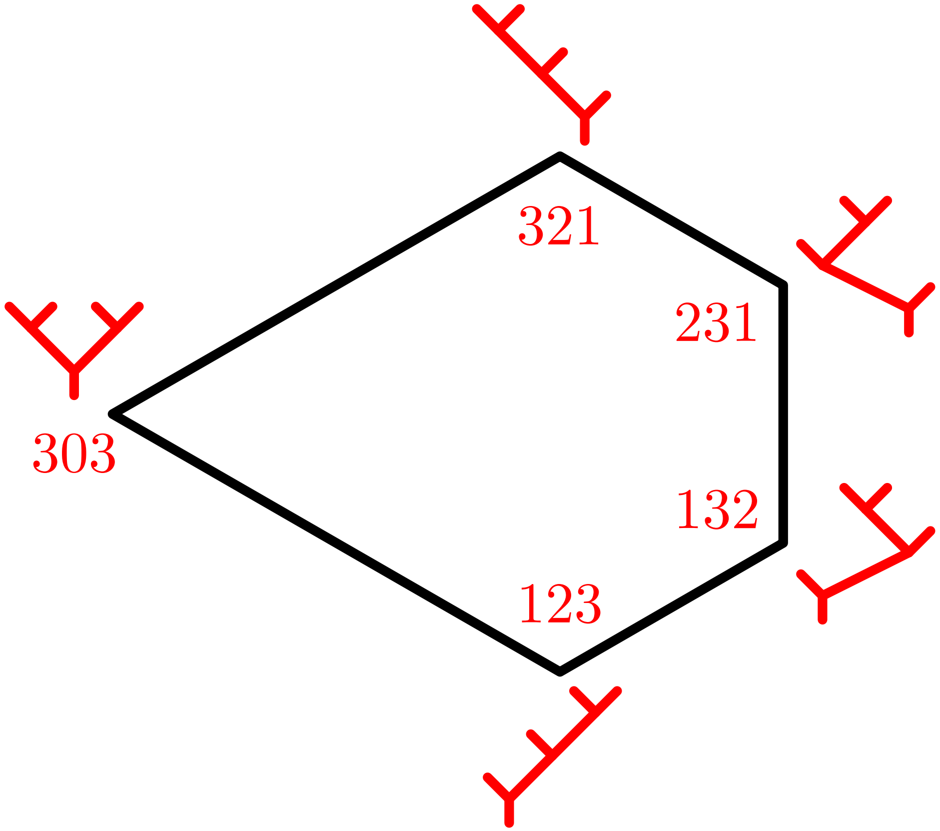

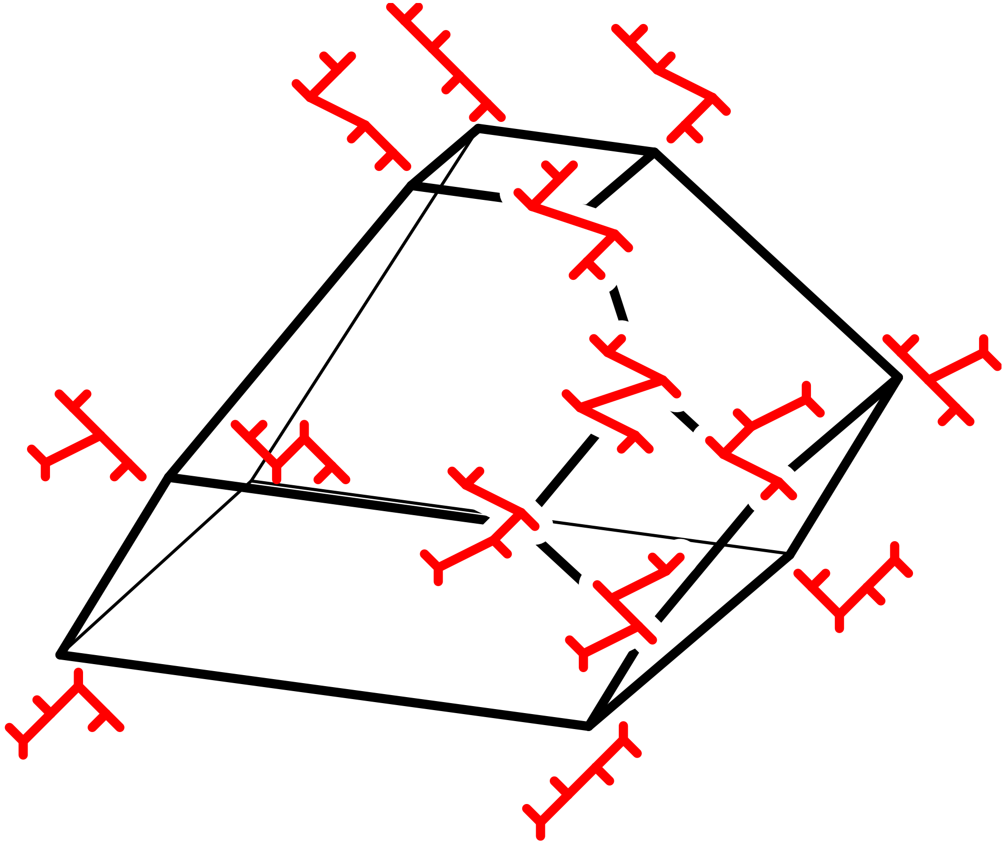

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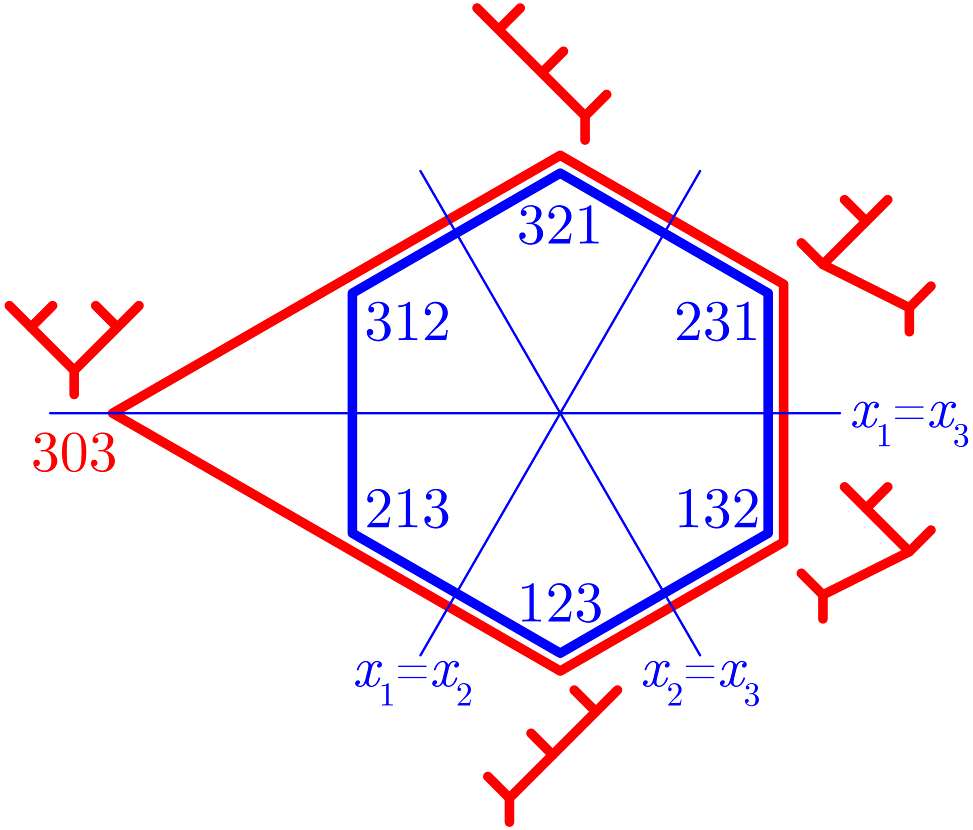
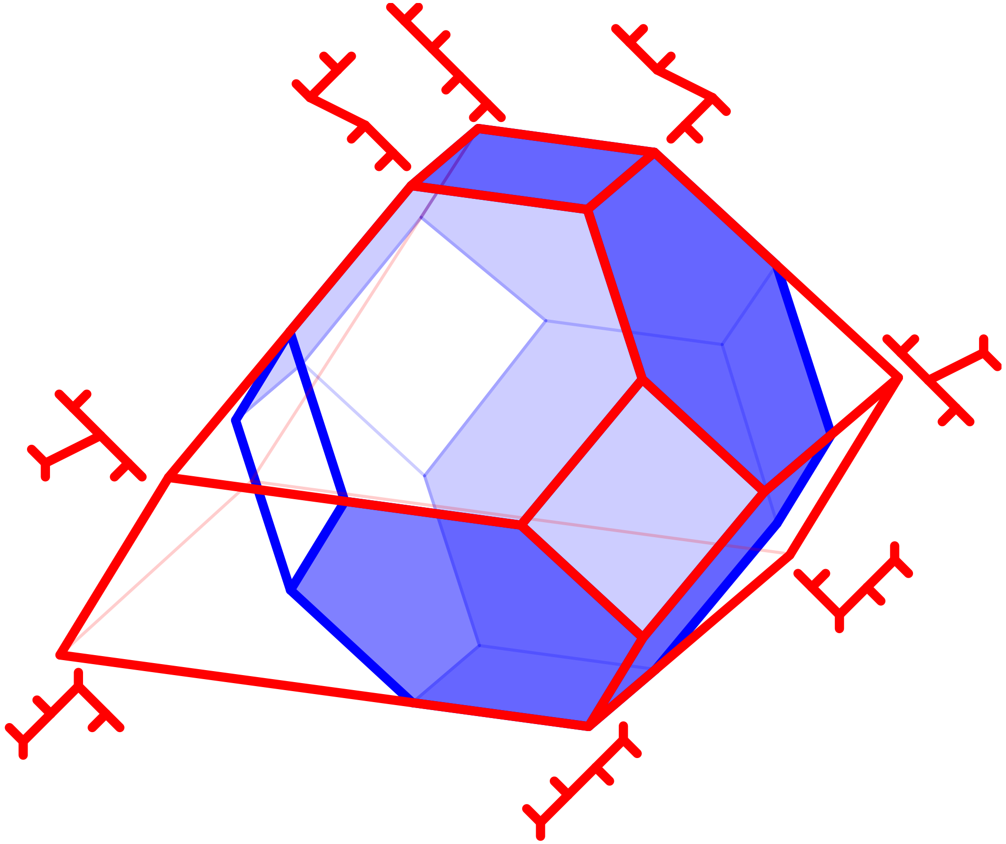
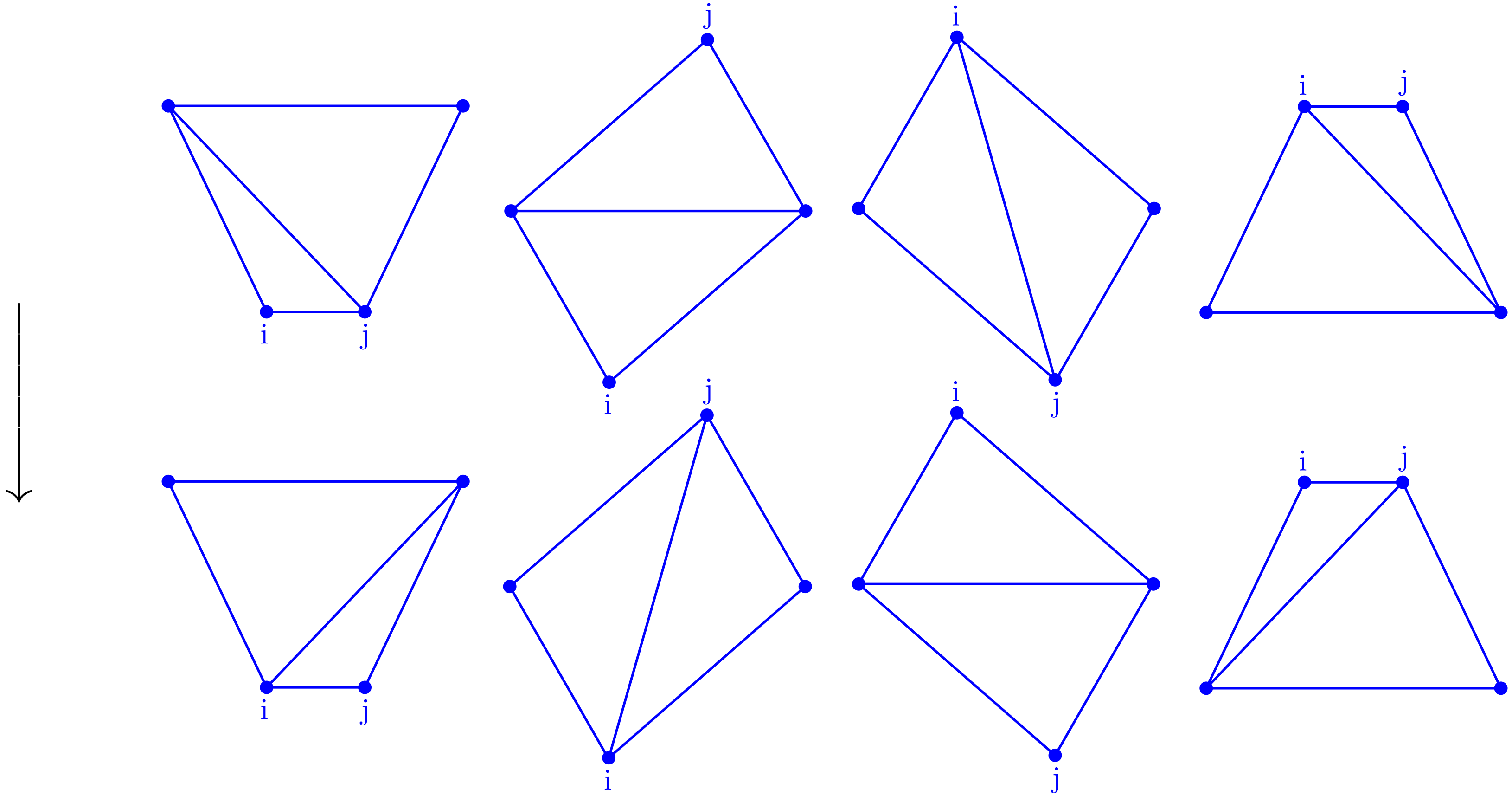




TYPE A CAMBRIAN LATTICES

Cambrian lattice $=$ slope increasing flips on triangulations

slope
increasing
flip

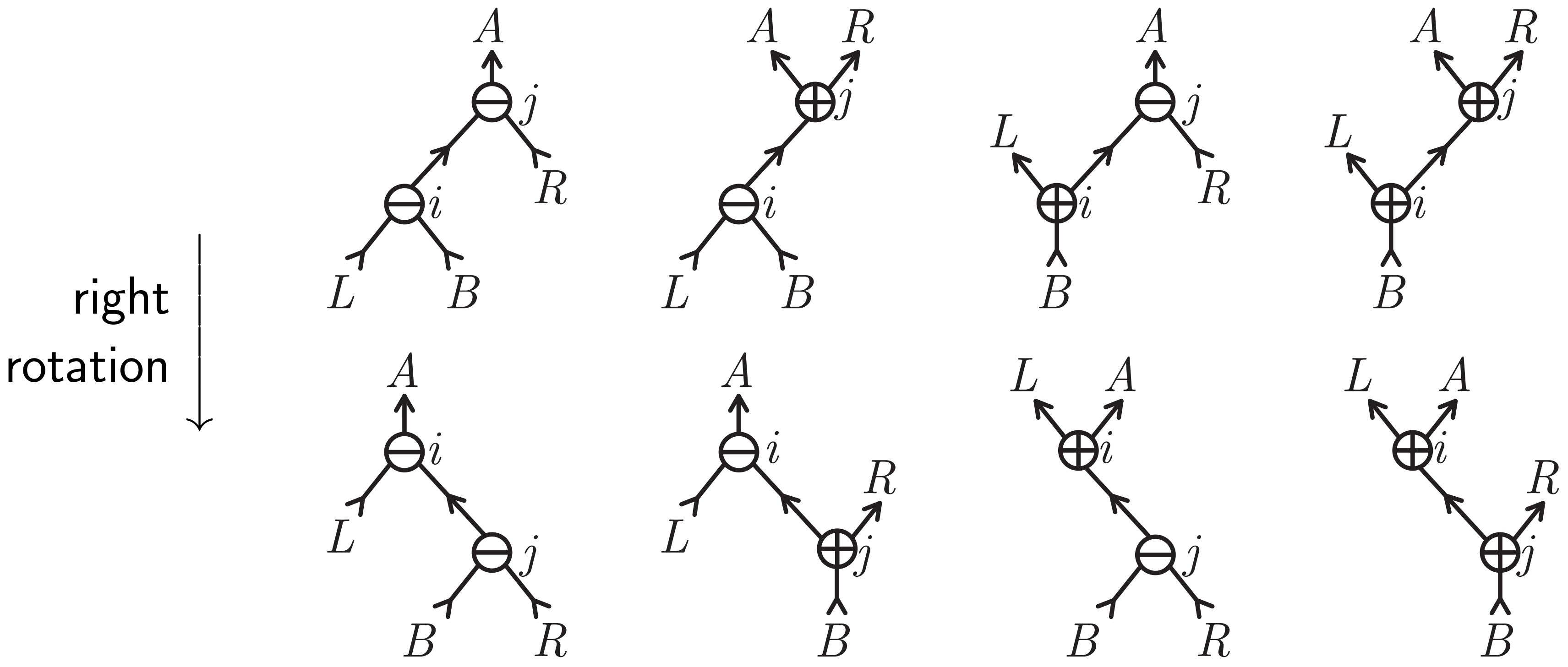


Reading, *Cambrian lattices* ('06)

TYPE A CAMBRIAN LATTICES

Cambrian lattice $=$ slope increasing flips on triangulations

$=$ right rotations on Cambrian trees



Reading, *Cambrian lattices* ('06)

TYPE A CAMBRIAN LATTICES

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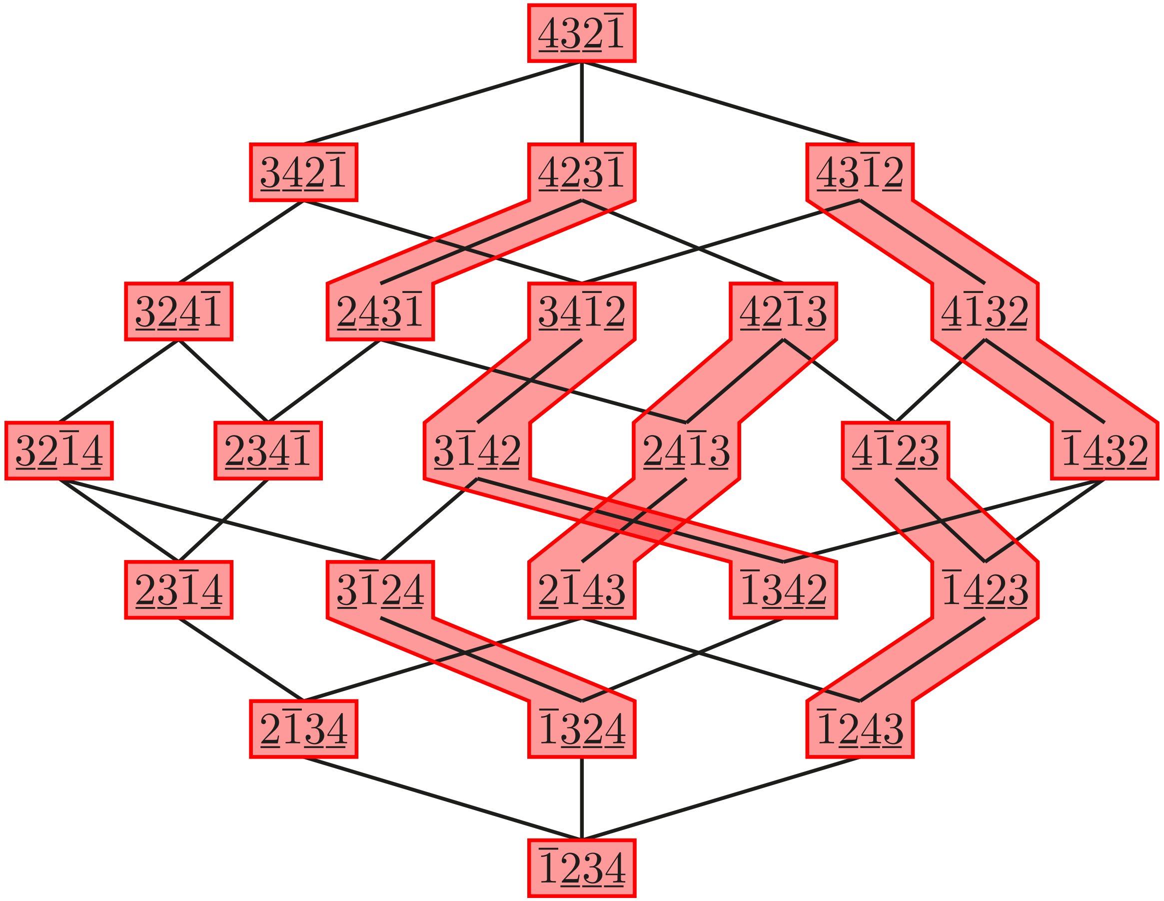
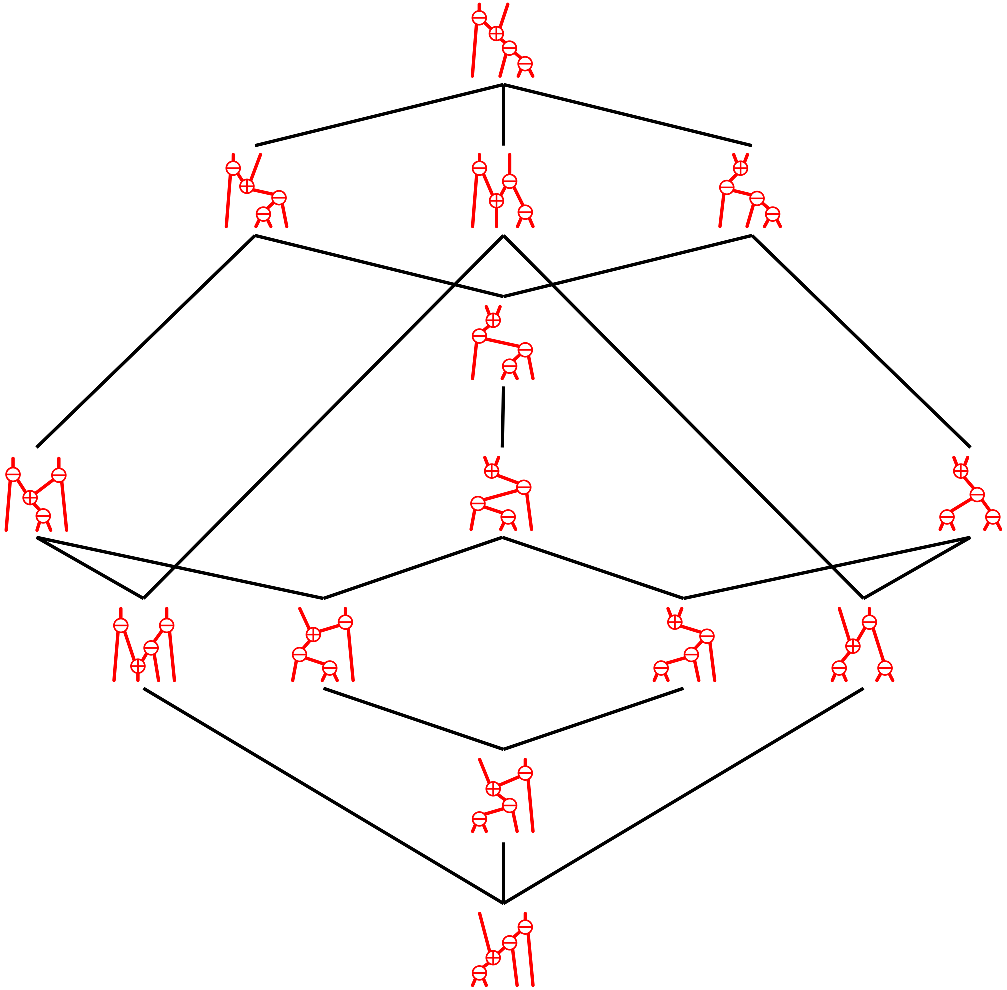
$=$ lattice quotient of the weak order by the Cambrian congruence





Reading, *Cambrian lattices* ('06)

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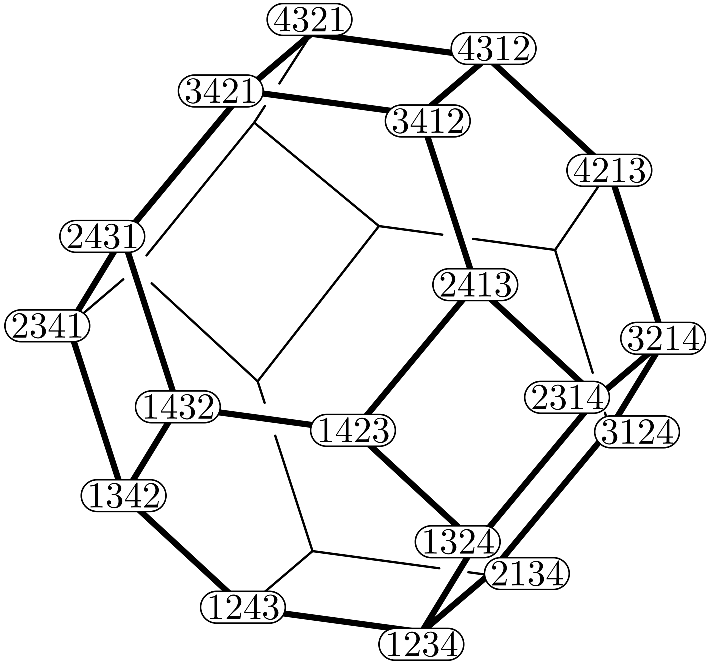
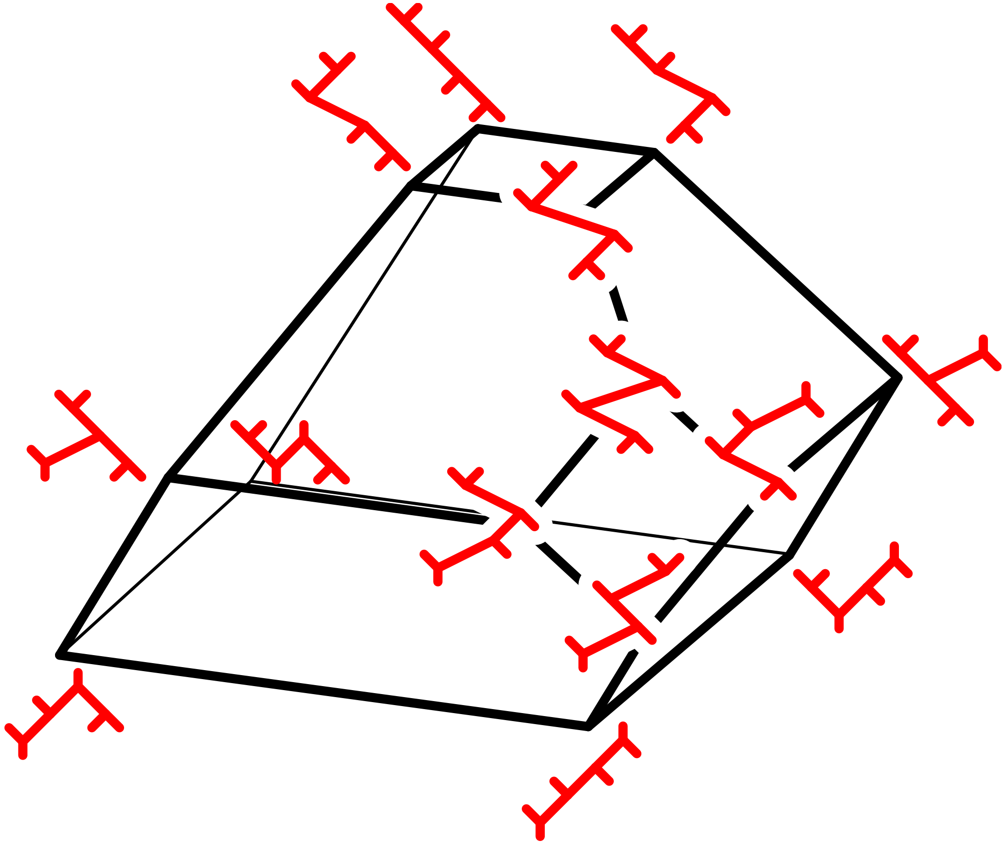
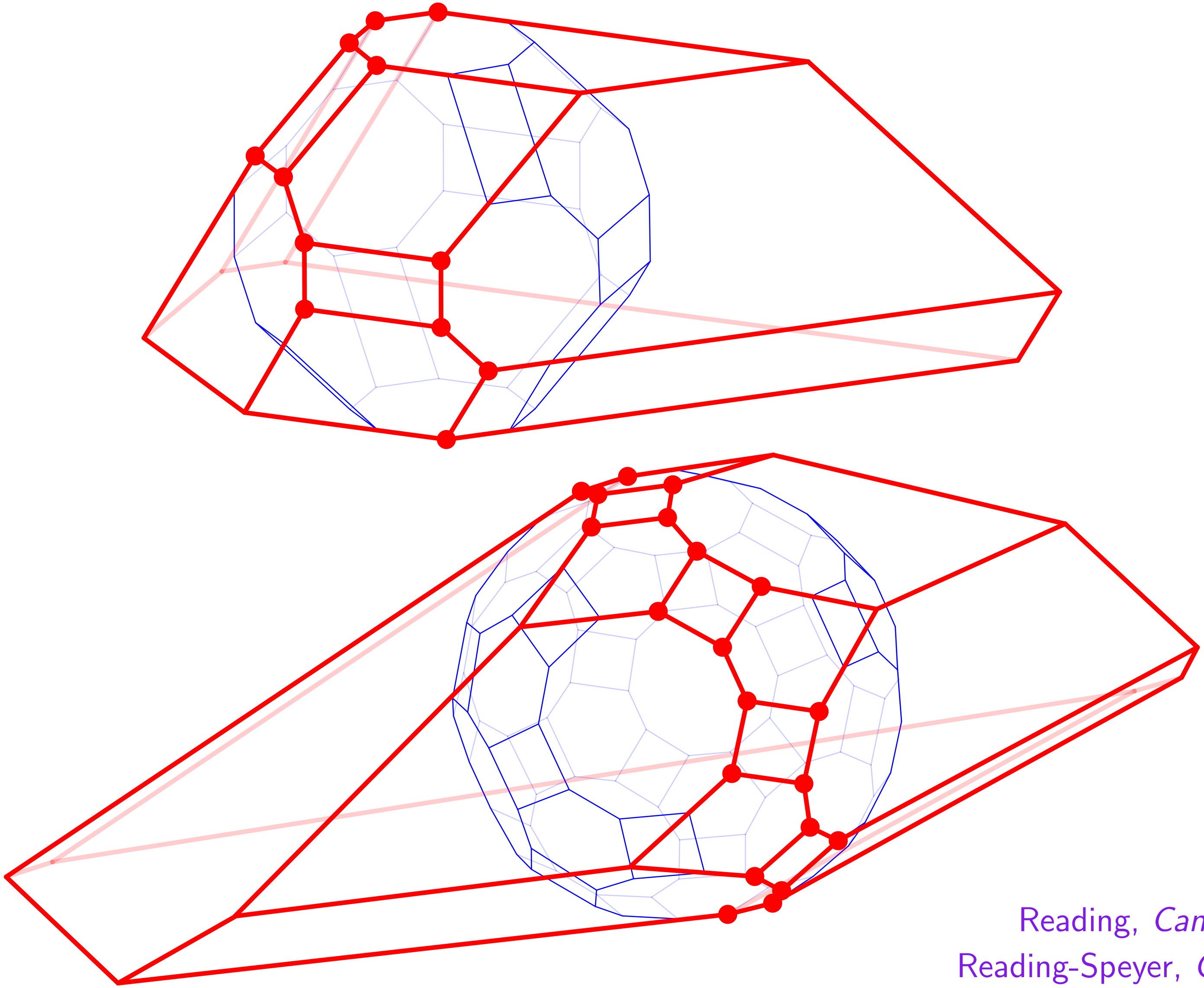




Reading, *Cambrian lattices* ('06)

CAMBRIAN FANS AND GENERALIZED ASSOCIAHEDRA

All this story extends to arbitrary finite Coxeter groups



Reading, *Cambrian lattices* ('06)

Reading-Speyer, *Cambrian fans* ('09)

Hohlweg-Lange-Thomas, *Permutahedra and generalized associahedra* ('11)

CAMBRIAN HOPF ALGEBRAS

Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)
Chatel-P., *Cambrian Hopf Algebras* ('15^+)

SHUFFLE AND CONVOLUTION

For $n, n' \in \mathbb{N}$, consider the set of perms of $\mathfrak{S}_{n+n'}$ with at most one descent, at position n:

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n')\}$$

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

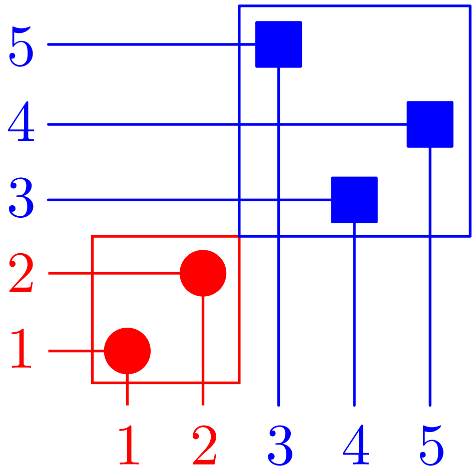
shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1)+n, \dots, \tau'(n')+n] \in \mathfrak{S}_{n+n'}$

shifted shuffle product $\tau \mathbin{\bar{\sqcup\!\sqcup}} \tau' = \left\{ (\tau\bar{\tau}') \circ \pi^{-1} \;\middle|\; \pi \in \mathfrak{S}^{(n,n')} \right\} \subset \mathfrak{S}_{n+n'}$

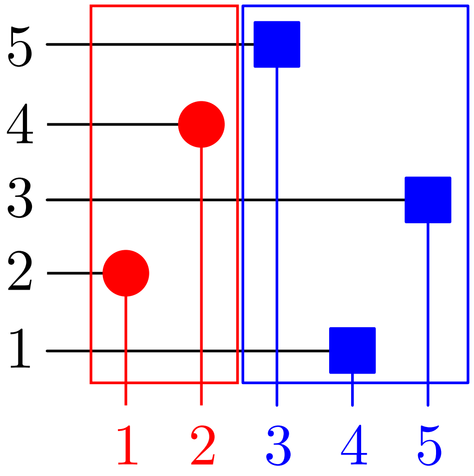
convolution product $\tau \star \tau' = \left\{ \pi \circ (\tau\bar{\tau}') \;\middle|\; \pi \in \mathfrak{S}^{(n,n')} \right\} \subset \mathfrak{S}_{n+n'}$

Exm: $12 \mathbin{\bar{\sqcup\!\sqcup}} 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

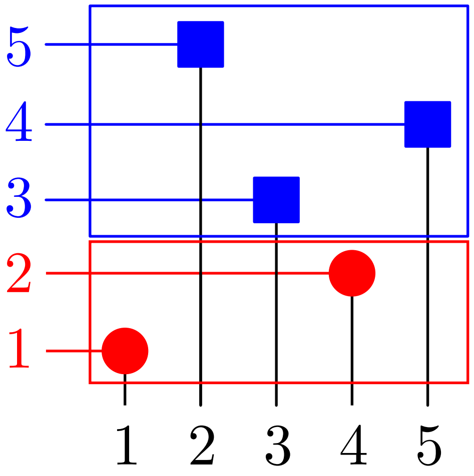
$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



concatenation



shuffle



convolution

MALVENUTO-REUTENAUER ALGEBRA

DEF. Combinatorial Hopf Algebra $=$ combinatorial vector space $\mathcal{B}$ endowed with

product $\cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$

coproduct $\triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$

which are “compatible”, ie.

$$\begin{tikzcd}
\mathcal{B} \otimes \mathcal{B} \arrow[r, "\cdot"] \arrow[d, "\triangle \otimes \triangle"'] & \mathcal{B} \arrow[r, "\triangle"] & \mathcal{B} \otimes \mathcal{B} \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[rr, "I \otimes \mathrm{swap} \otimes I"'] & & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[u, "\cdot \otimes \cdot"']
\end{tikzcd}$$

Malvenuto-Reutenauer algebra $=$ Hopf algebra FQSym with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ and where

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \shuffle \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

Malvenuto-Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)

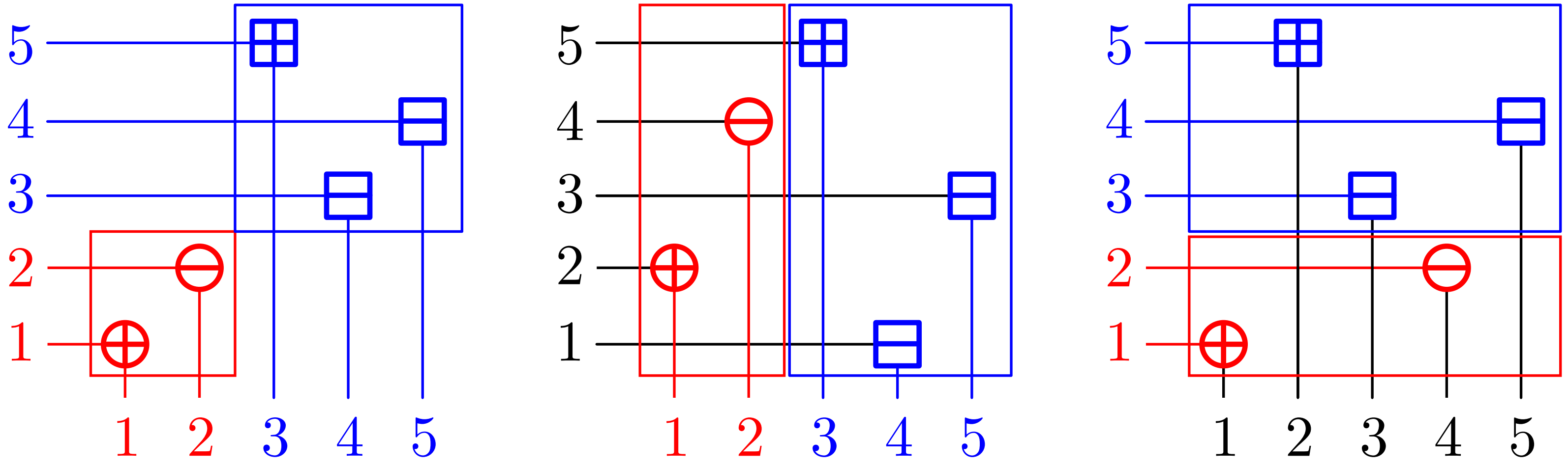
SIGNED VERSION

For signed permutations:

- signs are attached to values in the shuffle product
- signs are attached to positions in the convolution product

Exm: $\overline{1}\underline{2} \mathbin{\bar{\sqcup\!\sqcup}} \underline{23}\overline{1} = \{\overline{1}\underline{245}\overline{3}, \overline{1}\underline{425}\overline{3}, \overline{1}\underline{452}\overline{3}, \overline{1}\underline{45}\overline{3}\underline{2}, \underline{4}\overline{1}\underline{25}\overline{3}, \underline{4}\overline{1}\underline{52}\overline{3}, \underline{4}\overline{1}\underline{5}\overline{3}\underline{2}, \underline{45}\overline{1}\underline{2}\overline{3}, \underline{45}\overline{13}\underline{2}, \underline{45}\overline{31}\underline{2}\}$,

$\overline{1}\underline{2} \star \underline{23}\overline{1} = \{\overline{1}\underline{245}\overline{3}, \overline{1}\underline{345}\overline{2}, \overline{1}\underline{435}\overline{2}, \overline{1}\underline{534}\overline{2}, \overline{2}\underline{345}\overline{1}, \overline{2}\underline{435}\overline{1}, \overline{2}\underline{534}\overline{1}, \overline{3}\underline{425}\overline{1}, \overline{3}\underline{524}\overline{1}, \overline{4}\underline{523}\overline{1}\}$.



concatenation shuffle convolution

FQSym$_\pm$ = Hopf algebra with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}_\pm}$ and where

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \mathbin{\bar{\sqcup\!\sqcup}} \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

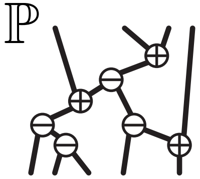
CAMBRIAN ALGEBRA AS SUBALGEBRA OF FQSym$_\pm$

Cambrian algebra = vector subspace Camb of FQSym$_\pm$ generated by

$$\mathbb{P}_{\mathrm{T}} := \sum_{\tau \in \mathcal{L}(\mathrm{T})} \mathbb{F}_\tau,$$

for all Cambrian trees T.

Exm: $\mathbb{P}$



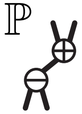
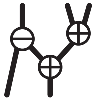
$$= \mathbb{F}_{\underline{2}\underline{1}\overline{3}\overline{7}\underline{5}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\underline{1}\overline{7}\overline{3}\underline{5}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\underline{1}\overline{7}\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{1}\overline{3}\underline{5}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{1}\underline{5}\overline{3}\underline{4}\overline{6}}$$
$$+ \mathbb{F}_{\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{2}\underline{1}\overline{3}\underline{5}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{2}\underline{1}\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{2}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{5}\underline{2}\underline{1}\overline{3}\underline{4}\overline{6}}$$

THEO. Camb is a subalgebra of FQSym$_\pm$

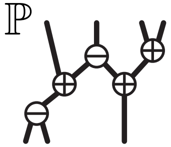
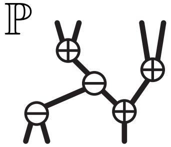
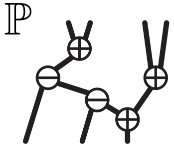
Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)
Hivert-Novelli-Thibon, *The algebra of binary search trees* ('05)
Chatel-P., *Cambrian Hopf Algebras* ('15$^+$)

GAME: Explain the product and coproduct directly on the Cambrian trees...

PRODUCT IN CAMBRIAN ALGEBRA

$\mathbb{P}$  $\cdot\ \mathbb{P}$  $= \mathbb{F}_{\underline{1}\overline{2}} \cdot \big(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{2}\overline{3}\underline{1}}\big)$

$$= \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{F}_{\underline{1}\overline{2}\overline{4}\overline{5}\underline{3}} + \mathbb{F}_{\underline{1}\overline{4}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\underline{1}\overline{4}\overline{2}\overline{5}\underline{3}} + \mathbb{F}_{\underline{1}\overline{4}\overline{5}\overline{2}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\overline{5}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\overline{2}\underline{3}} + \mathbb{F}_{\overline{4}\overline{5}\underline{1}\overline{2}\underline{3}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{2}\overline{5}} + \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{5}\overline{2}} \\ + \mathbb{F}_{\underline{1}\overline{4}\overline{5}\underline{3}\overline{2}} + \mathbb{F}_{\overline{4}\underline{1}\underline{3}\overline{2}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\underline{3}\overline{5}\overline{2}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\underline{3}\overline{2}} \\ + \mathbb{F}_{\overline{4}\overline{5}\underline{1}\underline{3}\overline{2}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\overline{4}\underline{3}\underline{1}\overline{2}\overline{5}} + \mathbb{F}_{\overline{4}\underline{3}\underline{1}\overline{5}\overline{2}} \\ + \mathbb{F}_{\overline{4}\underline{3}\overline{5}\underline{1}\overline{2}} + \mathbb{F}_{\overline{4}\overline{5}\underline{3}\underline{1}\overline{2}} \end{pmatrix}$$

$=\ \mathbb{P}$  $+\ \mathbb{P}$  $+\ \mathbb{P}$ 

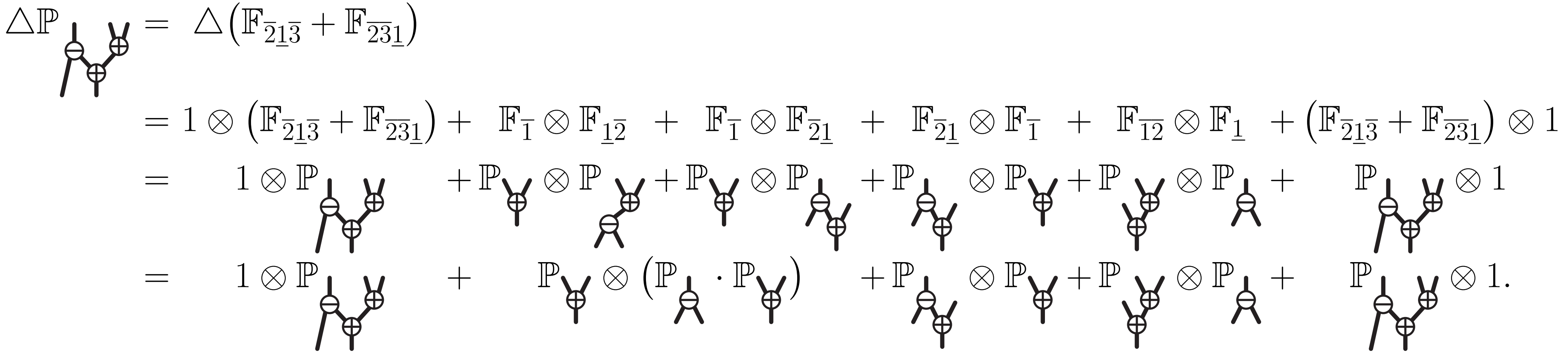
PROP. For any Cambrian trees T and T',

$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}}$$

where S runs over the interval $\Big[\mathrm{T} \nearrow \bar{\mathrm{T}}'\ ,\ \mathrm{T} \nwarrow \bar{\mathrm{T}}'\Big]$ in the $\varepsilon(\mathrm{T})\varepsilon(\mathrm{T}')$-Cambrian lattice

Chatel-P., *Cambrian Hopf Algebras* ('15^{+})

COPRODUCT IN CAMBRIAN ALGEBRA



$$\triangle \mathbb{P} = \triangle\left(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{2}\overline{3}\underline{1}}\right)$$

$$= 1 \otimes \left(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{2}\overline{3}\underline{1}}\right) + \mathbb{F}_{\overline{1}} \otimes \mathbb{F}_{\underline{1}\overline{2}} + \mathbb{F}_{\overline{1}} \otimes \mathbb{F}_{\overline{2}\underline{1}} + \mathbb{F}_{\overline{2}\underline{1}} \otimes \mathbb{F}_{\overline{1}} + \mathbb{F}_{\overline{1}\overline{2}} \otimes \mathbb{F}_{\underline{1}} + \left(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{2}\overline{3}\underline{1}}\right) \otimes 1$$

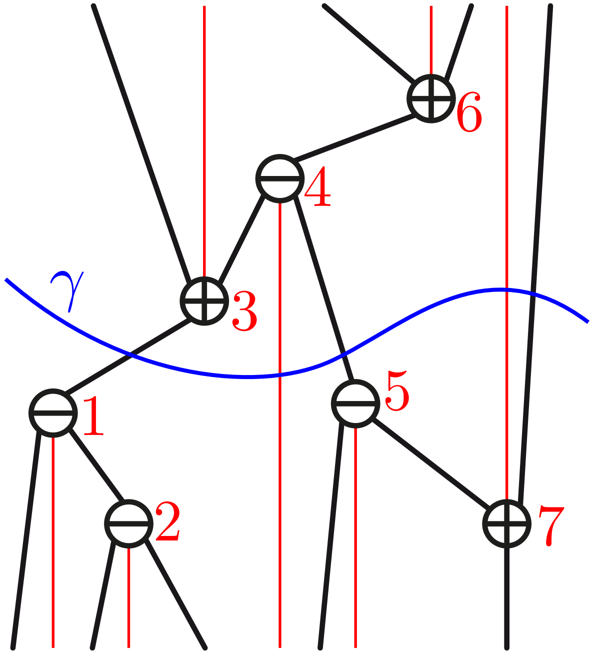
$$= 1 \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1$$

$$= 1 \otimes \mathbb{P} + \mathbb{P} \otimes \left(\mathbb{P} \cdot \mathbb{P}\right) + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1.$$

PROP. For any Cambrian tree S,

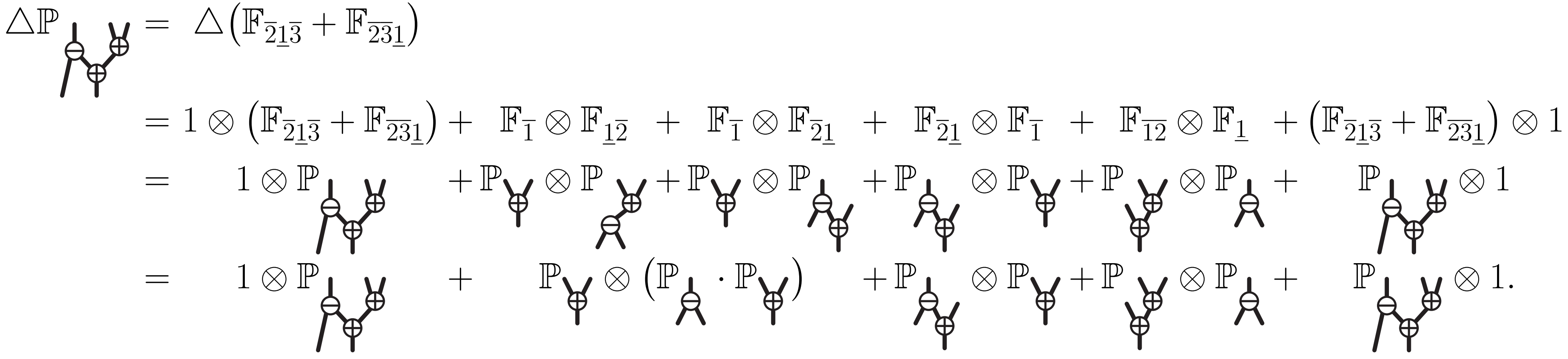
$$\triangle \mathbb{P}_{\mathrm{S}} = \sum_{\gamma} \Big(\prod_{\mathrm{T} \in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}} \Big) \otimes \Big(\prod_{\mathrm{T}' \in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'} \Big)$$

where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the Cambrian forests above and below γ respectively



Chatel-P., *Cambrian Hopf Algebras* ('15$^+$)

COPRODUCT IN CAMBRIAN ALGEBRA



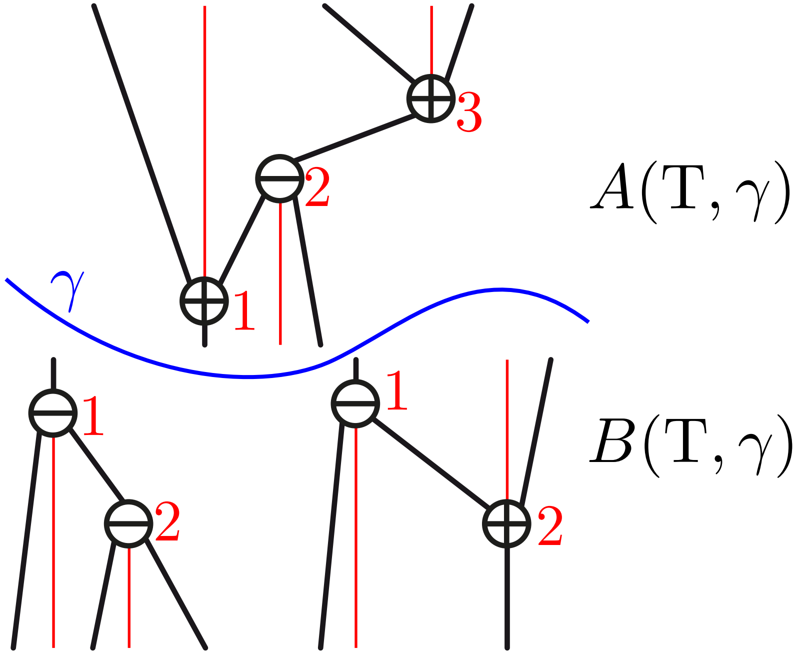
$$\triangle\mathbb{P} = \triangle\left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right)$$

$$= 1 \otimes \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\underline{1}\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\bar{2}\underline{1}} + \mathbb{F}_{\bar{2}\underline{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{\bar{1}\bar{2}} \otimes \mathbb{F}_{\underline{1}} + \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) \otimes 1$$

PROP. For any Cambrian tree S,

$$\triangle\mathbb{P}_{\mathrm{S}} = \sum_{\gamma}\Bigg(\prod_{\mathrm{T}\in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}}\Bigg) \otimes \Bigg(\prod_{\mathrm{T}'\in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'}\Bigg)$$

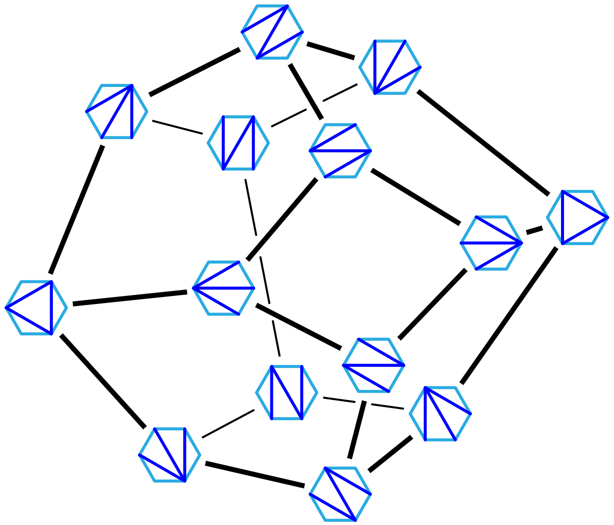
where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the Cambrian forests above and below γ respectively



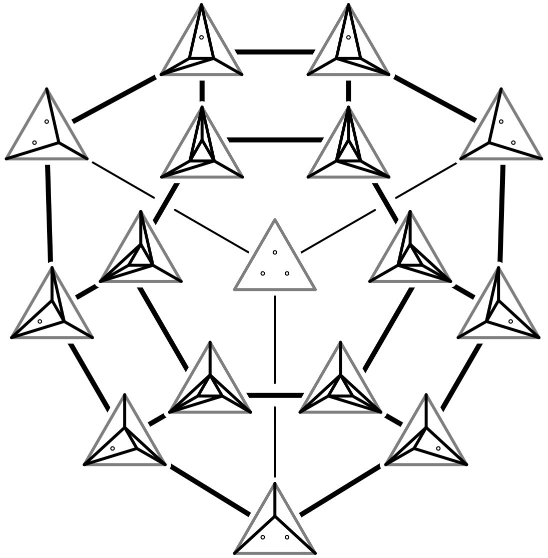
Chatel-P., *Cambrian Hopf Algebras* ('15$^+$)

THREE FAMILIES OF REALIZATIONS

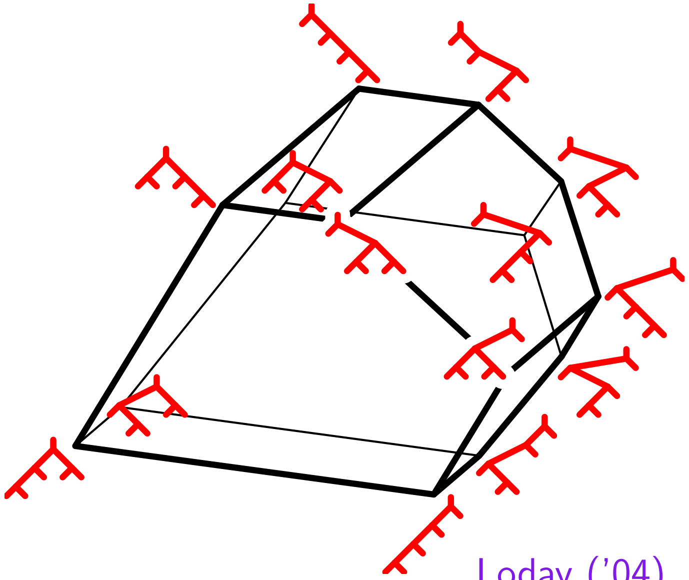
SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)



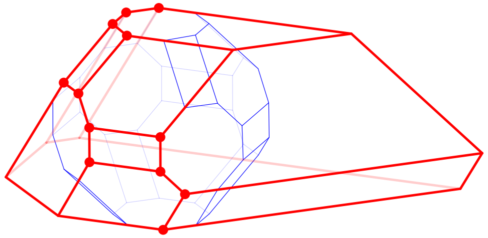
LODAY'S ASSOCIAHEDRON

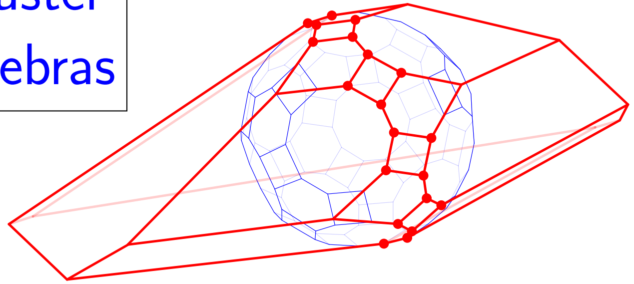


Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

Hopf algebra

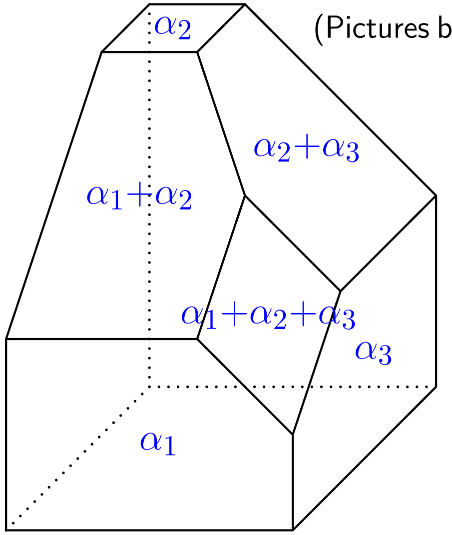
Cluster algebras





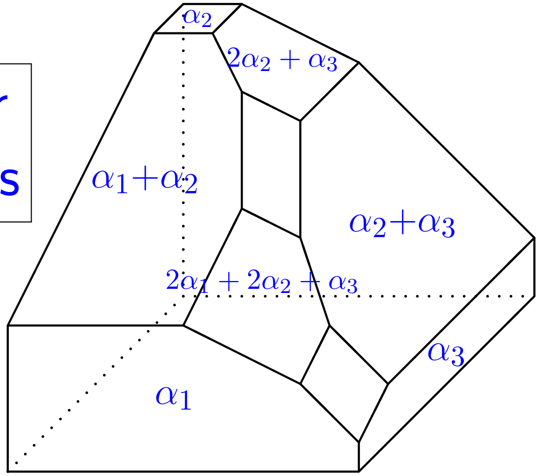
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras



SECONDARY FAN & POLYTOPE

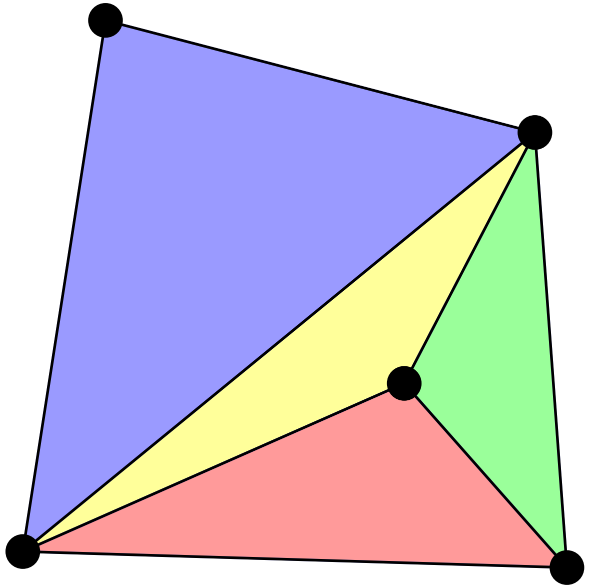
Gelfand-Kapranov-Zelevinsky,
Discriminants, resultants, and multidimensional determinants ('94)

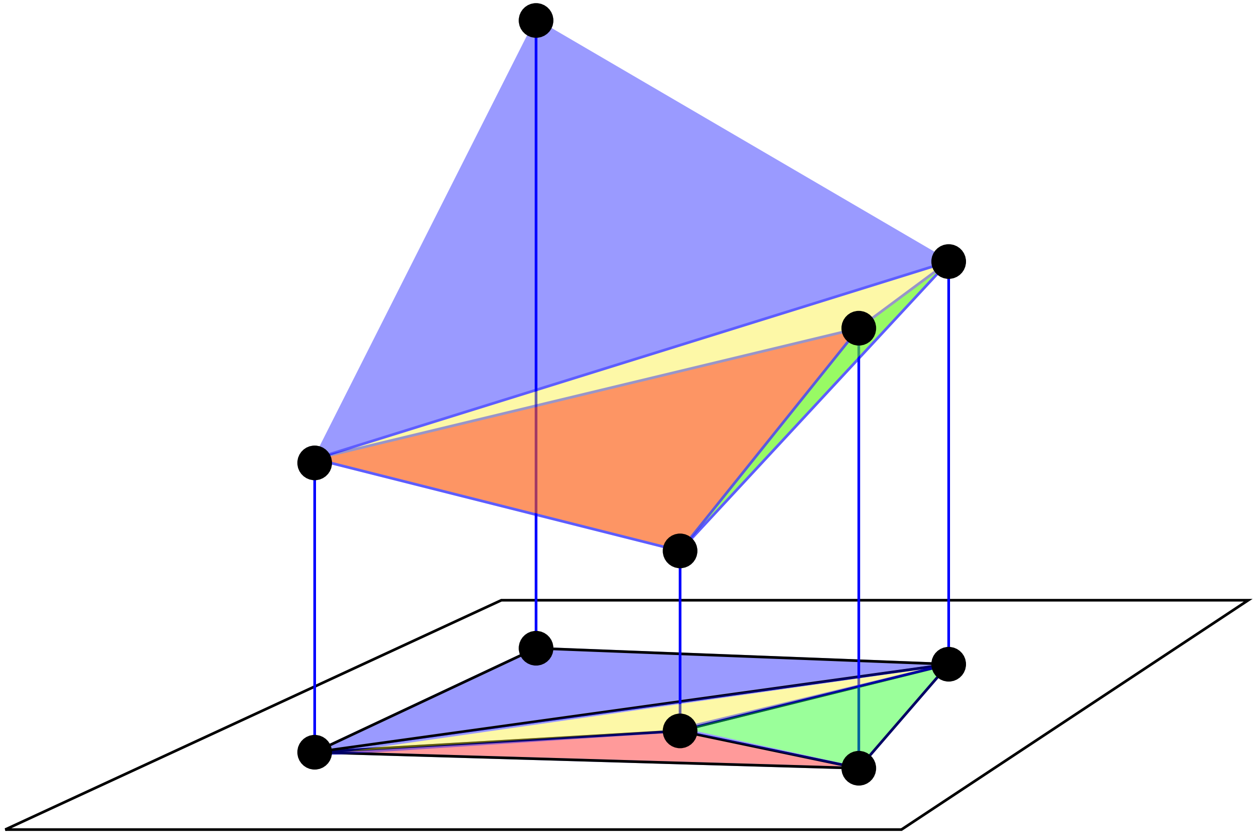
De Loera-Rambau-Santos, *Triangulations* ('10)

REGULAR SUBDIVISIONS

$\mathbf{P}$ point set in $\mathbb{R}^d$

Regular subdivision of $\mathbf{P}$ = projection of the lower envelope of a lifting of $\mathbf{P}$





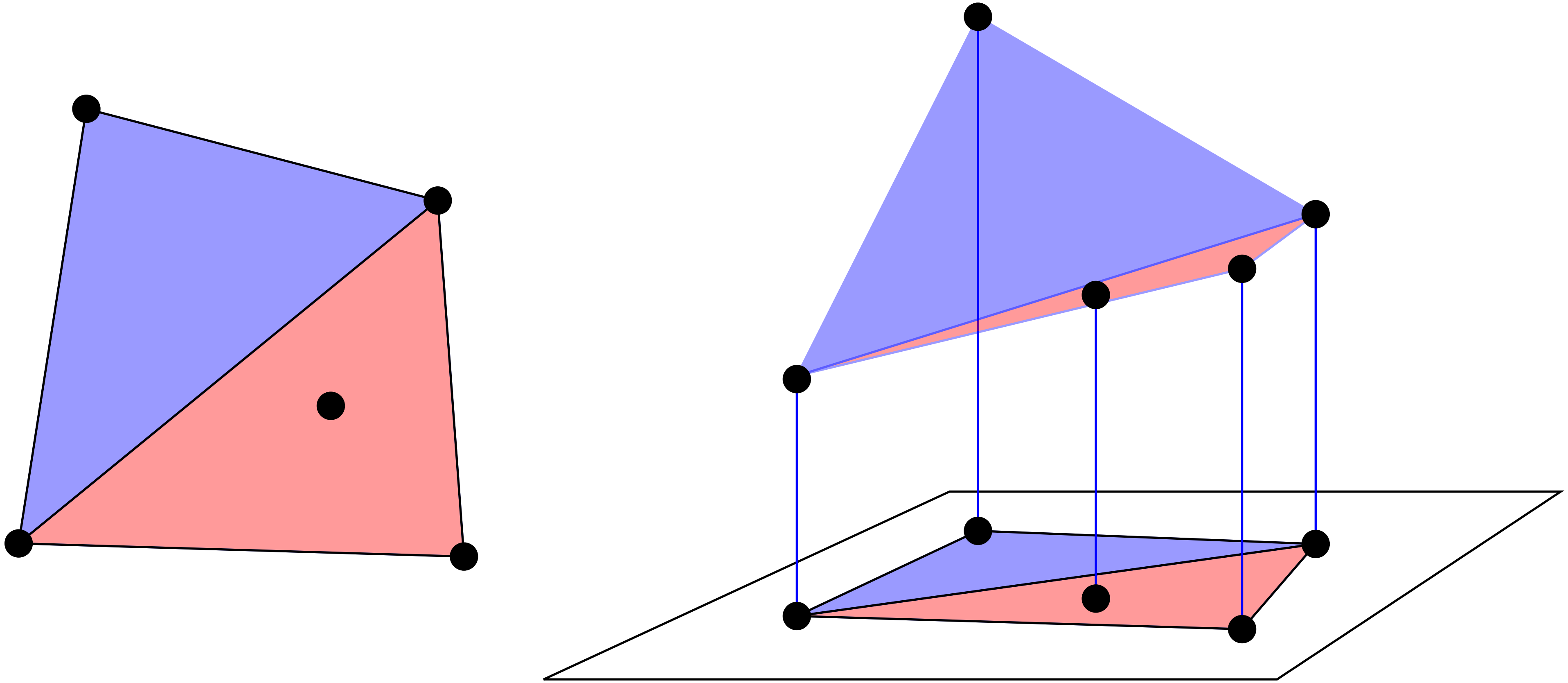
for a lifting function $\omega : \mathbf{P} \to \mathbb{R}$

$\mathrm{S}(\mathbf{P}, \omega)$ = projection of the lower envelope of $\big\{ \big(\mathbf{p}, \omega(\mathbf{p}) \big) \bigm| \mathbf{p} \in \mathbf{P} \big\}$

REGULAR SUBDIVISIONS

$\mathbf{P}$ point set in $\mathbb{R}^d$

Regular subdivision of $\mathbf{P}$ $=$ projection of the lower envelope of a lifting of $\mathbf{P}$



for a lifting function $\omega : \mathbf{P} \to \mathbb{R}$

$\mathrm{S}(\mathbf{P}, \omega) =$ projection of the lower envelope of $\big\{ \big(\mathbf{p}, \omega(\mathbf{p})\big) \;\big|\; \mathbf{p} \in \mathbf{P} \big\}$

REGULAR SUBDIVISIONS

$\mathbf{P}$ point set in $\mathbb{R}^d$

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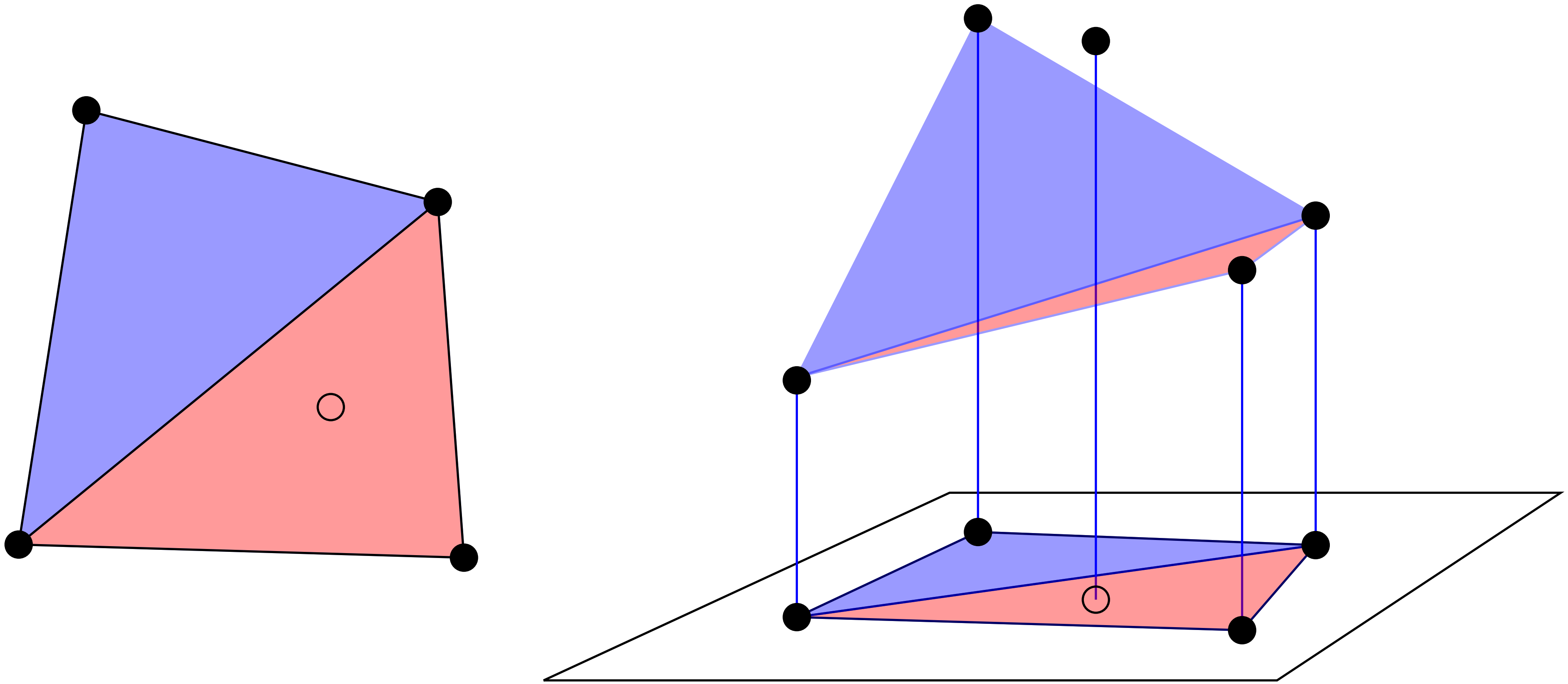


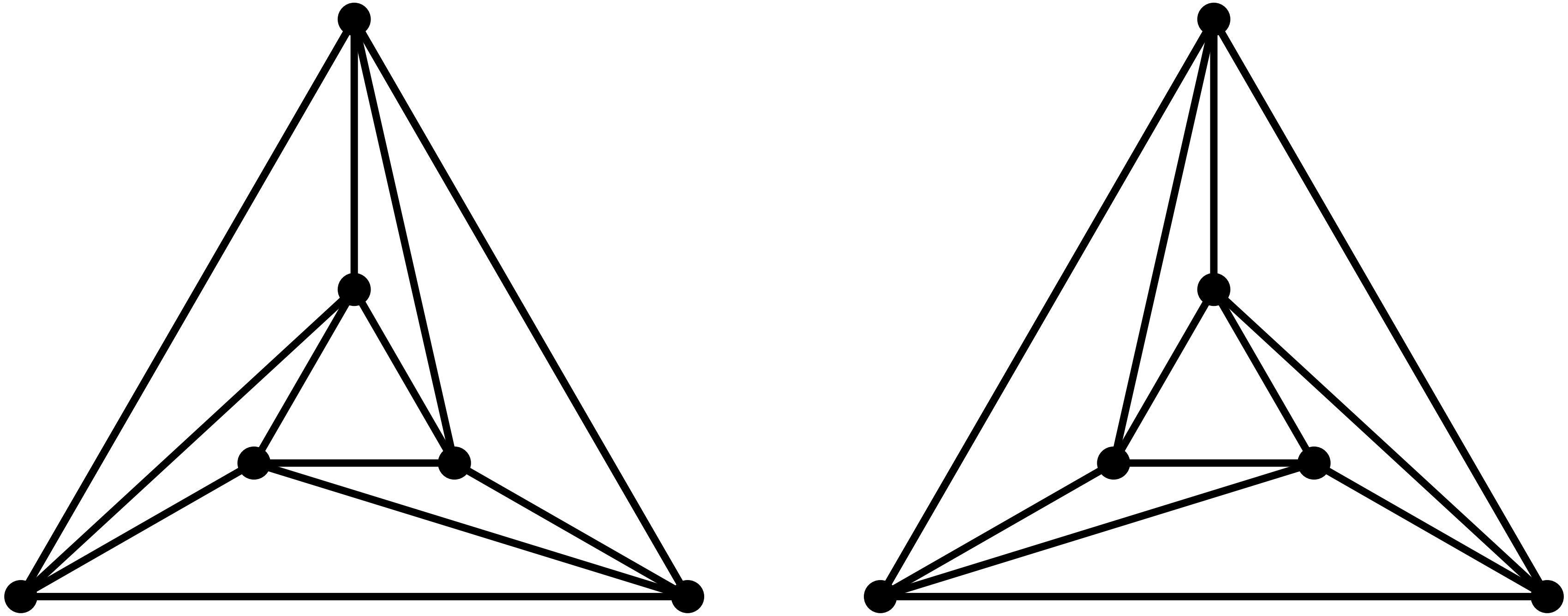
for a lifting function $\omega : \mathbf{P} \to \mathbb{R}$

$\mathrm{S}(\mathbf{P}, \omega)$ = projection of the lower envelope of $\left\{ \big(\mathbf{p}, \omega(\mathbf{p})\big) \;\middle|\; \mathbf{p} \in \mathbf{P} \right\}$

NON-REGULAR SUBDIVISIONS

$\mathbf{P}$ point set in $\mathbb{R}^d$

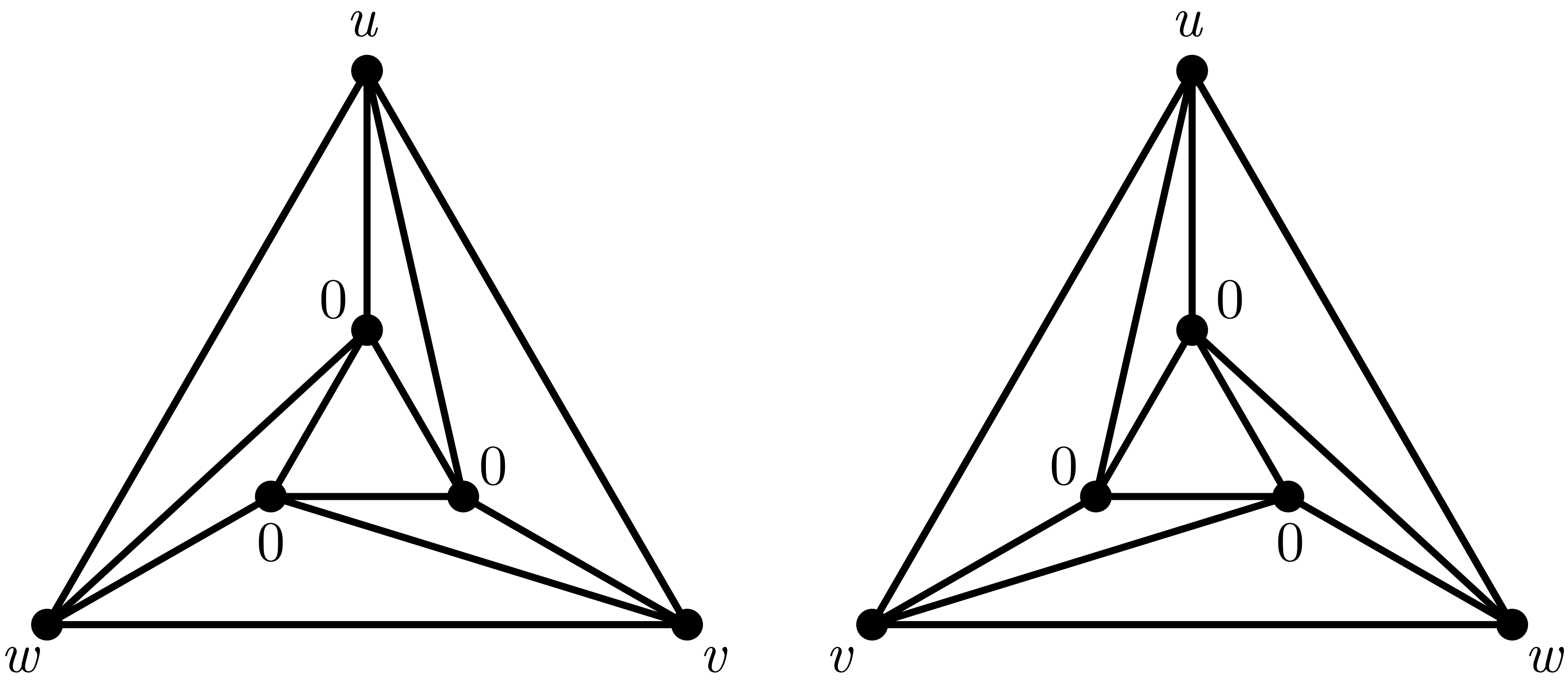
Regular subdivision of $\mathbf{P}$ $=$ projection of the lower envelope of a lifting of $\mathbf{P}$



NON-REGULAR SUBDIVISIONS

$\mathbf{P}$ point set in $\mathbb{R}^d$

Regular subdivision of $\mathbf{P}$ $=$ projection of the lower envelope of a lifting of $\mathbf{P}$

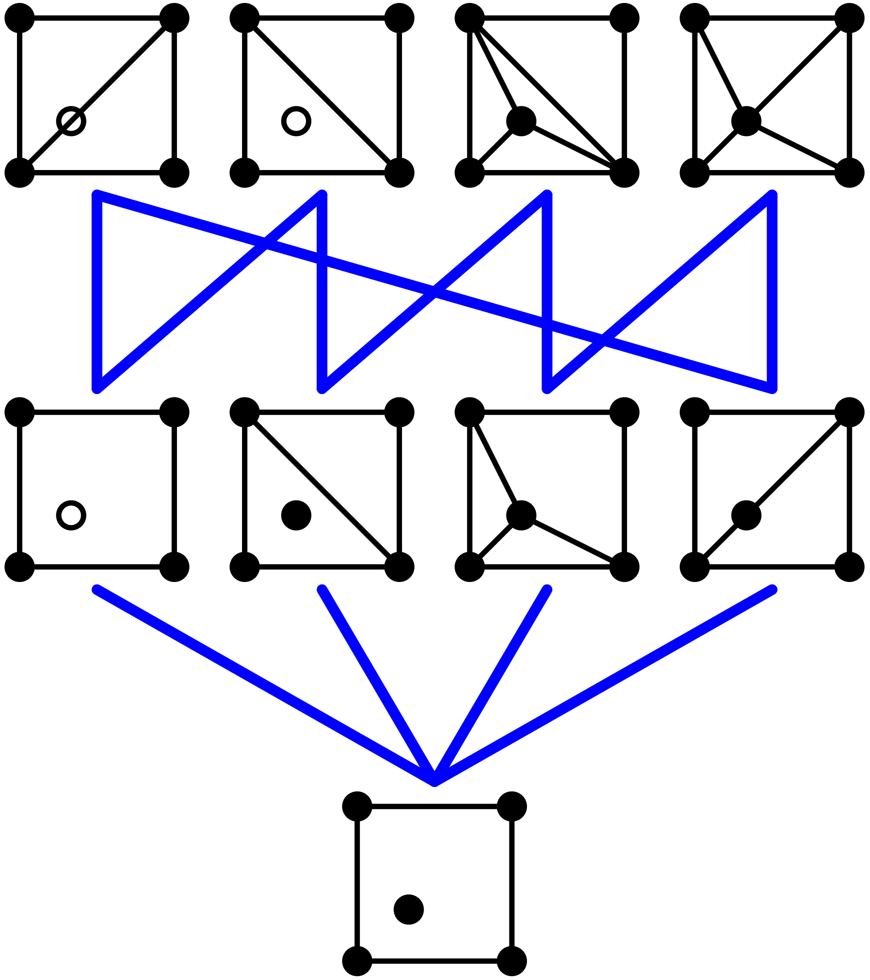


if regular, $\exists\, u, v, w \in \mathbb{R}$ such that $u < v < w < u$

REGULAR SUBDIVISIONS REFINEMENT LATTICE

$\mathbf{P}$ point set in $\mathbb{R}^d$

Regular subdivision of $\mathbf{P}$ = projection of the lower envelope of a lifting of $\mathbf{P}$

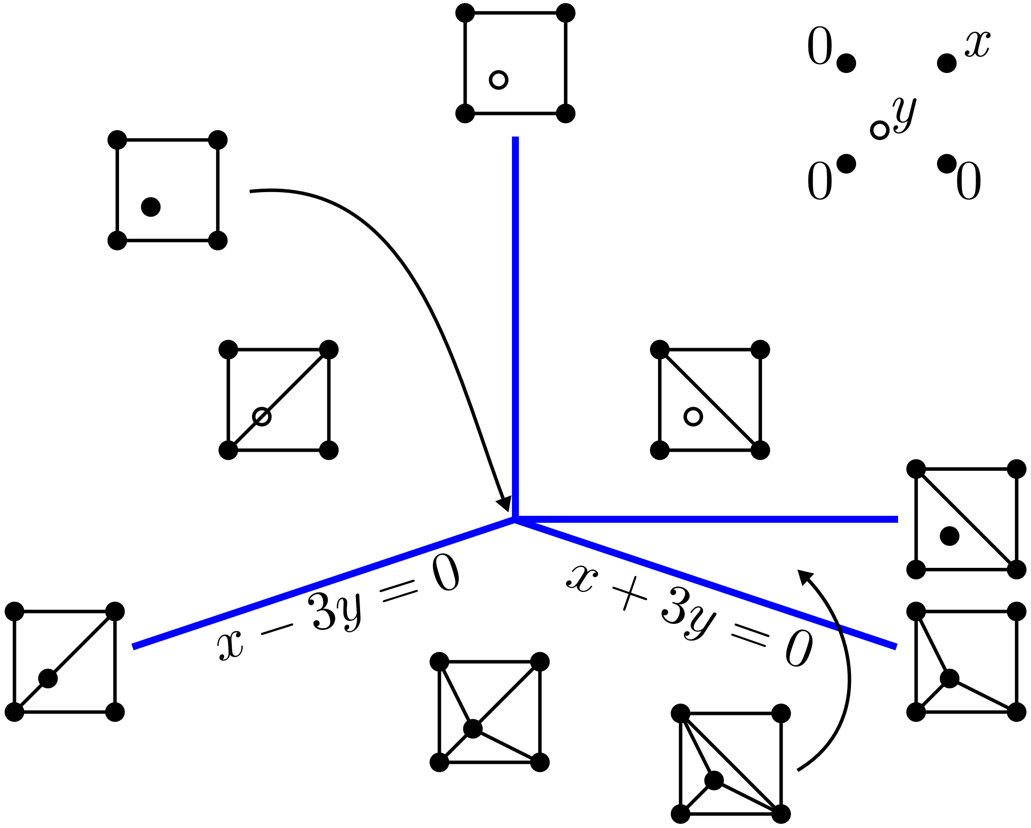


SECONDARY FAN

$\mathbf{P}$ point set in $\mathbb{R}^d$, $\quad S$ subdivision of $\mathbf{P}$

Secondary cone of $\mathrm{S} = C(\mathrm{S}) = \left\{\omega \in \mathbb{R}^{\mathbf{P}} \;\middle|\; \mathrm{S} \text{ refines } \mathrm{S}(\mathbf{P}, \omega)\right\}$

Secondary fan of $\mathbf{P} = \Sigma\mathrm{Fan}(\mathbf{P}) = \left\{C(\mathrm{S}) \;\middle|\; \mathrm{S} \text{ subdivision of } \mathbf{P}\right\}$



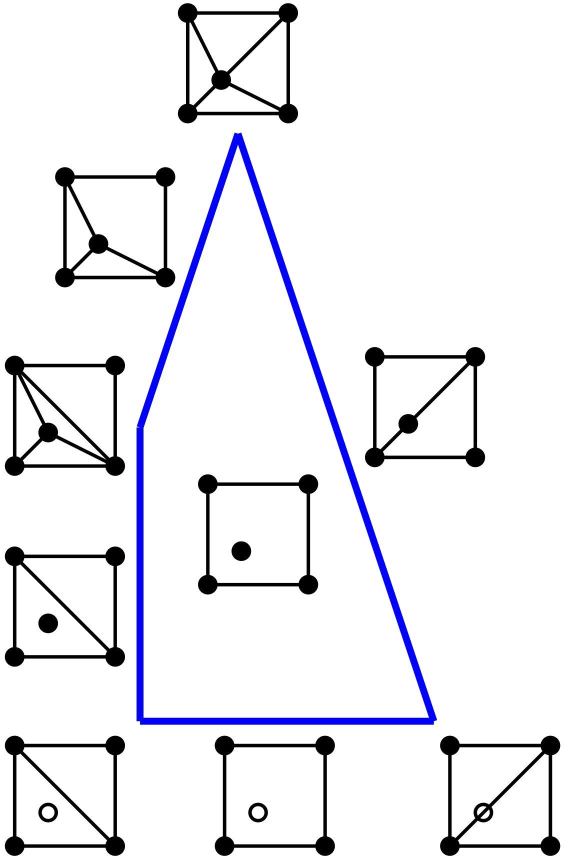
Gelfand-Kapranov-Zelevinsky, *Discriminants, resultants, and multidimensional determinants* ('94)

SECONDARY POLYTOPE

$\mathbf{P}$ point set in $\mathbb{R}^d$, T triangulation of $\mathbf{P}$

Volume vector of $\mathrm{T} = \Phi(\mathrm{T}) = \left(\sum_{\mathbf{p} \in \triangle \in \mathrm{T}} \mathrm{vol}(\triangle) \right)_{\mathbf{p} \in \mathbf{P}}$

Secondary polytope of $\mathbf{P}$ = ΣPoly$(\mathbf{P}) = \mathrm{conv} \left\{ \Phi(\mathrm{T}) \;\middle|\; \mathrm{T} \text{ triangulation of } \mathbf{P} \right\}$



Gelfand-Kapranov-Zelevinsky, *Discriminants, resultants, and multidimensional determinants* ('94)

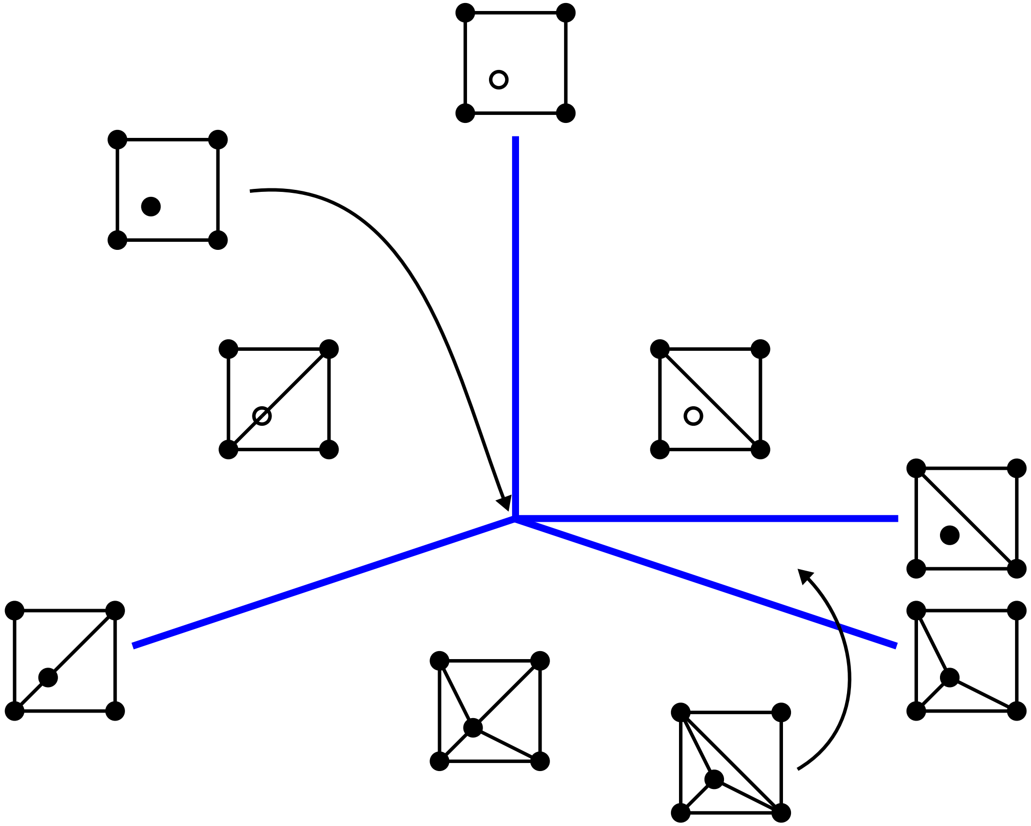
SECONDARY FAN & POLYTOPE

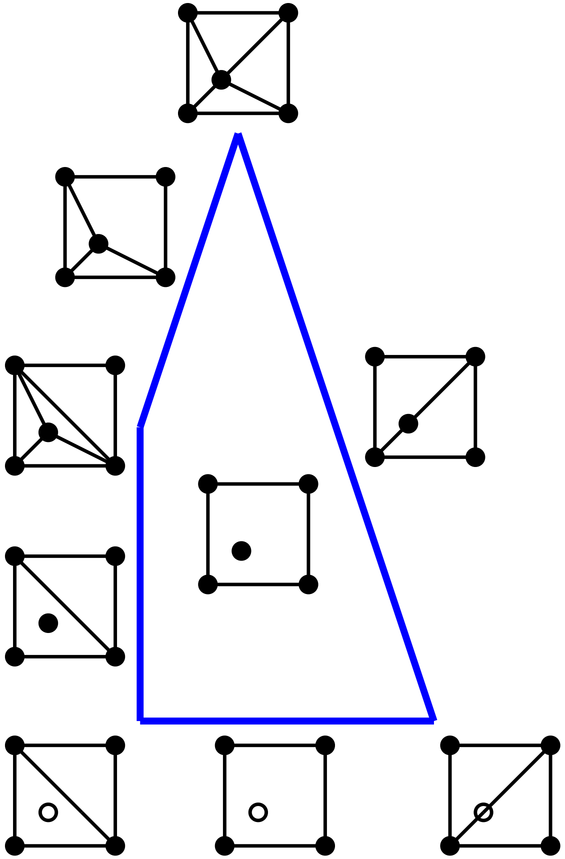
Secondary fan of $\mathbf{P}$ = $\Sigma\text{Fan}(\mathbf{P}) = \left\{ C(\mathrm{S}) \;\middle|\; \mathrm{S} \text{ subdivision of } \mathbf{P} \right\}$

Secondary polytope of $\mathbf{P}$ = $\Sigma\text{Poly}(\mathbf{P}) = \operatorname{conv} \left\{ \Phi(\mathrm{T}) \;\middle|\; \mathrm{T} \text{ triangulation of } \mathbf{P} \right\}$

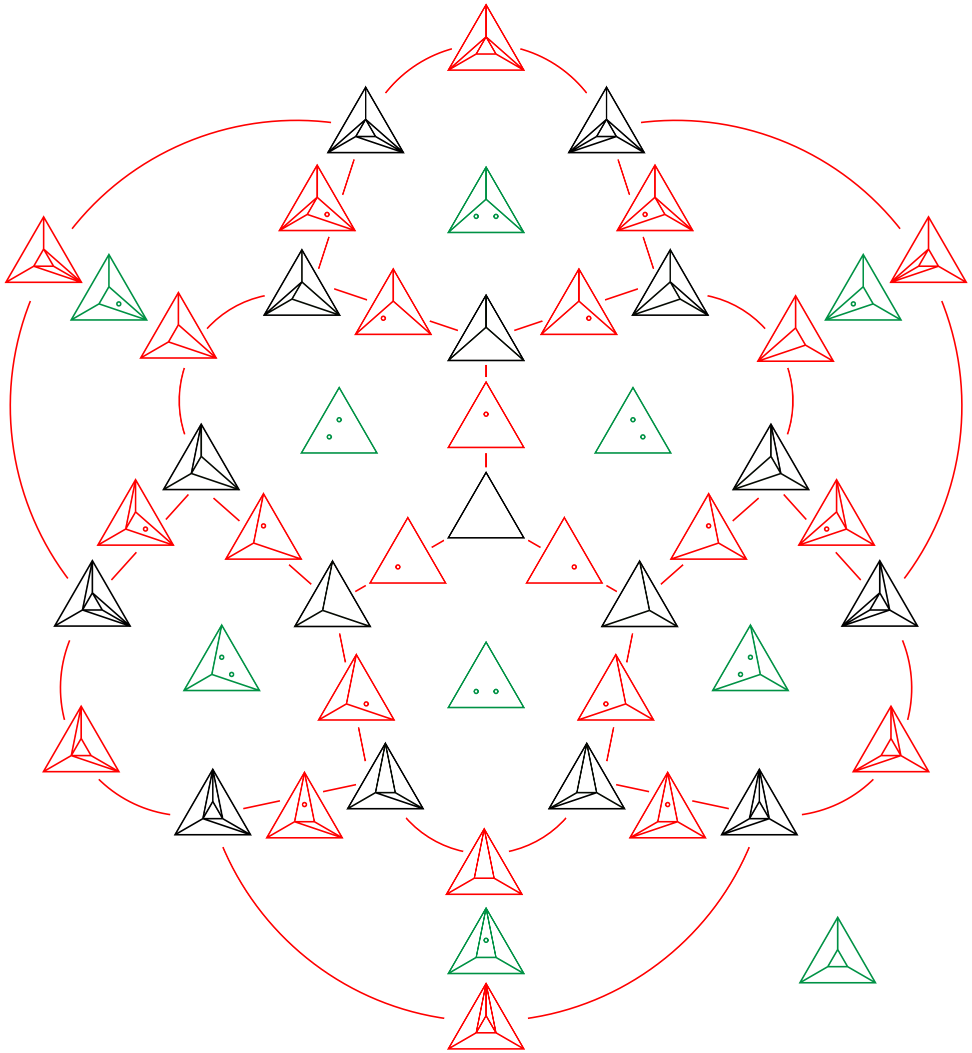
THM.
- dimension of $\Sigma\text{Poly}(\mathbf{P}) = |\mathbf{P}| - d - 1$
- $\Sigma\text{Fan}(\mathbf{P})$ is the inner normal fan of $\Sigma\text{Poly}(\mathbf{P})$
- face lattice of $\Sigma\text{Poly}(\mathbf{P})$ = refinement poset of regular subdivisions of $\mathbf{P}$

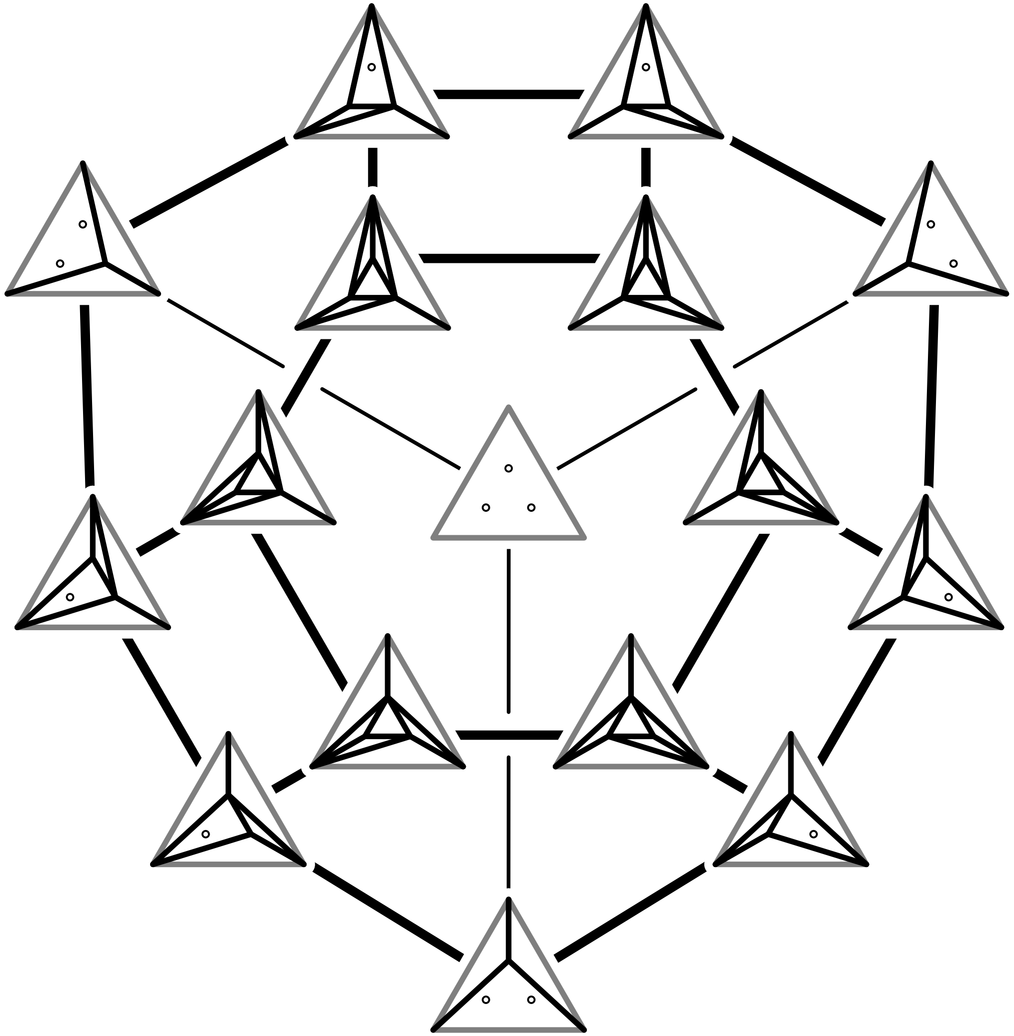
Gelfand-Kapranov-Zelevinsky, *Discriminants, resultants, and multidimensional determinants* ('94)





SECONDARY POLYTOPE

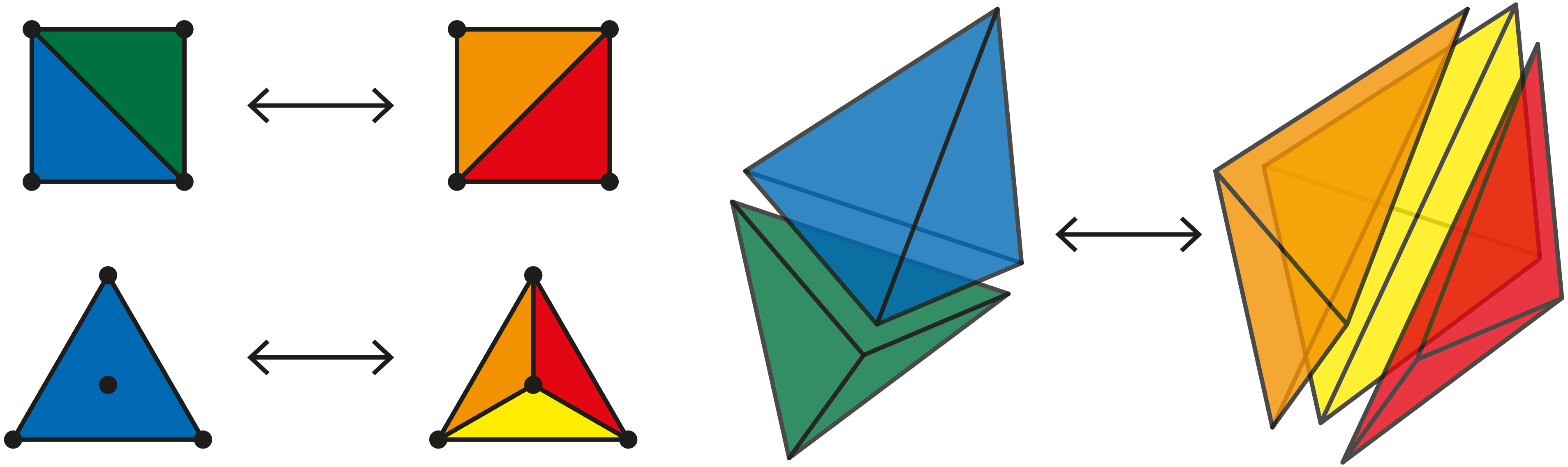




HIGH DIMENSIONAL FLIP GRAPHS

Triangulation of $\mathbf{P} \subseteq \mathbb{R}^d$ = well-behaved cover of $\operatorname{conv}(\mathbf{P})$ with simplices

Bistellar flip = $\big\{ \operatorname{conv}(\mathbf{X} \smallsetminus \{\mathbf{x}\}) \;\big|\; \mathbf{x} \in \mathbf{X}^+ \big\} \quad \longleftrightarrow \quad \big\{ \operatorname{conv}(\mathbf{X} \smallsetminus \{\mathbf{x}\}) \;\big|\; \mathbf{x} \in \mathbf{X}^+ \big\}$
where $|\mathbf{X}| = d+2$ and $\mathbf{X} = \mathbf{X}^+ \sqcup \mathbf{X}^-$ Radon partition of $\mathbf{X}$



CORO. The graph of bistellar flips on regular triangulations of $\mathbf{P}$ is connected for any $\mathbf{P}$.

HIGH DIMENSIONAL FLIP GRAPHS

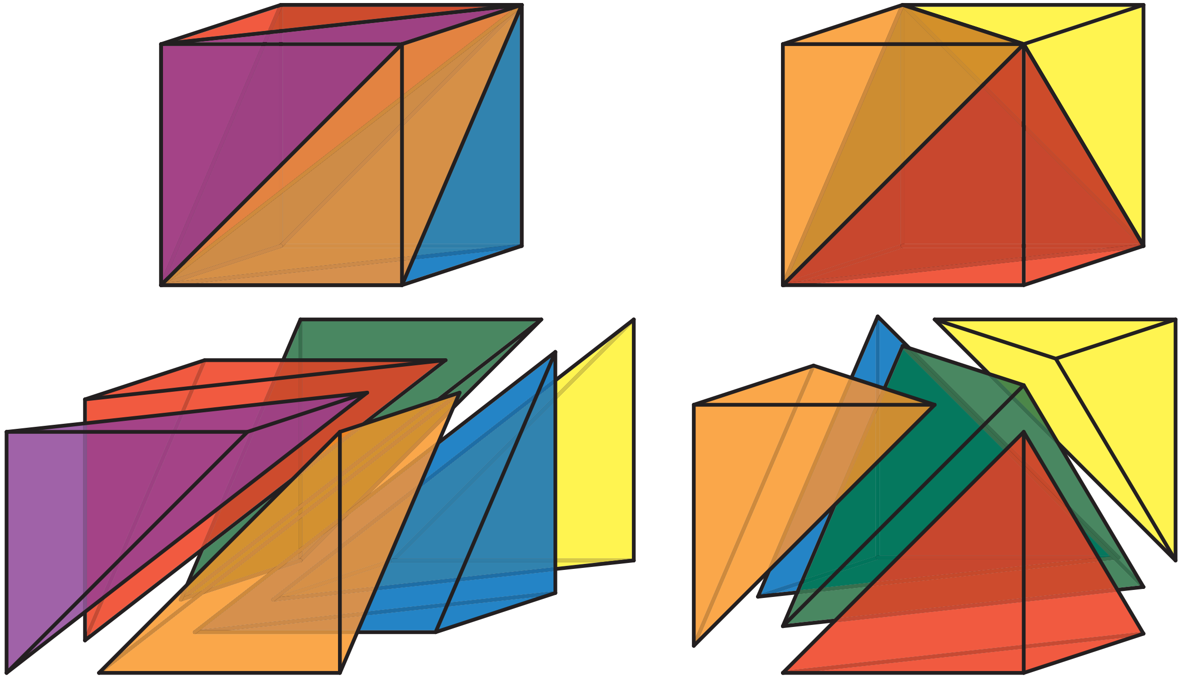
Triangulation of $\mathbf{P} \subseteq \mathbb{R}^d$ = well-behaved cover of $\operatorname{conv}(\mathbf{P})$ with simplices

Bistellar flip = $\big\{ \operatorname{conv}(\mathbf{X} \smallsetminus \{\mathbf{x}\}) \;\big|\; \mathbf{x} \in \mathbf{X}^+ \big\} \quad \longleftrightarrow \quad \big\{ \operatorname{conv}(\mathbf{X} \smallsetminus \{\mathbf{x}\}) \;\big|\; \mathbf{x} \in \mathbf{X}^+ \big\}$
where $|\mathbf{X}| = d + 2$ and $\mathbf{X} = \mathbf{X}^+ \sqcup \mathbf{X}^-$ Radon partition of $\mathbf{X}$

CORO. The graph of bistellar flips on regular triangulations of $\mathbf{P}$ is connected for any $\mathbf{P}$.

THEO. The graph of bistellar flips on all triangulations of $\mathbf{P}$ is not always connected.

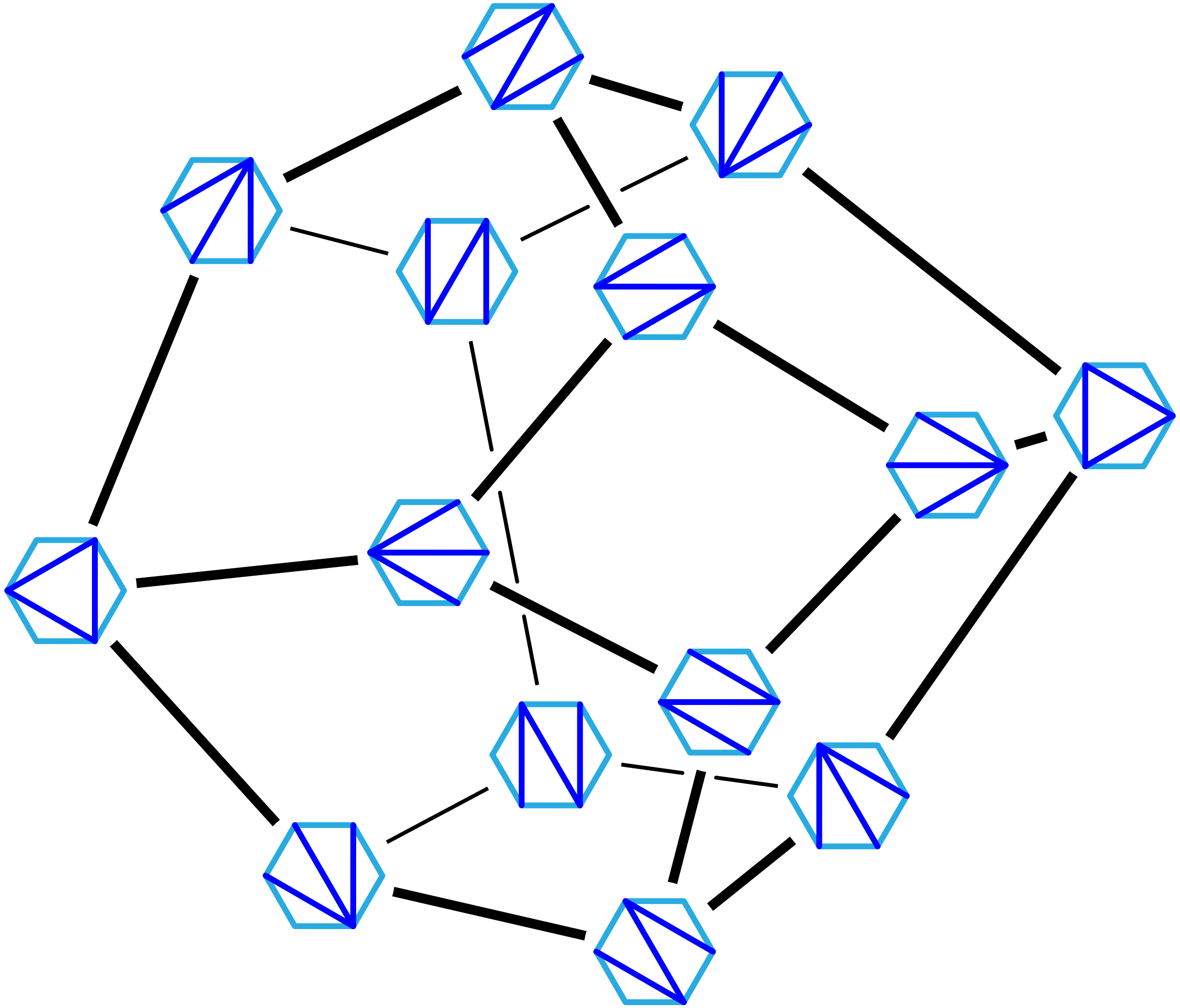
Santos, *A point set whose space of triangulations is disconnected* ('00)

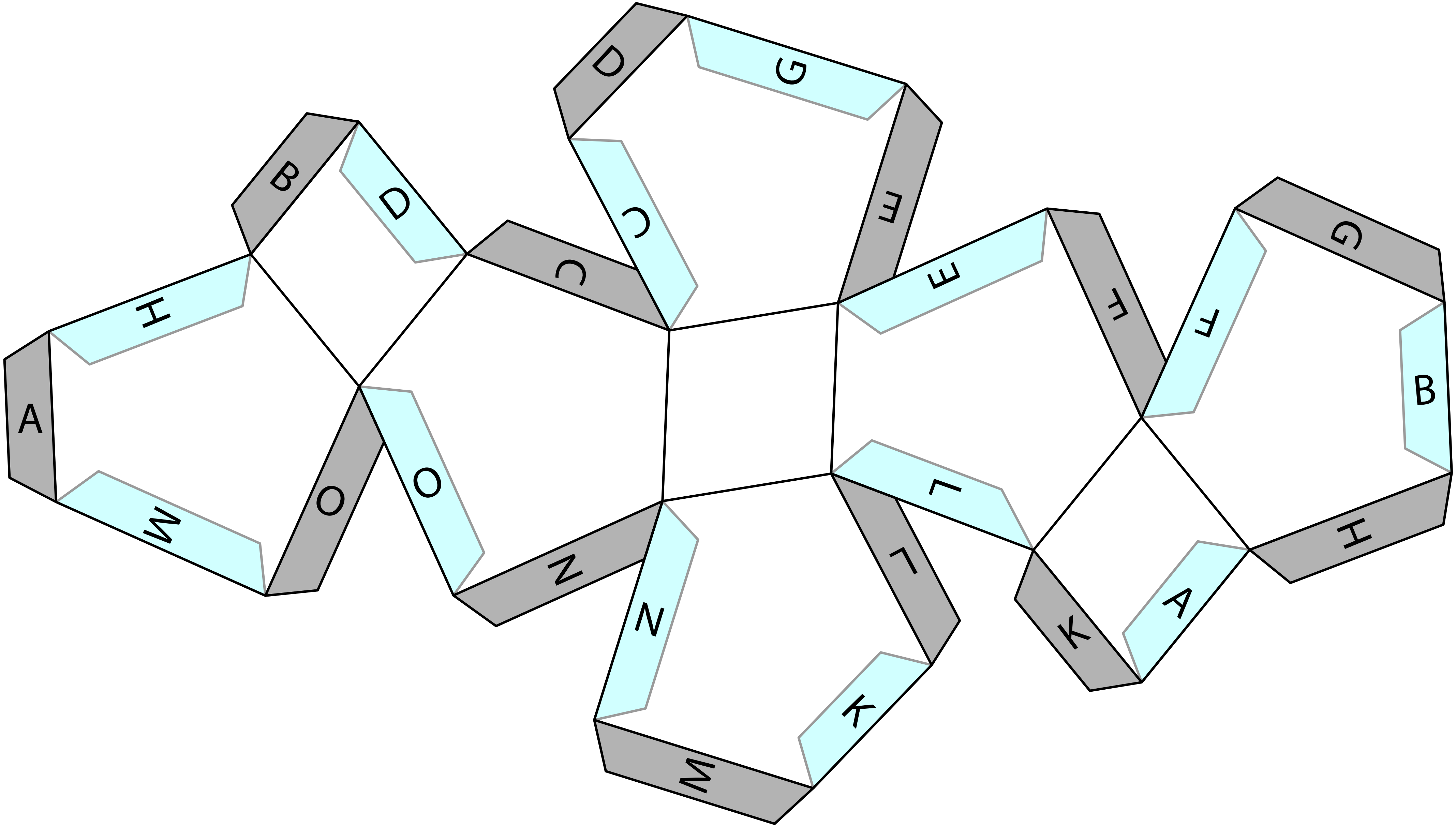


BACK TO THE ASSOCIAHEDRON

All dissections of a 2-dimensional convex point $\mathbf{P}$ set are regular

CORO. $\Sigma\mathrm{Poly}(\mathbf{P})$ is an associahedron

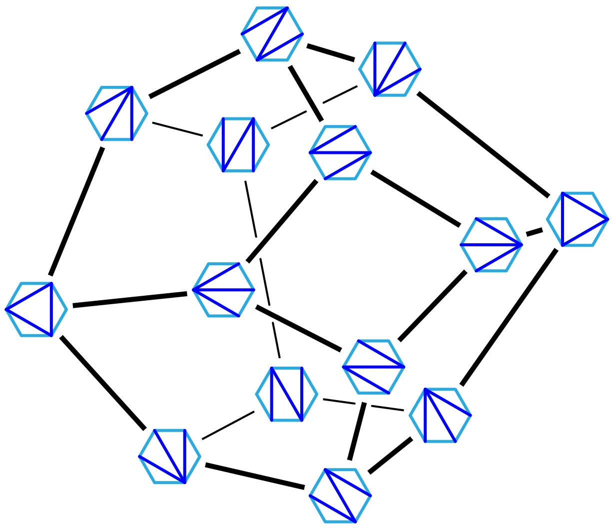




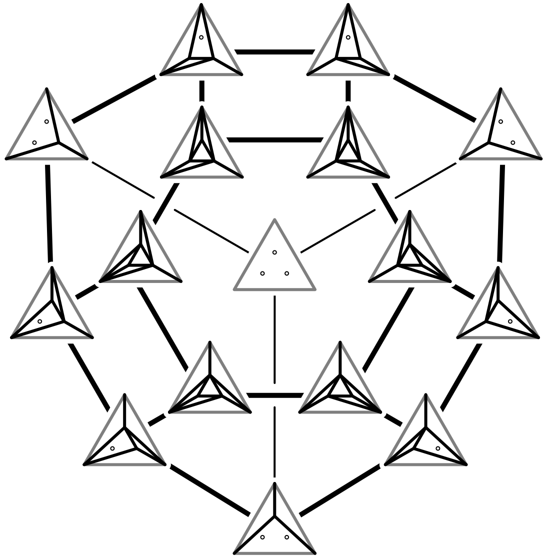
SECONDARY POLYTOPE

THREE FAMILIES OF REALIZATIONS

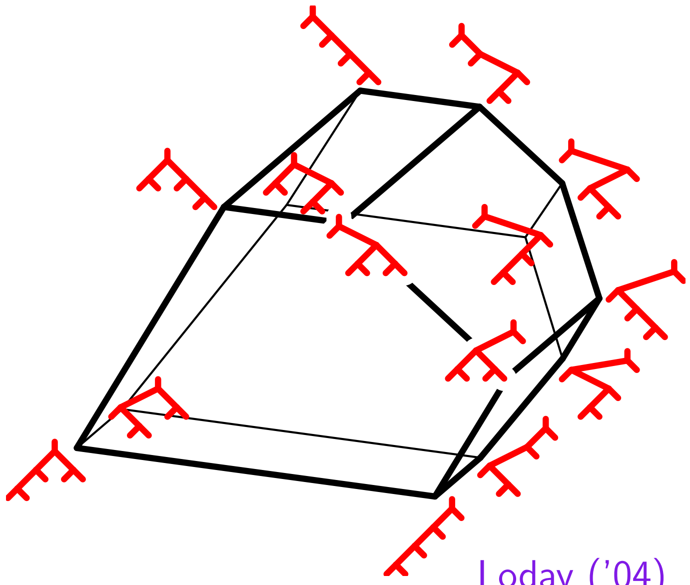
SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)



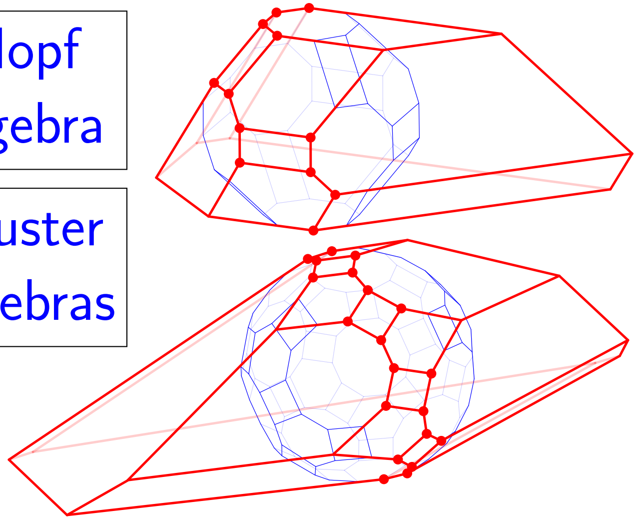
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

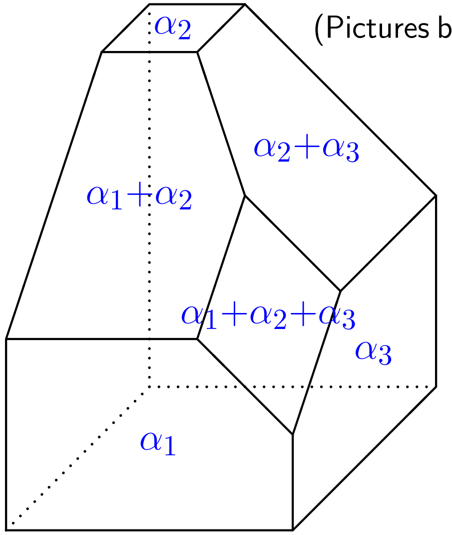
Hopf algebra

Cluster algebras



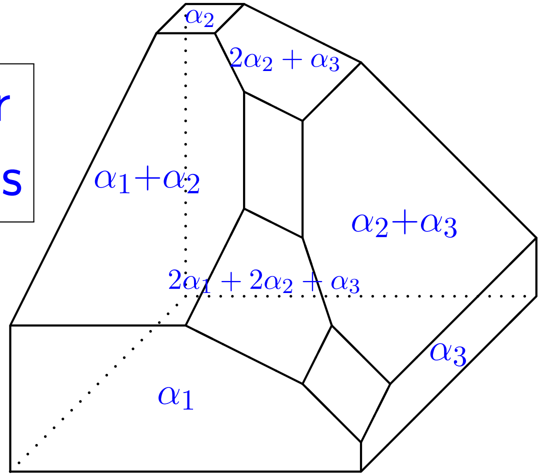
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)

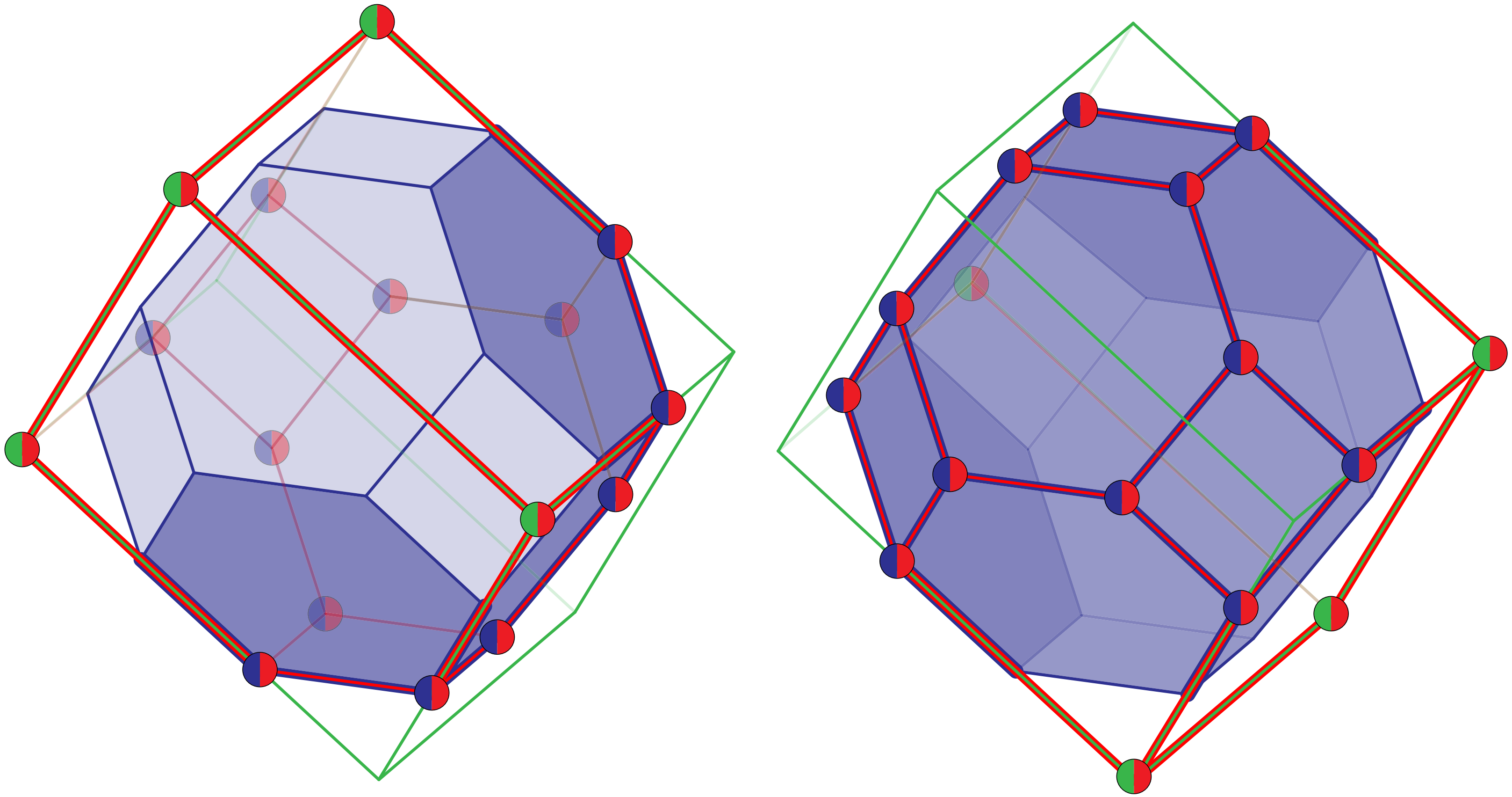


Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras



II. GRAPH ASSOCIAHEDRA



GRAPHICAL NESTED COMPLEX

Carr-Devadoss, *Coxeter complexes and graph associahedra* ('06)
Postnikov, *Permutohedra, associahedra, and beyond* ('09)
Zelevinsky, *Nested complexes and their polyhedral realizations* ('06)

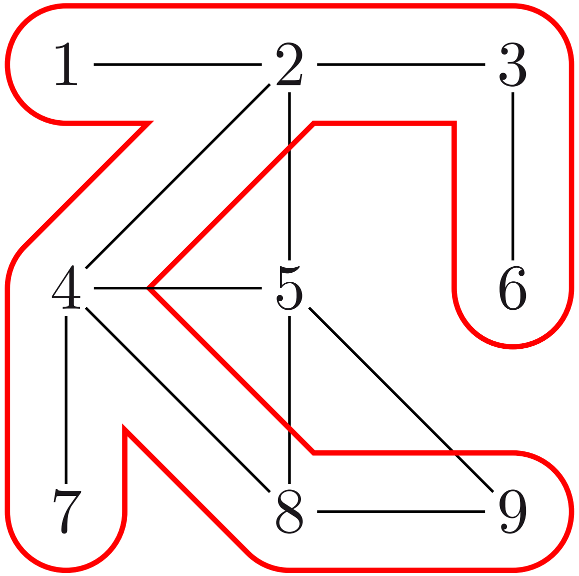
NESTED COMPLEX AND GRAPH ASSOCIAHEDRON

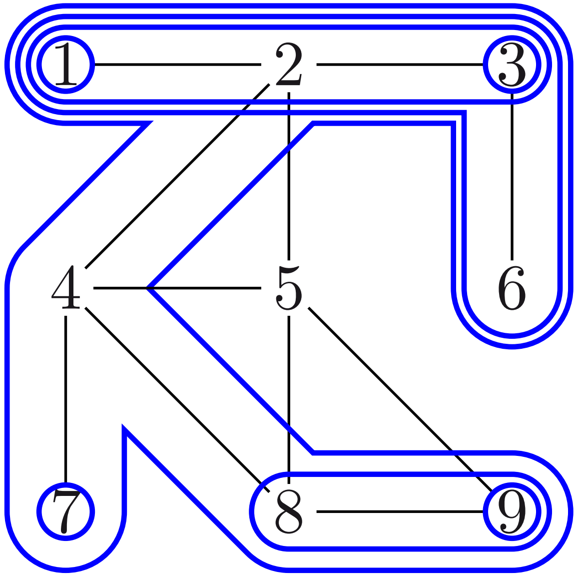
G graph on ground set V

Tube of G = connected induced subgraph of G

Compatible tubes = nested, or disjoint and non-adjacent

Tubing on G = collection of pairwise compatible tubes of G





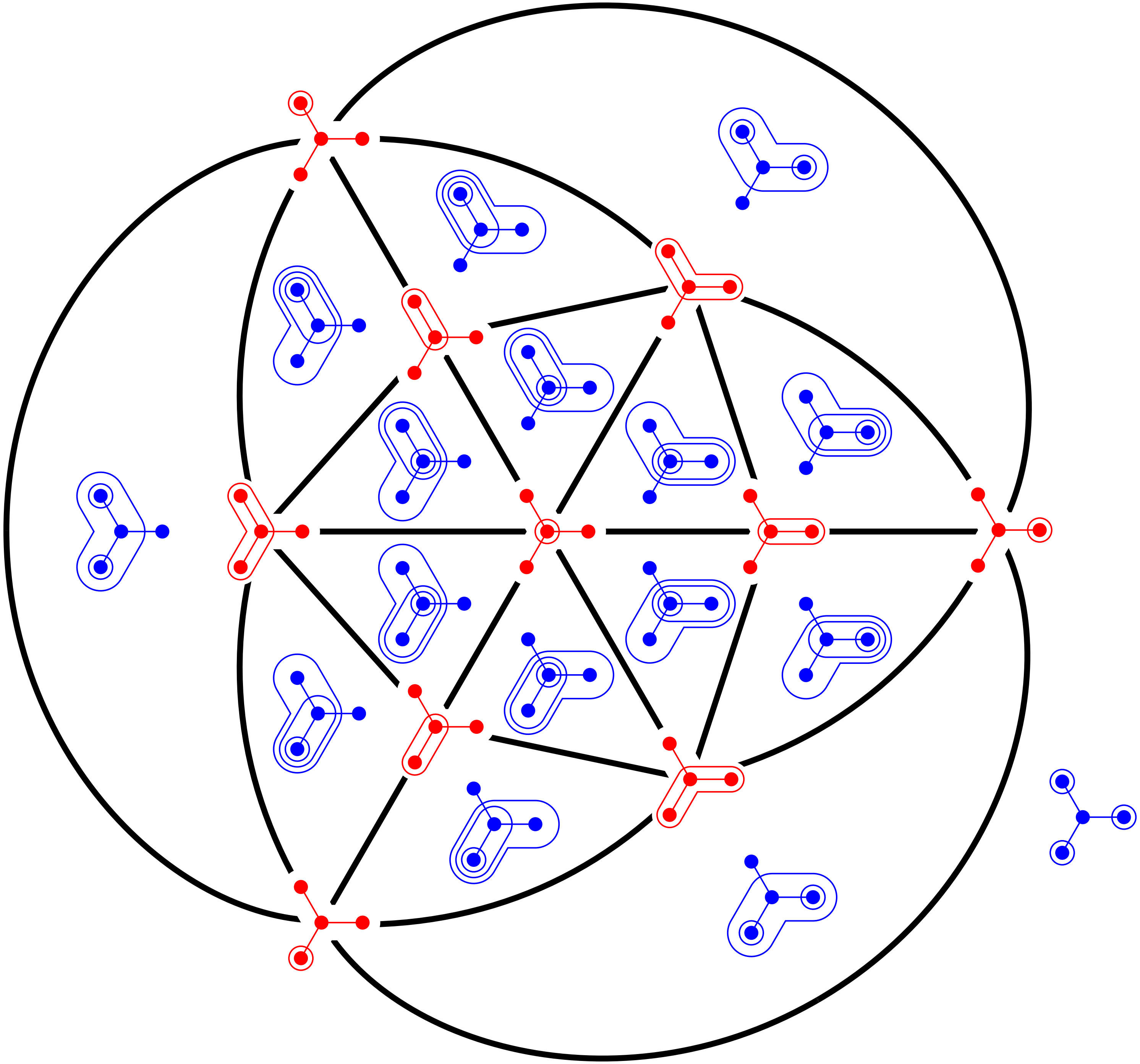
Nested complex $\mathcal{N}(\mathrm{G})$ = simplicial complex of tubings on G

= clique complex of the compatibility relation on tubes

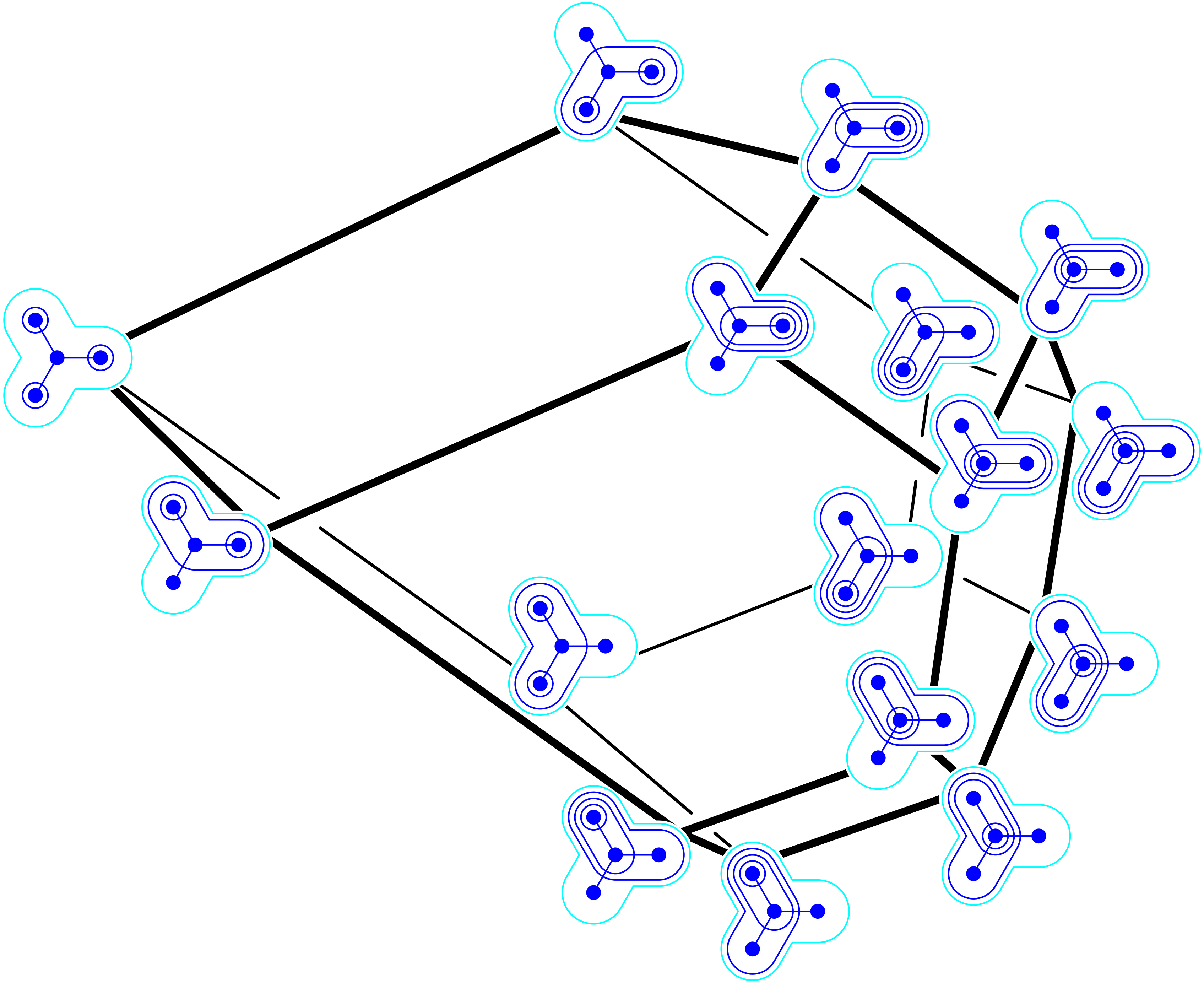
G-associahedron = polytopal realization of the nested complex on G

Carr-Devadoss, *Coxeter complexes and graph associahedra* ('06)

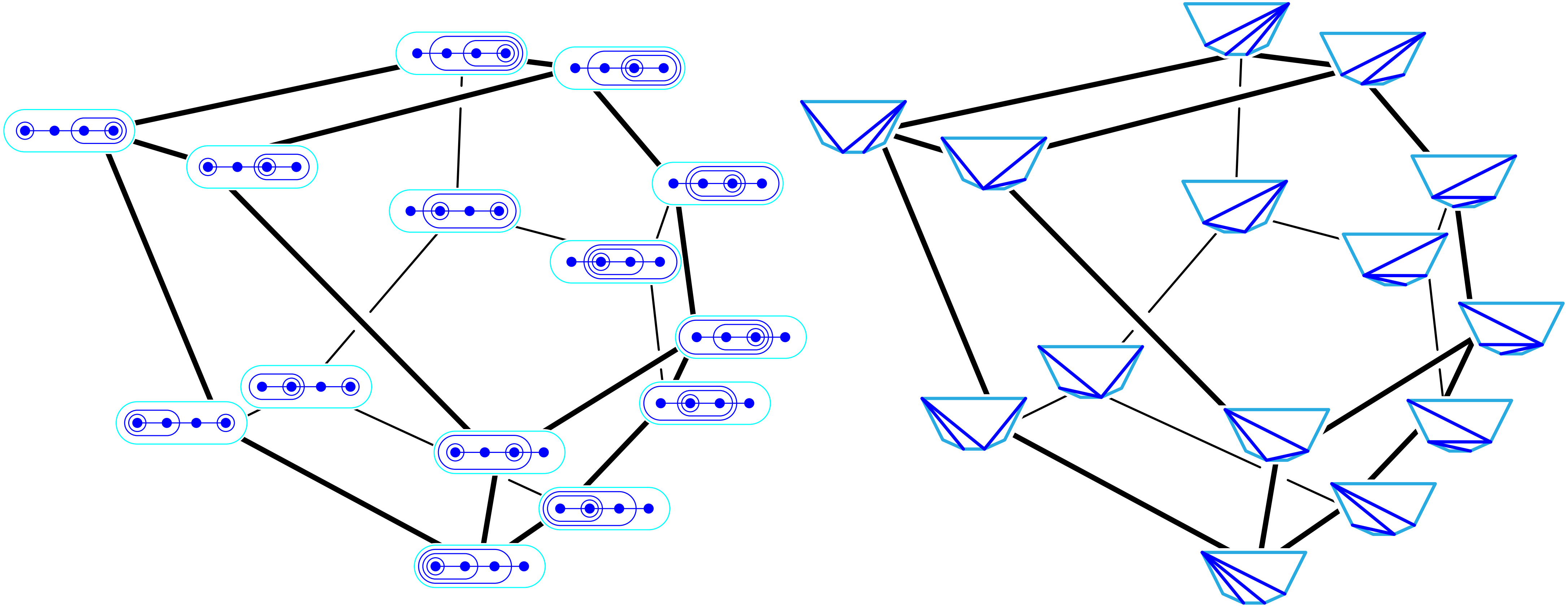
EXM: NESTED COMPLEX



EXM: GRAPH ASSOCIAHEDRON

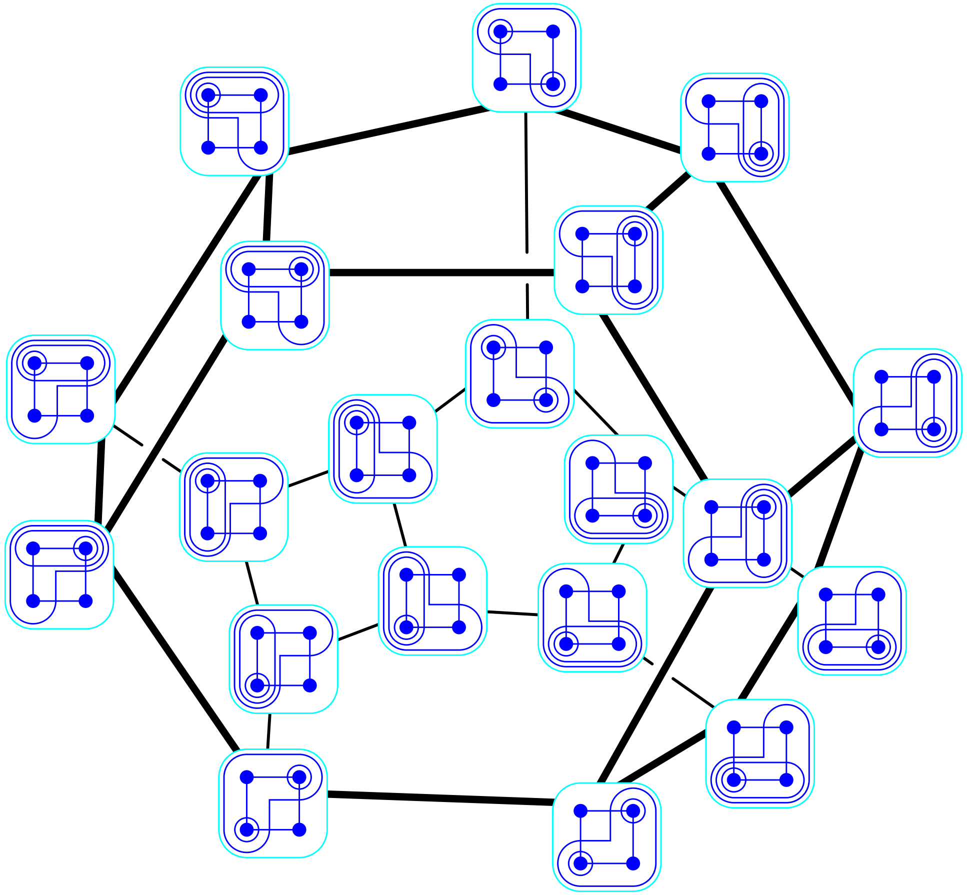


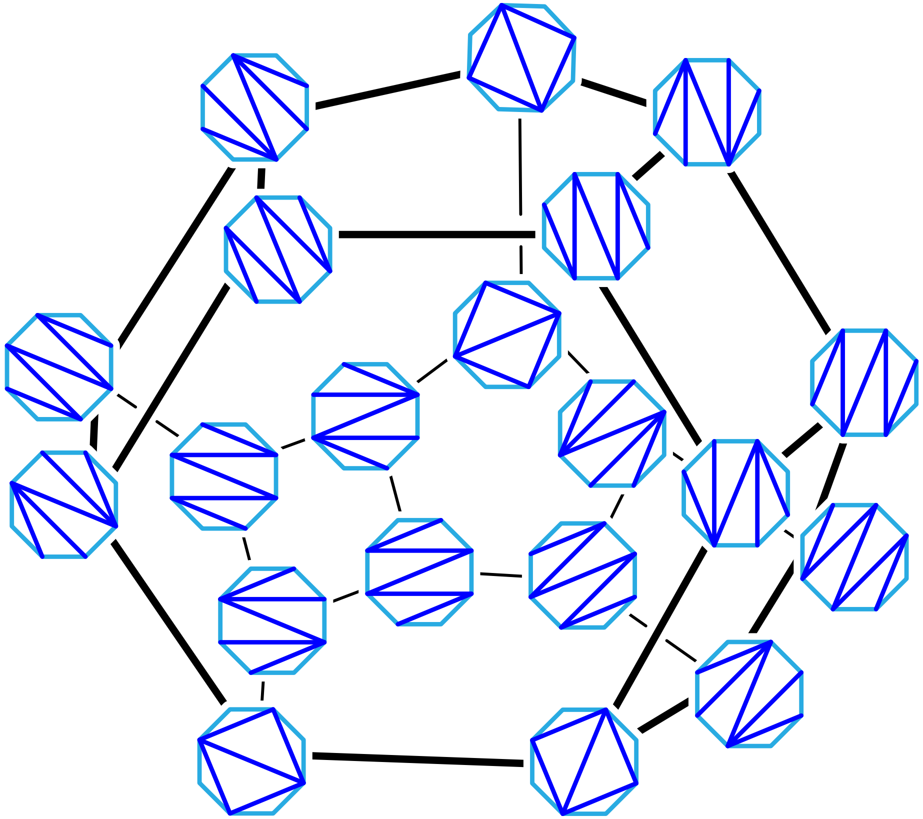
SPECIAL GRAPH ASSOCIAHEDRA



path associahedron = associahedron

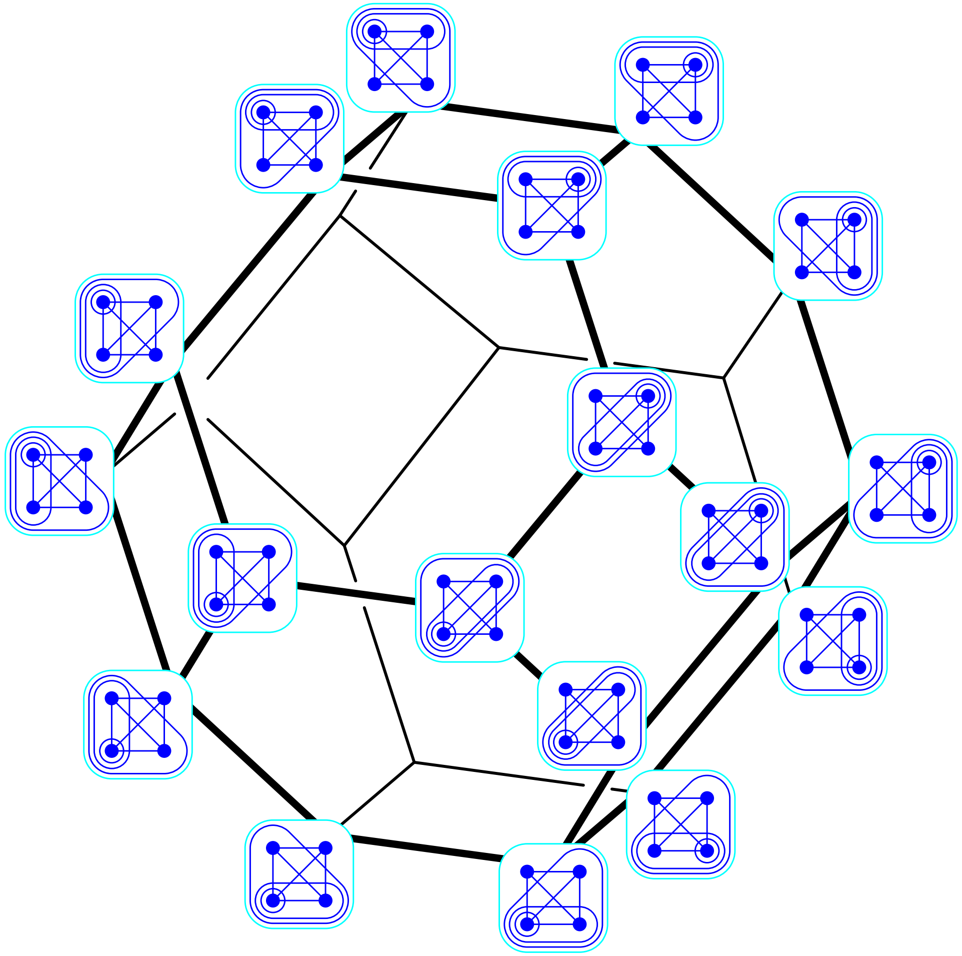
SPECIAL GRAPH ASSOCIAHEDRA

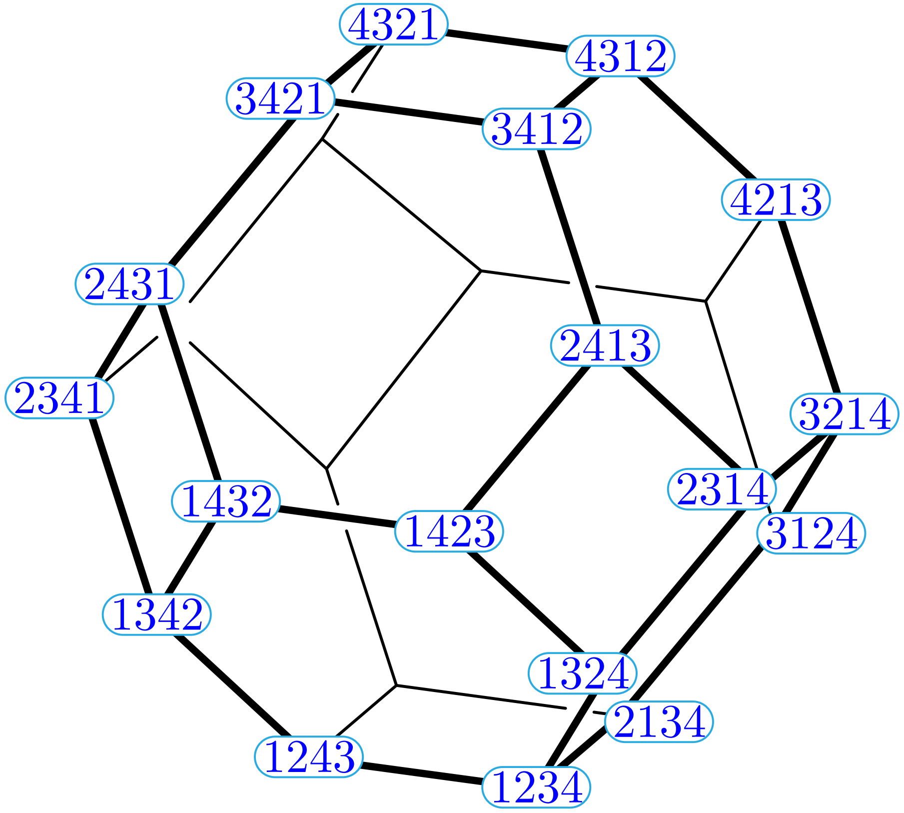




cycle associahedron = cyclohedron

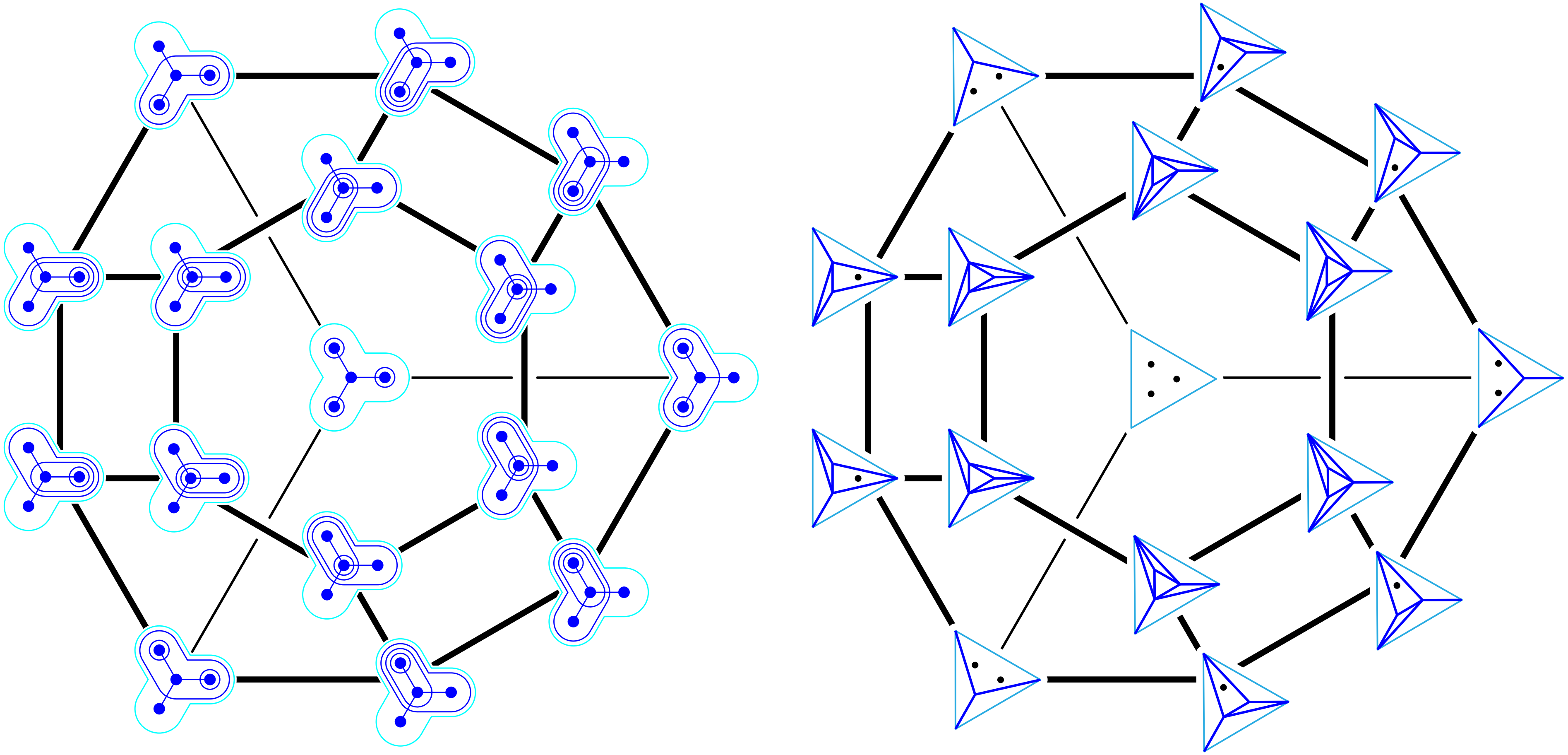
SPECIAL GRAPH ASSOCIAHEDRA





complete graph associahedron $=$ permutahedron

SPECIAL GRAPH ASSOCIAHEDRA



star associahedron = secondary polytope of dilated simplices

LINEAR LAURENT PHENOMENON ALGEBRAS

Laurent Phenomenon Algebra = commut. ring gen. by cluster variables grouped in clusters

seed = pair $(\mathbf{x}, \mathbf{F})$ where

- $\mathbf{x} = \{x_1, \dots, x_n\}$ cluster variables
- $\mathbf{F} = \{F_1, \dots, F_n\}$ exchange polynomials

seed mutation = $(\mathbf{x}, \mathbf{F}) \longmapsto \mu_i(\mathbf{x}, \mathbf{F}) = (\mathbf{x}', \mathbf{F}')$ where

- $x'_i = \hat{F}_i / x_i$ while $x'_j = x_j$ for $j \neq i$
- F'_j obtained from F_j by eliminating x_i

THM. Every cluster variable in a LP algebra is a Laurent polynomial in the cluster variables of any seed.

Lam-Pylyavskyy, *Laurent Phenomenon Algebras* ('12)

Connection to graph associahedra: Any (directed) graph G defines a linear LP algebra whose cluster complex contains the nested complex of G

Lam-Pylyavskyy, *Linear Laurent Phenomenon Algebras* ('12)

COMPATIBILITY FANS
FOR GRAPHICAL NESTED COMPLEXES

Manneville-P., `arXiv:1501.07152`

COMPATIBILITY FANS FOR ASSOCIAHEDRA

T° an initial triangulation
δ, δ' two internal diagonals

compatibility degree between δ and δ':

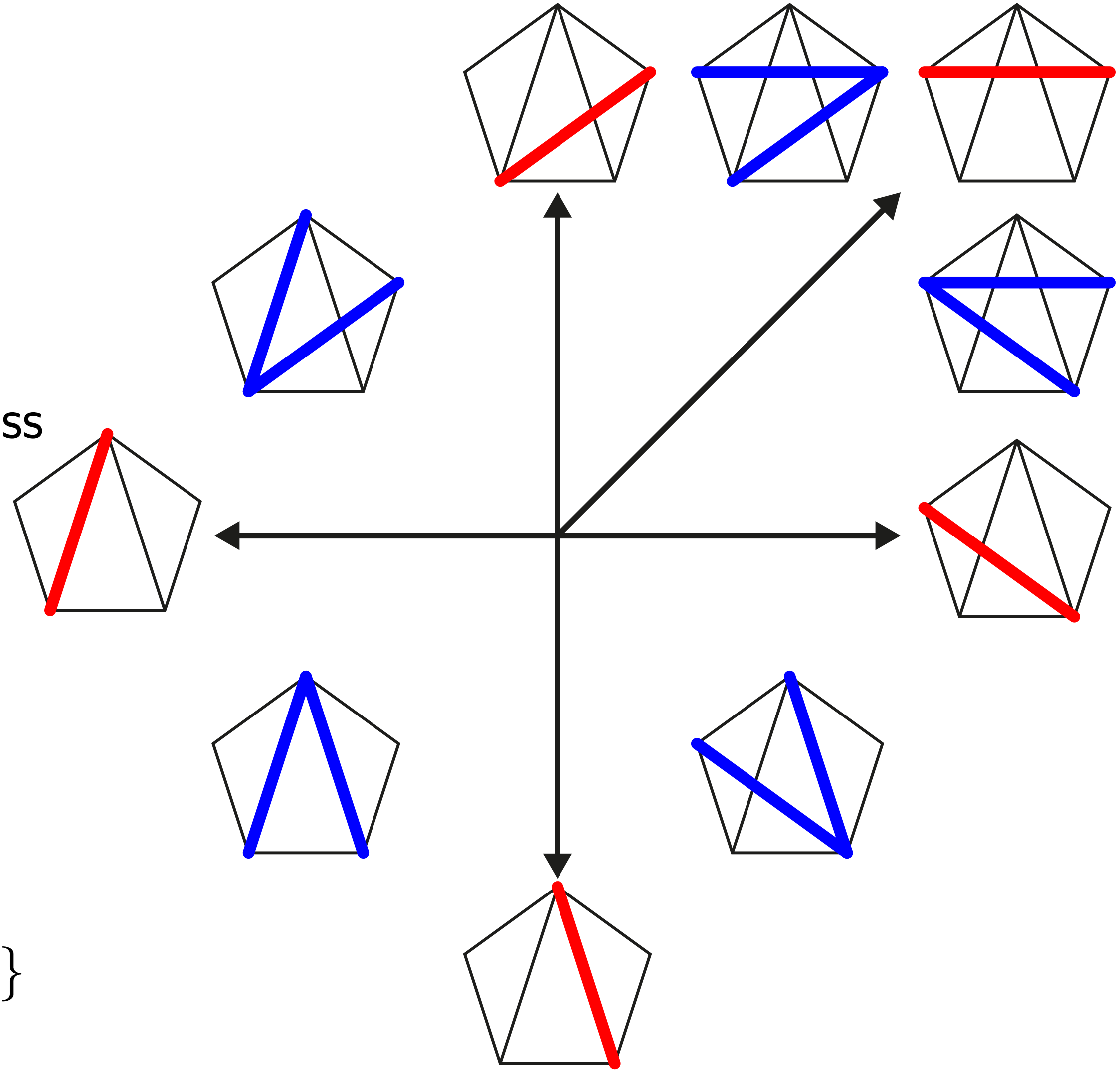
$$(\delta \parallel \delta') = \begin{cases} -1 & \text{if } \delta = \delta' \\ 0 & \text{if } \delta \text{ and } \delta' \text{ do not cross} \\ 1 & \text{if } \delta \text{ and } \delta' \text{ cross} \end{cases}$$

compatibility vector of δ wrt T°:

$$\mathbf{d}(\mathrm{T}^\circ, \delta) = \big[(\delta^\circ \parallel \delta) \big]_{\delta^\circ \in \mathrm{T}^\circ}$$

compatibility fan wrt T°:

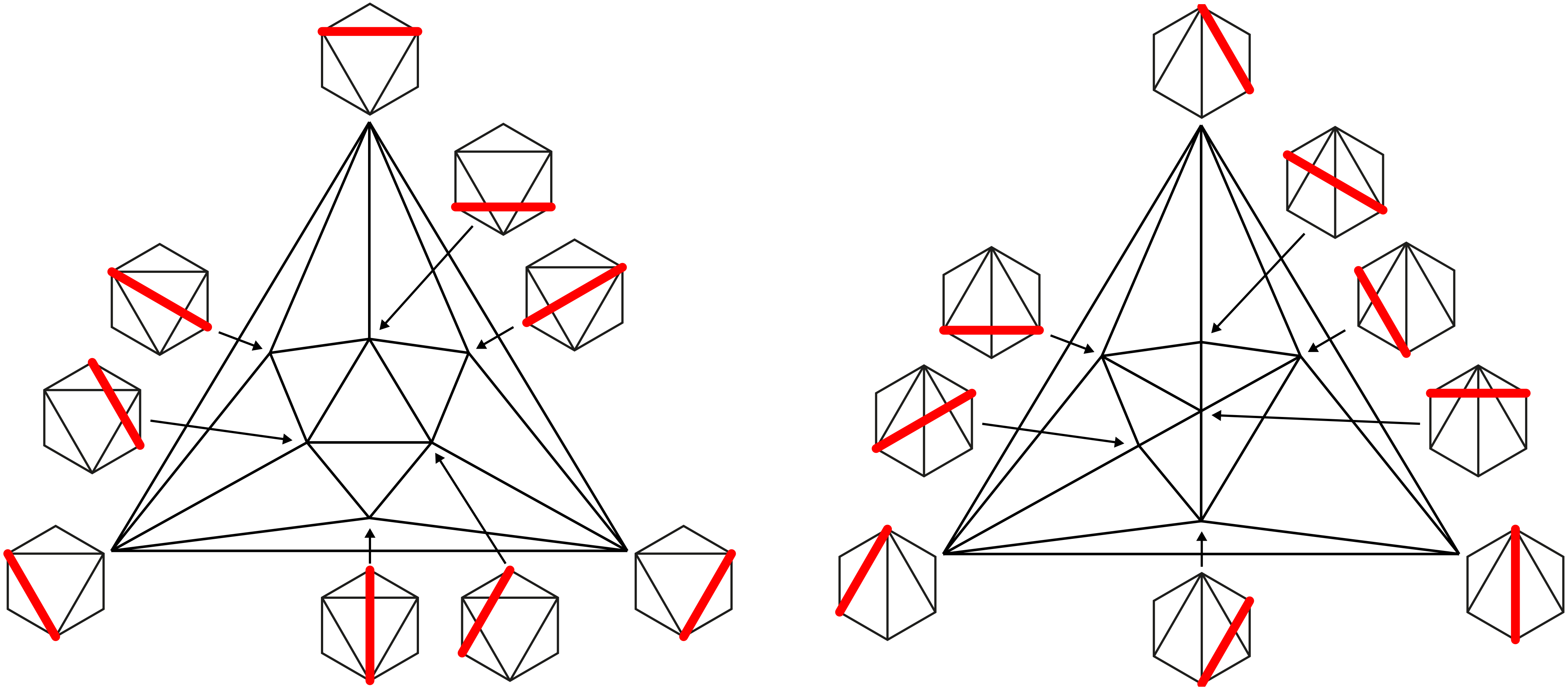
$$\mathcal{D}(\mathrm{T}^\circ) = \{ \mathbb{R}_{\geq 0} \, \mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D} \text{ dissection} \}$$



Fomin-Zelevinsky, *Y-Systems and generalized associahedra* ('03)
Fomin-Zelevinsky, *Cluster algebras II: Finite type classification* ('03)
Chapoton-Fomin-Zelevinsky, *Polytopal realizations of generalized associahedra* ('02)
Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11)

COMPATIBILITY FANS FOR ASSOCIAHEDRA

Different initial triangulations T° yield different realizations



THM. For any initial triangulation T°, the cones $\{\mathbb{R}_{\geq 0}\,\mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D}$ dissection$\}$ form a complete simplicial fan. Moreover, this fan is always polytopal.

Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11)

COMPATIBILITY FANS FOR GRAPHICAL NESTED COMPLEXES

T° an initial maximal tubing on G

t, t' two tubes of G

compatibility degree between t and t'

$$(\mathrm{t} \,\|\, \mathrm{t}') = \begin{cases} -1 & \text{if } \mathrm{t} = \mathrm{t}' \\ 0 & \text{if } \mathrm{t}, \mathrm{t}' \text{ are compatible} \\ |\{\text{neighbors of } \mathrm{t} \text{ in } \mathrm{t}' \smallsetminus \mathrm{t}\}| & \text{otherwise} \end{cases}$$

compatibility vector of t wrt T°:

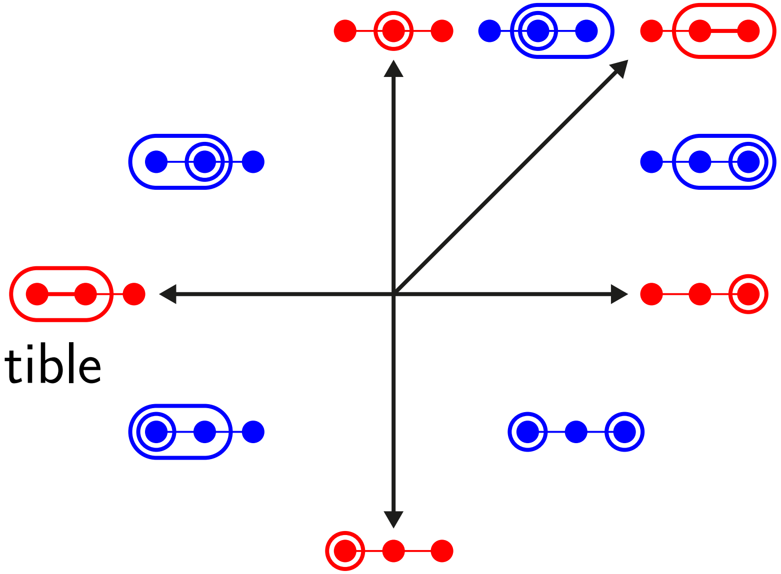
$$\mathbf{d}(\mathrm{T}^\circ, \mathrm{t}) = \big[(\mathrm{t}^\circ \,\|\, \mathrm{t})\big]_{\mathrm{t}^\circ \in \mathrm{T}^\circ}$$

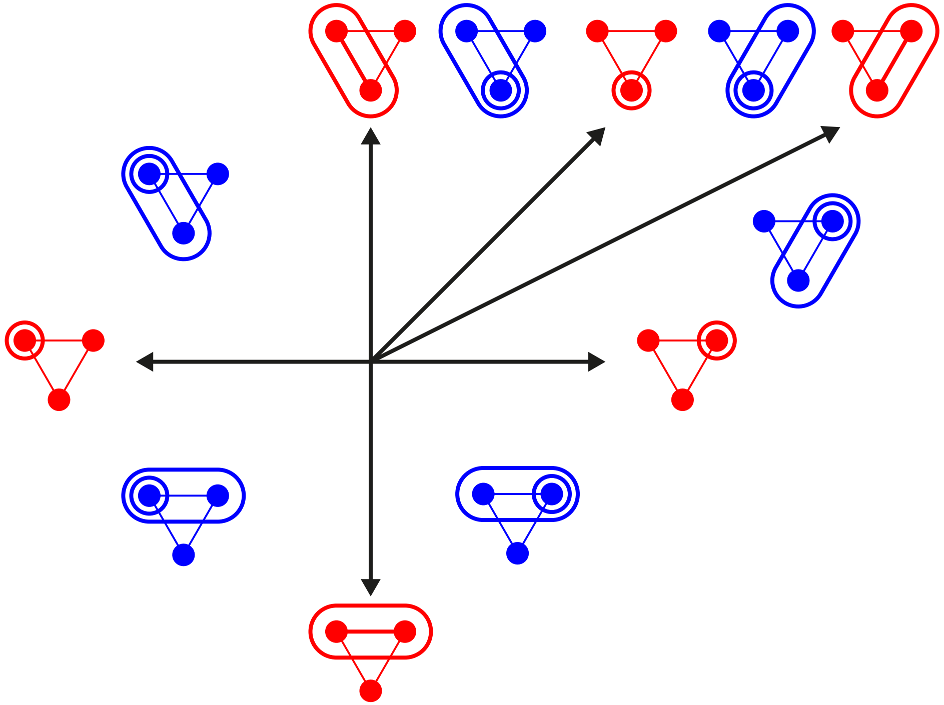
THM. For any initial maximal tubing T° on G, the collection of cones

$$\mathcal{D}(\mathrm{G}, \mathrm{T}^\circ) = \{\mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}^\circ, \mathrm{T}) \mid \mathrm{T} \text{ tubing on } \mathrm{G}\}$$

forms a complete simplicial fan, called compatibility fan of G.

Manneville-P., *Compatibility fans for graphical nested complexes*

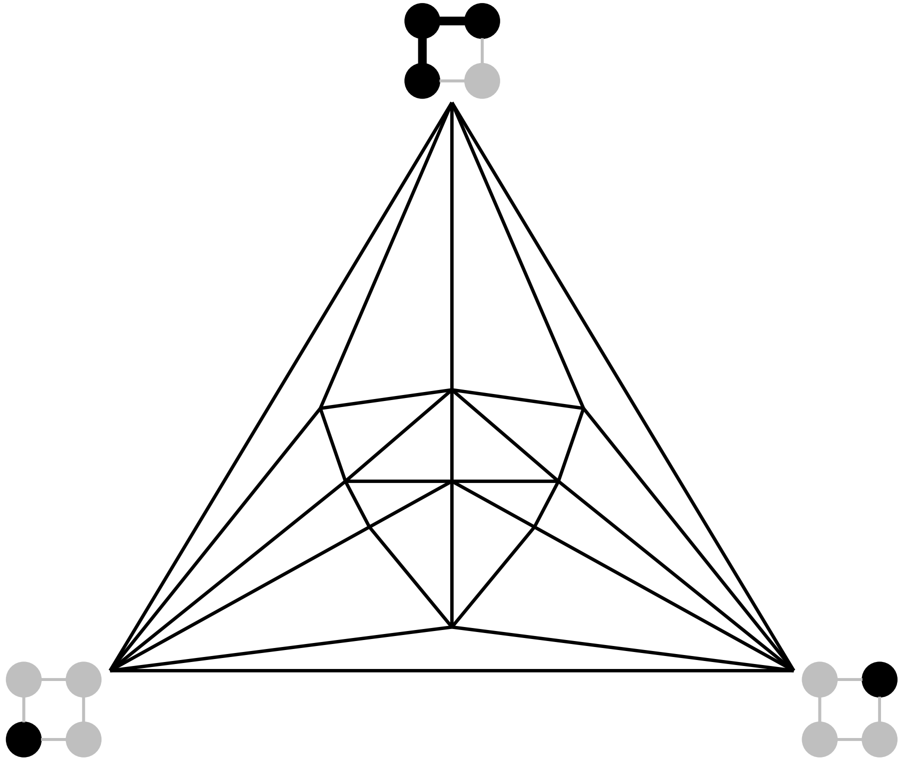




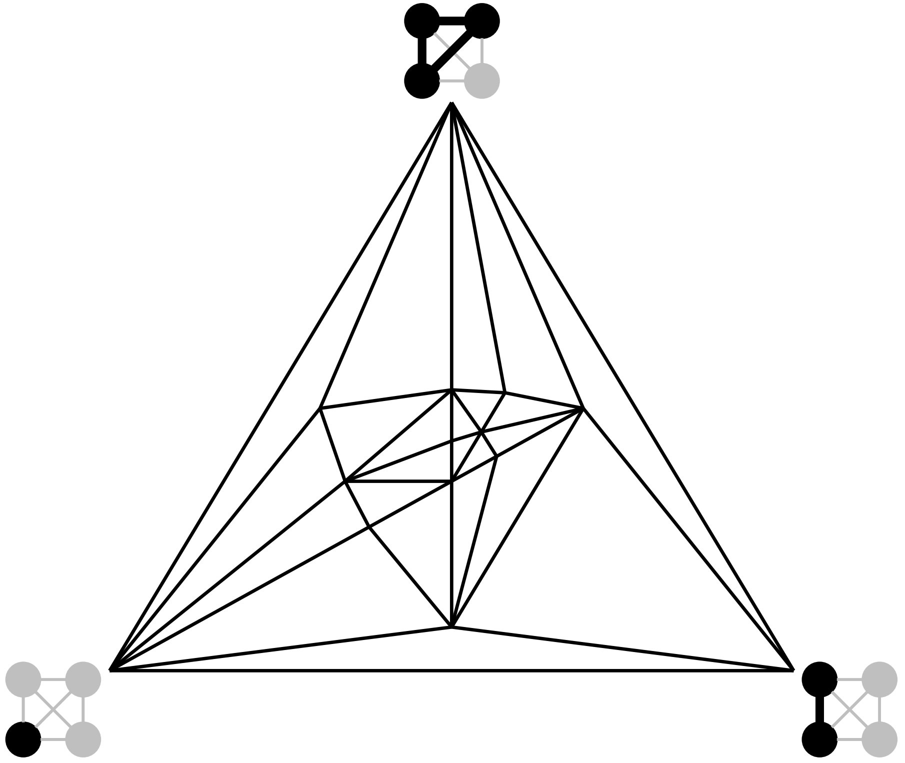
COMPATIBILITY FANS FOR GRAPHICAL NESTED COMPLEXES

THM. $\{\mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}^\circ, \mathrm{T}) \mid \mathrm{T} \text{ tubing on } \mathrm{G}\}$ forms a complete simplicial fan for any T°.

Manneville-P., *Compatibility fans for graphical nested complexes* ('15^+)



Cyclohedron



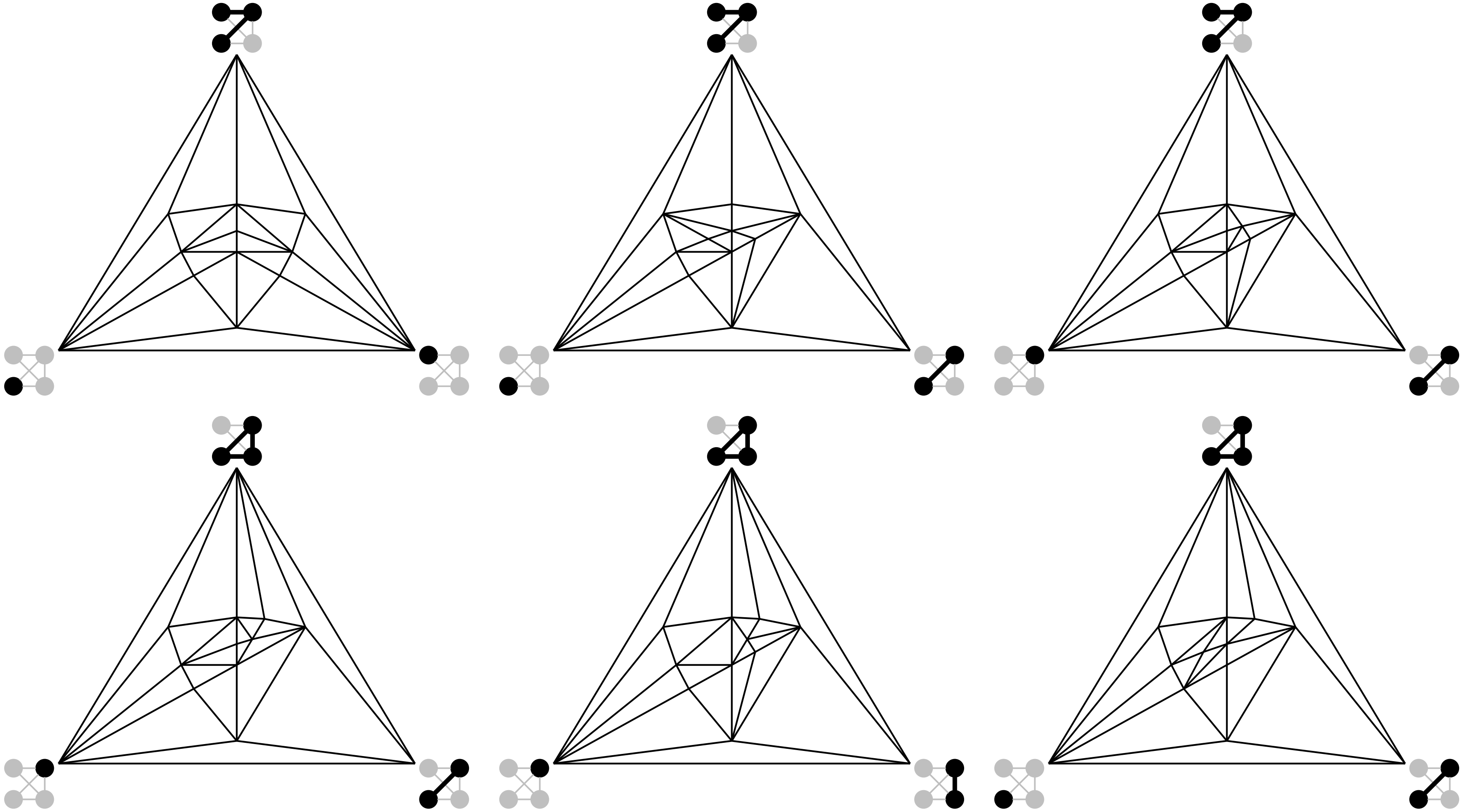
Permutahedron

GRAPH CATALAN MANY SIMPLICIAL FAN REALIZATIONS

THM. When none of the connected components of G is a spider,

$$\begin{aligned} &\#\ \text{linear isomorphism classes of compatibility fans of } \mathrm{G} \\ = &\#\ \text{orbits of maximal tubings on } \mathrm{G} \text{ under graph automorphisms of } \mathrm{G}. \end{aligned}$$

Manneville-P., *Compatibility fans for graphical nested complexes* ('15^{+})



POLYTOPALITY?

QU. Are all compatibility fans polytopal?

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Polytopality of a complete simplicial fan $\iff$ Feasibility of a linear program

Exm: We check that the compatibility fan on the complete graph K_7 is polytopal by solving a linear program on 126 variables and $17\,640$ inequalities

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$\Longrightarrow$ All compatibility fans on complete graphs of ≤ 7 vertices are polytopal...
$\Longrightarrow$ All compatibility fans on graphs of ≤ 4 vertices are polytopal...

POLYTOPALITY?

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$\Longrightarrow$ All compatibility fans on complete graphs of ≤ 7 vertices are polytopal...
$\Longrightarrow$ All compatibility fans on graphs of ≤ 4 vertices are polytopal...

To go further, we need to understand better the linear dependences between the compatibility vectors of the tubes involved in a flip

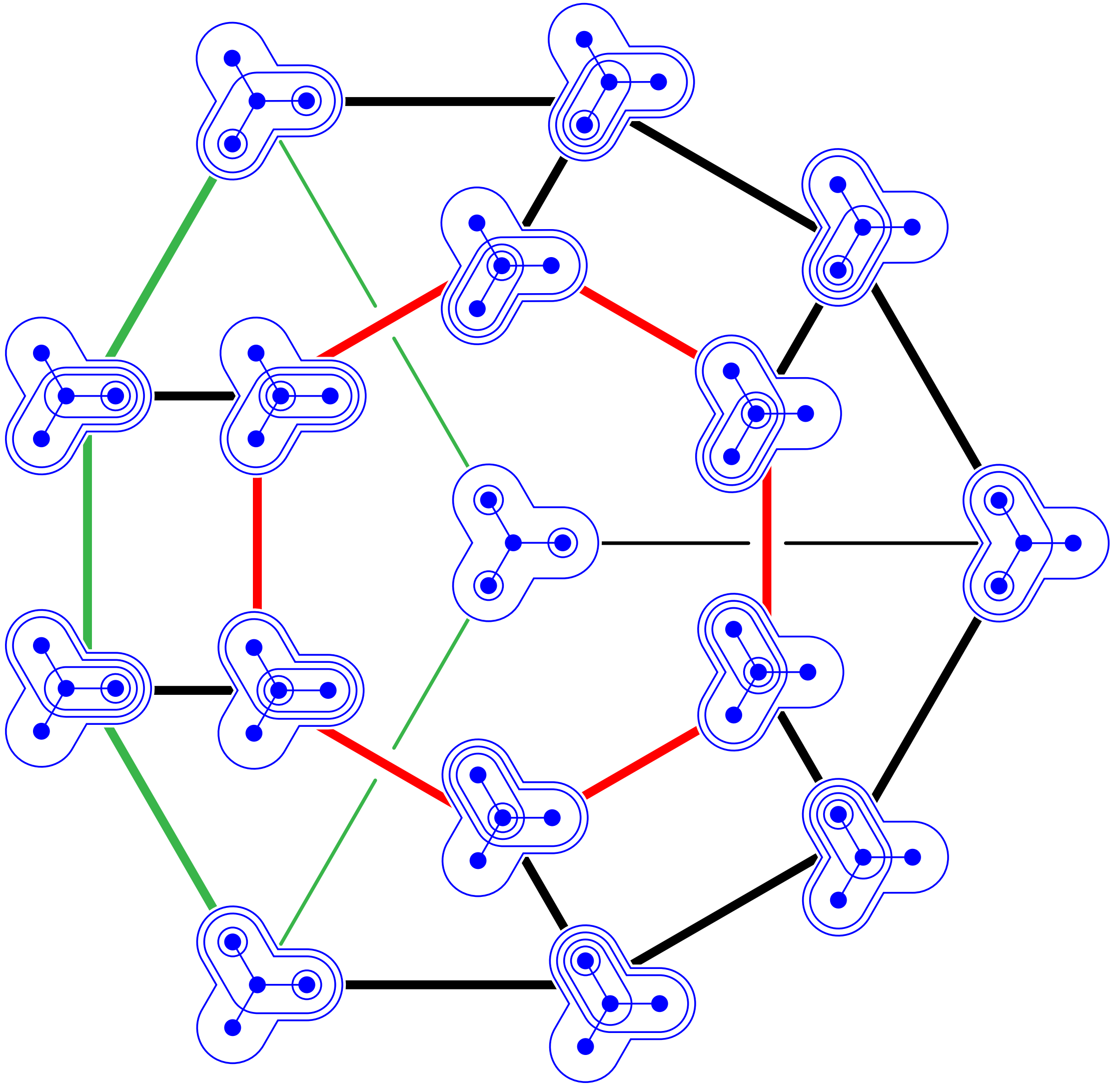
THM. All compatibility fans on the paths and cycles are polytopal

Ceballos-Santos-Ziegler, *Many non-equivalent realizations of the associahedron* ('11)
Manneville-P., *Compatibility fans for graphical nested complexes* ('15^{+})

POLYTOPALITY?

QU. Are all compatibility fans polytopal?

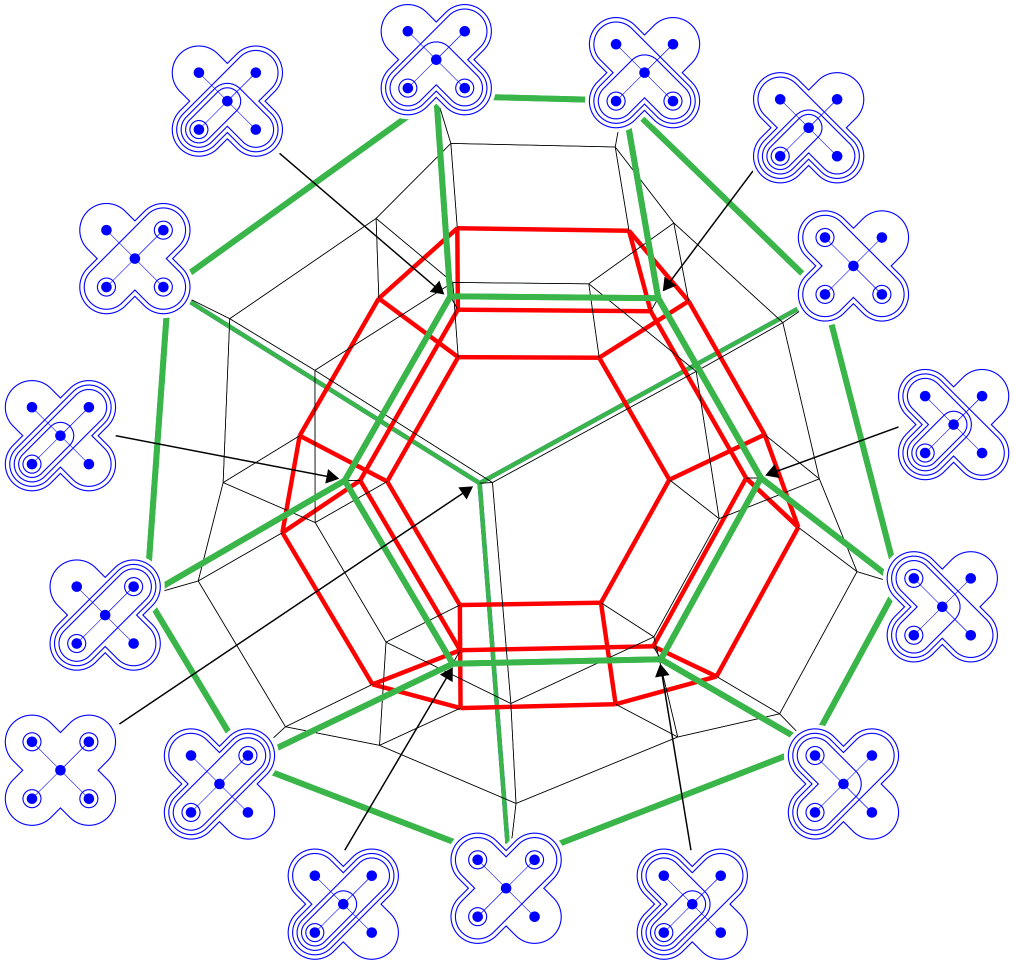
Remarkable realizations of the stellohedra

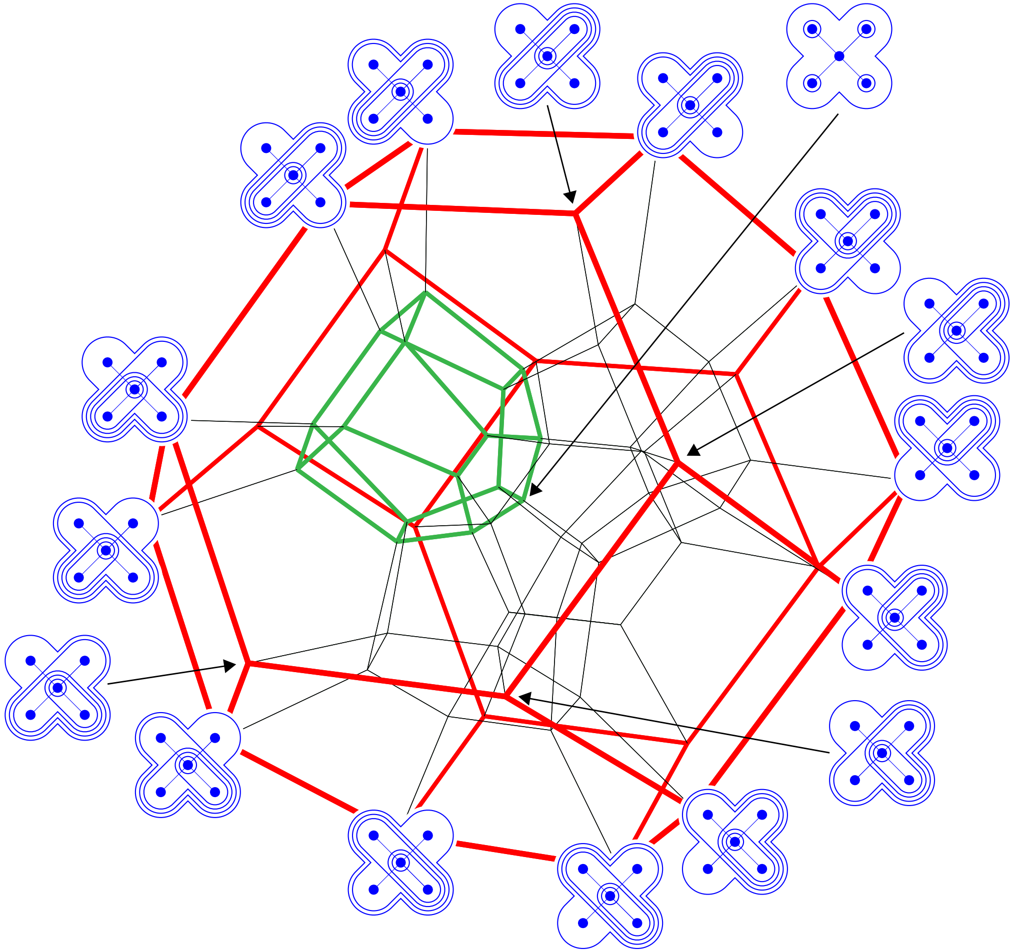


POLYTOPALITY?

QU. Are all compatibility fans polytopal?

Remarkable realizations of the stellohedra





Convex hull of the orbits under coordinate permutations of the set $\left\{\sum_{i>k} i\,\mathbf{e}_i \;\middle|\; 0 \le k \le n\right\}$

SIGNED TREE ASSOCIAHEDRA

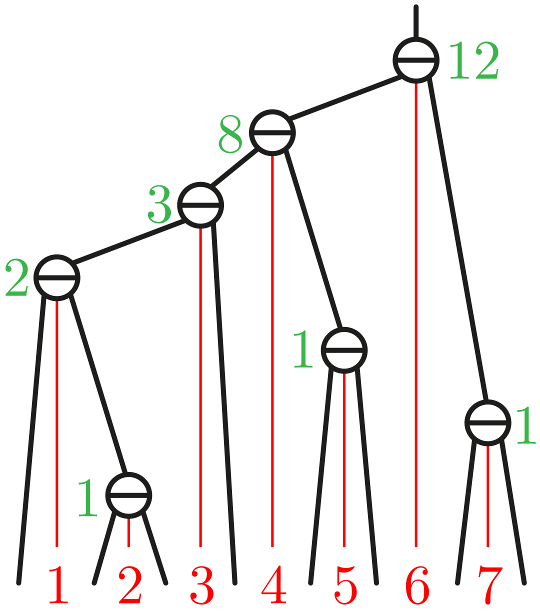
arXiv:1309.5222

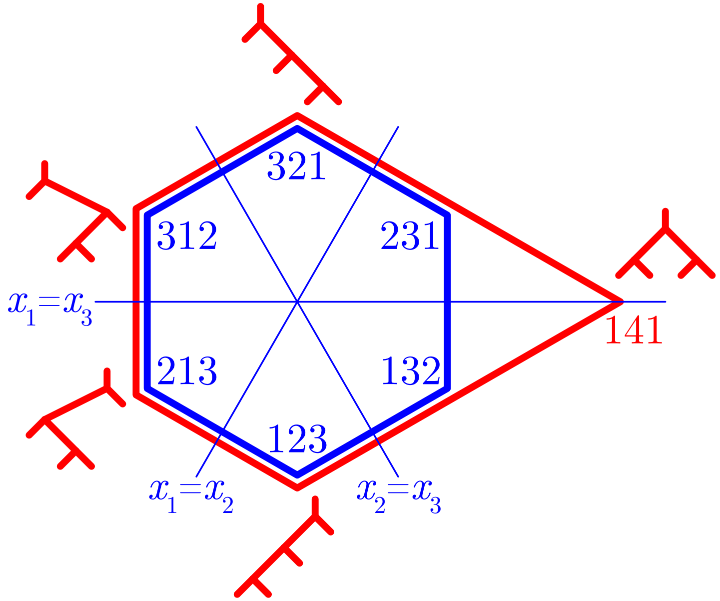
LODAY'S ASSOCIAHEDRON

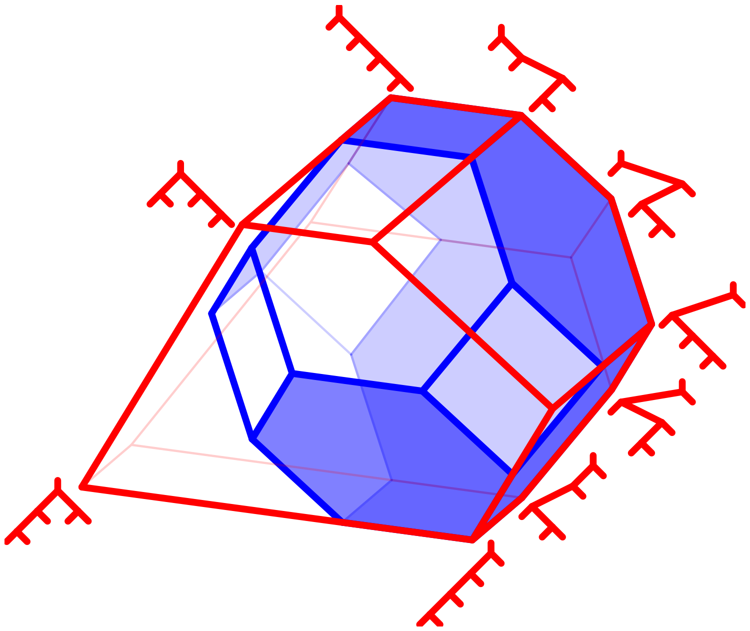
$$\text{Asso}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1\le i\le j\le n+1} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i)\cdot r(\mathrm{T},i)\big]_{i\in[n+1]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{\mathbf{x}\in\mathbb{R}^{n+1} \;\middle|\; \sum_{i\le k\le j} x_i \ge \binom{j-i+2}{2}\right\}$$

Loday, *Realization of the Stasheff polytope* ('04)



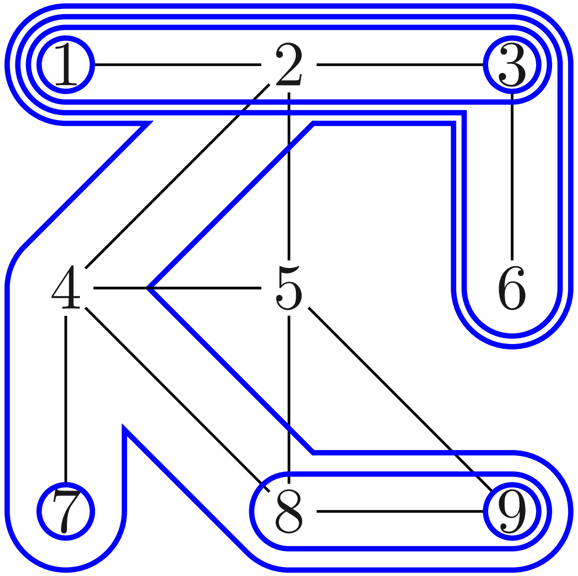


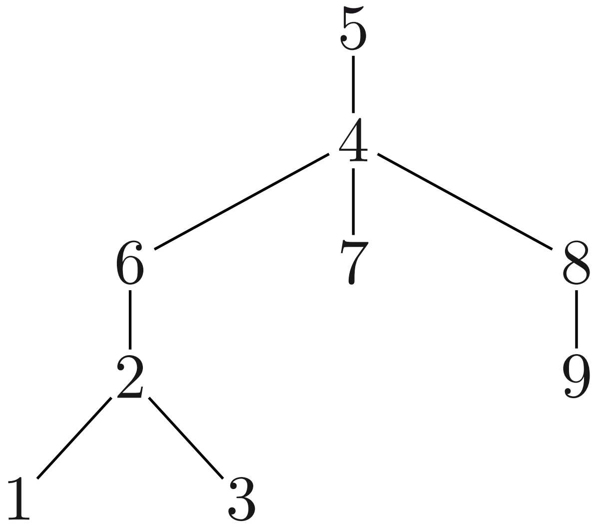


- Asso(n) obtained by deleting inequalities in the facet description of the permutahedron
- normal cone of $\mathbf{L}(\mathrm{T})$ in $\text{Asso}(n) = \{\mathbf{x}\in\mathbb{H} \mid x_i < x_j \text{ for all } i \to j \text{ in } \mathrm{T}\}$
  $= \bigcup_{\sigma\in\mathcal{L}(\mathrm{T})}$ normal cone of σ in $\text{Perm}(n)$

SPINES

spine of a tubing T = Hasse diagram of the inclusion poset of T





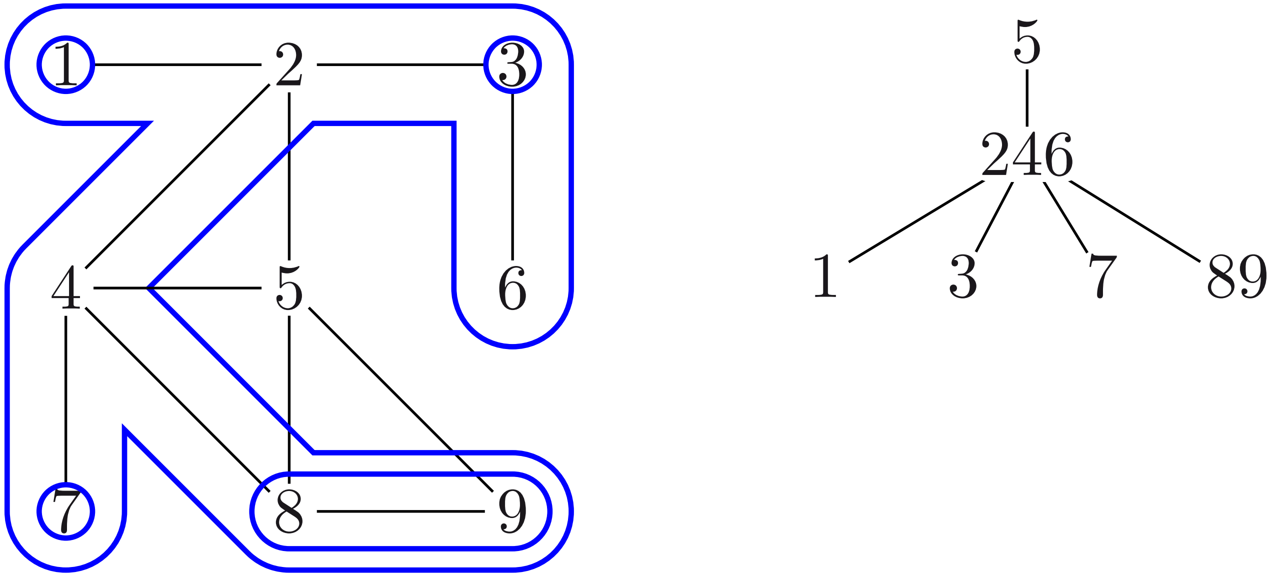
tube t of the tubing T $\longmapsto$ node $\mathrm{s(t)}$ of the spine S labeled by $\mathrm{t} \smallsetminus \bigcup \{\mathrm{t}' \mid \mathrm{t}' \in \mathrm{T}, \mathrm{t}' \subsetneq \mathrm{t}\}$

tube $\mathrm{t(s)} \coloneqq \bigcup \{\mathrm{s}' \mid \mathrm{s}' \le \mathrm{s}$ in $\mathrm{S}\}$ of the tubing T $\longleftarrow$ node s of the spine S

S spine on G $\iff$ for each node s of S with children $\mathrm{s}_1 \dots \mathrm{s}_k$, the tubes $\mathrm{t(s_1)} \dots \mathrm{t(s}_k)$ lie in distinct connected components of $\mathrm{G}\big[\mathrm{t(s)} \smallsetminus \mathrm{s}\big]$

SPINES

spine of a tubing T = Hasse diagram of the inclusion poset of T



tube t of the tubing T $\longmapsto$ node $\mathrm{s}(\mathrm{t})$ of the spine S labeled by $\mathrm{t} \smallsetminus \bigcup \{\mathrm{t}' \mid \mathrm{t}' \in \mathrm{T}, \mathrm{t}' \subsetneq \mathrm{t}\}$

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NESTED FANS AND GRAPH ASSOCIAHEDRA

THM. The collection of cones $\Big\{ \{\mathbf{x} \in \mathbb{H} \mid x_i < x_j \text{ for all } i \to j \text{ in } \mathrm{T}\} \;\Big|\; \mathrm{T} \text{ tubing on } \mathrm{G} \Big\}$ forms a complete simplicial fan, called the nested fan of G. This fan is always polytopal.

Carr-Devadoss, *Coxeter complexes and graph associahedra* ('06)
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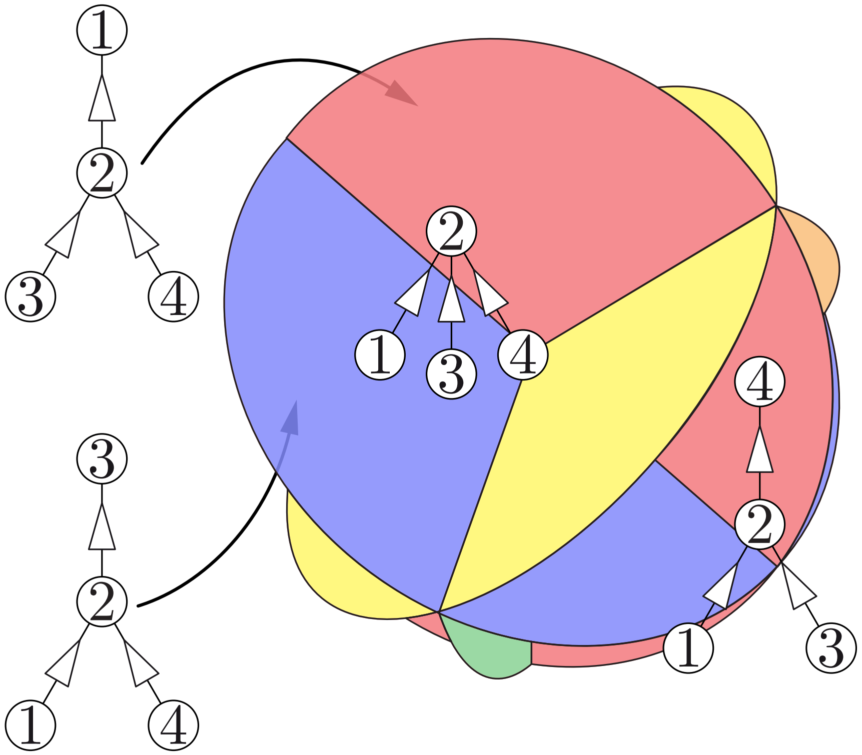
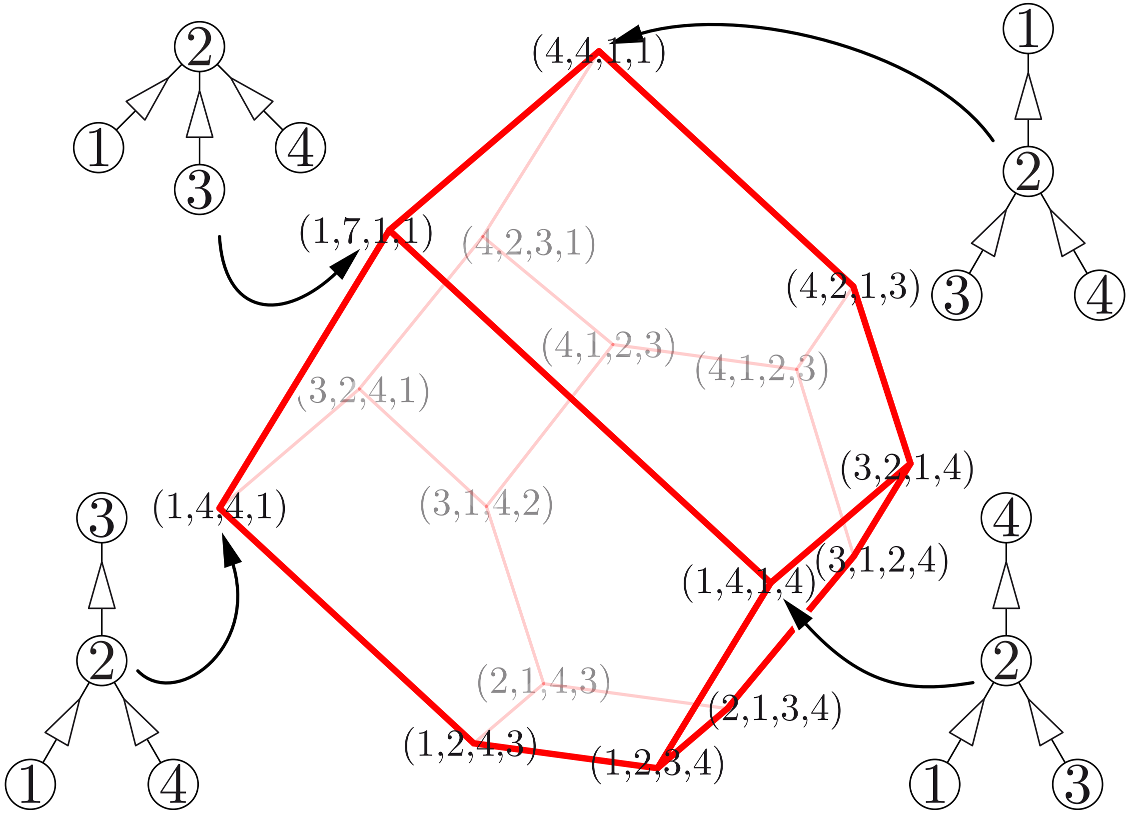
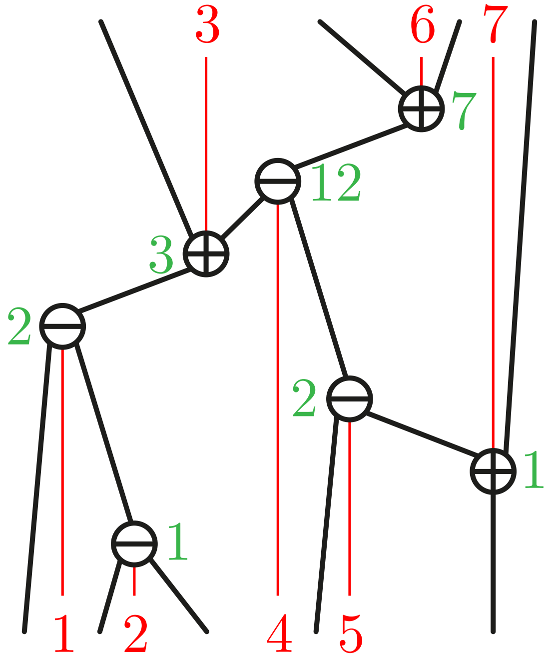




HOHLWEG-LANGE'S ASSOCIAHEDRA

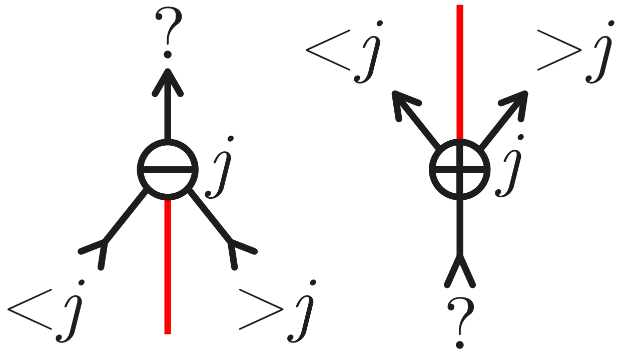


for an arbitrary signature $\varepsilon \in \pm^{n+1}$,

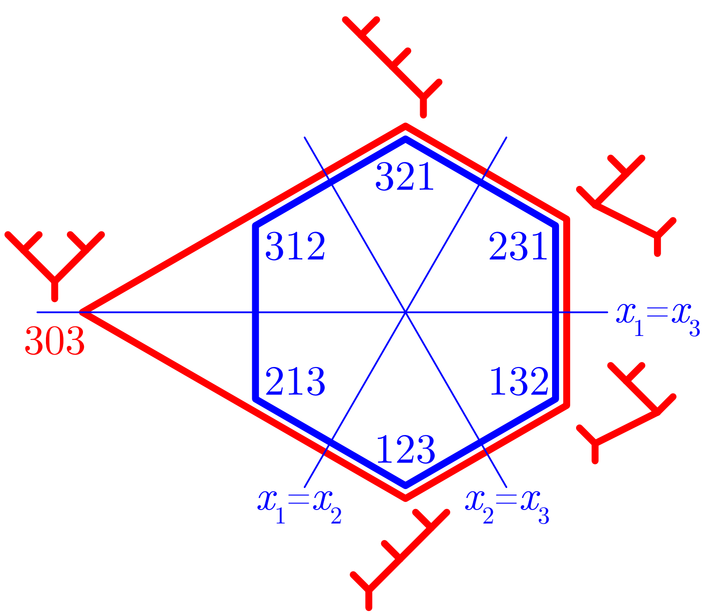
$$\mathsf{Asso}(\varepsilon) \;\coloneqq\; \operatorname{conv}\{\mathbf{HL}(\mathrm{T}) \mid \mathrm{T}\ \varepsilon\text{-Cambrian tree}\}$$

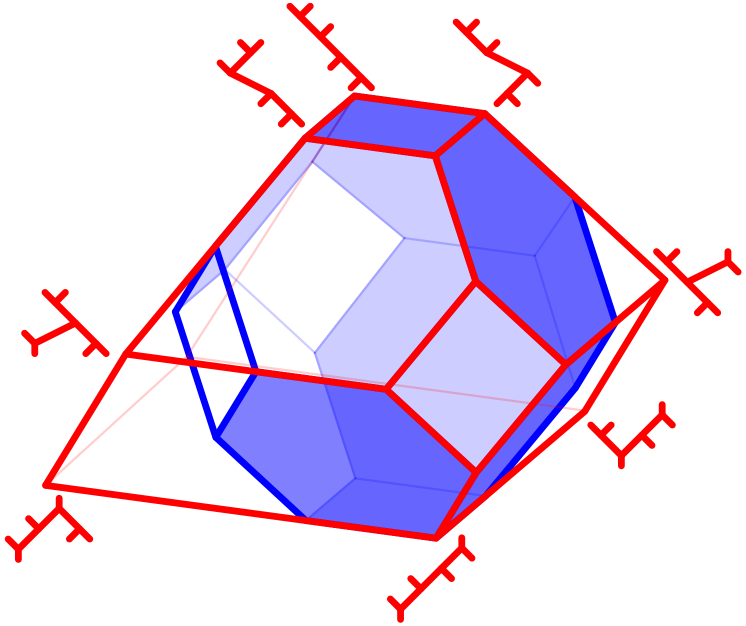
$$\text{with } \mathbf{HL}(\mathrm{T})_j \coloneqq \begin{cases} \ell(\mathrm{T}, j) \cdot r(\mathrm{T}, j) & \text{if } \varepsilon(j) = - \\ n + 2 - \ell(\mathrm{T}, j) \cdot r(\mathrm{T}, j) & \text{if } \varepsilon(j) = + \end{cases}$$

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)
Lange-P., *Using spines to revisit a construction of the associahedron* ('15)



Rule for Cambrian trees





- Asso(n) obtained by deleting inequalities in the facet description of the permutahedron
- normal cone of $\mathbf{HL}(\mathrm{T})$ in $\mathsf{Asso}(\varepsilon) = \{\mathbf{x} \in \mathbb{H} \mid x_i < x_j$ for all $i \to j$ in $\mathrm{T}\}$

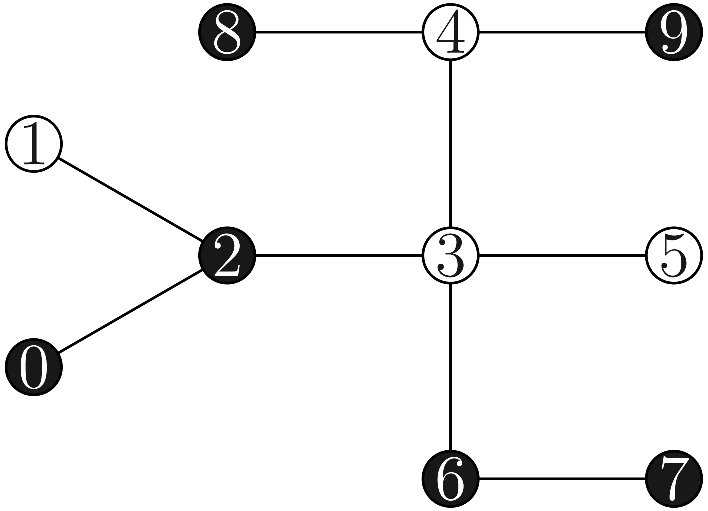
SIGNED SPINES ON SIGNED TREES

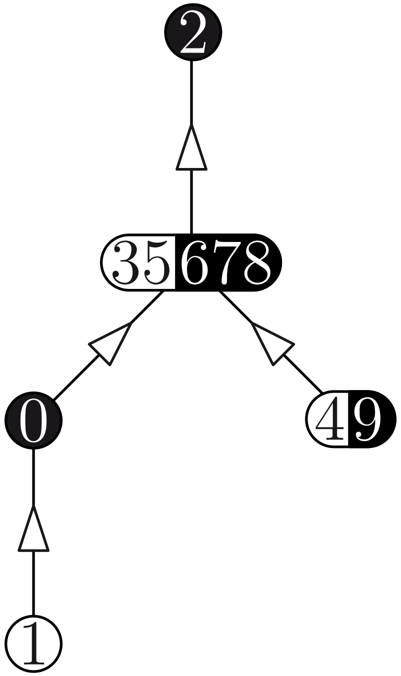
T tree on the signed ground set $\mathrm{V} = \mathrm{V}^- \sqcup \mathrm{V}^+$ (negative in white, positive in black)

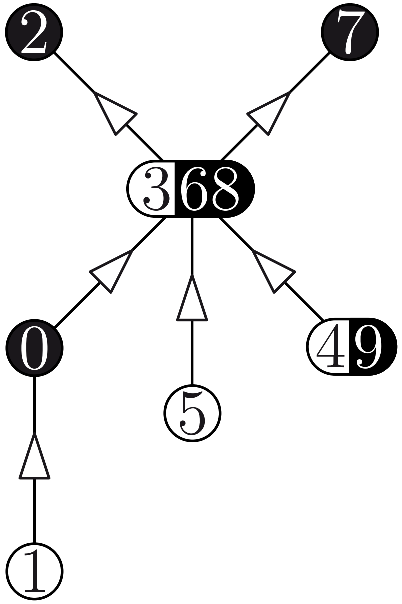
Signed spine on T = directed and labeled tree S st

(i) the labels of the nodes of S form a partition of the signed ground set V

(ii) at a node of S labeled by $U = U^- \sqcup U^+$, the source label sets of the different incoming arcs are subsets of distinct connected components of $\mathrm{T} \smallsetminus U^-$, while the sink label sets of the different outgoing arcs are subsets of distinct connected components of $\mathrm{T} \smallsetminus U^+$

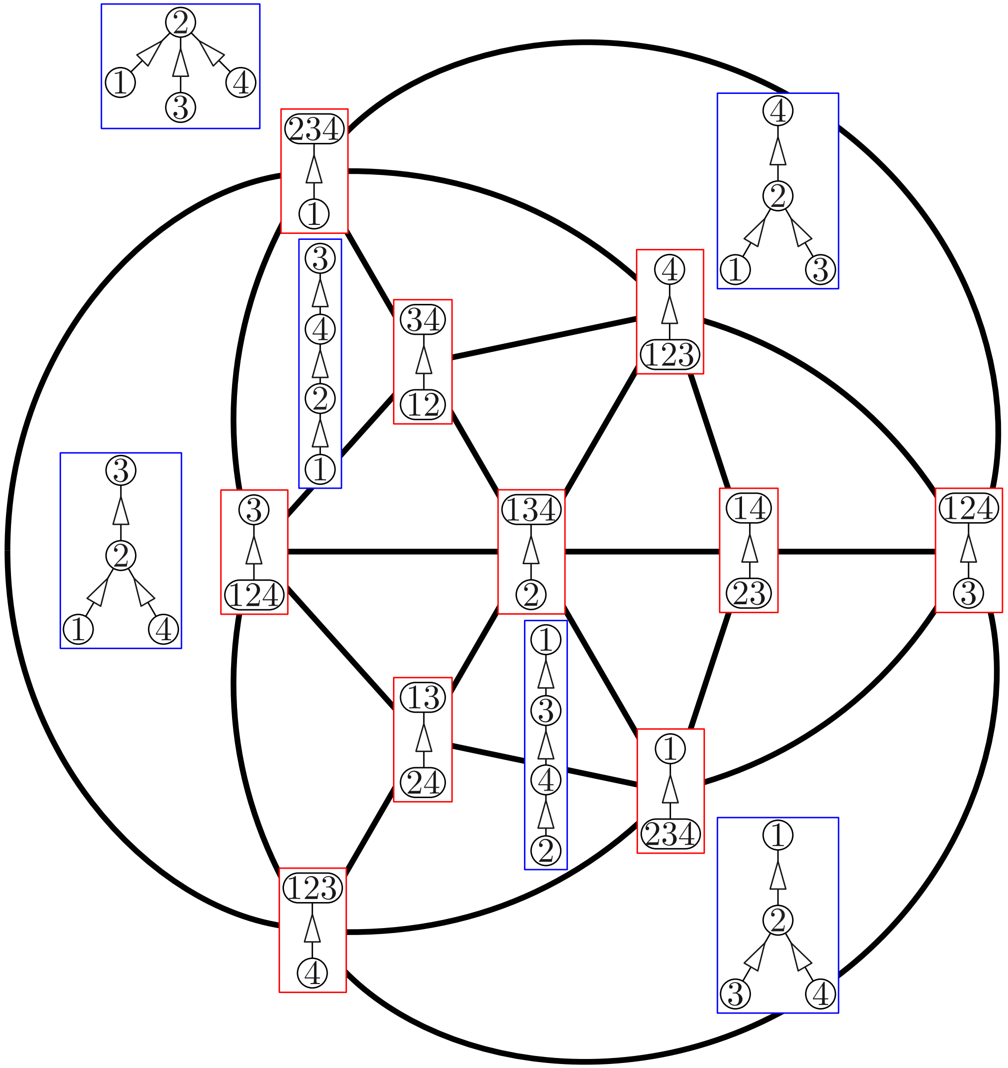


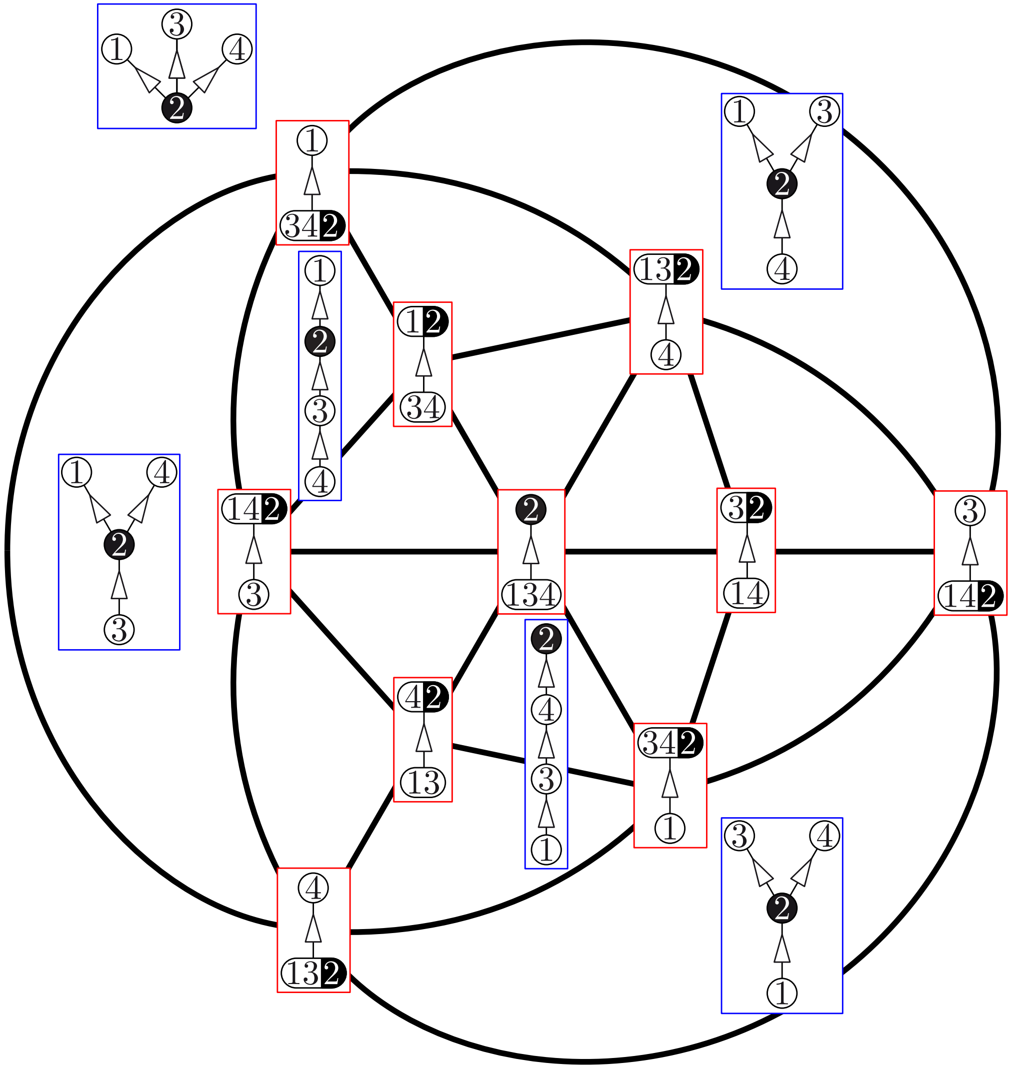




SPINE COMPLEX

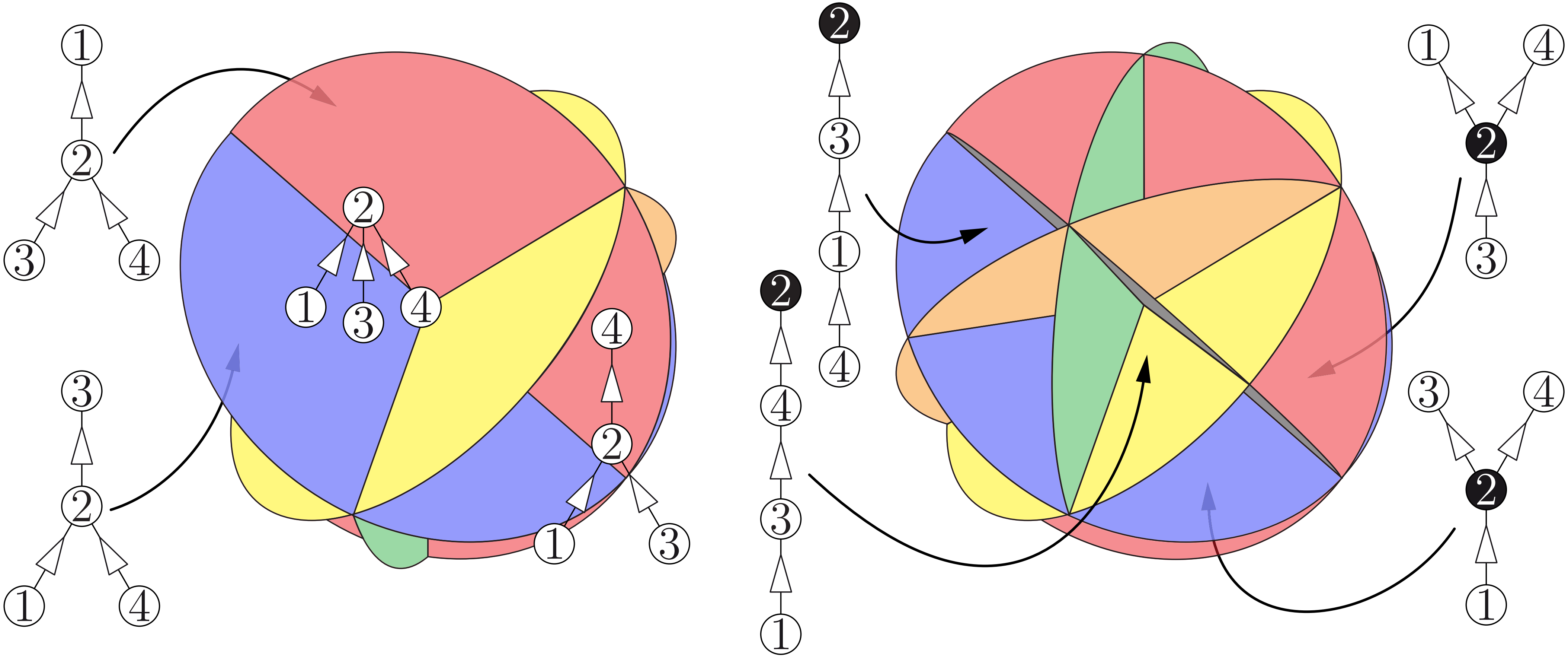
Signed spine complex $\mathcal{S}(\mathrm{T})$ = simplicial complex whose inclusion poset is isomorphic to the poset of edge contractions on the signed spines of T





SPINE FAN

For S spine on T, define $\mathrm{C}(\mathrm{S}) \coloneqq \{\mathbf{x} \in \mathbb{H} \mid x_u \leq x_v, \text{ for all arcs } u \to v \text{ in } \mathrm{S}\}$



THEO. The collection of cones $\mathcal{F}(\mathrm{T}) \coloneqq \{\mathrm{C}(\mathrm{S}) \mid \mathrm{S} \in \mathcal{S}(\mathrm{T})\}$ defines a complete simplicial fan on $\mathbb{H}$, which we call the spine fan

P., *Signed tree associahedra* ('13$^+$)

SIGNED TREE ASSOCIAHEDRON

Signed tree associahedron Asso(T) = convex polytope with

(i) a vertex $\mathbf{a}(\mathrm{S}) \in \mathbb{R}^{\mathrm{V}}$ for each maximal signed spine $\mathrm{S} \in \mathcal{S}(\mathrm{T})$, with coordinates

$$\mathbf{a}(\mathrm{S})_v = \begin{cases} \big|\, \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \,\big| & \text{if } v \in \mathrm{V}^- \\ |\mathrm{V}| + 1 - \big|\, \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \,\big| & \text{if } v \in \mathrm{V}^+ \end{cases}$$

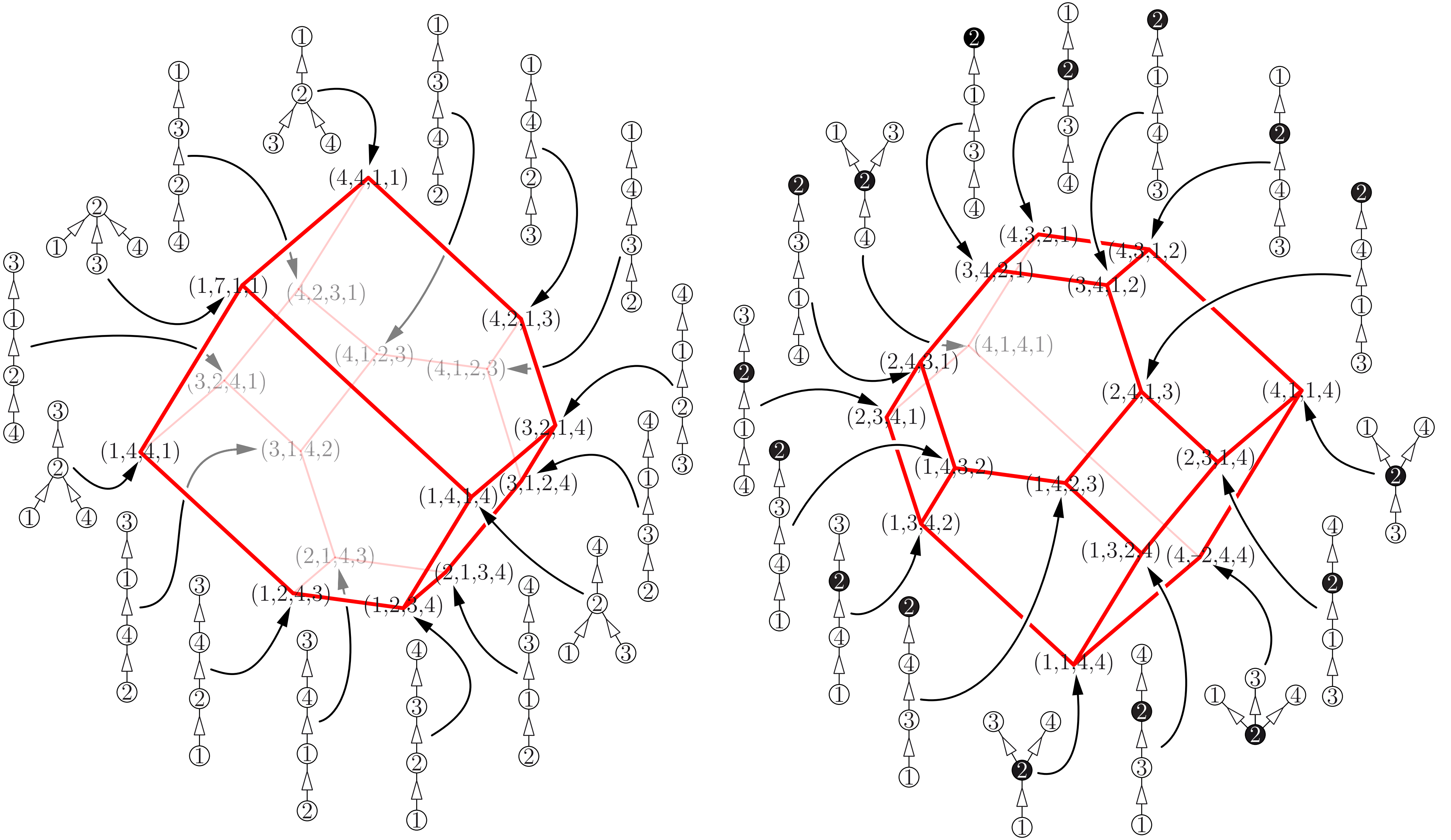
where r_v = unique incoming (resp. outgoing) arc when $v \in \mathrm{V}^-$ (resp. when $v \in \mathrm{V}^+$)
$\Pi(\mathrm{S})$ = set of all (undirected) paths in S, including the trivial paths

(ii) a facet defined by the half-space

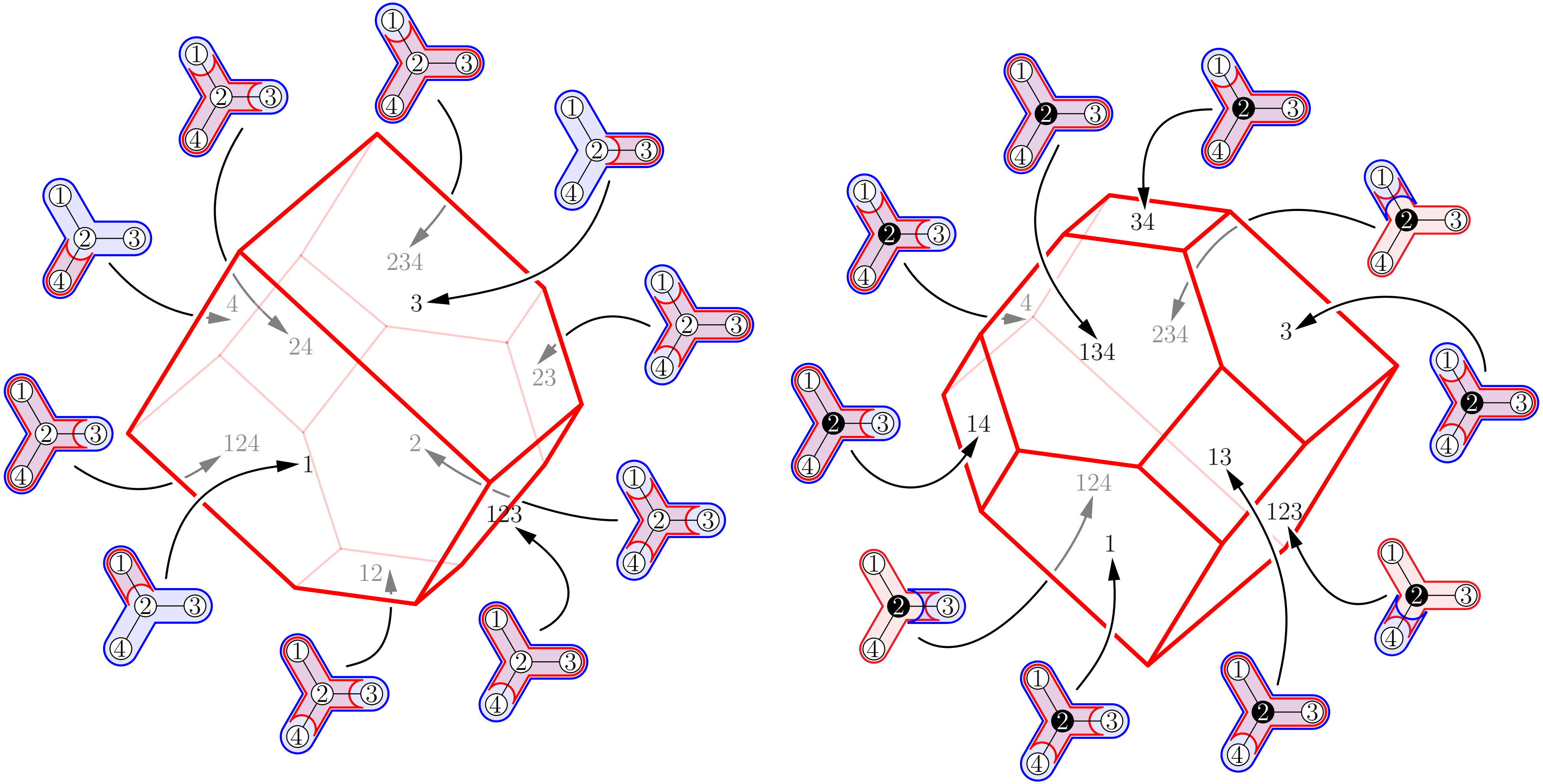
$$\mathbf{H}^{\geq}(B) := \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{V}} \;\middle|\; \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}$$

for each signed building block $B \in \mathcal{B}(\mathrm{T})$

EXM: VERTEX DESCRIPTION



EXM: FACET DESCRIPTION



MAIN RESULT

THM. The spine fan $\mathcal{F}(\mathrm{T})$ is the normal fan of the signed tree associahedron $\mathrm{Asso}(\mathrm{T})$, defined equivalently as

(i) the convex hull of the points

$$\mathbf{a}(\mathrm{S})_v = \begin{cases} \big| \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathrm{V}^- \\ |\mathrm{V}| + 1 - \big| \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathrm{V}^+ \end{cases}$$

for all maximal signed spines $\mathrm{S} \in \mathcal{S}(\mathrm{T})$

(ii) the intersection of the hyperplane $\mathbb{H}$ with the half-spaces

$$\mathbf{H}^{\geq}(B) := \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{V}} \;\middle|\; \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}$$

for all signed building blocks $B \in \mathcal{B}(\mathrm{T})$

P., *Signed tree associahedra* ('13$^+$)

CORO. The signed tree associahedron $\mathrm{Asso}(\mathrm{T})$ realizes the signed nested complex $\mathcal{N}(\mathrm{T})$

SKETCH OF THE PROOF

STEP 1. We have

$$\sum_{v \in \mathrm{V}} \mathbf{a}(\mathrm{S})_v = \binom{|\mathrm{V}|+1}{2} \qquad \text{and} \qquad \sum_{v \in \mathrm{sc}(r)} \mathbf{a}(\mathrm{S})_v = \binom{|\mathrm{sc}(r)|+1}{2}$$

for any arc r of S. In other words, “each vertex $\mathbf{a}(\mathrm{S})$ belongs to the hyperplanes $\mathbf{H}^{=}(B)$ it is supposed to”. Proof by double counting

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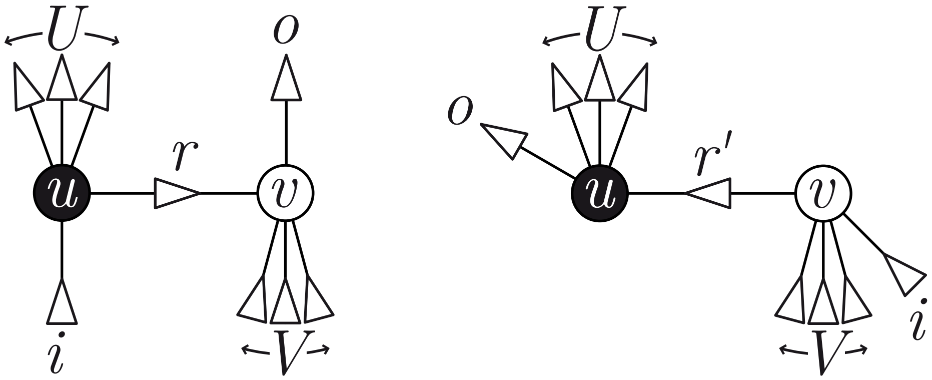
STEP 2. If S and S' are two adjacent maximal spines on T, such that S' is obtained from S by flipping an arc joining node u to node v, then

$$\mathbf{a}(\mathrm{S}') - \mathbf{a}(\mathrm{S}) \in \mathbb{R}_{>0} \cdot (e_u - e_v)$$



$$\mathbf{a}(\mathrm{S}') - \mathbf{a}(\mathrm{S}) = (|U|+1) \cdot (|V|+1) \cdot (e_u - e_v)$$

SKETCH OF THE PROOF

STEP 1. We have

$$\sum_{v\in \mathrm{V}} \mathbf{a}(\mathrm{S})_v = \binom{|\mathrm{V}|+1}{2} \qquad \text{and} \qquad \sum_{v\in \mathrm{sc}(r)} \mathbf{a}(\mathrm{S})_v = \binom{|\mathrm{sc}(r)|+1}{2}$$

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STEP 3. A general theorem concerning realizations of simplicial fan by polytopes

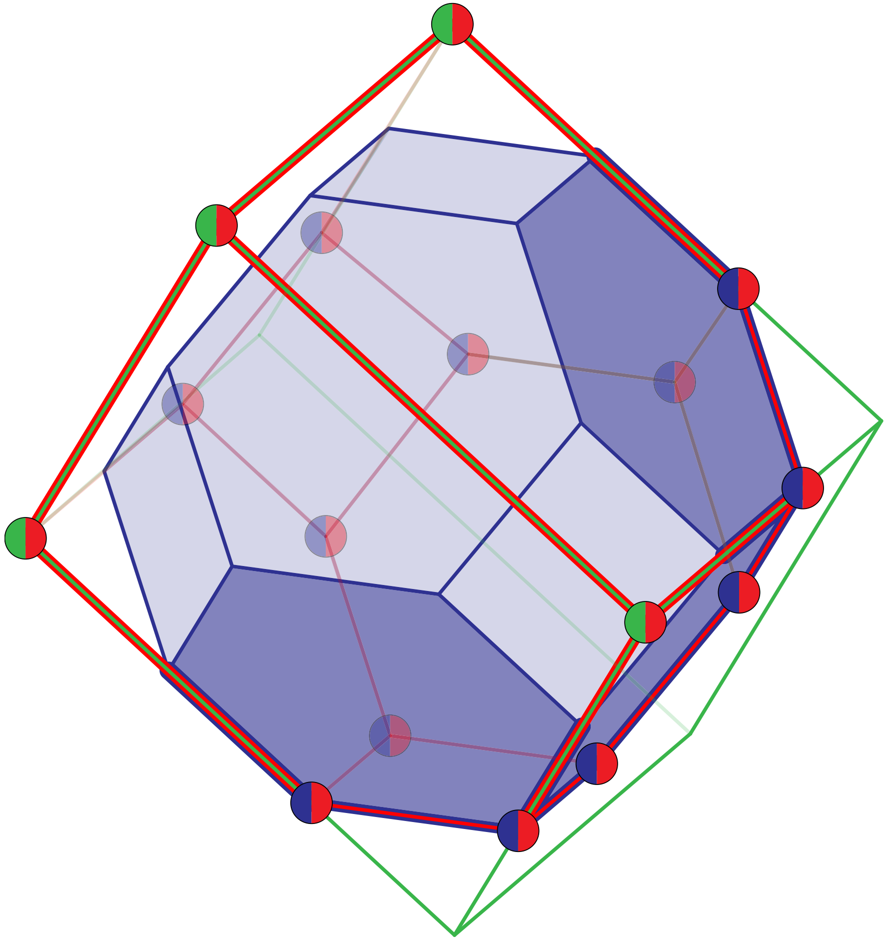
In other words, a characterization of when is a simplicial fan regular

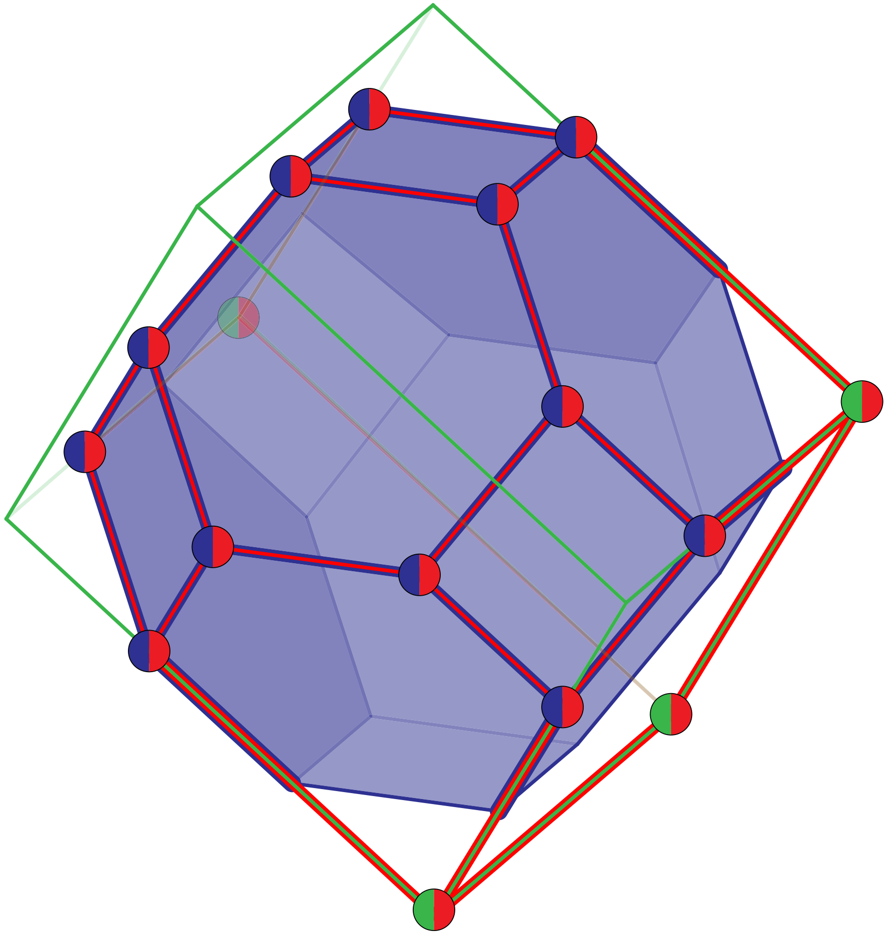
Hohlweg-Lange-Thomas, *Permutahedra and generalized associahedra* ('11)
De Loera-Rambau-Santos, *Triangulations: Structures for Algorithms and Applications* ('10)

FURTHER GEOMETRIC PROPERTIES

PROP. The signed tree associahedron $\mathrm{Asso}(\mathrm{T})$ is sandwiched between the permutahedron $\mathrm{Perm}(\mathrm{V})$ and the parallelepiped $\mathrm{Para}(\mathrm{T})$

$$\sum_{u \neq v \in \mathrm{V}} [e_u, e_v] = \mathrm{Perm}(\mathrm{T}) \quad \subset \quad \mathrm{Asso}(\mathrm{T}) \quad \subset \quad \mathrm{Para}(\mathrm{T}) = \sum_{u - v \in \mathrm{T}} \pi(u - v) \cdot [e_u, e_v]$$





FURTHER GEOMETRIC PROPERTIES

PROP. The signed tree associahedron Asso(T) is sandwiched between the permutahedron Perm(V) and the parallelepiped Para(T)

$$\sum_{u \neq v \in \mathrm{V}} [e_u, e_v] = \text{Perm}(\mathrm{T}) \quad \subset \quad \text{Asso}(\mathrm{T}) \quad \subset \quad \text{Para}(\mathrm{T}) = \sum_{u - v \in \mathrm{T}} \pi(u - v) \cdot [e_u, e_v]$$

Common vertices of Asso(T) and Para(T) $\equiv$ orientations of T which are spines on T

Common vertices of Asso(T) and Perm(T) $\equiv$ linear orders on V which are spines on T

$\Rightarrow$ no common vertex of the three polytopes except if T is a signed path

FURTHER GEOMETRIC PROPERTIES

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Common vertices of Asso(T) and Para(T) $\equiv$ orientations of T which are spines on T
Common vertices of Asso(T) and Perm(T) $\equiv$ linear orders on V which are spines on T
$\Rightarrow$ no common vertex of the three polytopes except if T is a signed path

PROP. Asso(T) and Asso(T') isometric $\iff$ T and T' isomorphic or anti-isomorphic, up to the sign of their leaves, ie. $\exists$ bijection $\theta : \mathrm{V} \to \mathrm{V}'$ st. $\forall u, v \in \mathrm{V}$

- $u - v$ edge in T $\iff$ $\theta(u) - \theta(v)$ edge in T'
- if u is not a leaf of T, the signs of u and $\theta(u)$ coincide (resp. are opposite)

GRAPH PROPERTIES OF GRAPH ASSOCIAHEDRA

Manneville-P., *Graph properties of graph associahedra* (SLC'15)

GRAPH PROPERTIES OF GRAPH ASSOCIAHEDRA

THM. The diameter $\delta(\text{Asso}(\mathrm{G}))$ of the skeleton of the graph associahedron

- is non-decreasing: $\mathrm{G} \subseteq \mathrm{G}' \Longrightarrow \delta(\text{Asso}(\mathrm{G})) \leq \delta(\text{Asso}(\mathrm{G}'))$
- satisfies the non-leaving-face property (all geodesics between two vertices of a face F of Asso(G) stay in F)
- is bounded by $\max(e, 2n-18) \leq \delta(\text{Asso}(\mathrm{G})) \leq \binom{n+1}{2}$

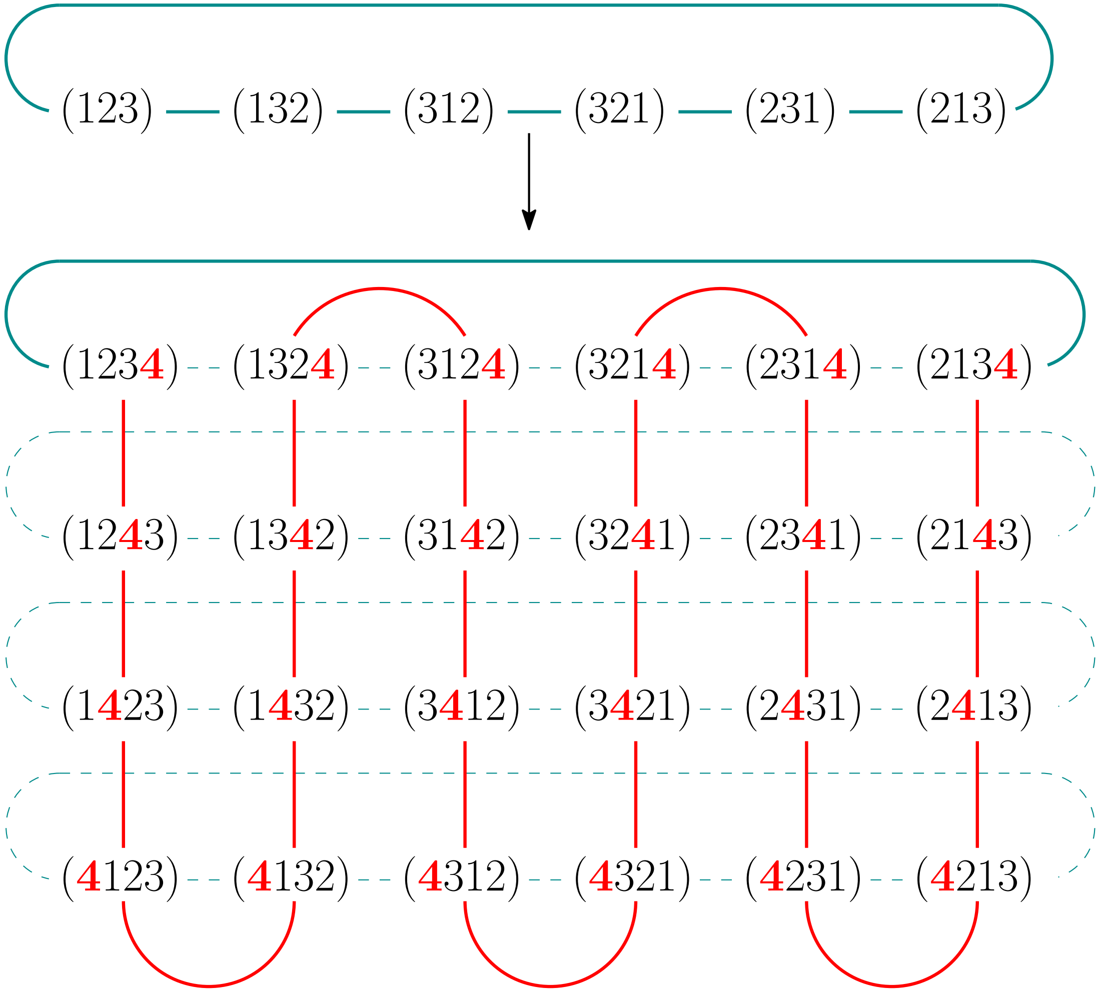
Sleator-Tarjan-Thurston, *Rotation distance, triangulations, and hyperbolic geometry* ('88)

Pournin, *The diameter of associahedra* ('14)

Manneville-P., *Graph properties of graph associahedra* (SLC'15)

GRAPH PROPERTIES OF GRAPH ASSOCIAHEDRA

1. the graph of the permutahedron is Hamiltonian



Trotter, *Algorithm 115: Perm. Commun. ACM* ('62)
Johnson, *Generation of permutations by adjacent transposition* ('63)
Steinhaus, *One hundred problems in elementary mathematics* ('64)

GRAPH PROPERTIES OF GRAPH ASSOCIAHEDRA

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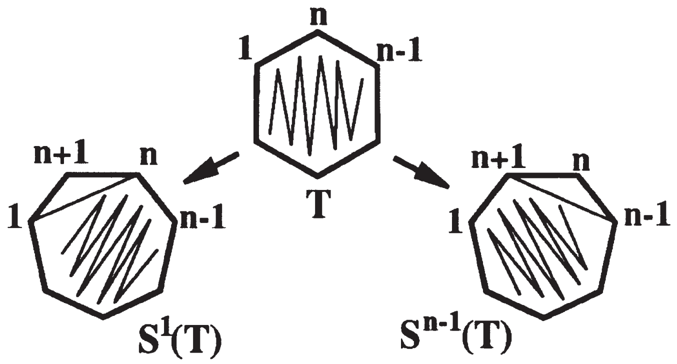
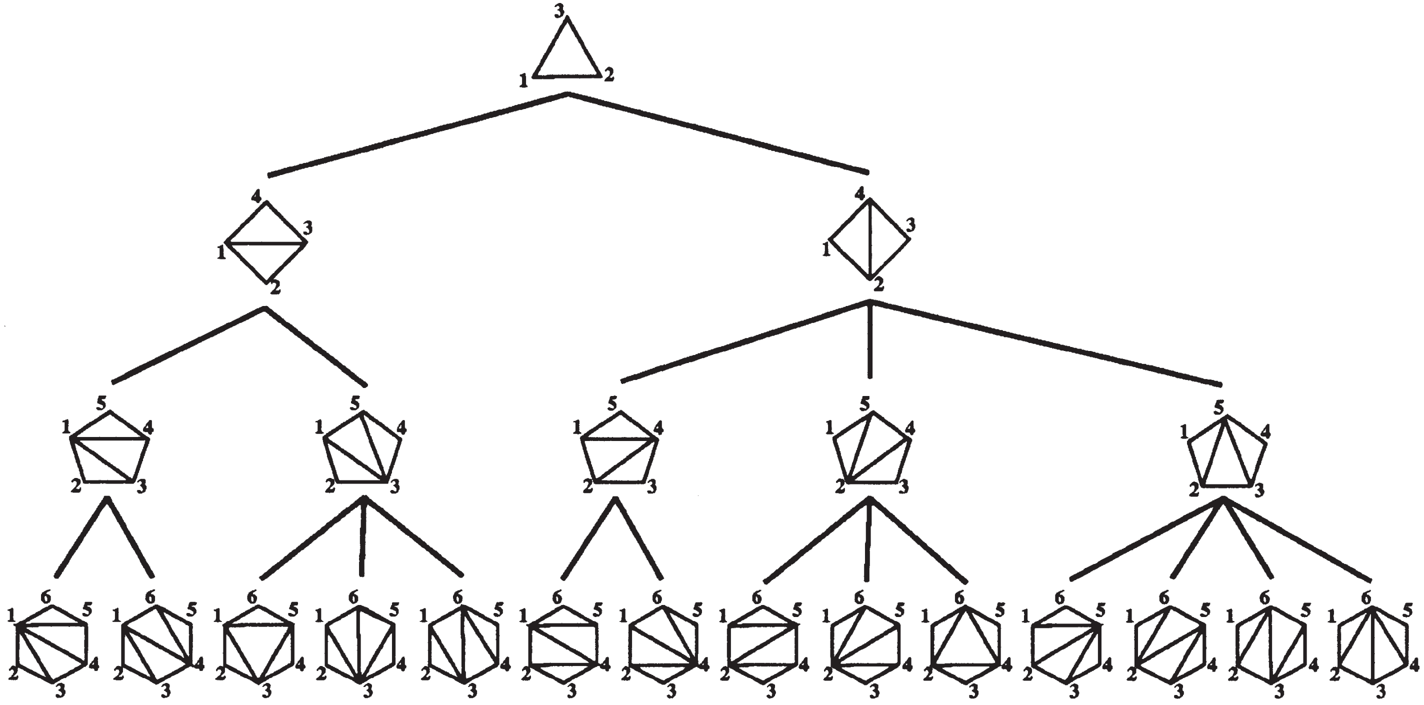
Johnson, *Generation of permutations by adjacent transposition* ('63)
Steinhaus, *One hundred problems in elementary mathematics* ('64)

2. the graph of the associahedron is Hamiltonian

Lucas, *The rotation graph of binary trees is Hamiltonian* ('87)
Hurtado-Noy, *Graph of triangulations of a convex polygon and tree of triangulations* ('99)

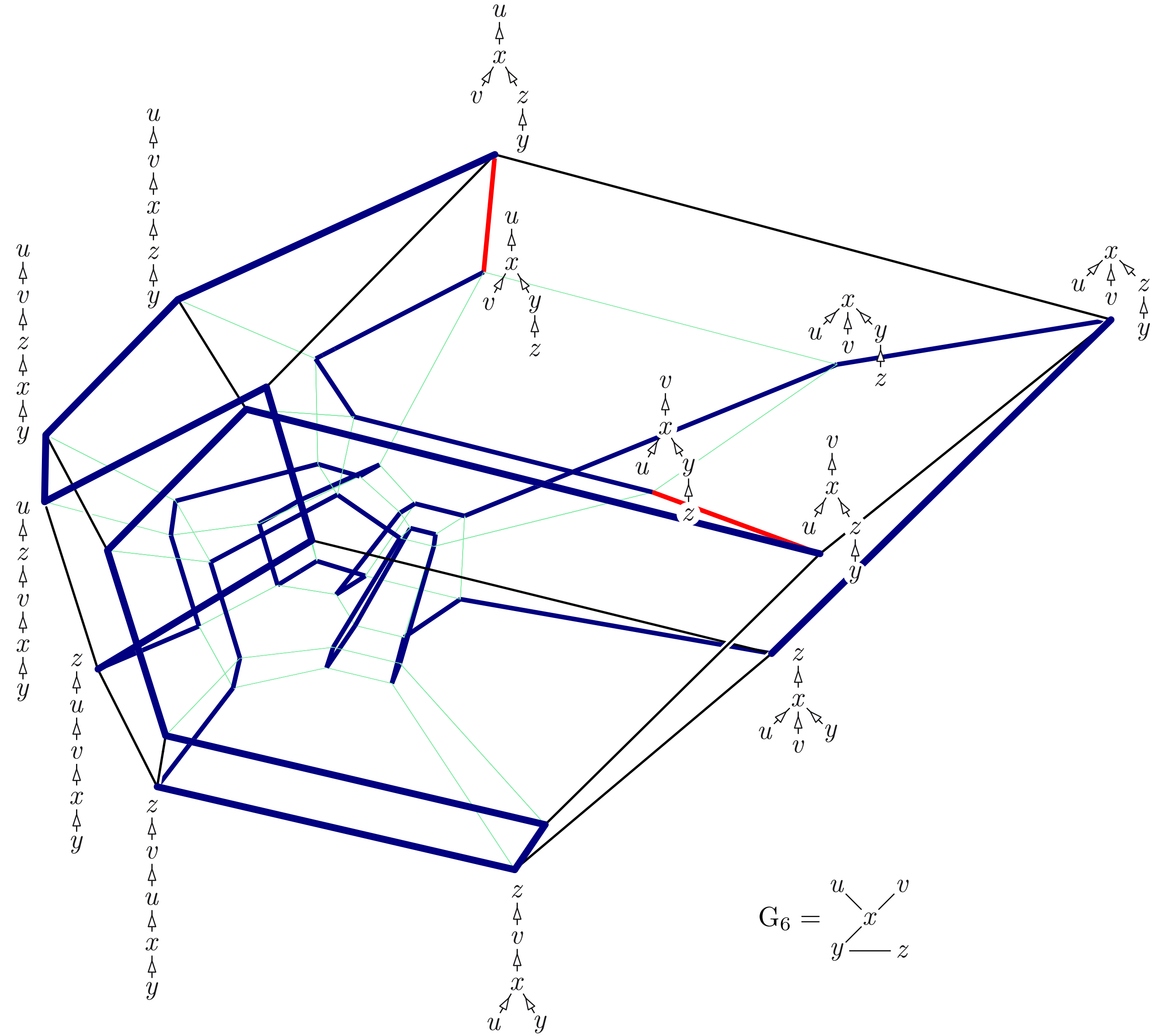




CONSTRUCTING AN HAMILTONIAN CIRCUIT...

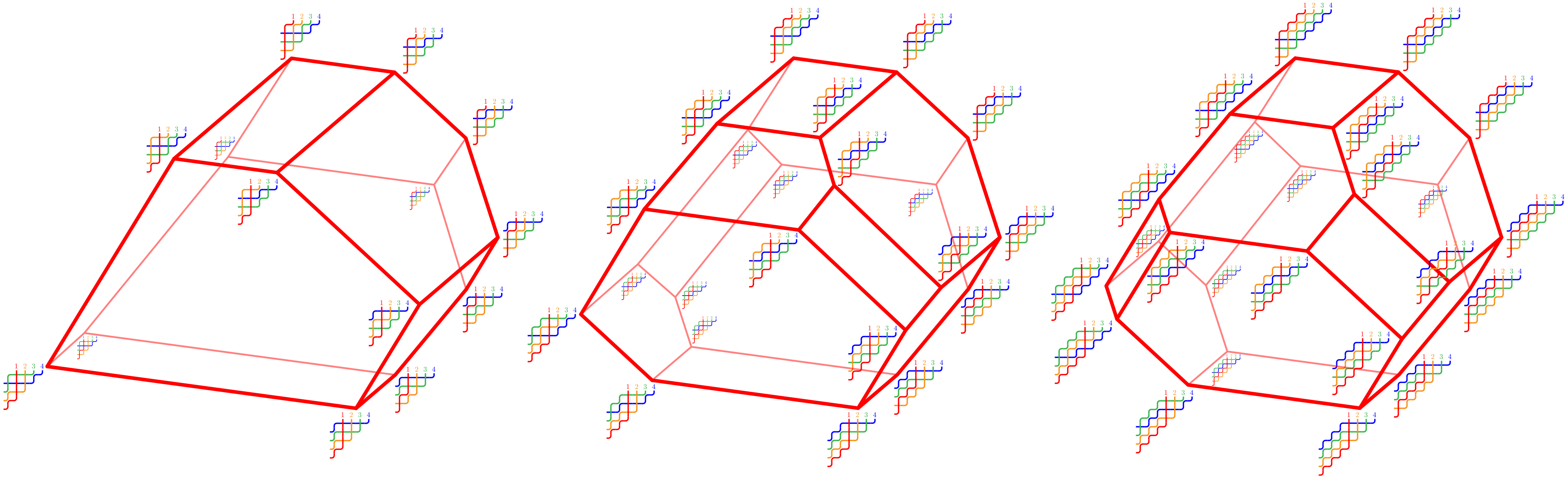
THM. When G has at least 2 edges, the skeleton of Asso(G) is Hamiltonian.

Manneville-P., *Graph properties of graph associahedra* (SLC'15)



$\mathrm{G}_6 =$

III. BRICK POLYTOPES



THREE GEOMETRIC FAMILIES

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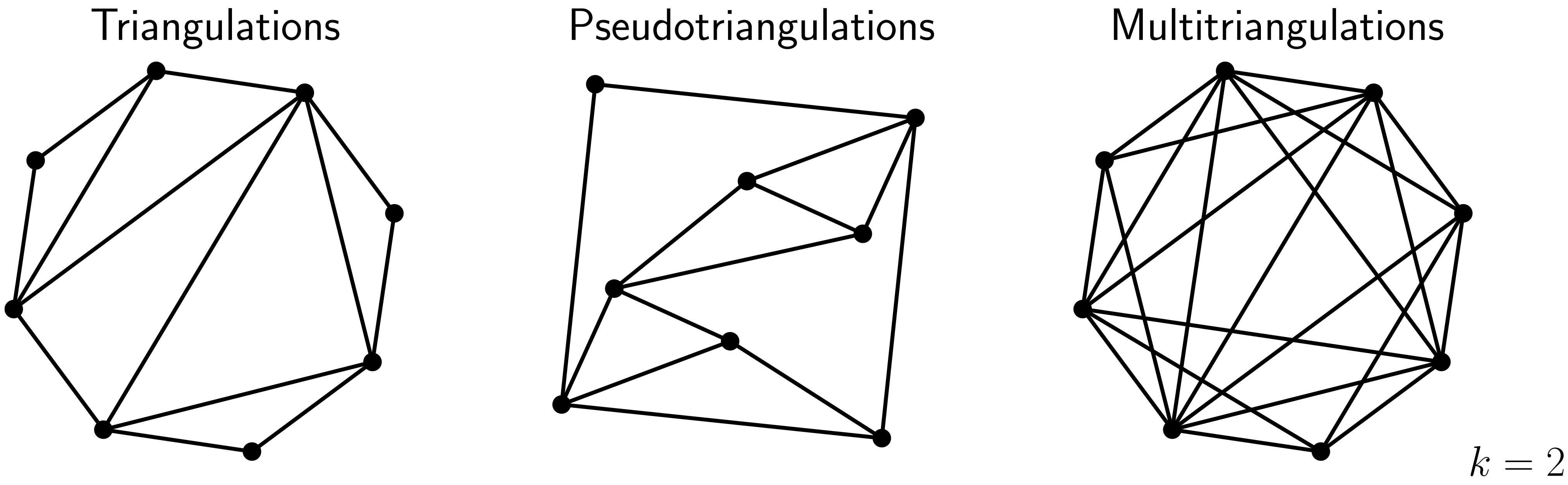


triangulation $=$ maximal crossing-free set of edges

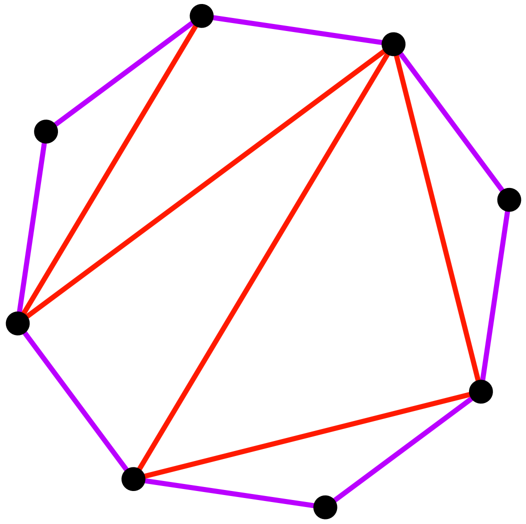
pseudotriangulation $=$ maximal crossing-free pointed set of edges

k-triangulation $=$ maximal $(k+1)$-crossing-free set of edges

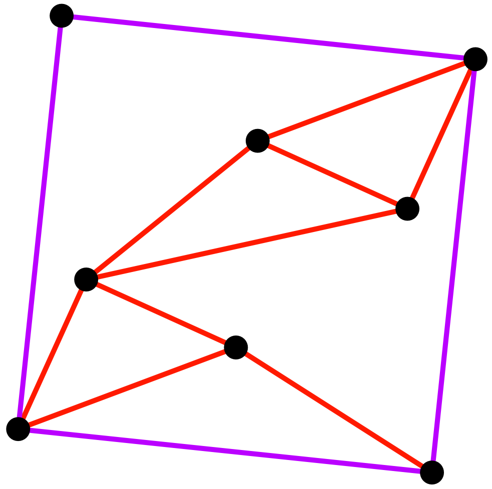
Pocchiola-Vegter, *Topologically sweeping visibility complexes via pseudotriangulations* ('96)
Capoyleas-Pach, *A Turán-type theorem on chords of a convex polygon* ('92)

RELEVANT EDGES

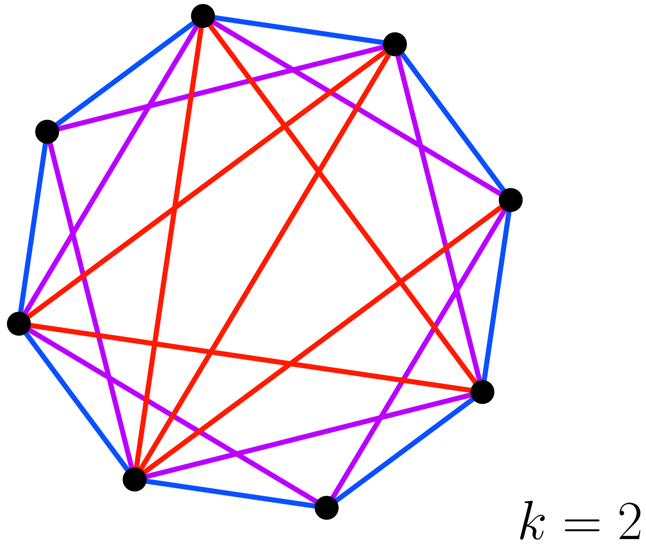
Triangulations



Pseudotriangulations



Multitriangulations



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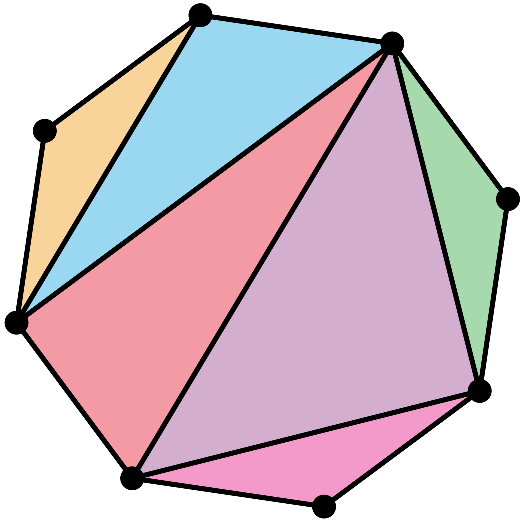
TRIANGLES – PSEUDOTRIANGLES – STARS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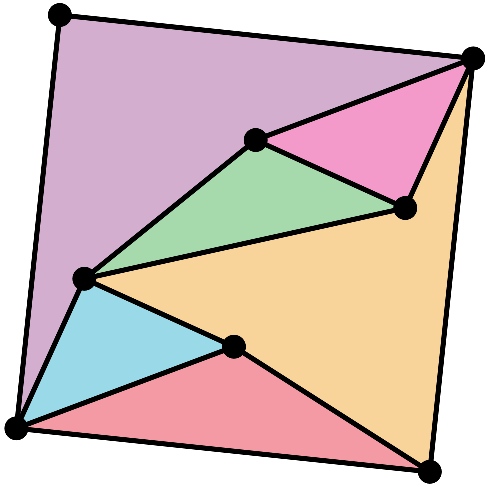
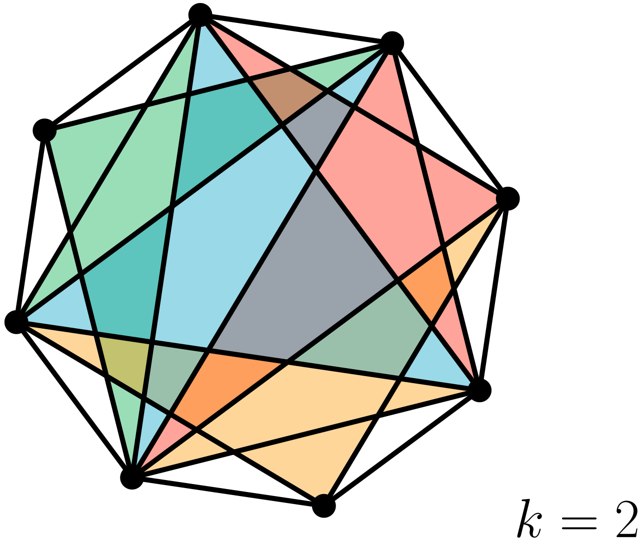

Pseudotriangulations


Multitriangulations

triangulation = maximal crossing-free set of edges
= decomposition into triangles

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k-triangulation = maximal $(k+1)$-crossing-free set of edges
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Pocchiola-Vegter, *Topologically sweeping visibility complexes via pseudotriangulations* ('96)
P.-Santos, *Multitriangulations as complexes of star polygons* ('09)

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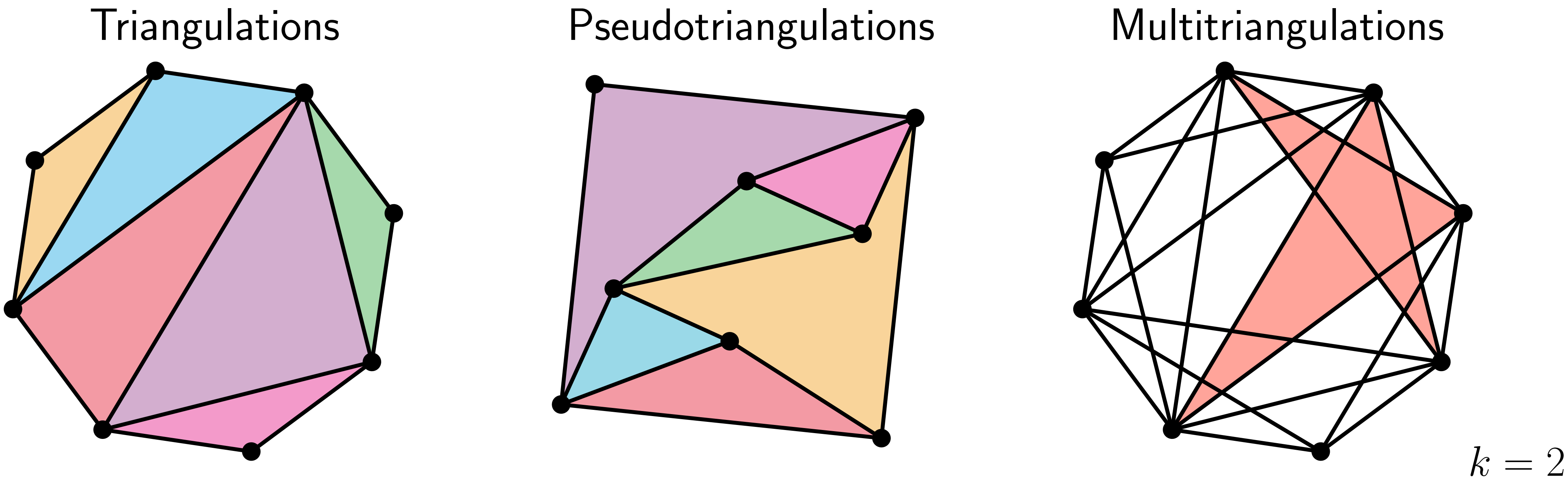

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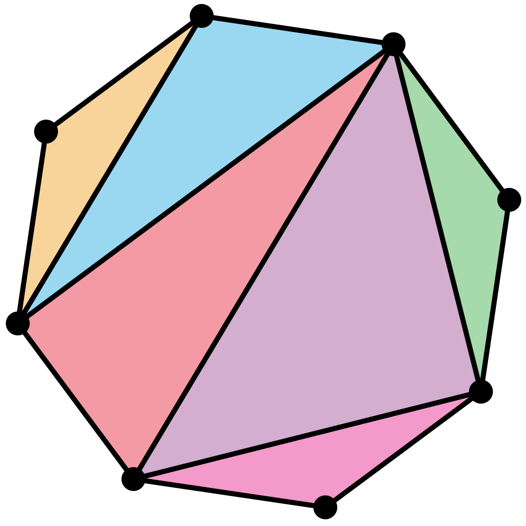
TRIANGLES – PSEUDOTRIANGLES – STARS

Triangulations



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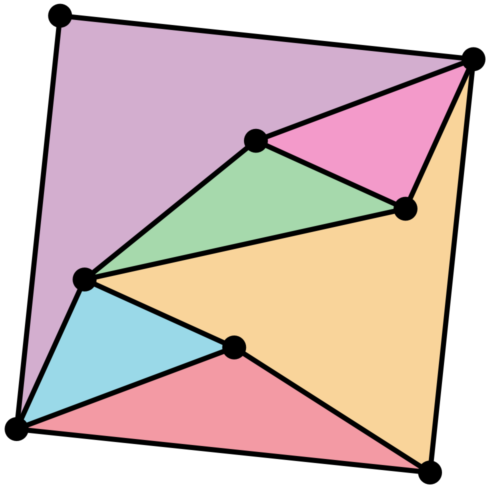
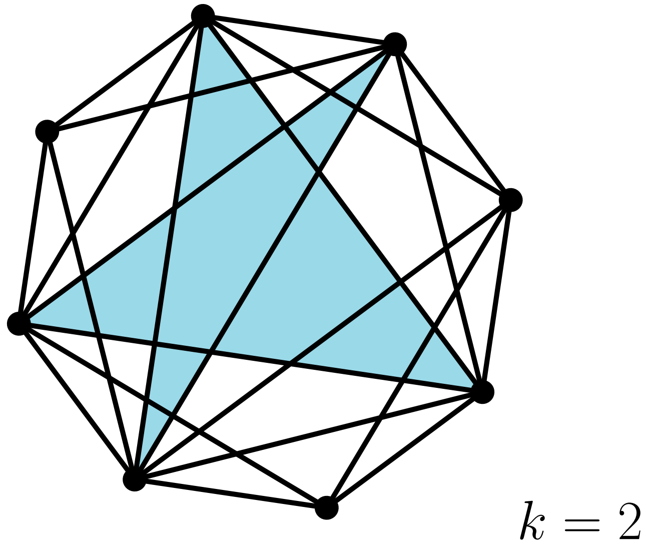


Multitriangulations



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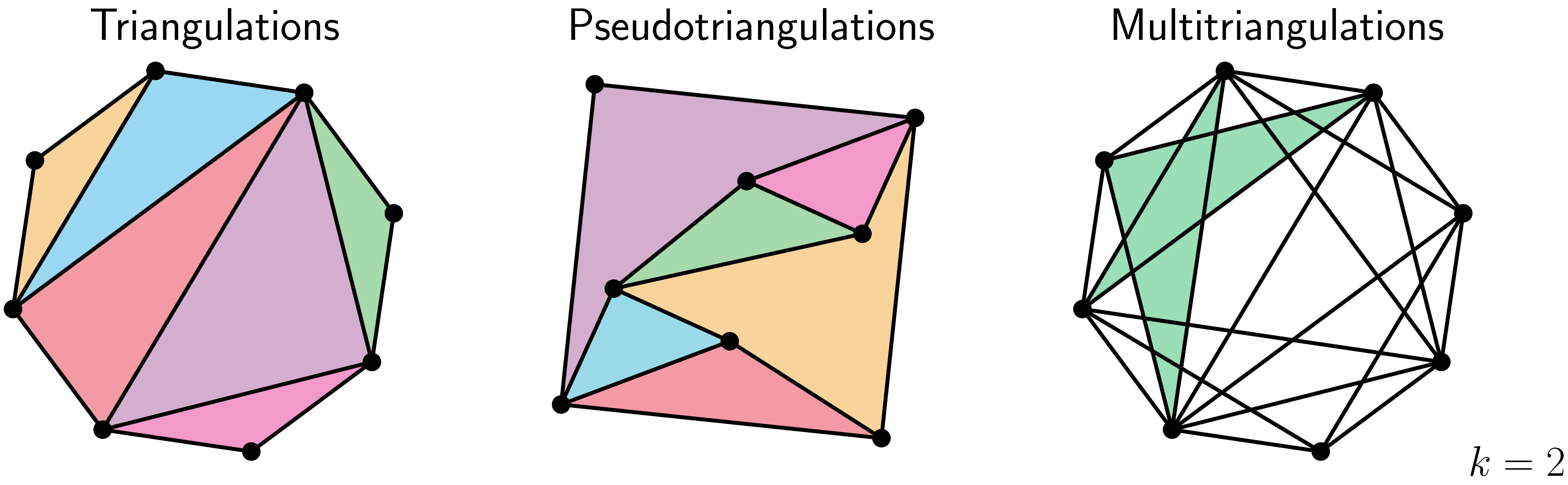

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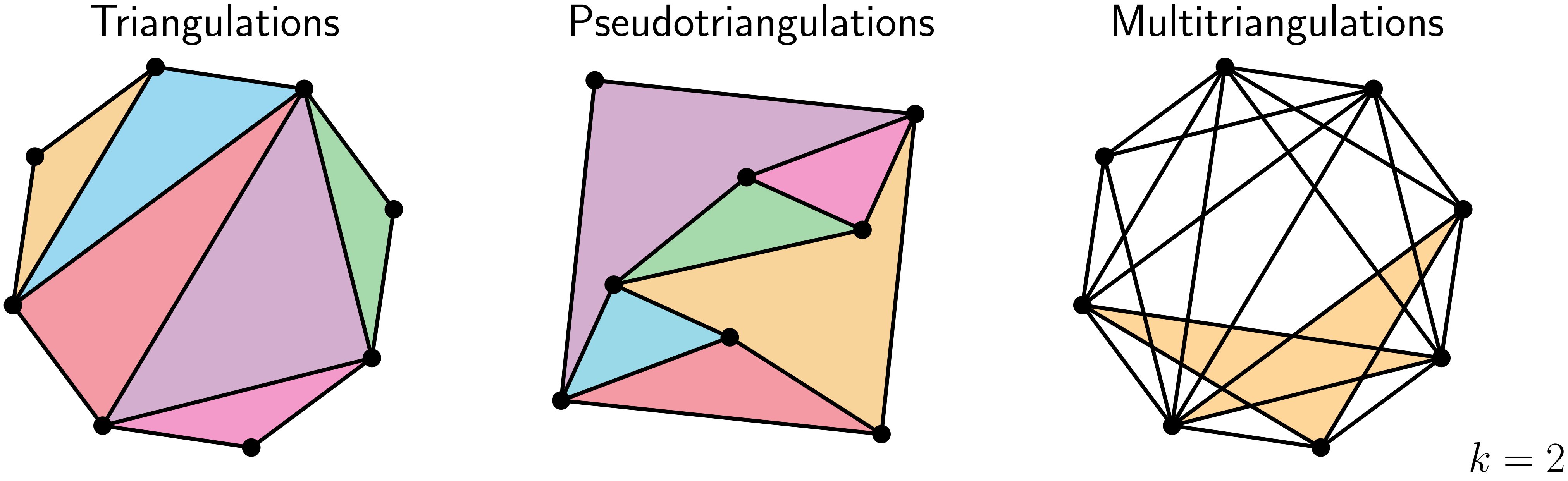

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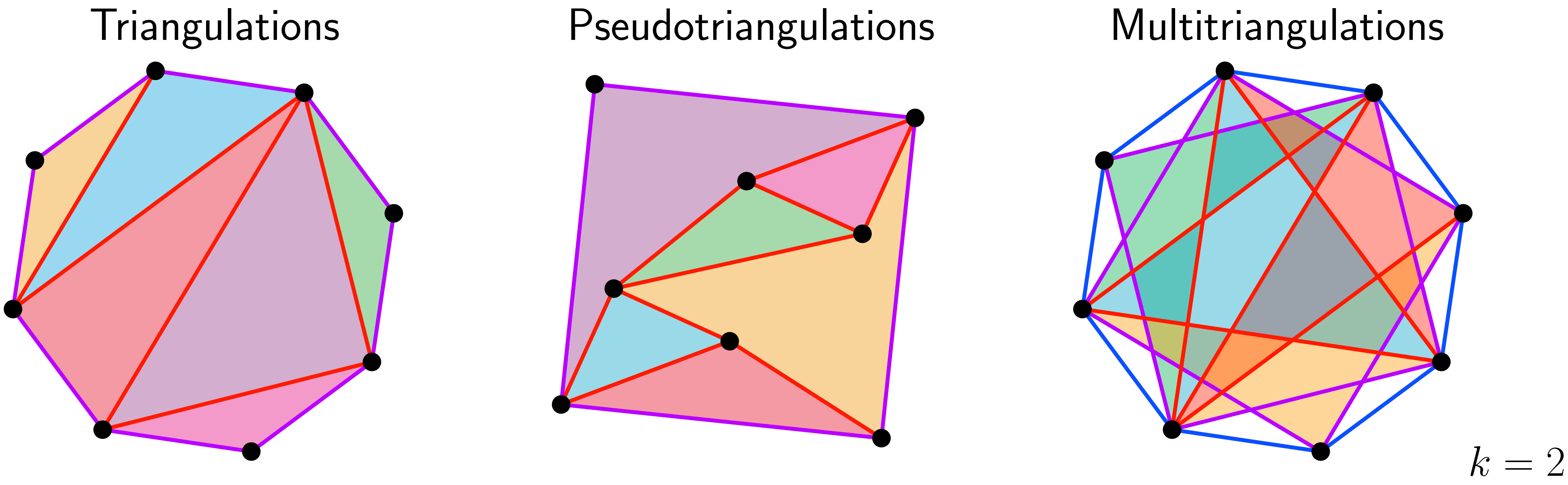
Pocchiola-Vegter, *Topologically sweeping visibility complexes via pseudotriangulations* ('96)
P.-Santos, *Multitriangulations as complexes of star polygons* ('09)

TRIANGLES – PSEUDOTRIANGLES – STARS



triangulation $=$ maximal crossing-free set of edges
$=$ decomposition into triangles

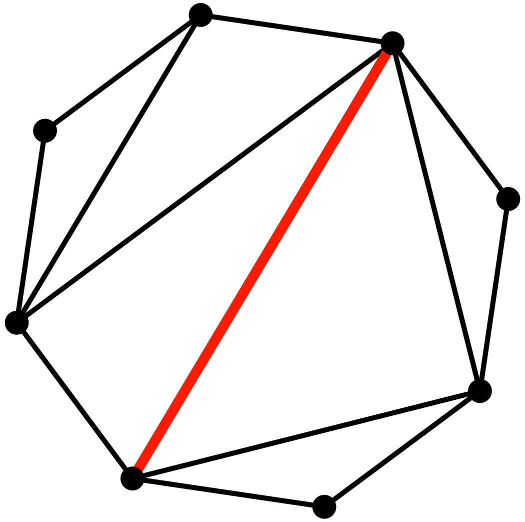
pseudotriangulation $=$ maximal crossing-free pointed set of edges
$=$ decomposition into pseudotriangles

k-triangulation $=$ maximal $(k+1)$-crossing-free set of edges
$=$ decomposition into k-stars

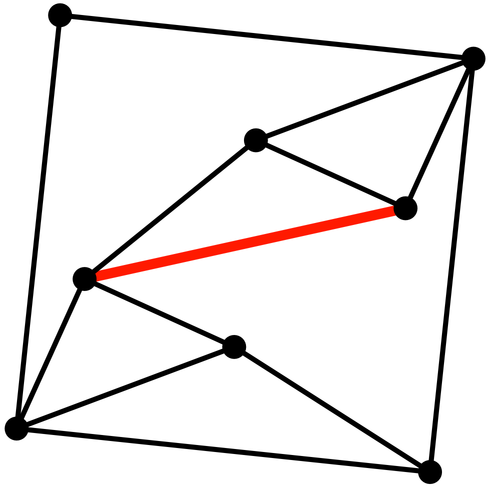
Pocchiola-Vegter, *Topologically sweeping visibility complexes via pseudotriangulations* ('96)
P.-Santos, *Multitriangulations as complexes of star polygons* ('09)

FLIPS

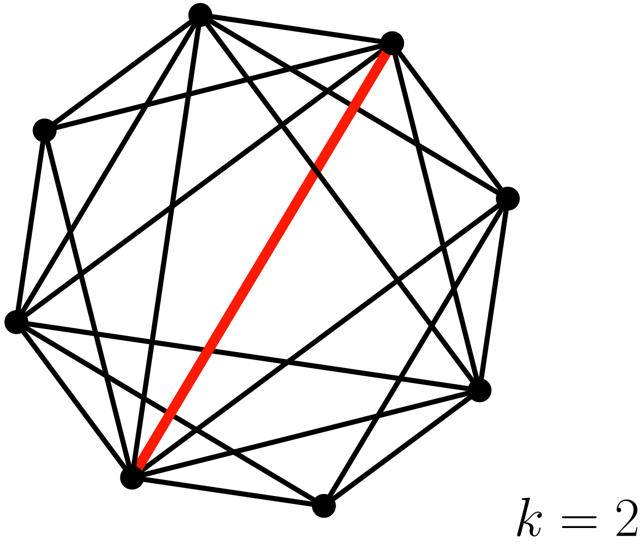
Triangulations



Pseudotriangulations



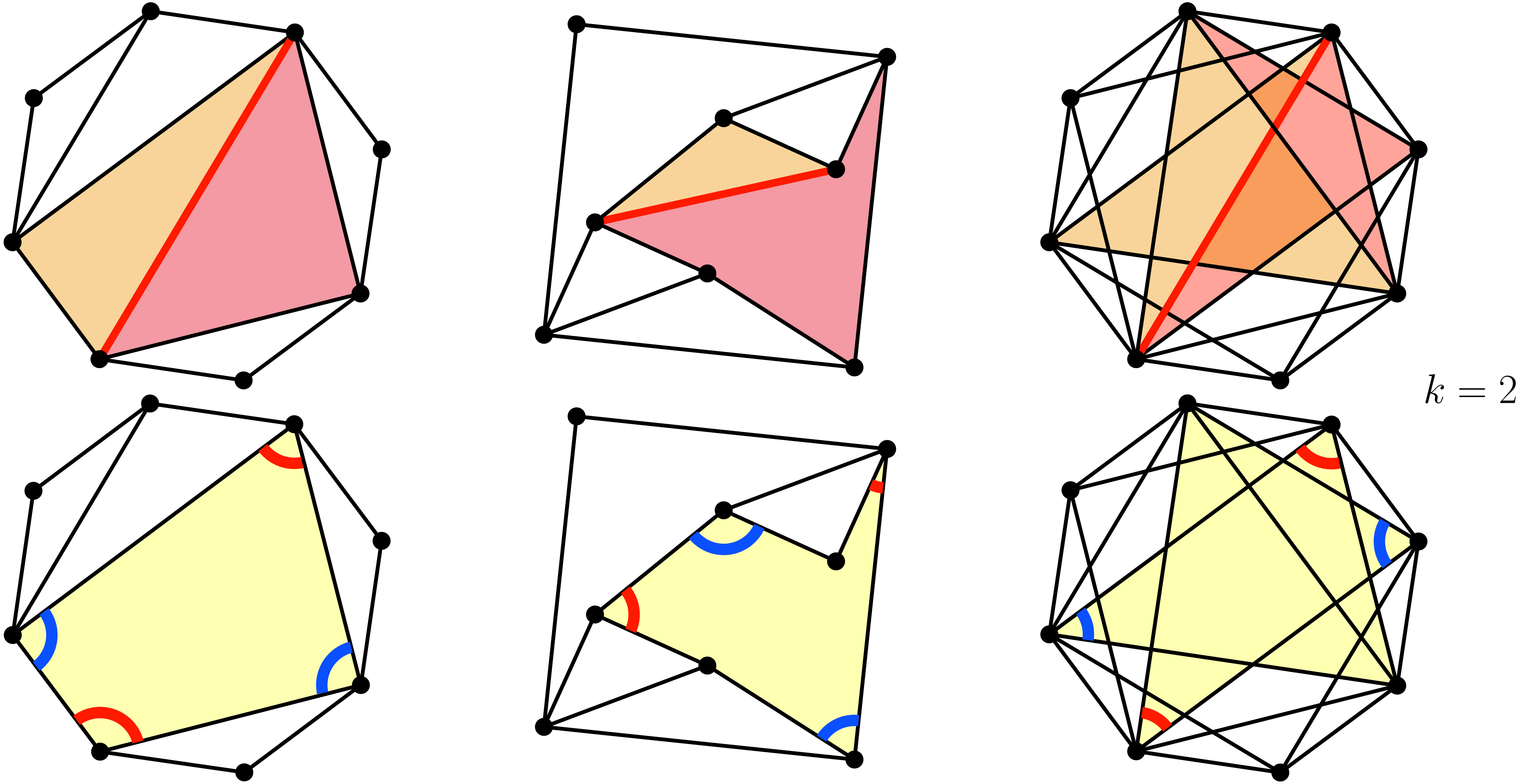
Multitriangulations



flip = exchange an internal edge with the common bisector of the two adjacent cells

FLIPS

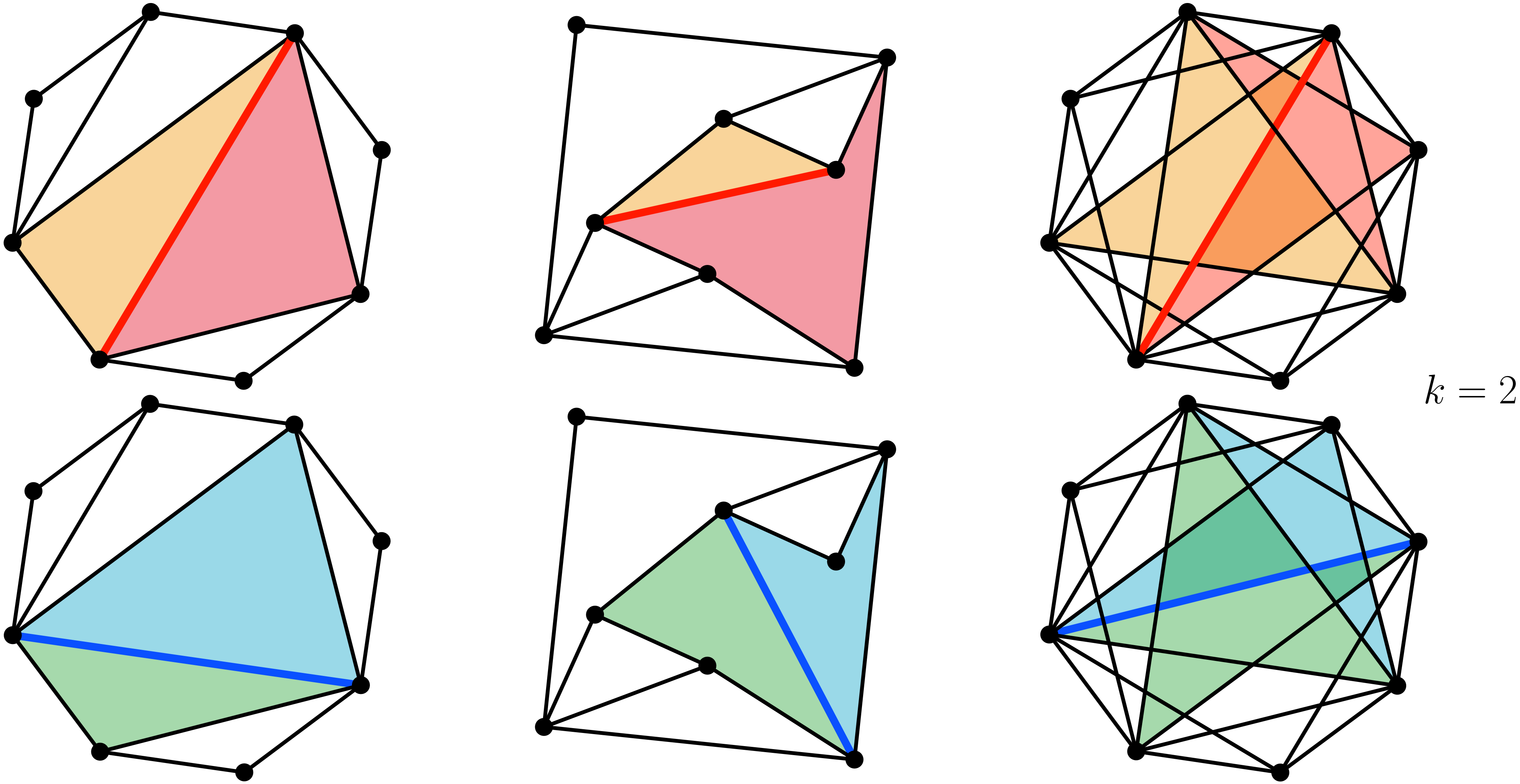
Triangulations Pseudotriangulations Multitriangulations



flip = exchange an internal edge with the common bisector of the two adjacent cells

FLIPS

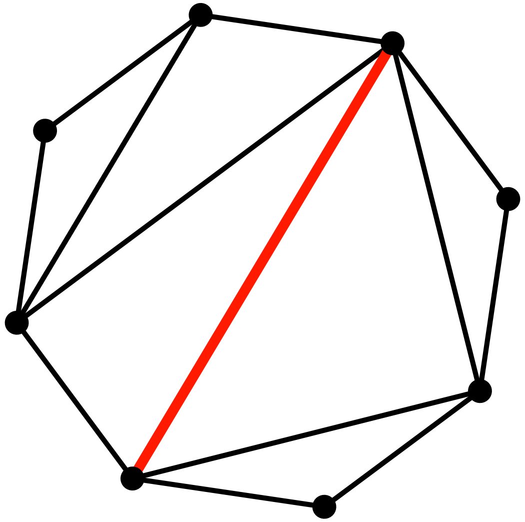
Triangulations Pseudotriangulations Multitriangulations



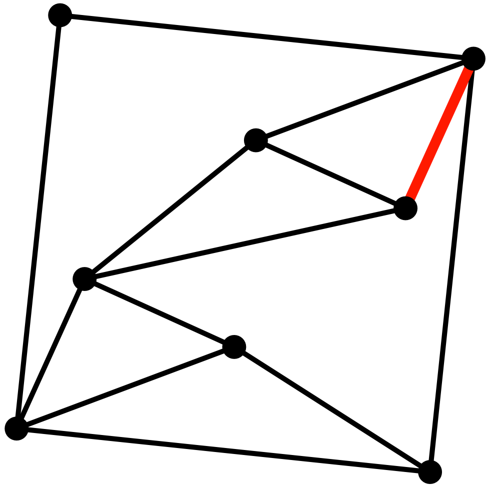
flip $=$ exchange an internal edge with the common bisector of the two adjacent cells

FLIPS

Triangulations



Pseudotriangulations



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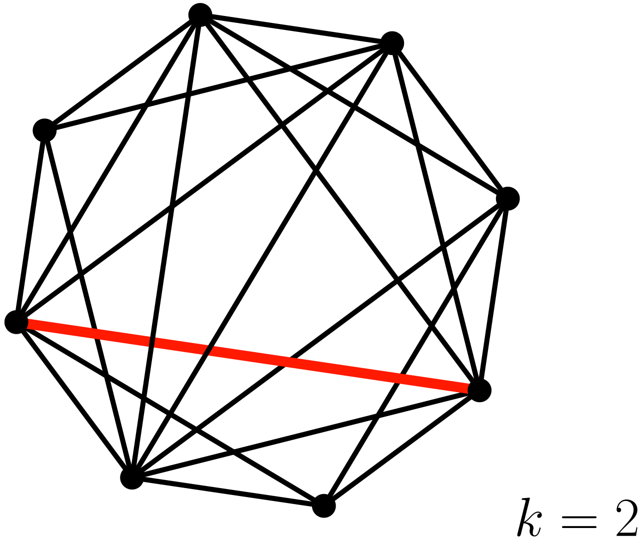
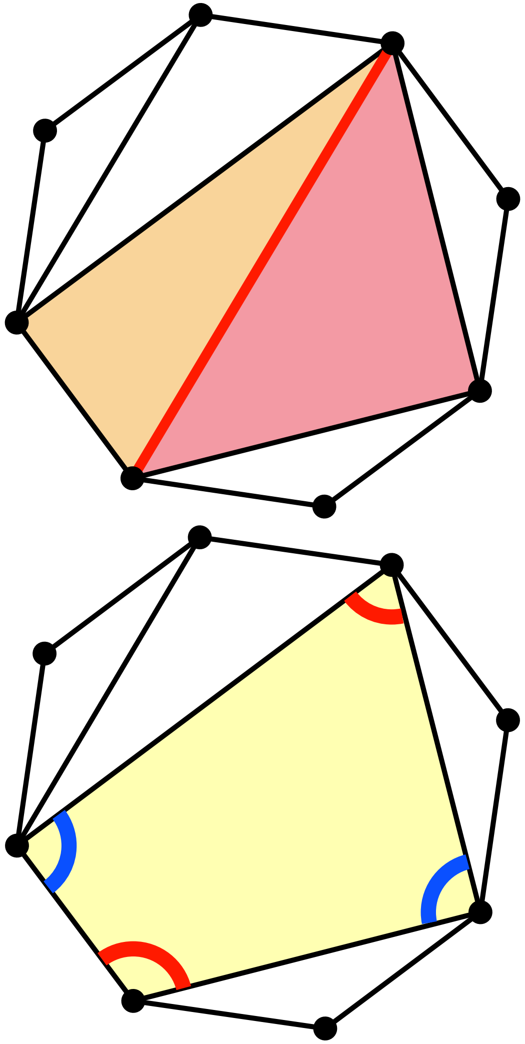


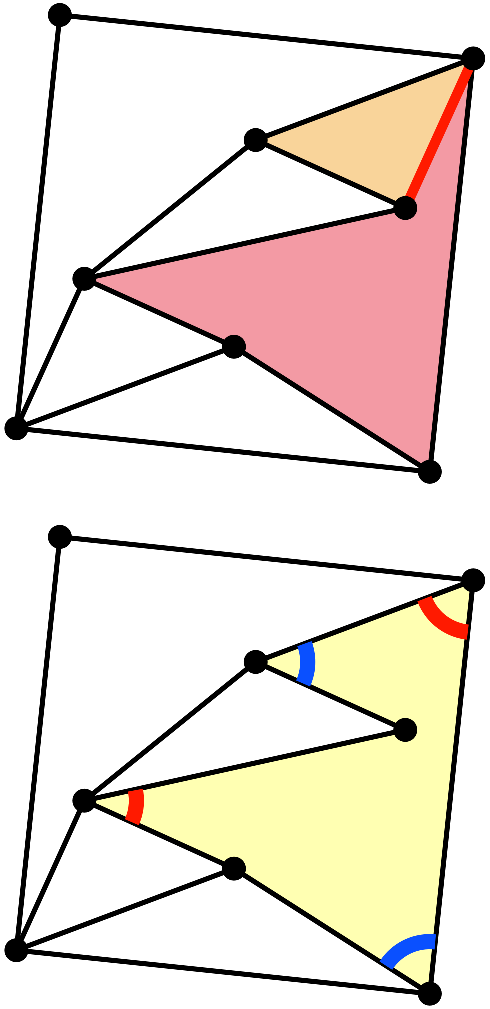
flip $=$ exchange an internal edge with the common bisector of the two adjacent cells

FLIPS

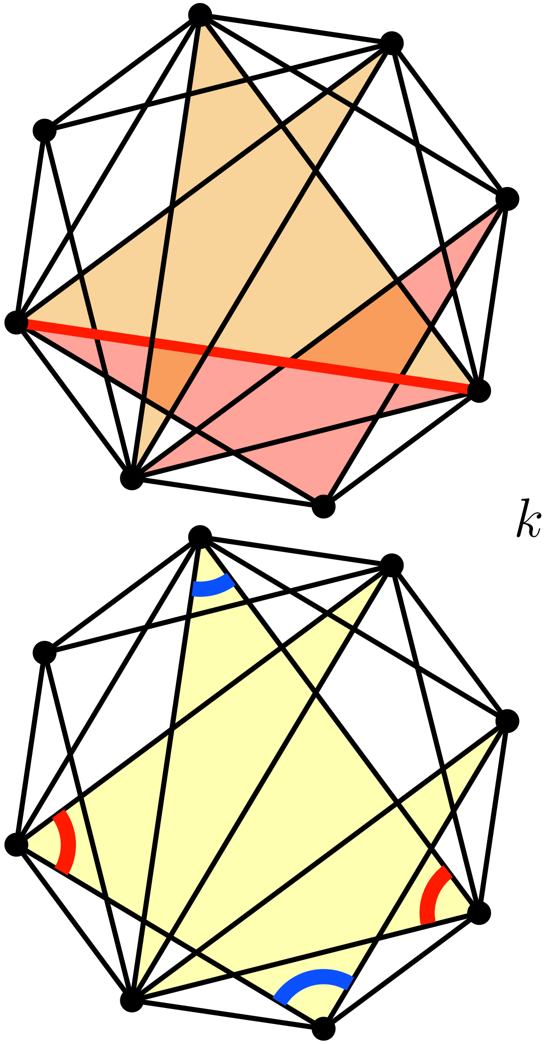
Triangulations



Pseudotriangulations



Multitriangulations

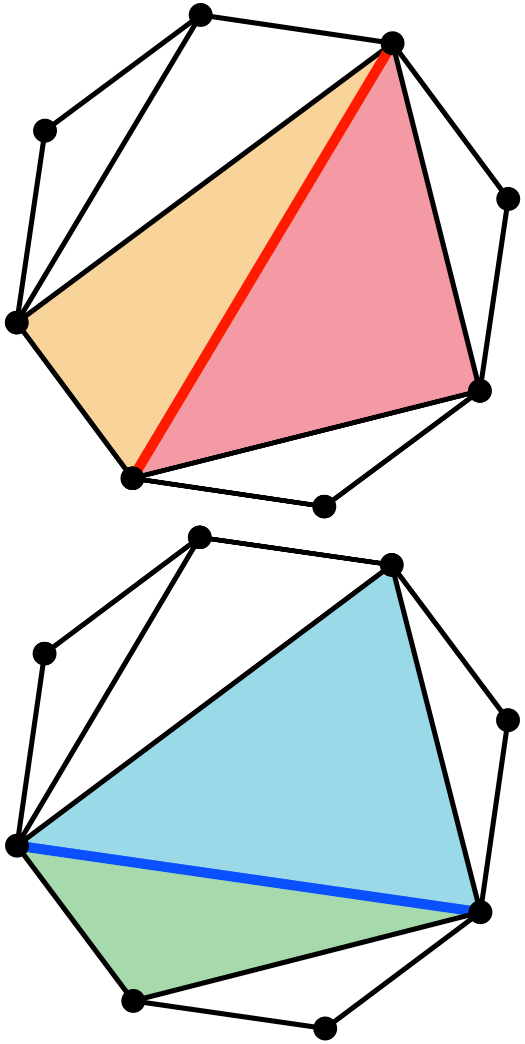


$k = 2$

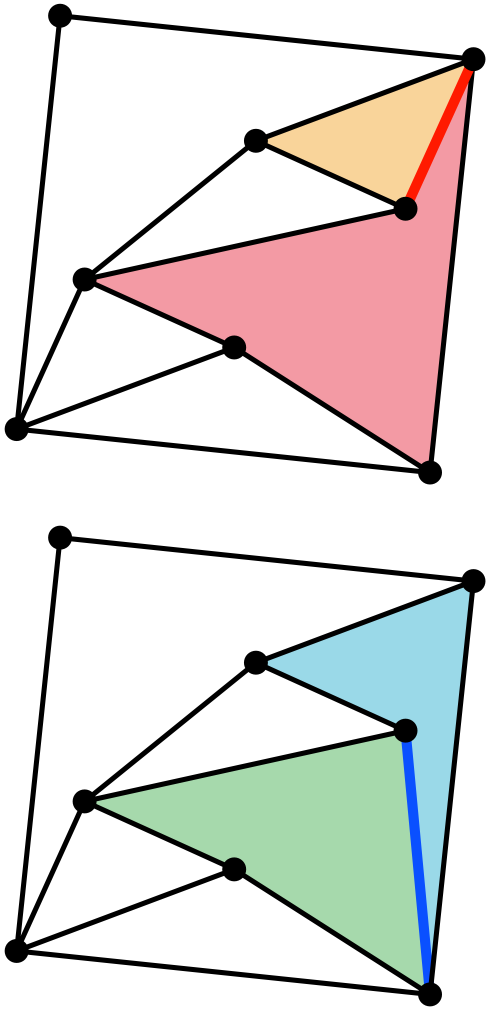
flip $=$ exchange an internal edge with the common bisector of the two adjacent cells

FLIPS

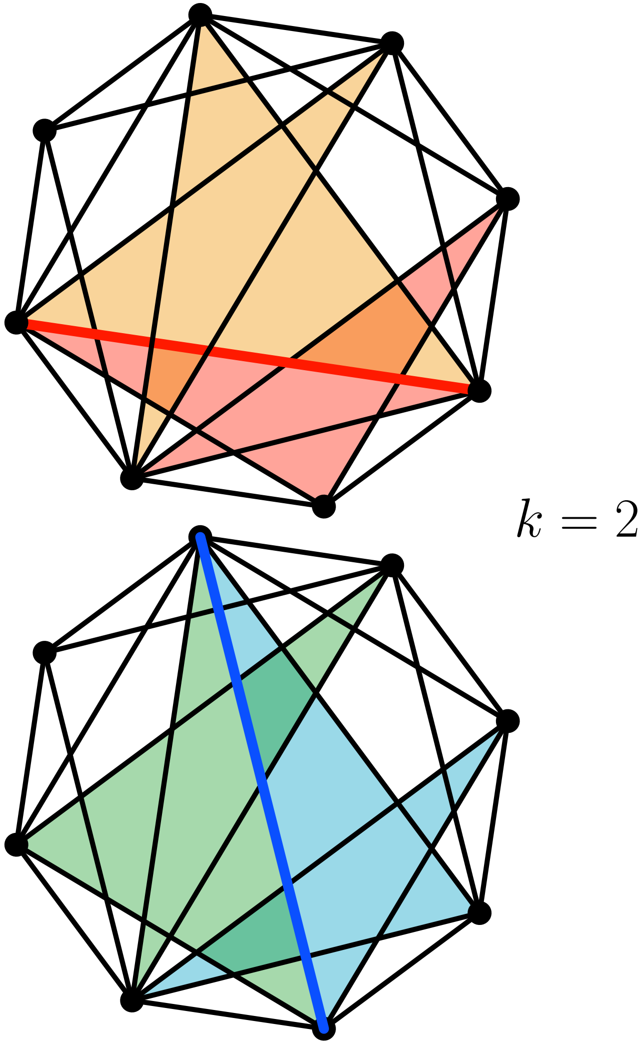
Triangulations



Pseudotriangulations



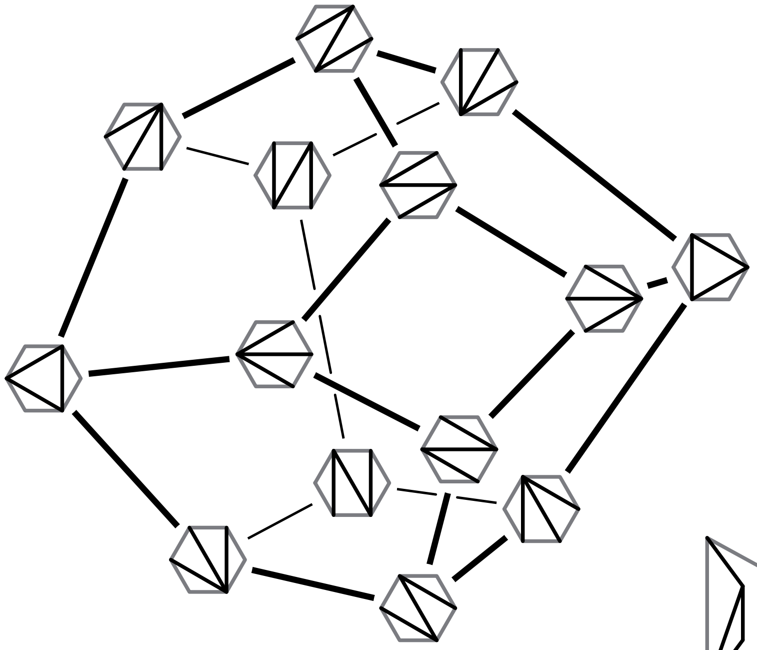
Multitriangulations



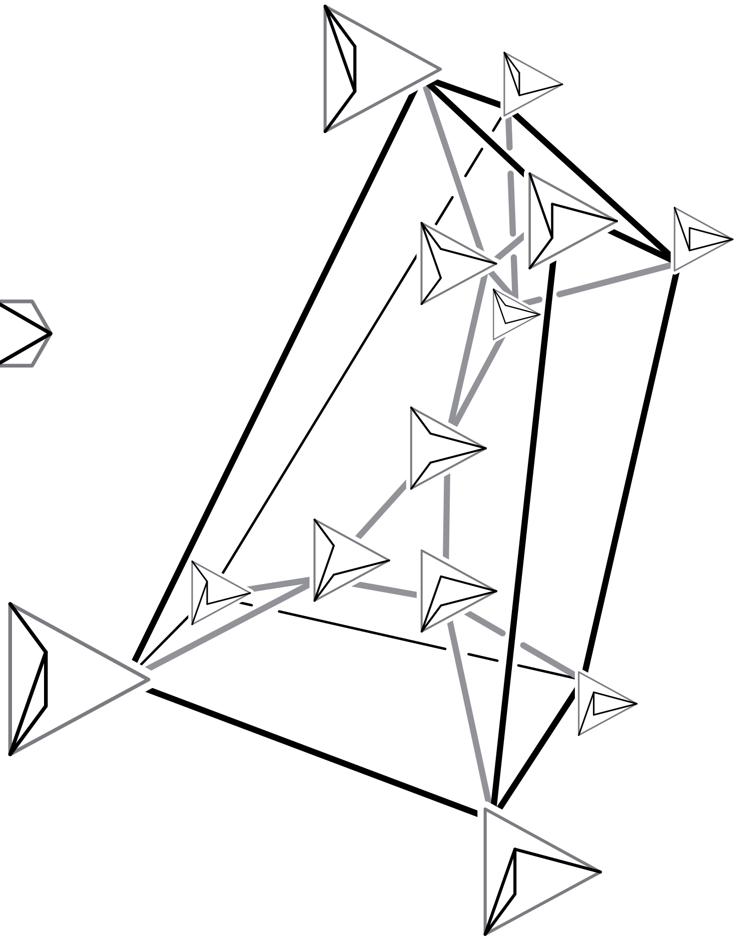
flip $=$ exchange an internal edge with the common bisector of the two adjacent cells

THREE GEOMETRIC STRUCTURES

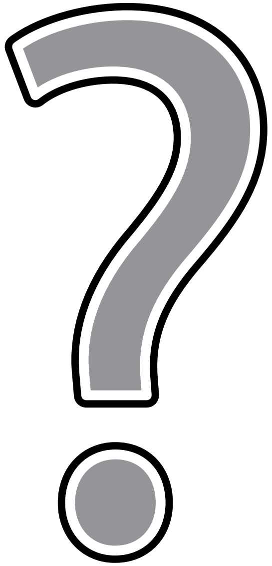
Triangulations



Pseudotriangulations



Multitriangulations



associahedron $\longleftrightarrow$ crossing-free sets of internal edges

pseudotriangulations polytope $\longleftrightarrow$ pointed crossing-free sets of internal edges

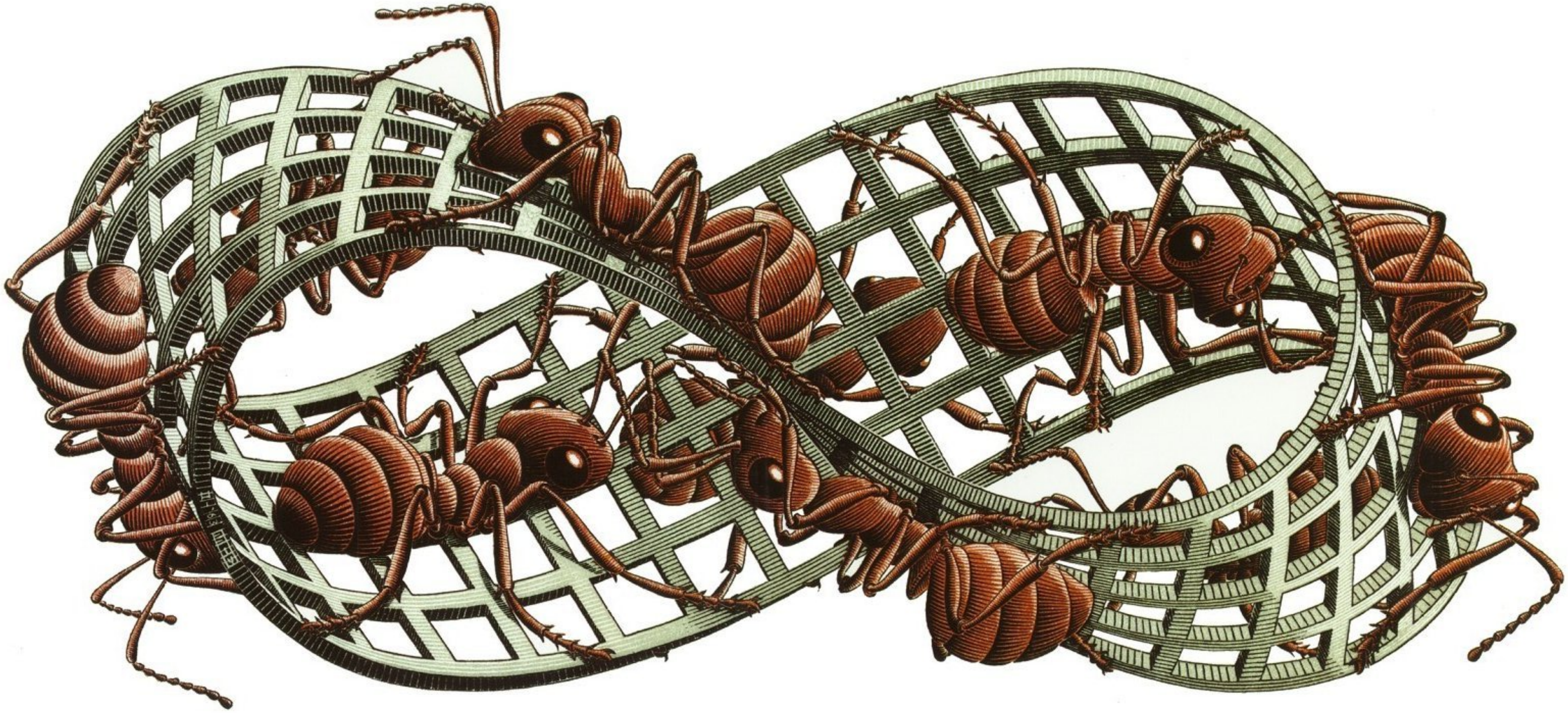
multiassociahedron $\longleftrightarrow$ $(k+1)$-crossing-free sets of k-internal edges

Rote-Santos-Streinu, *Expansive motions and the polytope of pointed pseudo-triangulations* ('03)

DUALITY

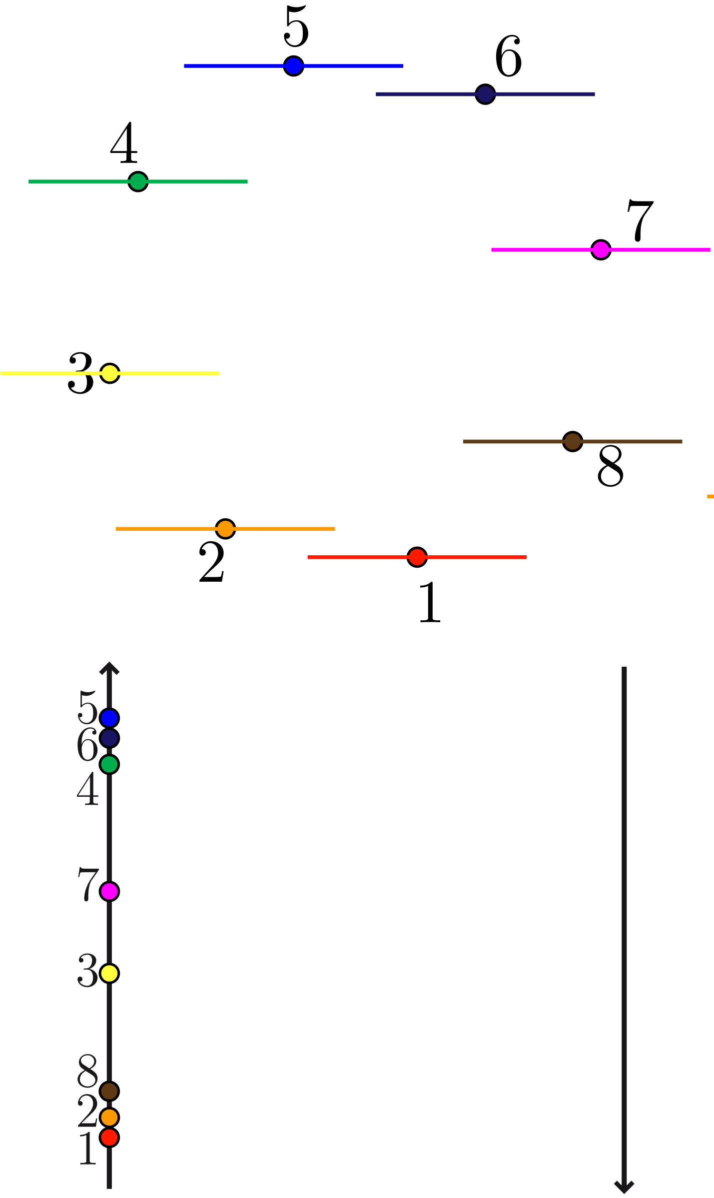
P.-Pocchiola, *Multitriang., pseudotriang. and primitive sorting networks* ('12)

LINE SPACE OF THE PLANE = MÖBIUS STRIP

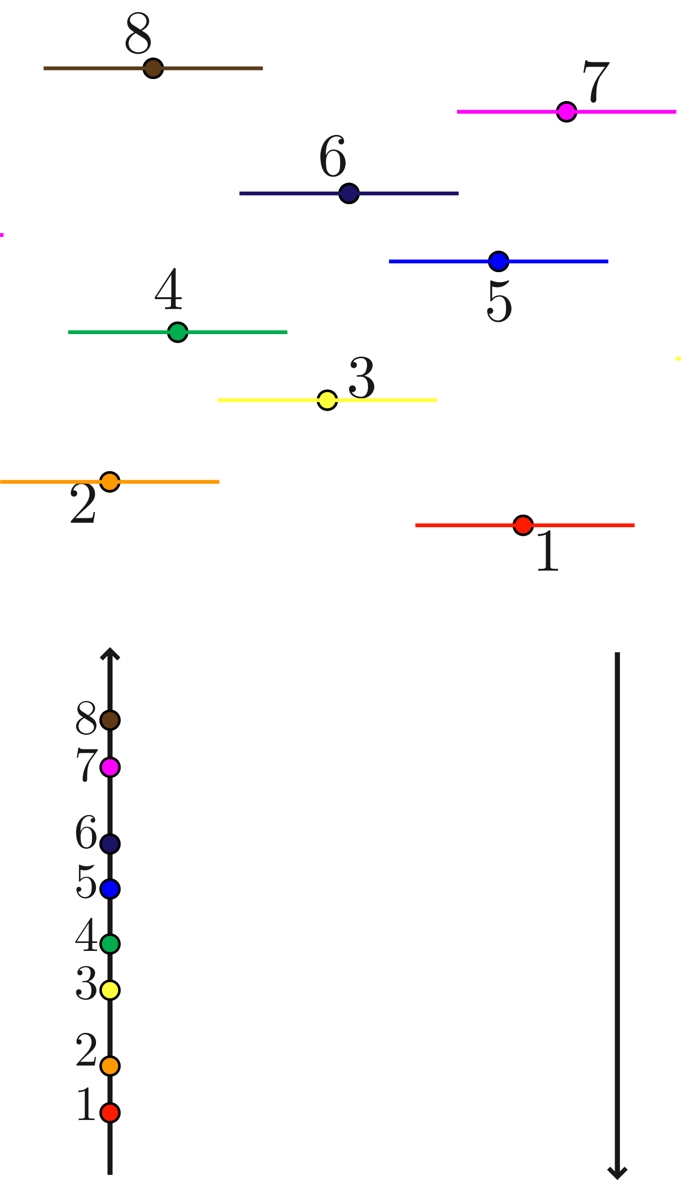


DUALITY

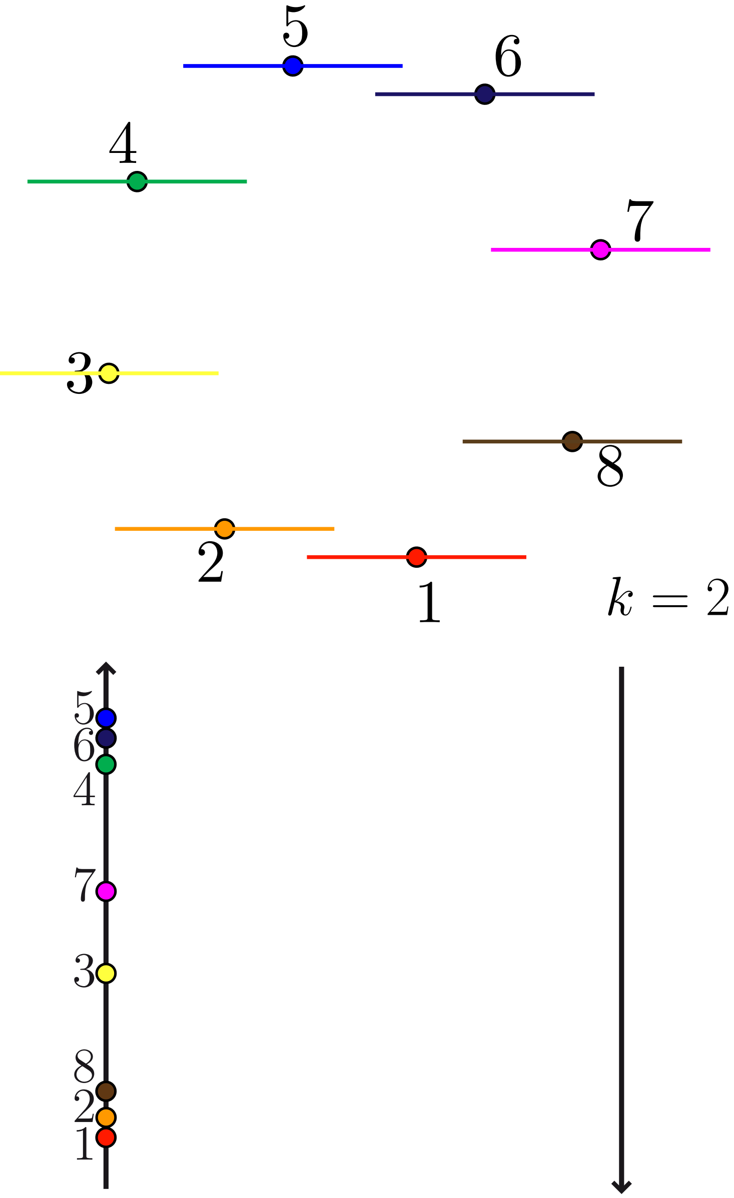
Triangulations



Pseudotriangulations

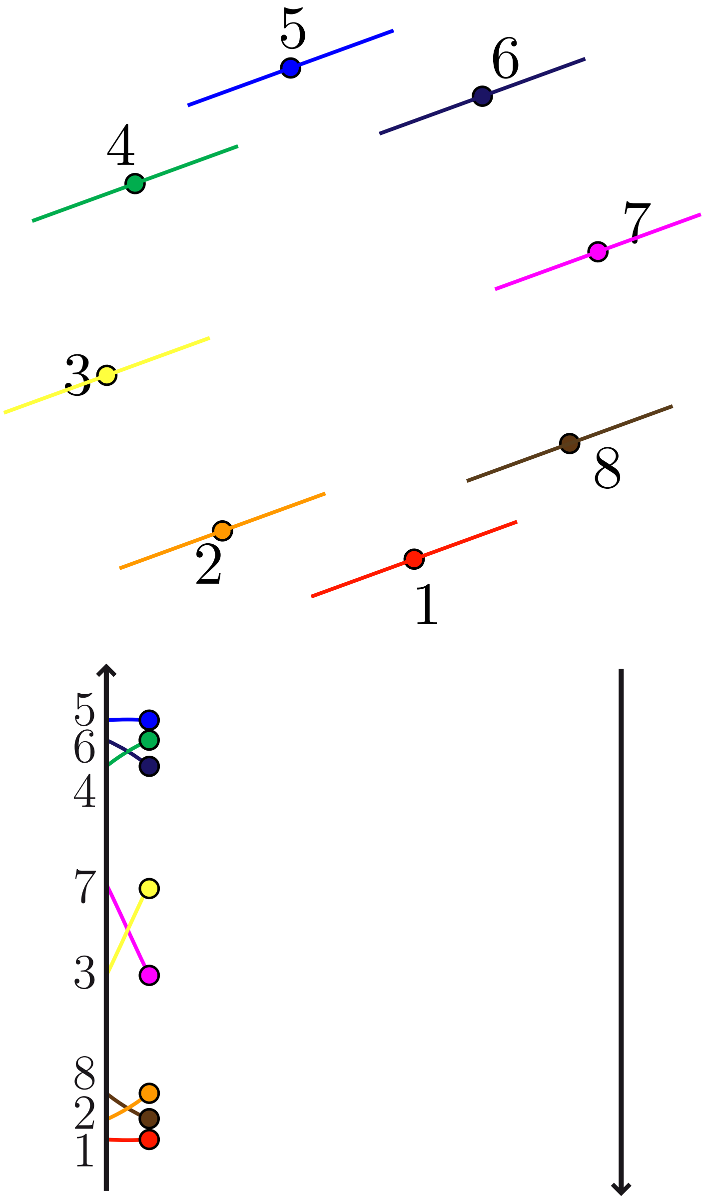


Multitriangulations

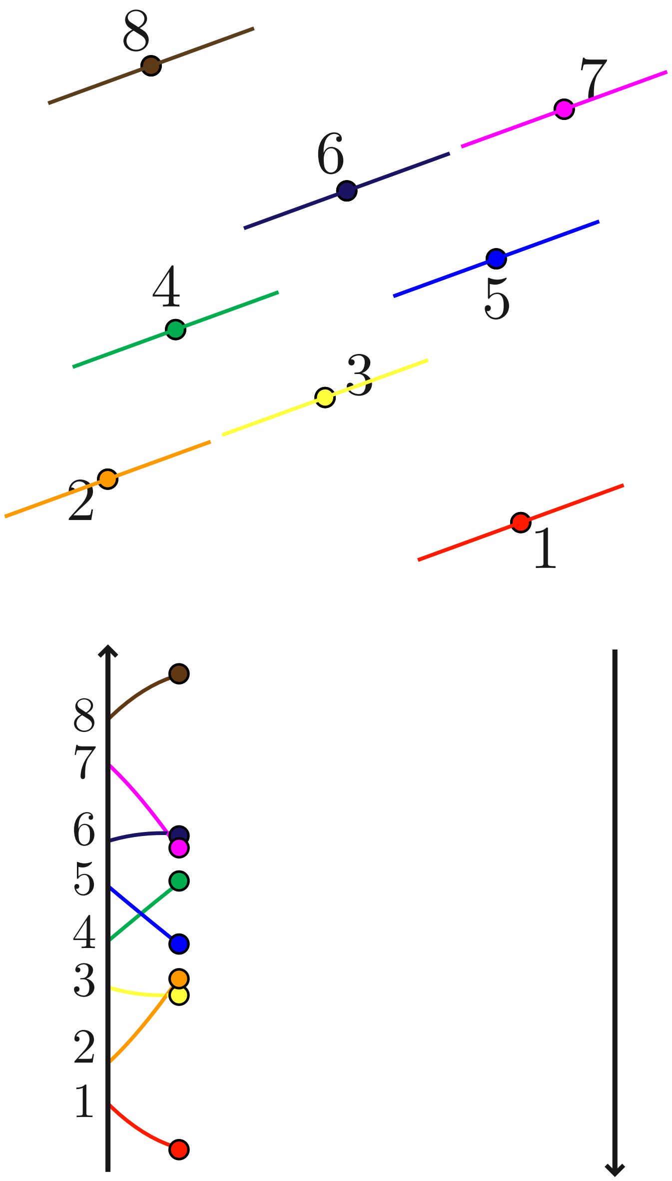


DUALITY

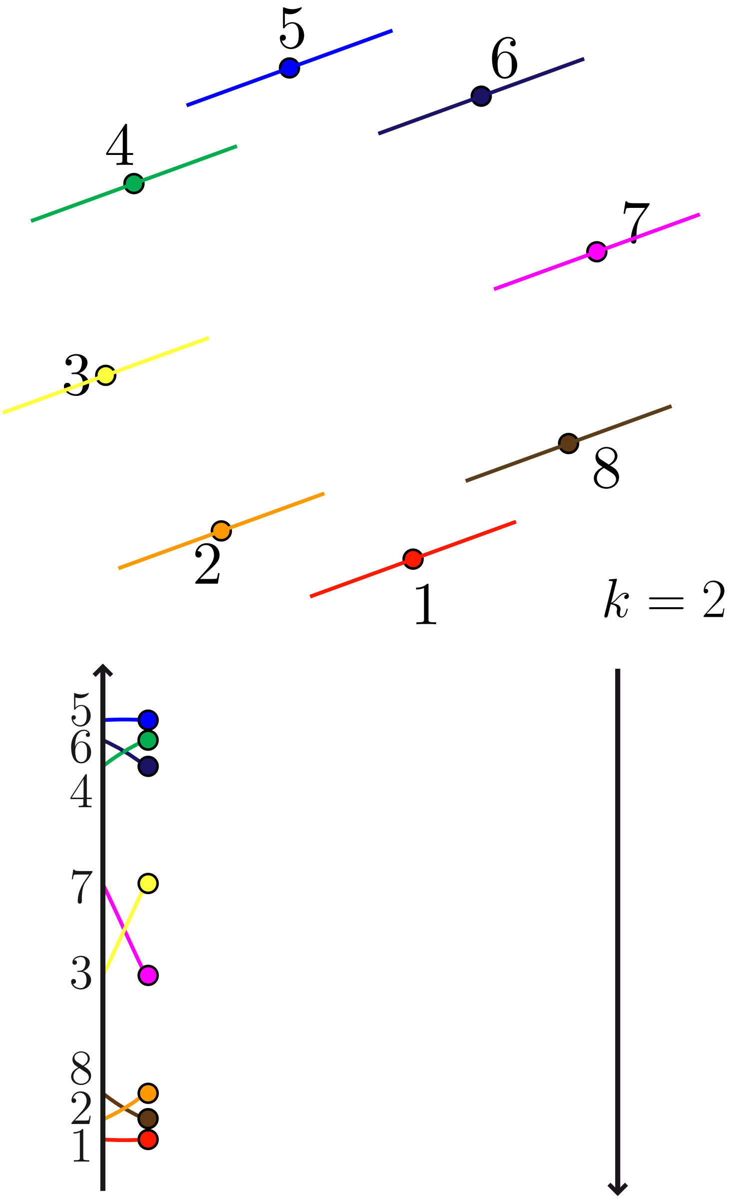
Triangulations



Pseudotriangulations

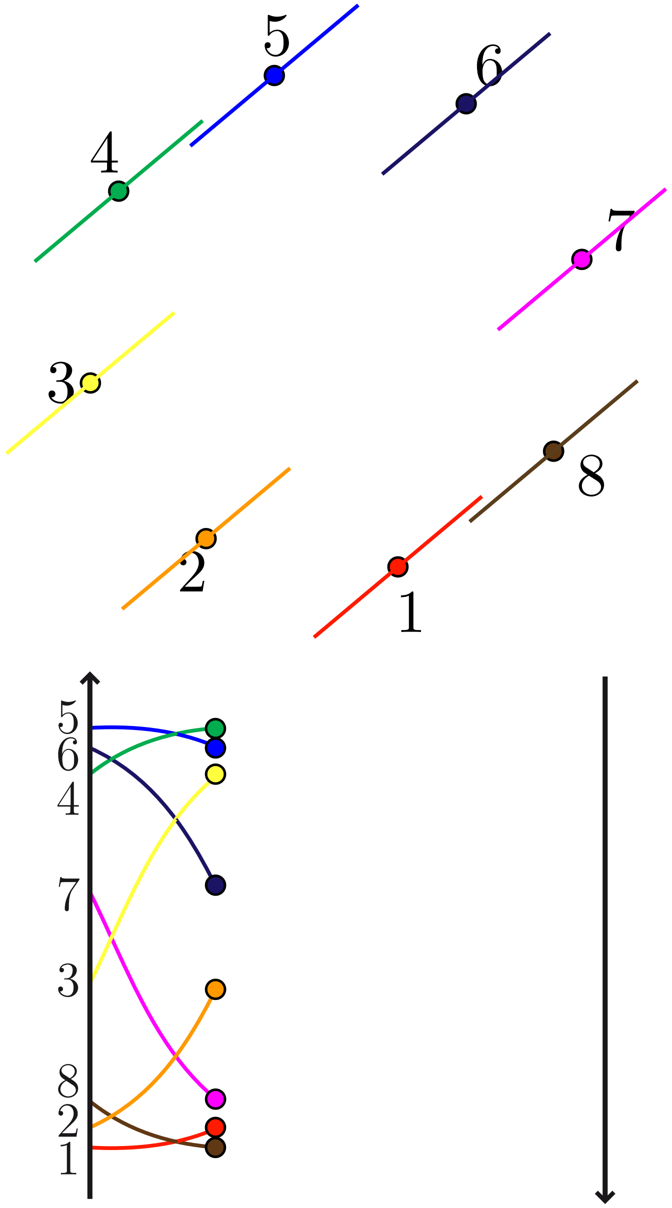


Multitriangulations



DUALITY

Triangulations



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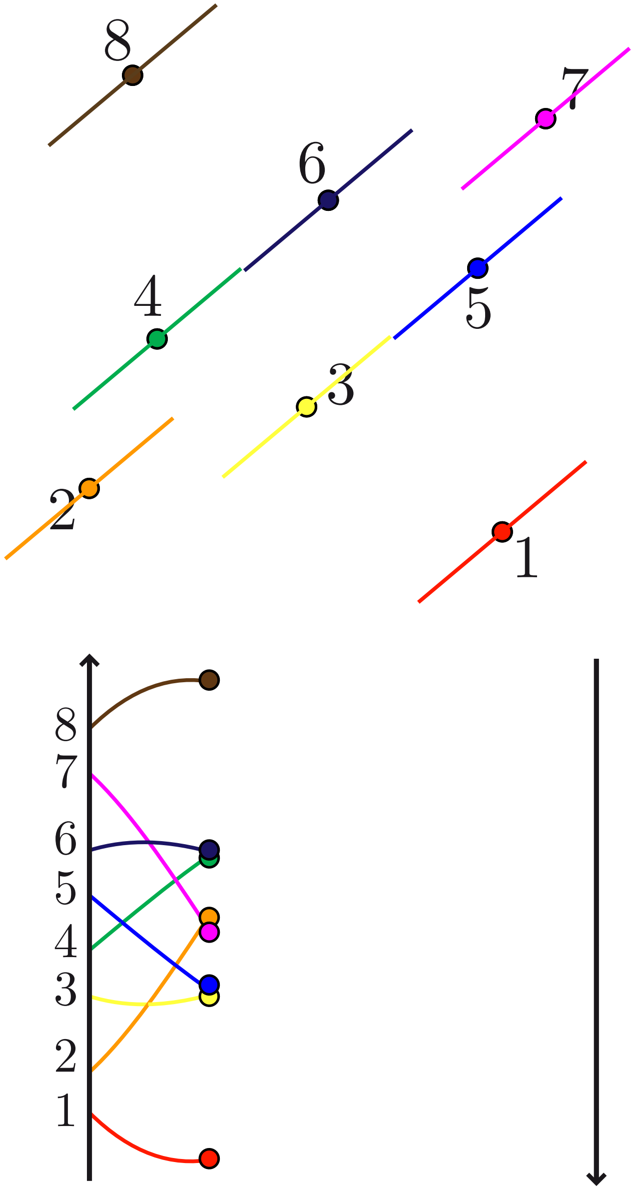

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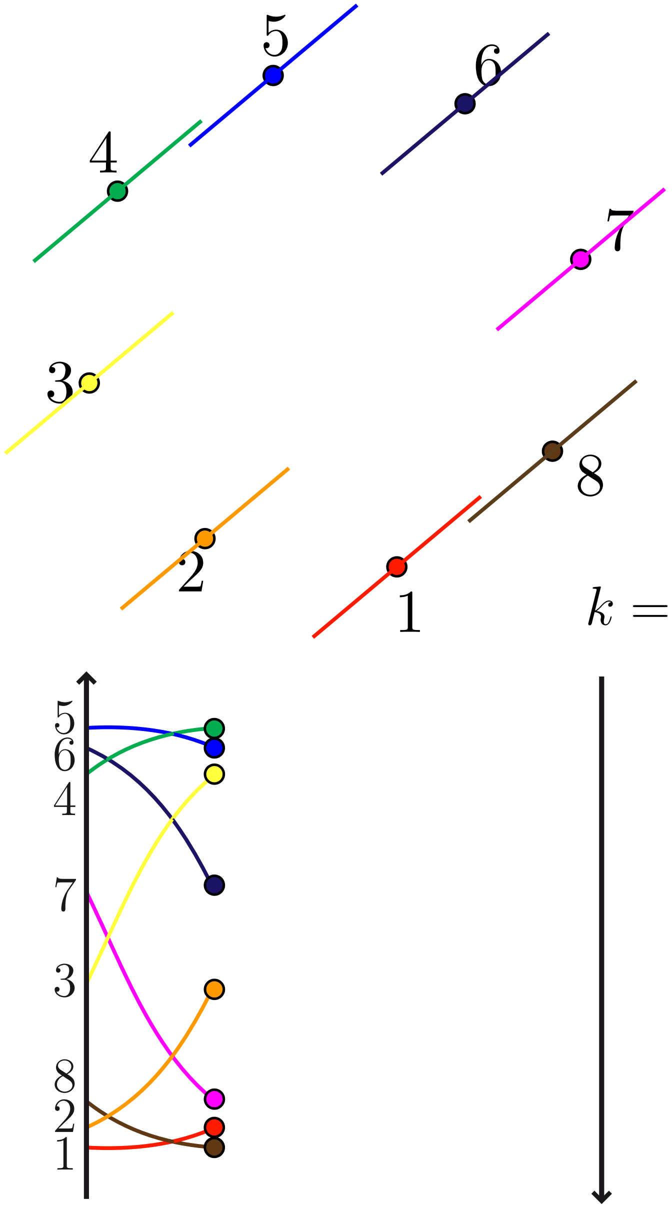
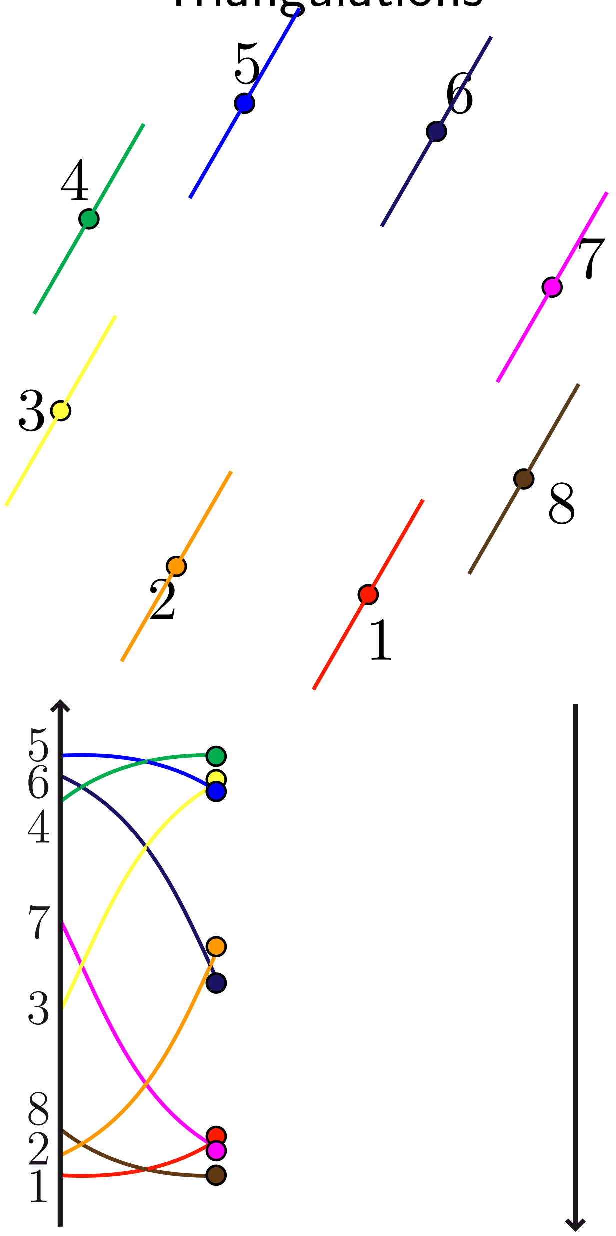


$k = 2$

DUALITY

Triangulations



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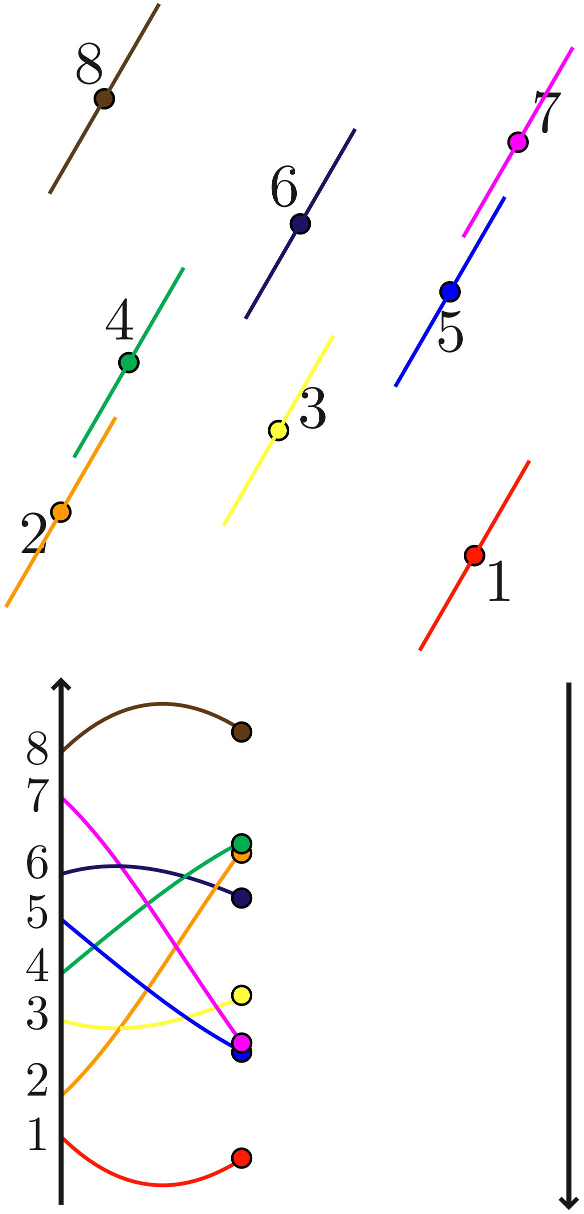

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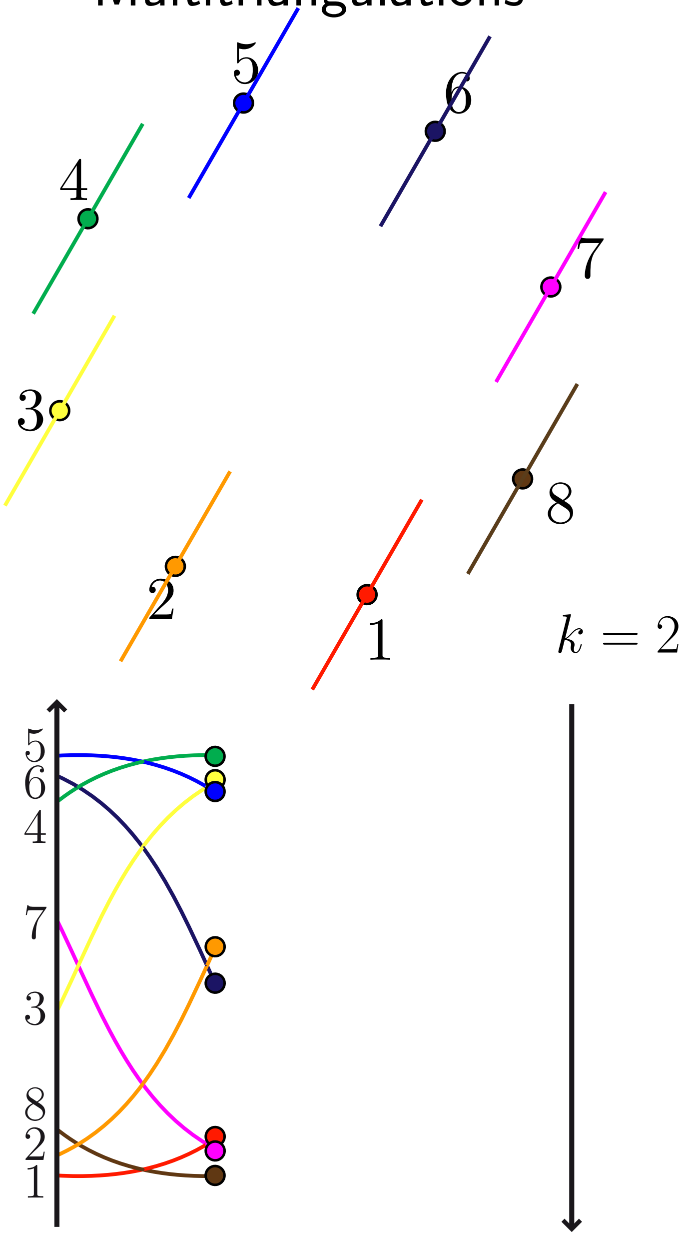

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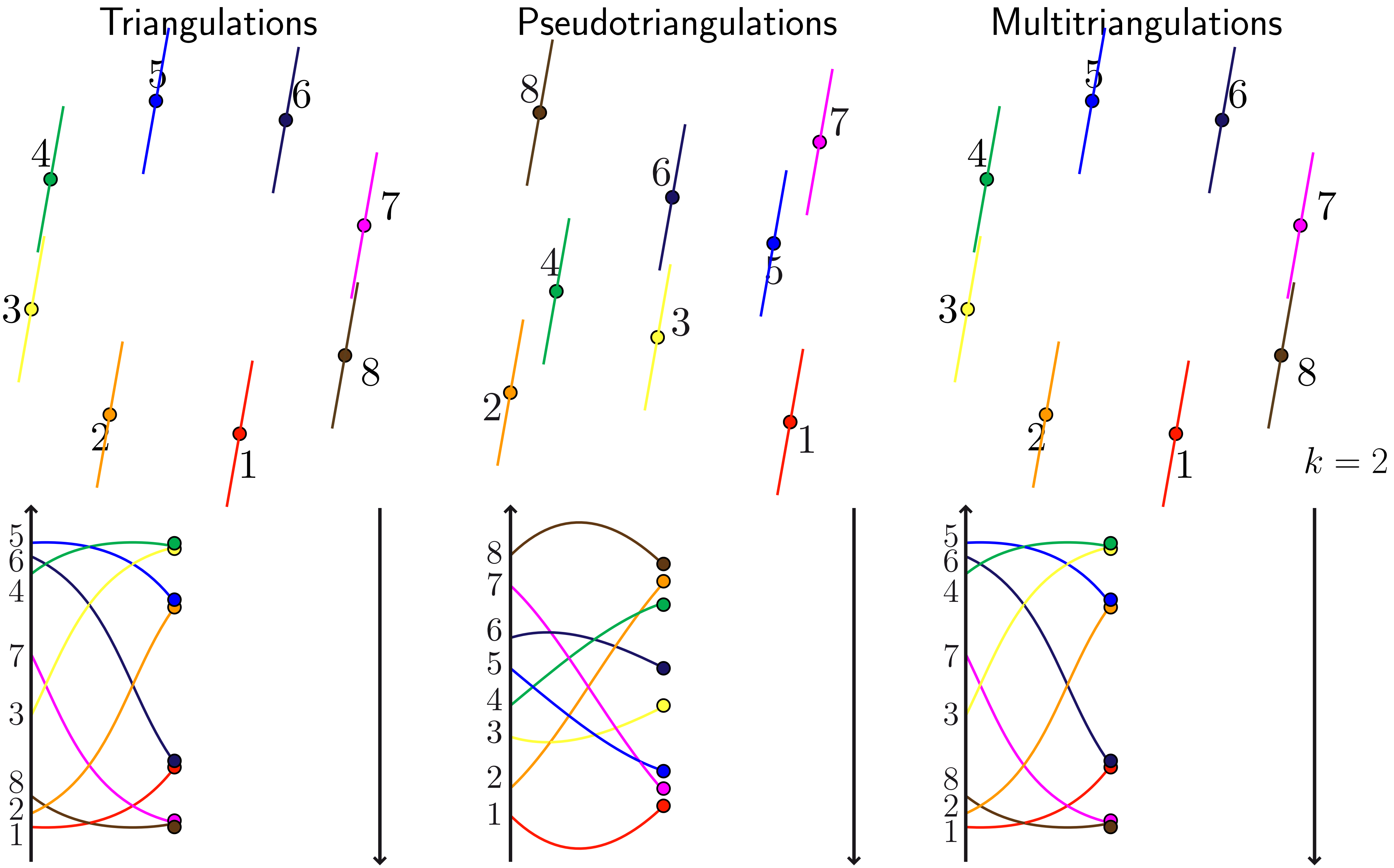
Triangulations
Pseudotriangulations
Multitriangulations
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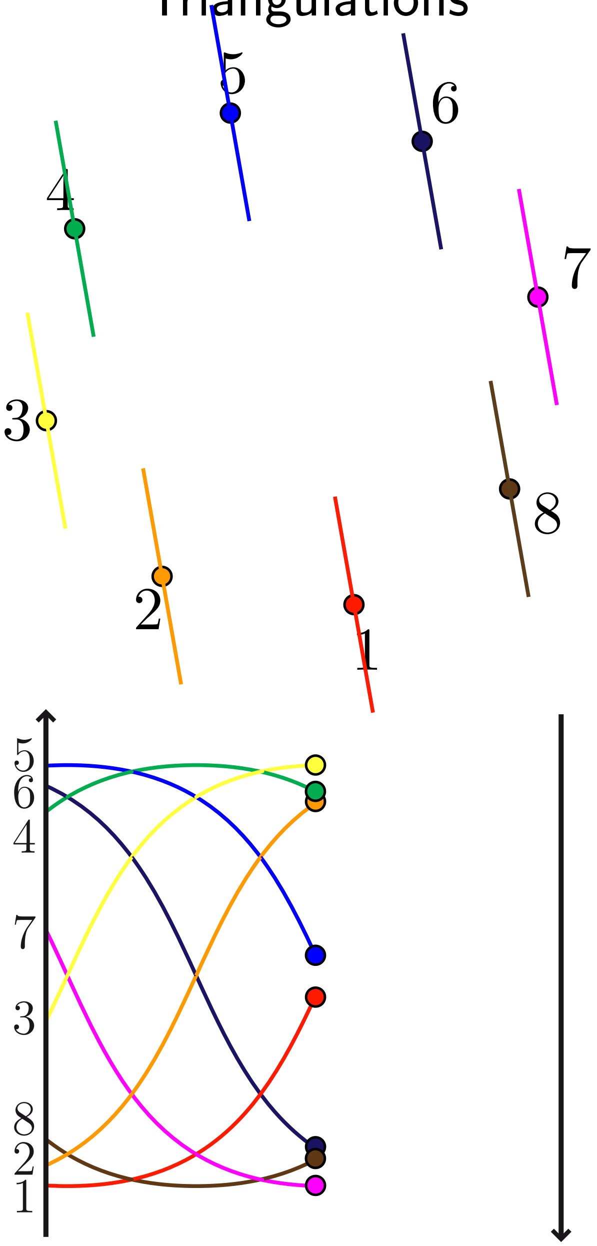
DUALITY

Triangulations



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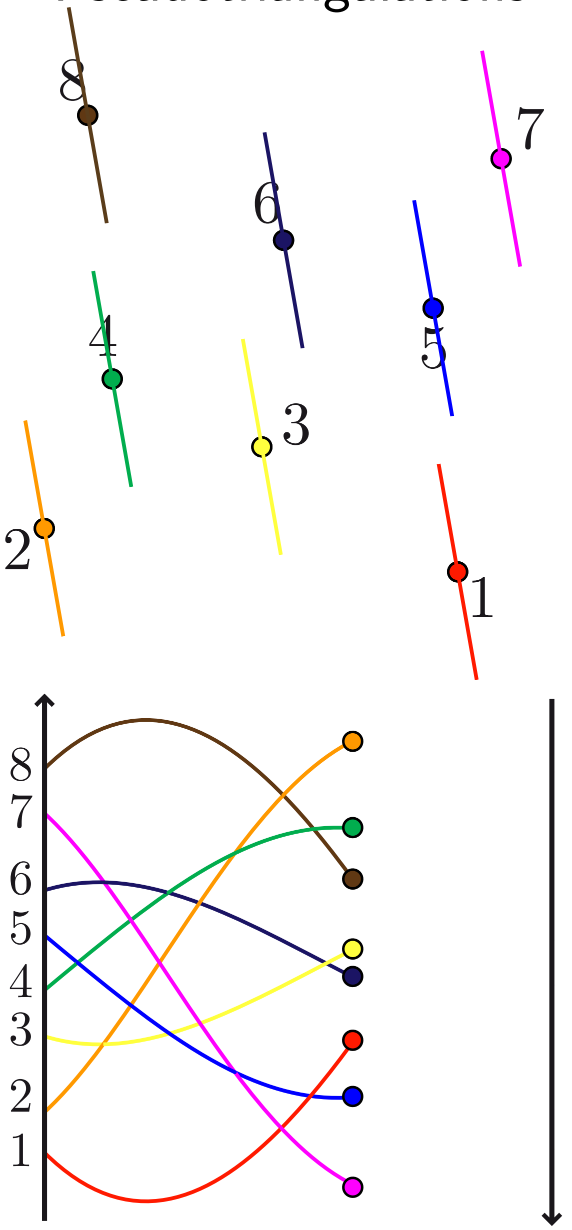

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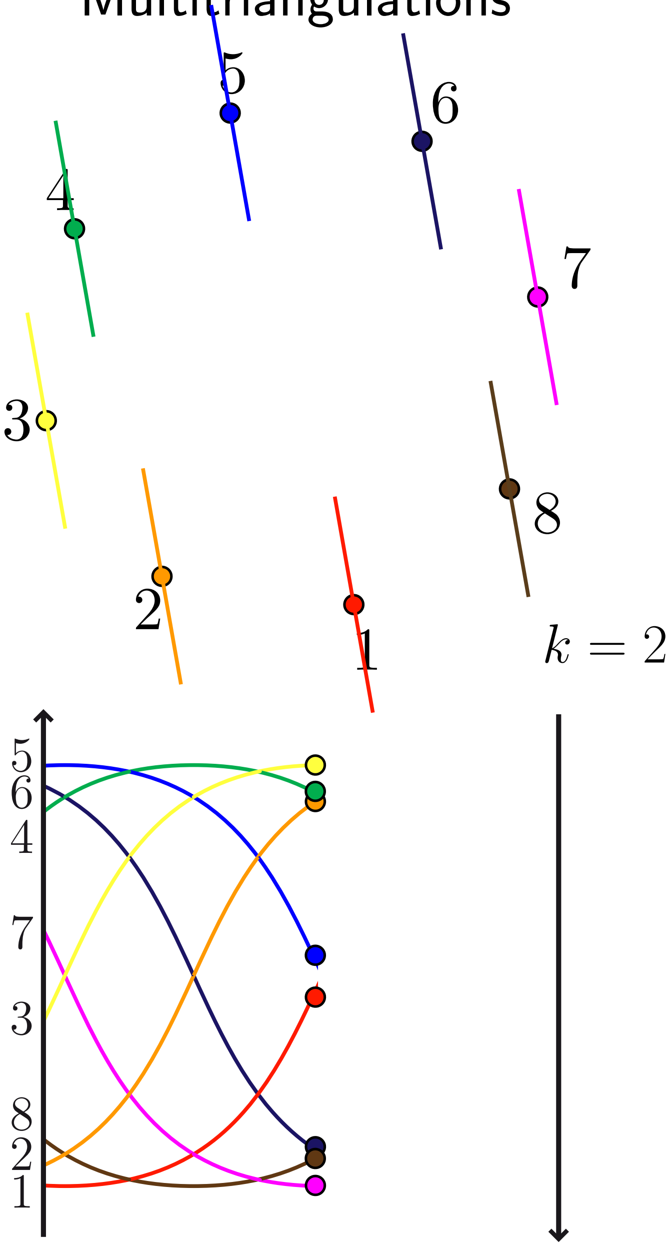

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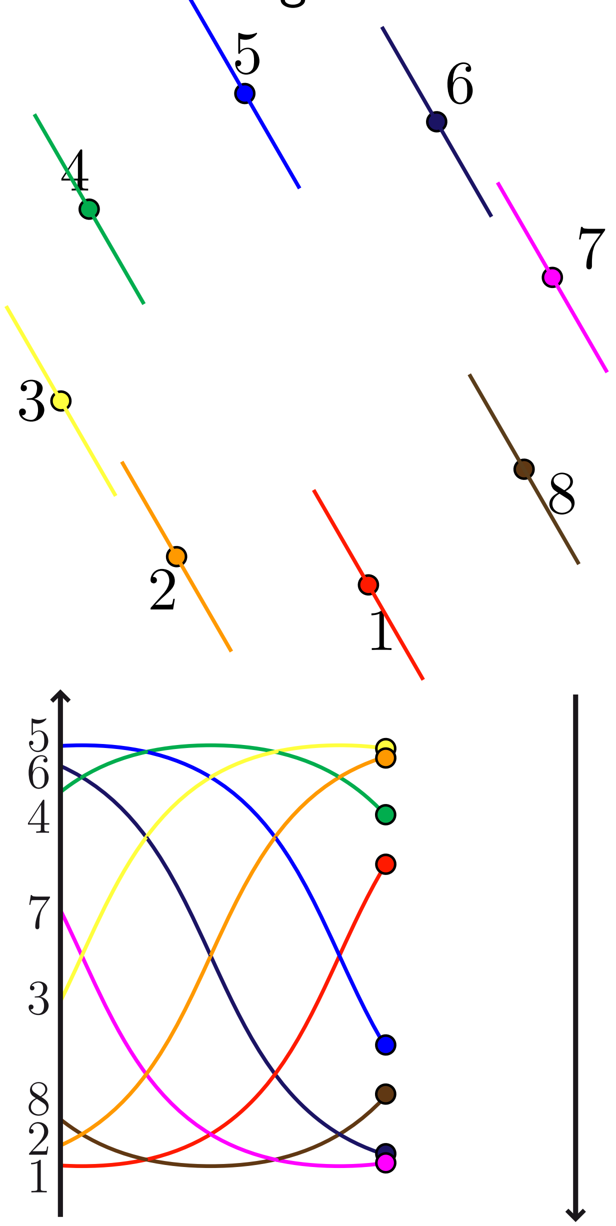
Triangulations



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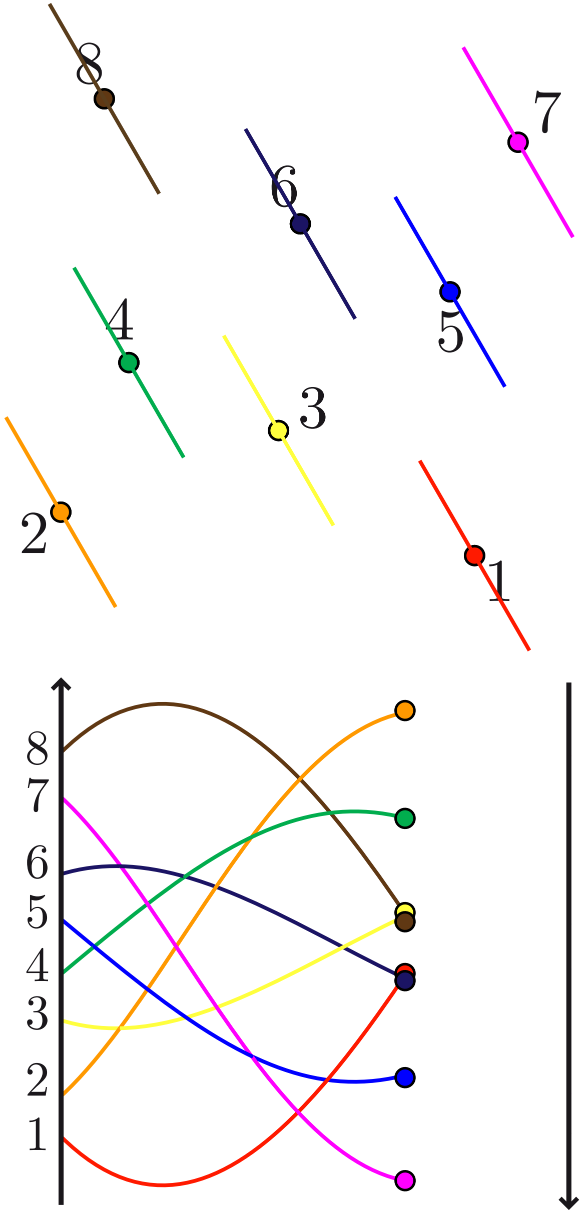

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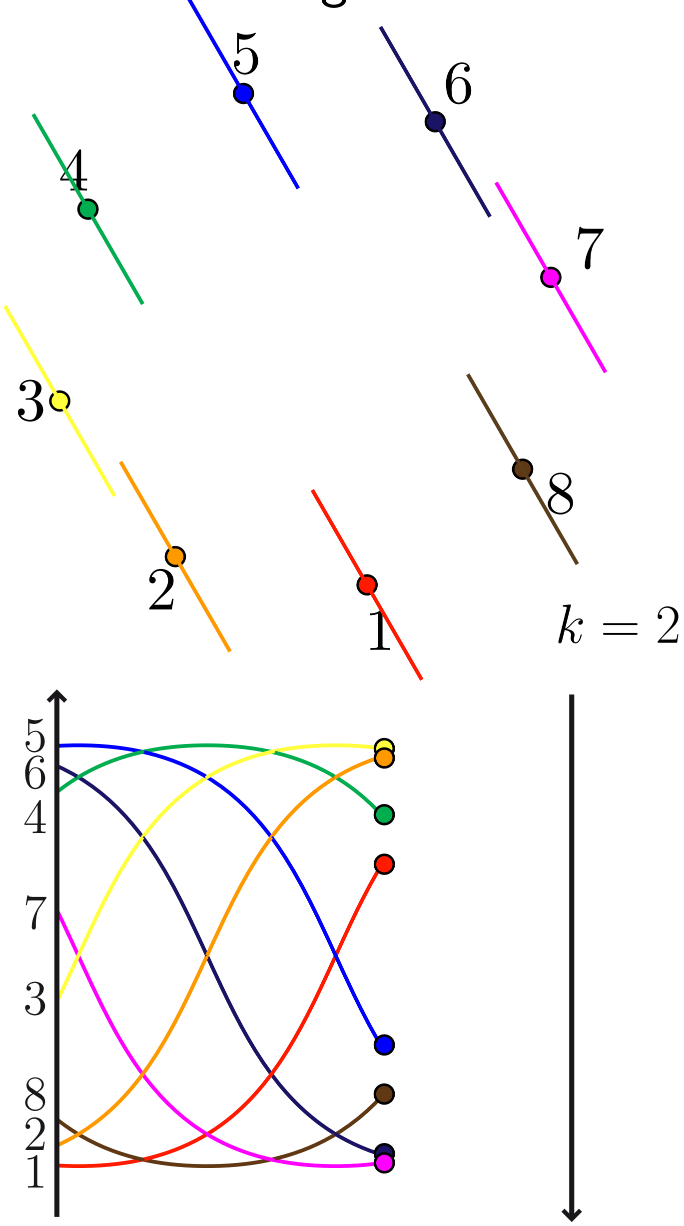

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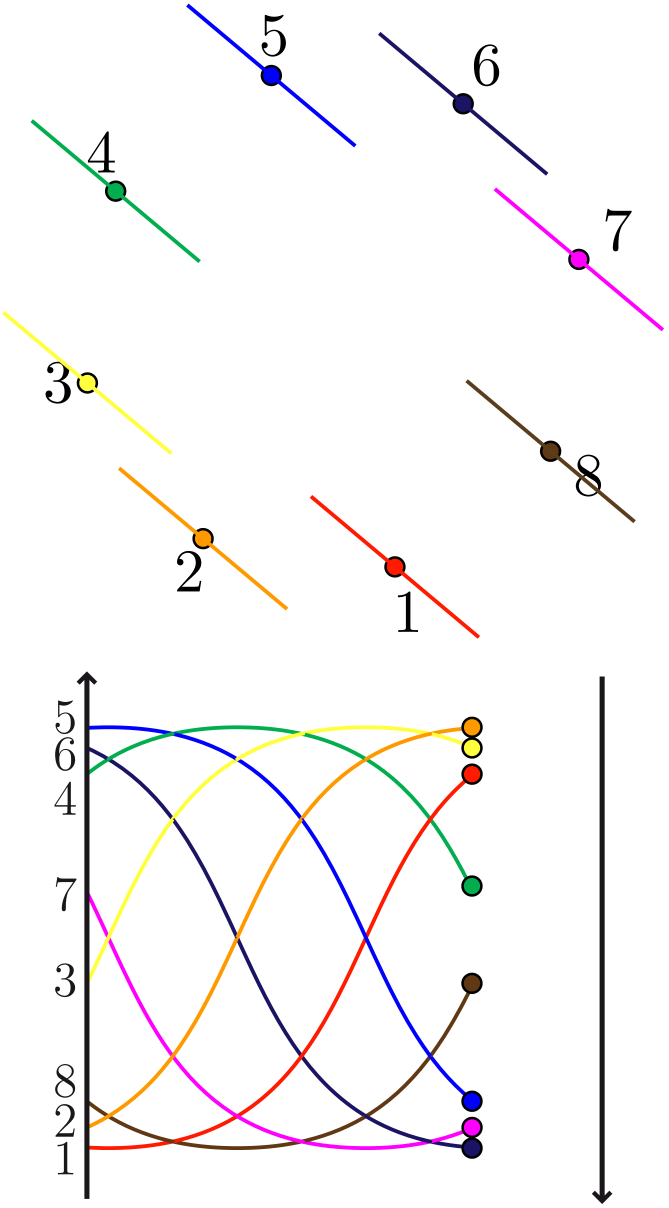
Triangulations



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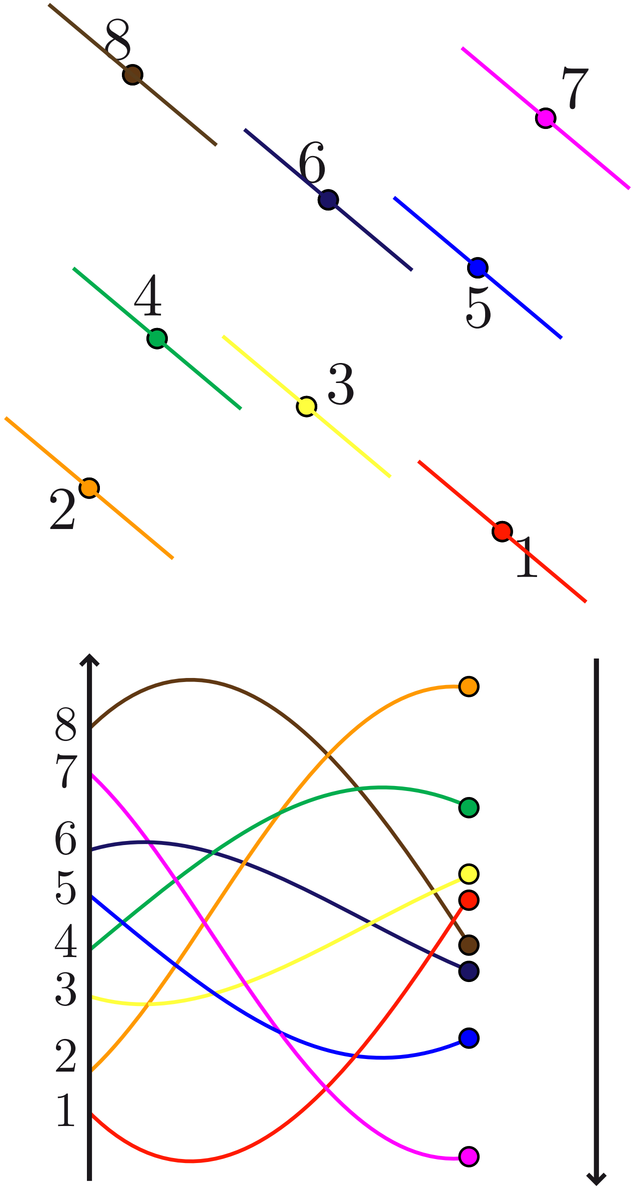

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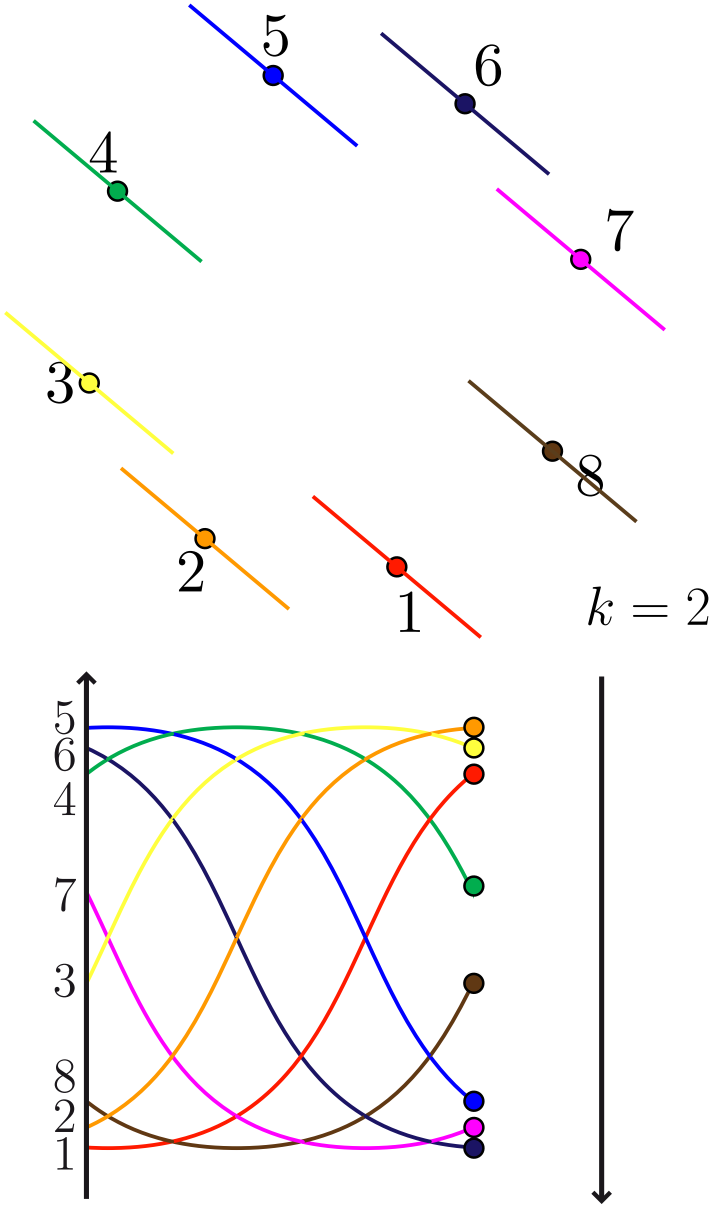

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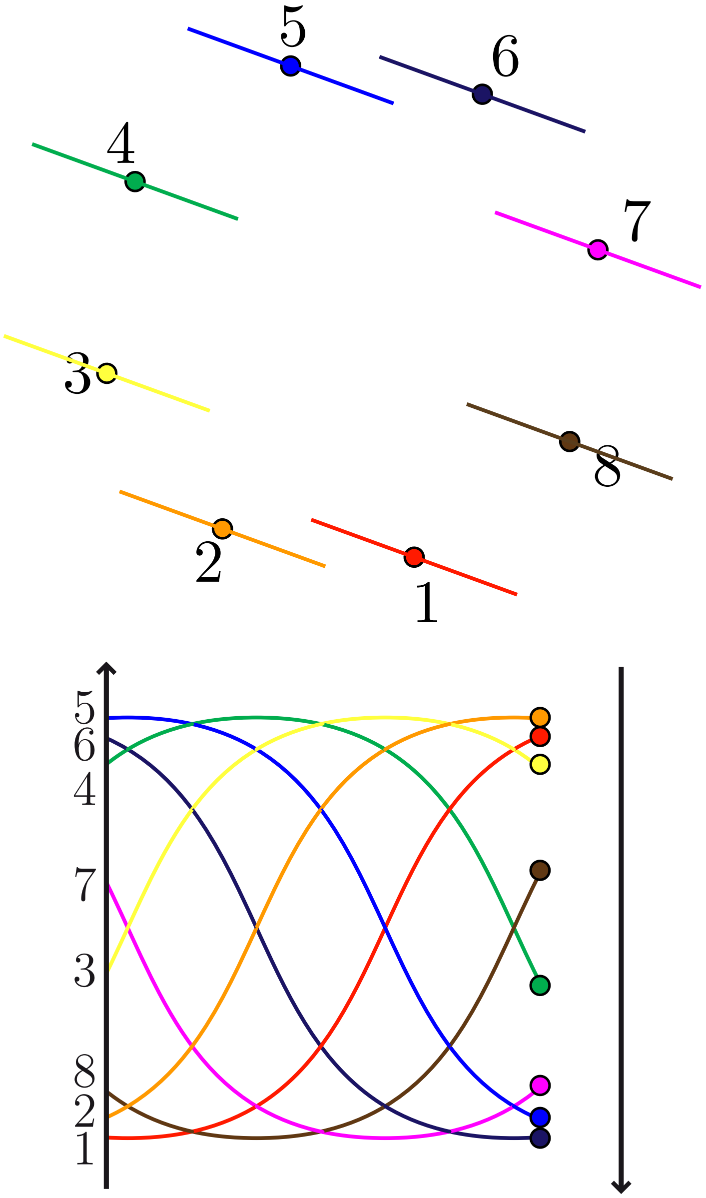
Triangulations



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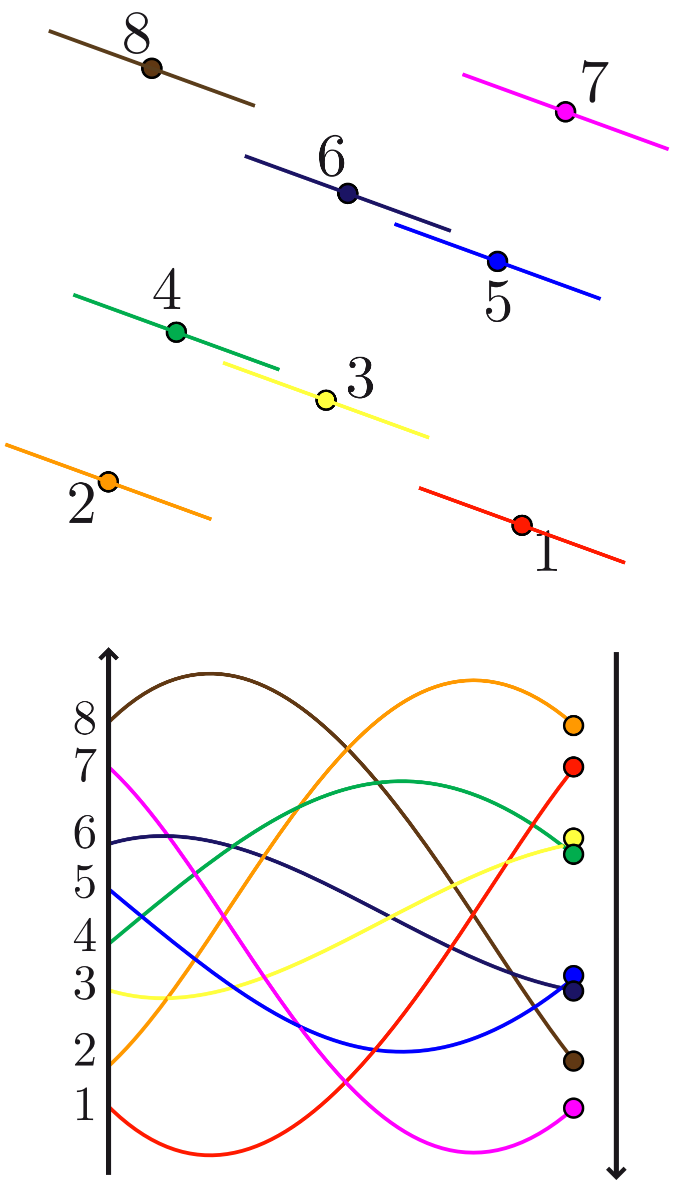

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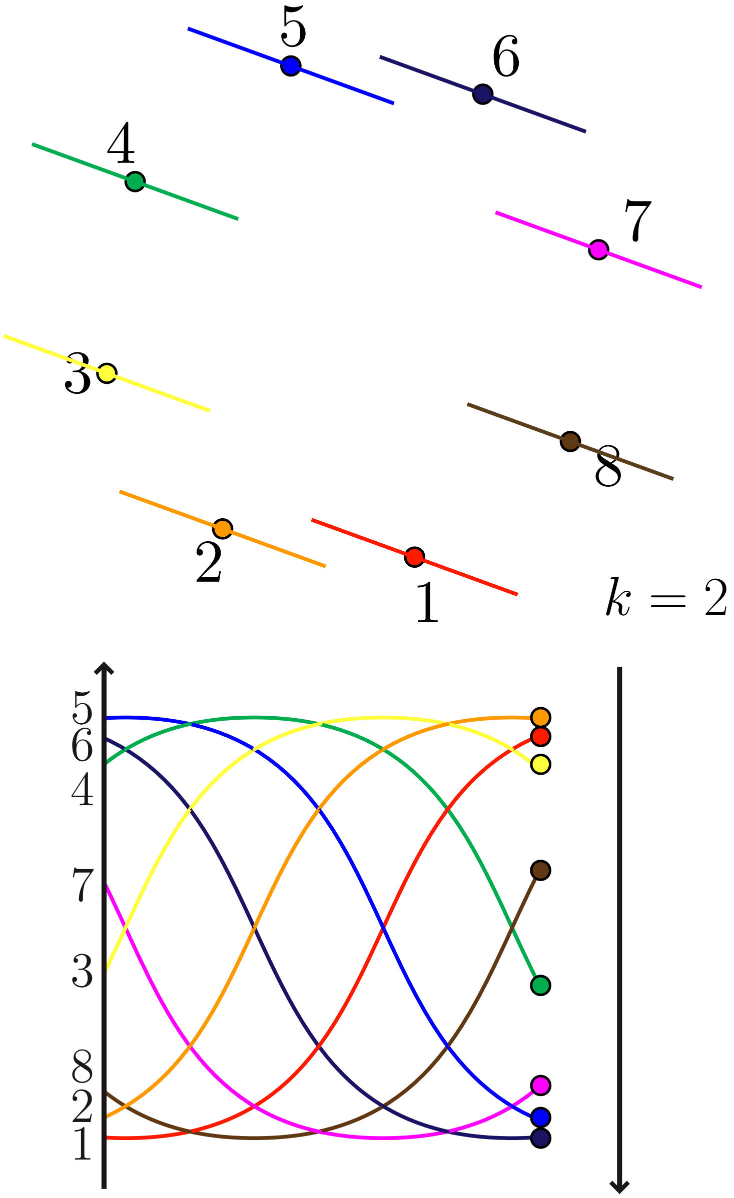

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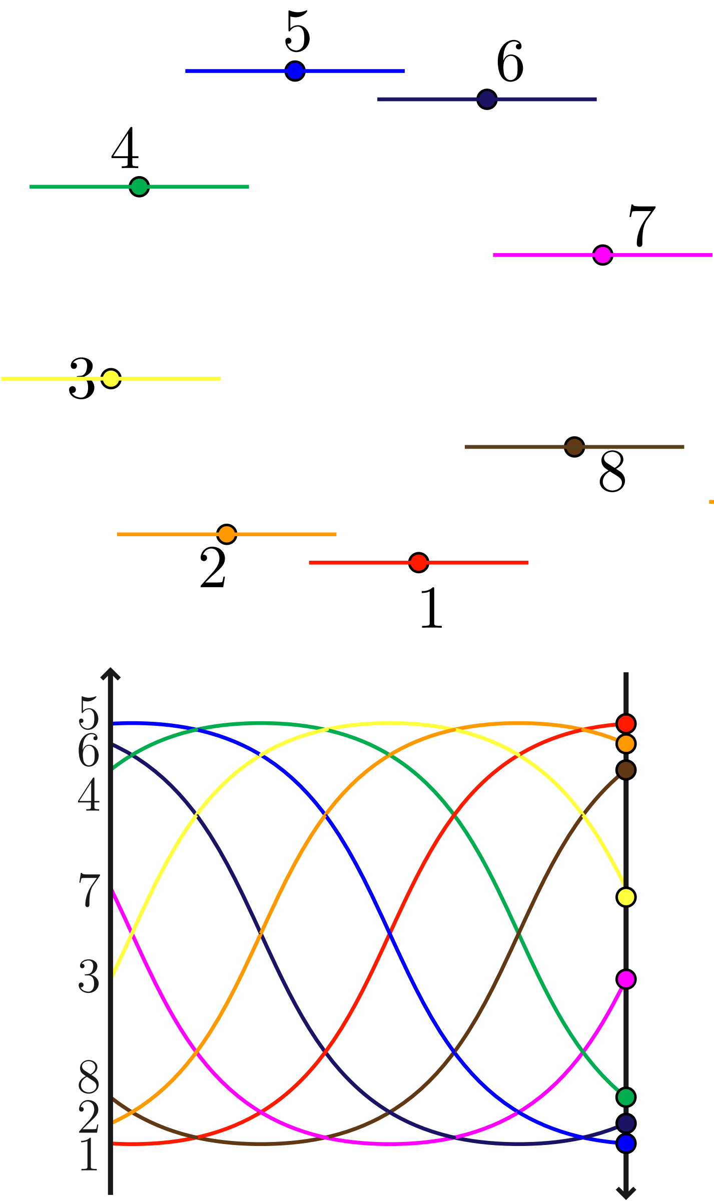
Triangulations



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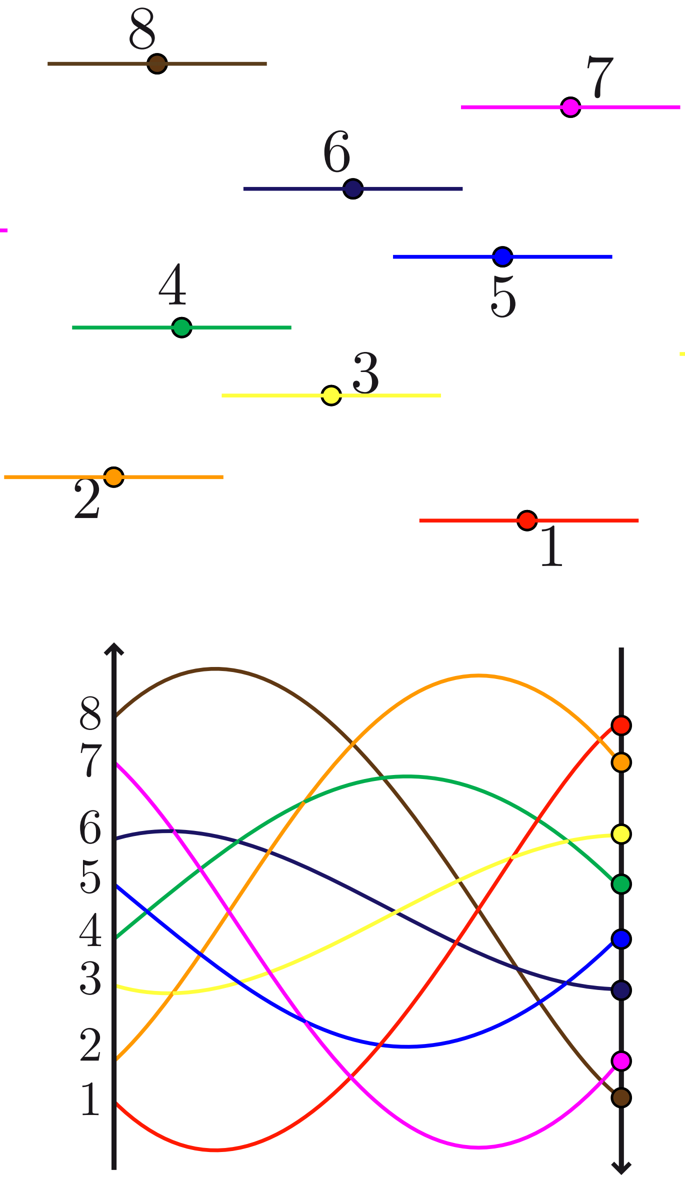
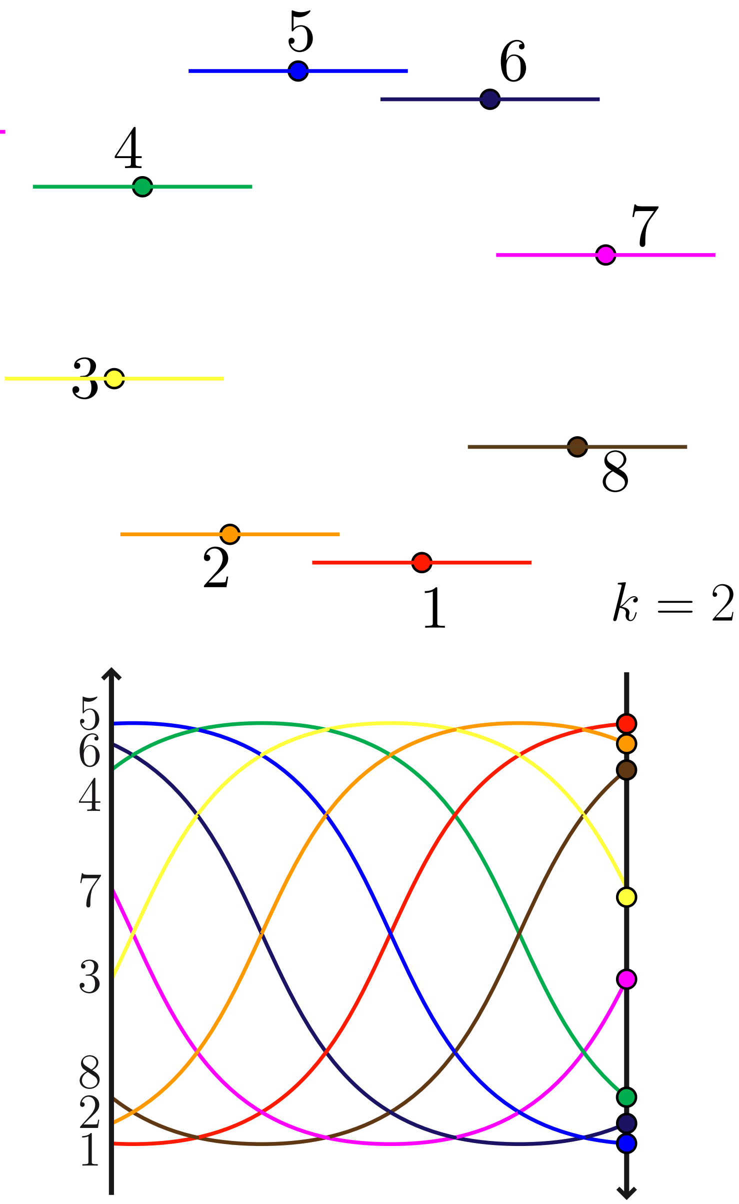
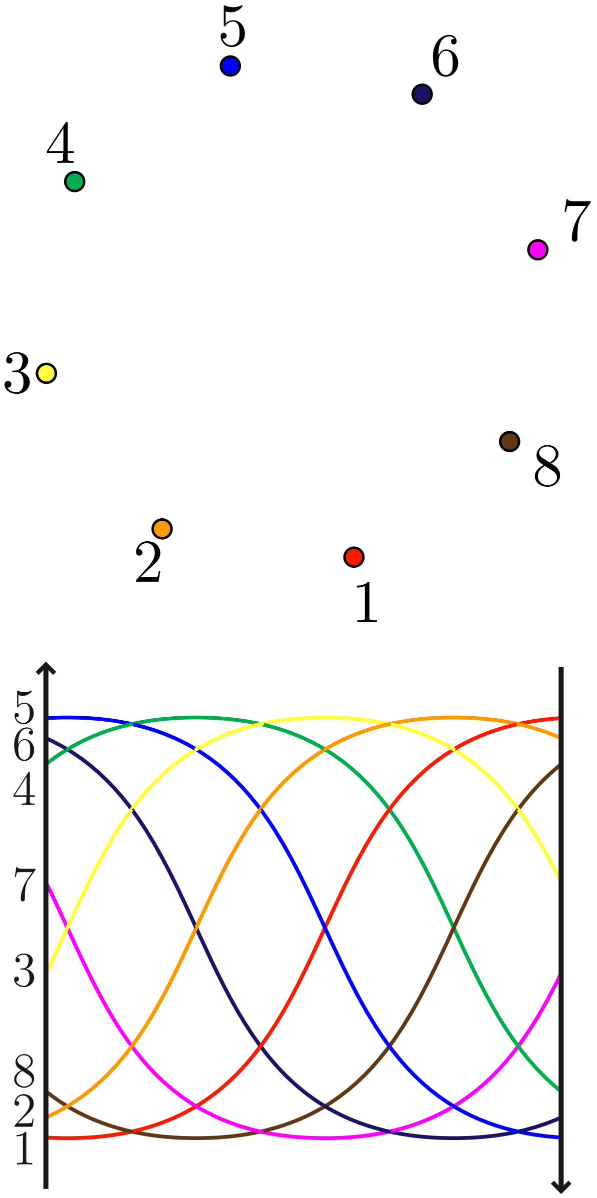


Multitriangulations



DUALITY

Triangulations



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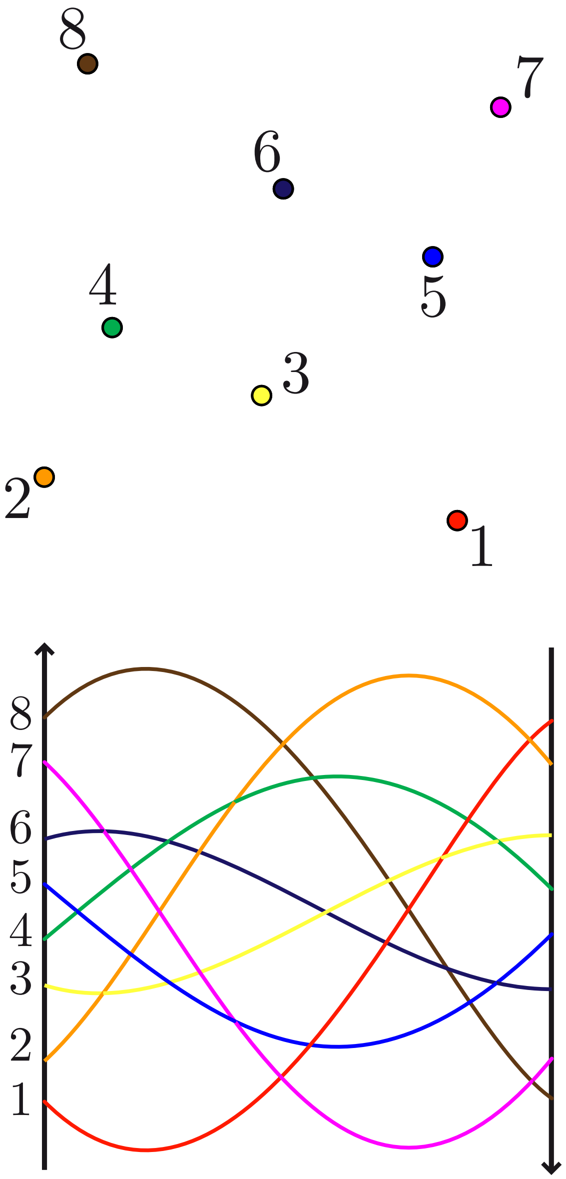

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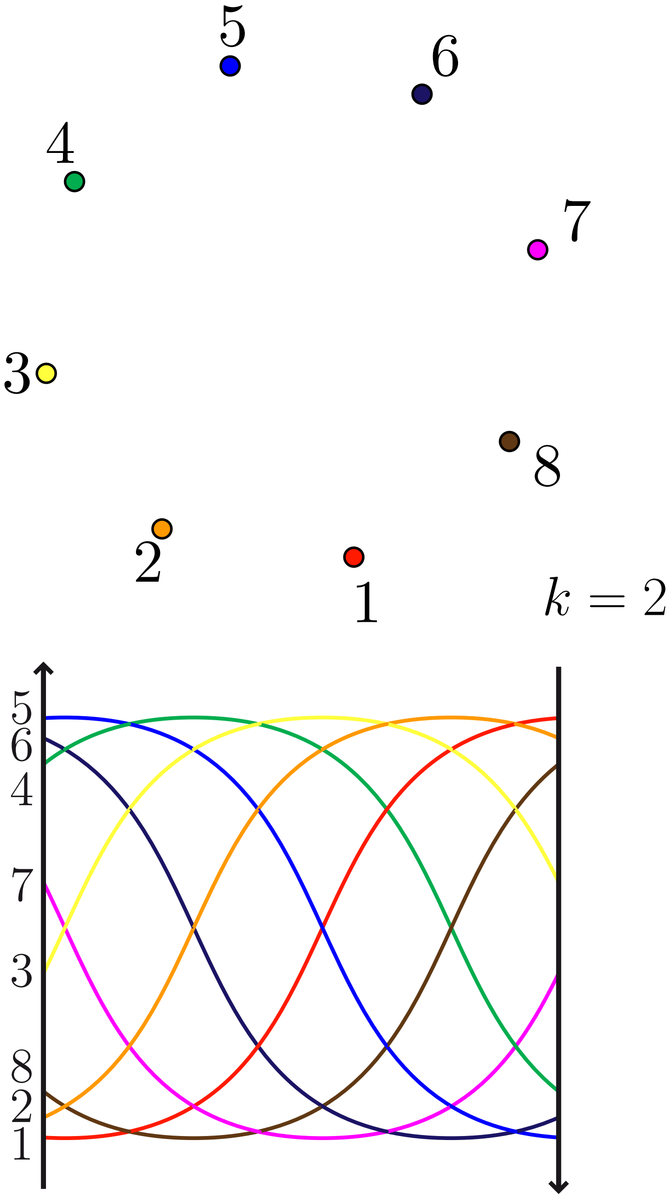
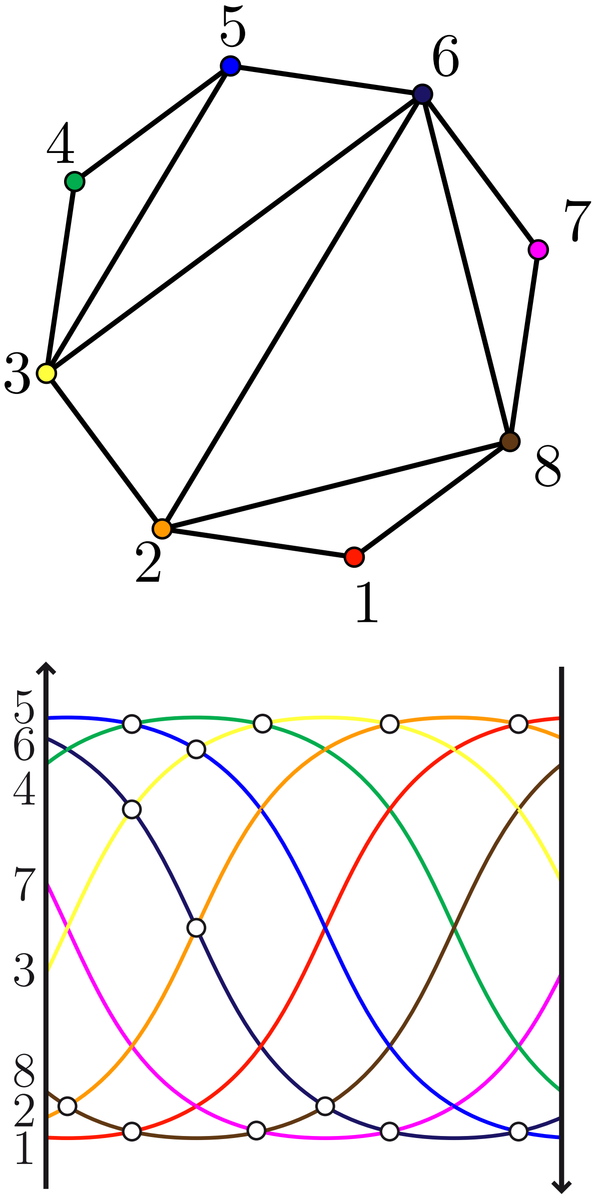


DUALITY

Triangulations



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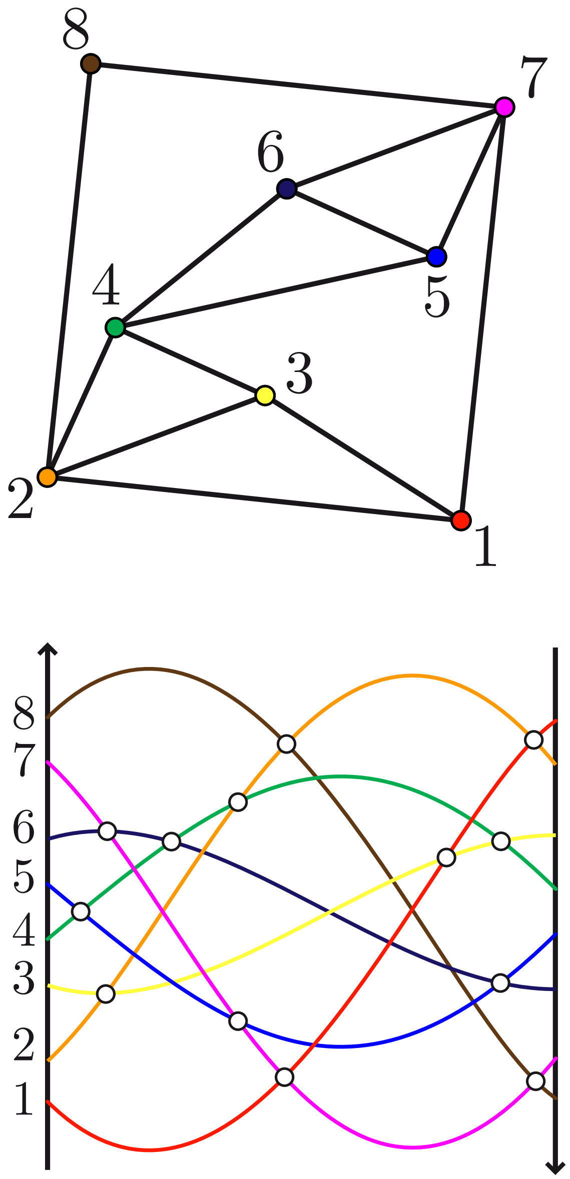

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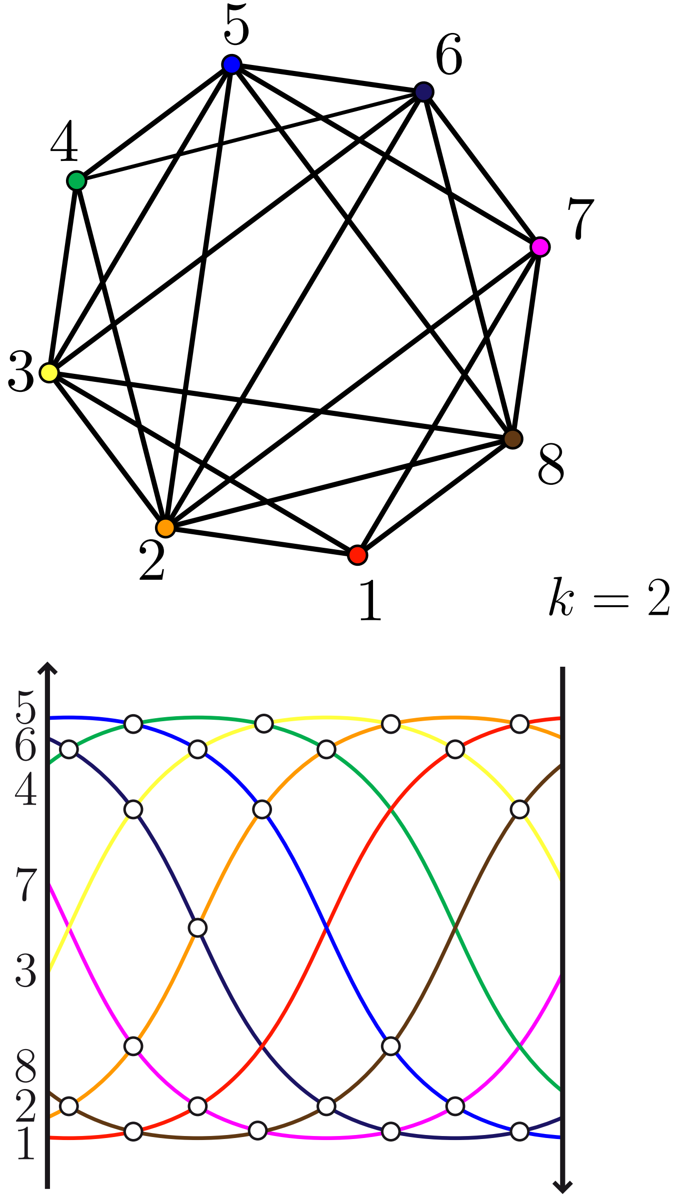
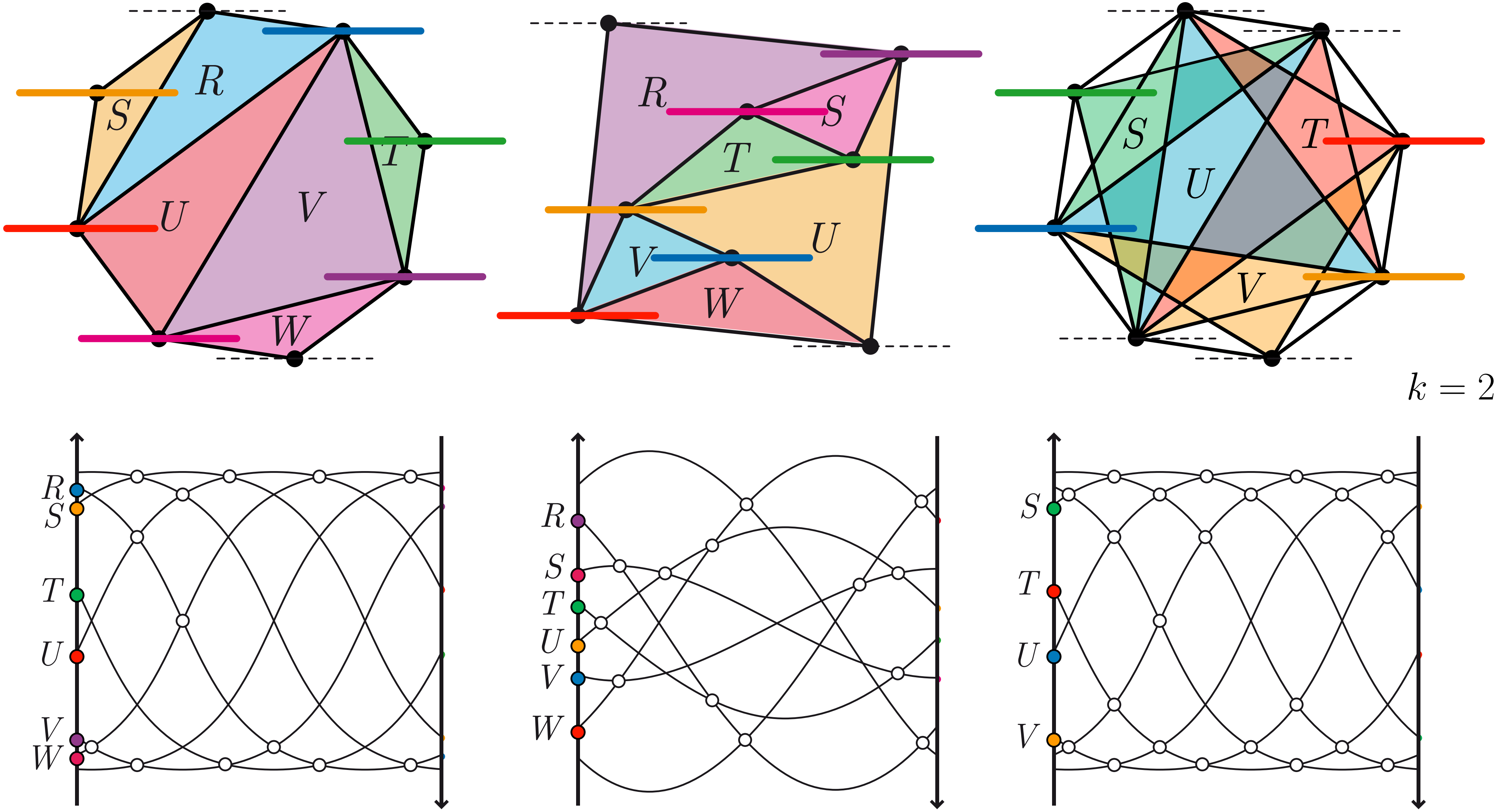


DUALITY

Triangulations

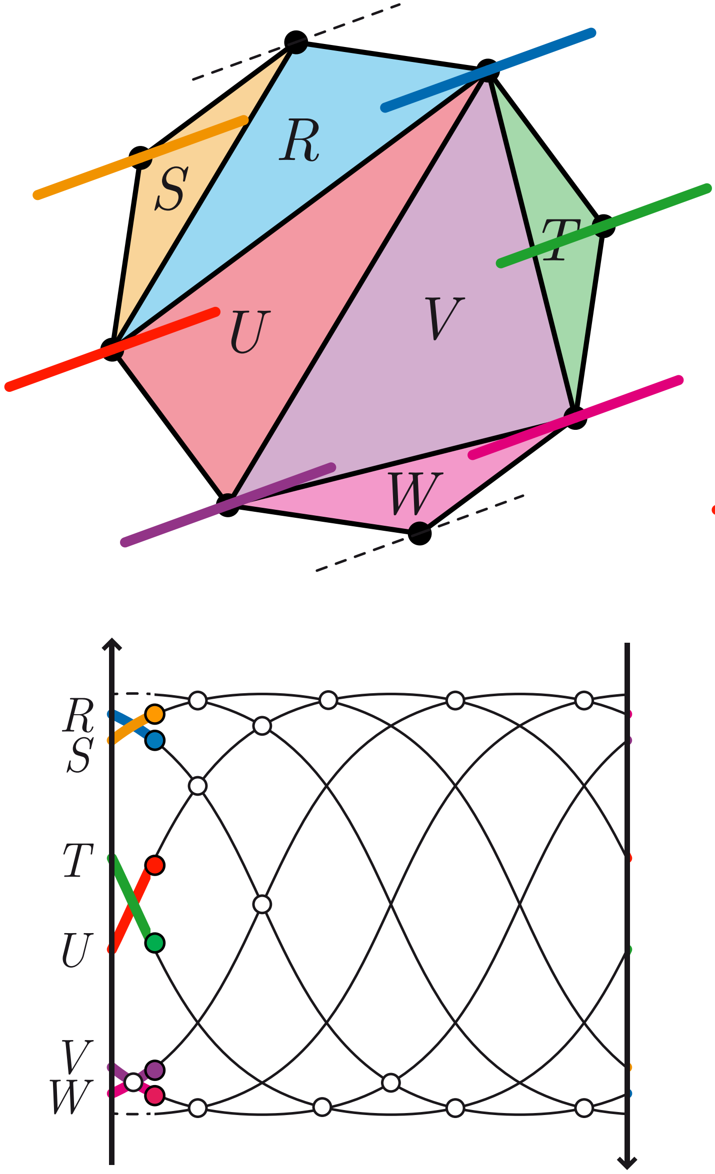
Pseudotriangulations

Multitriangulations

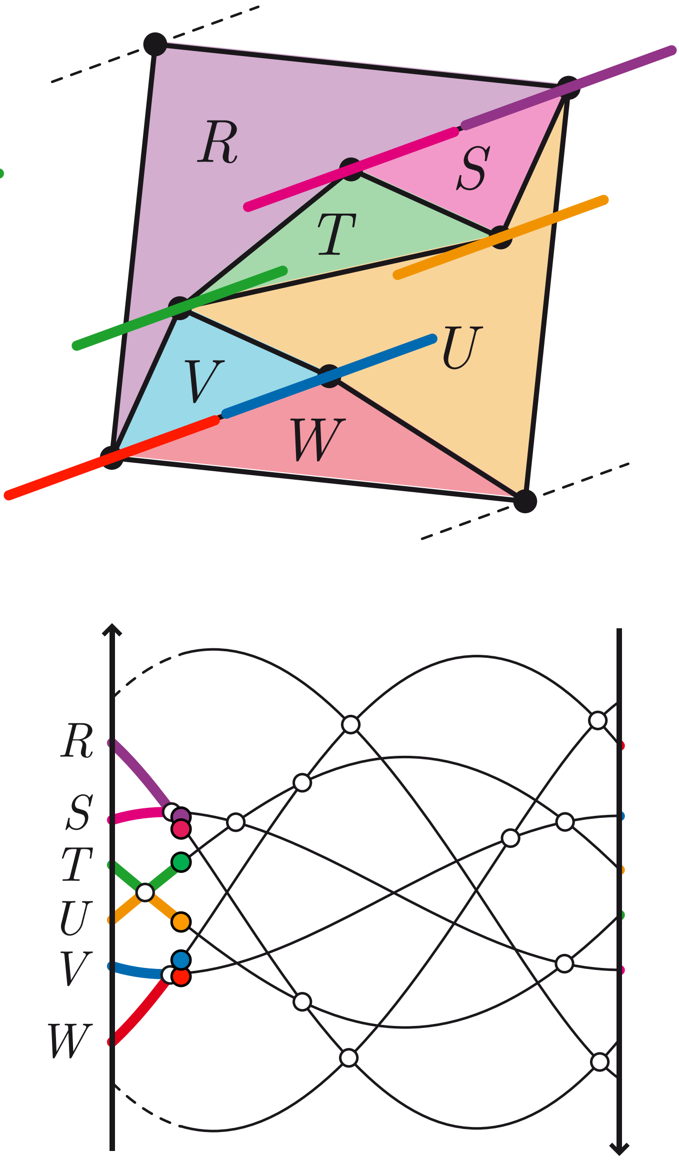


DUALITY

Triangulations



Pseudotriangulations



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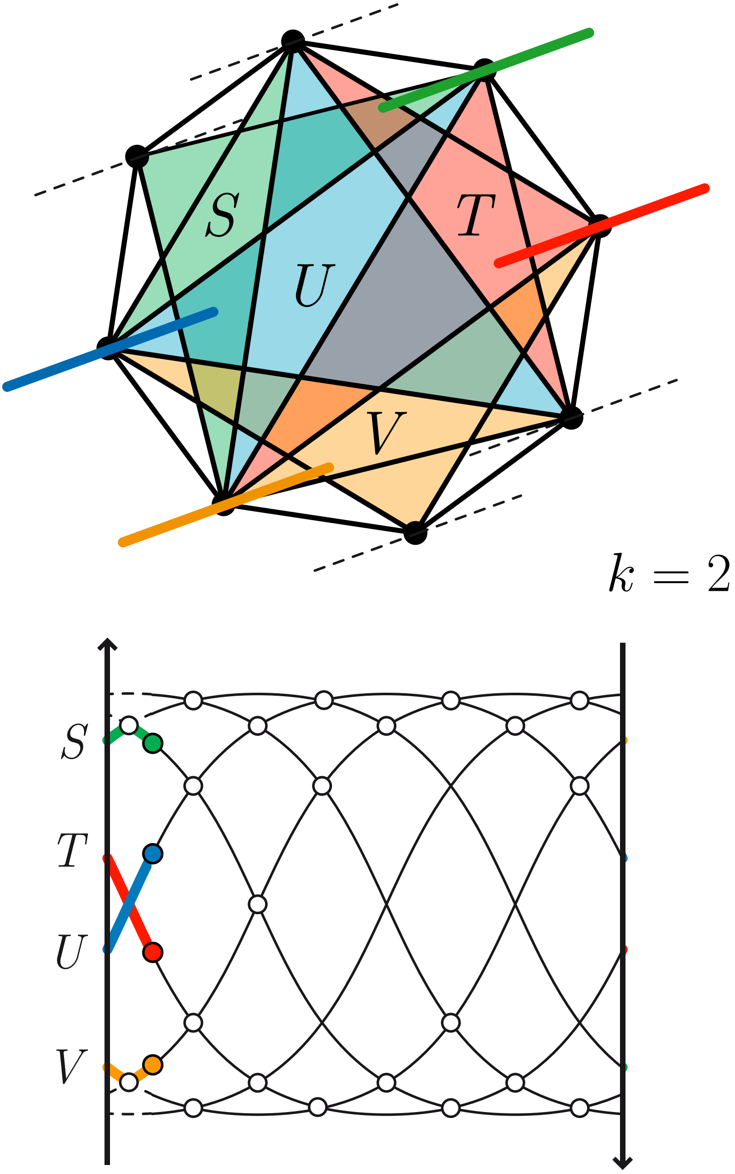
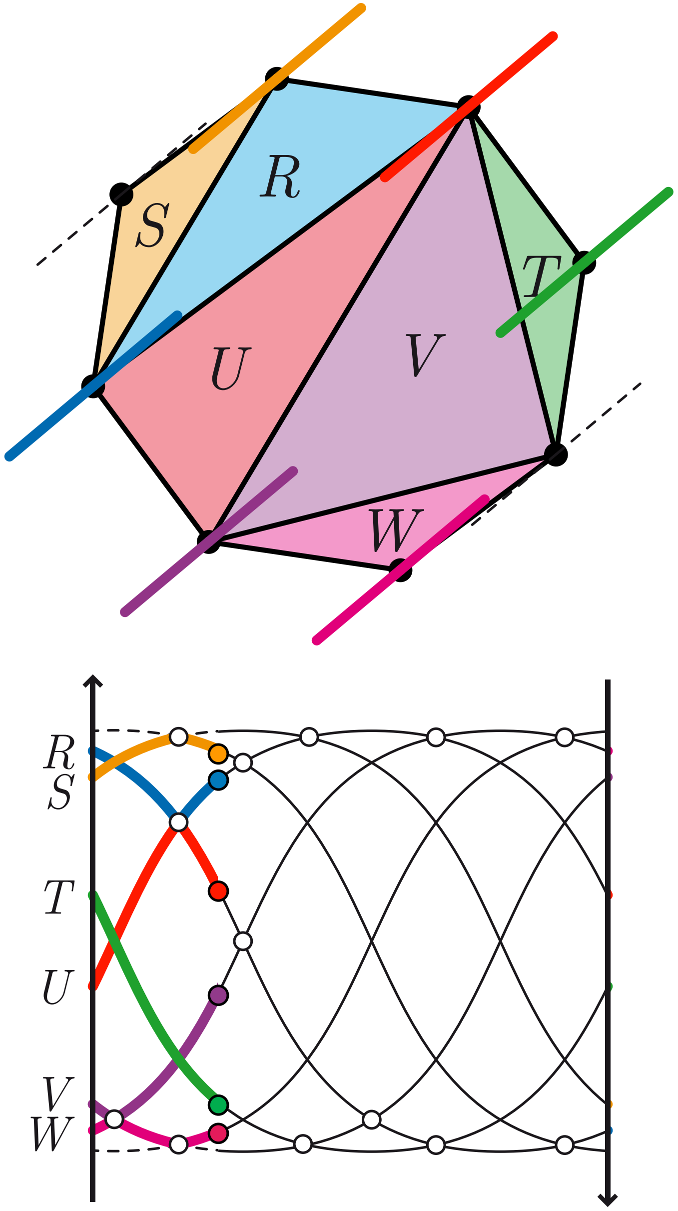
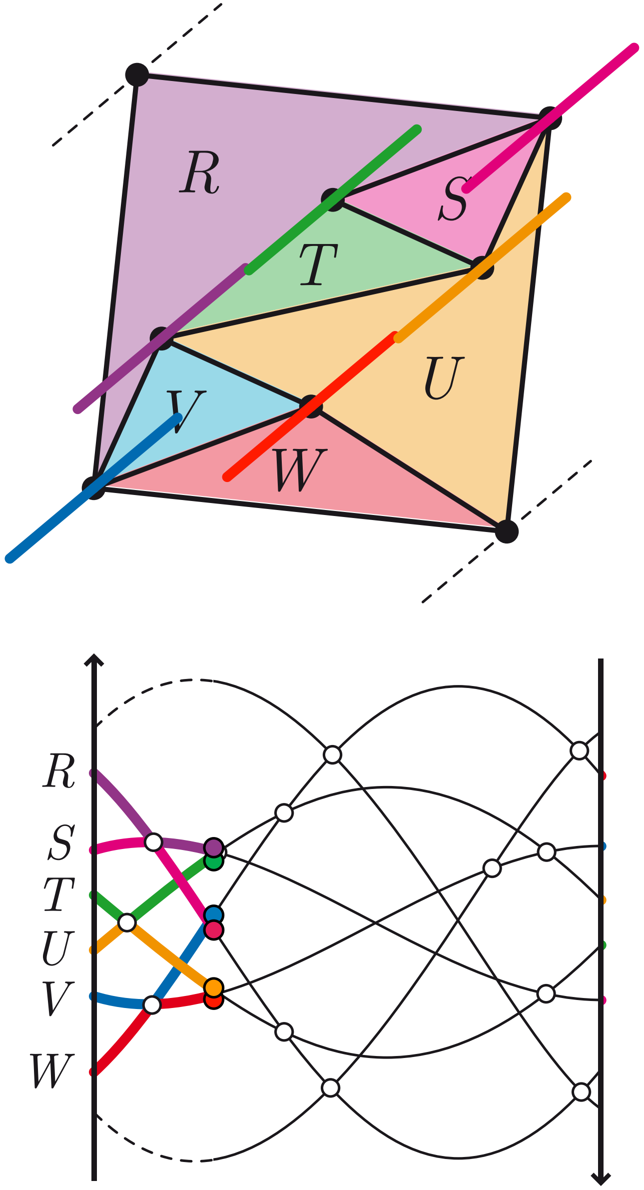


DUALITY

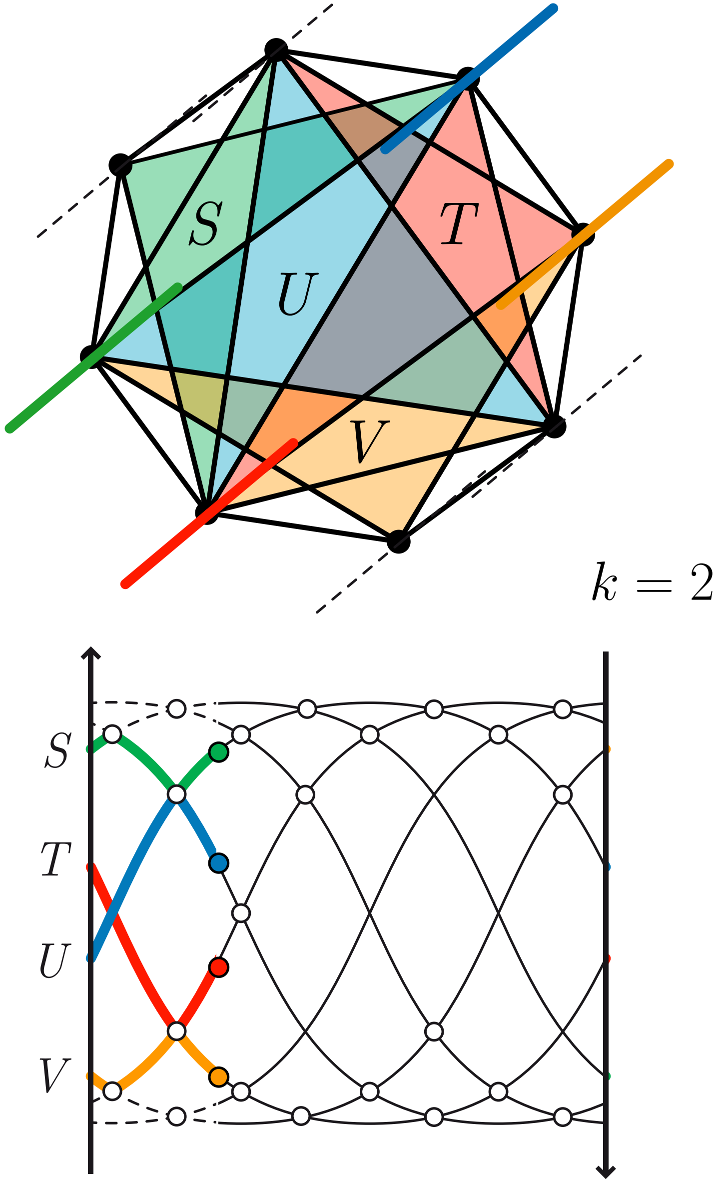
Triangulations



Pseudotriangulations

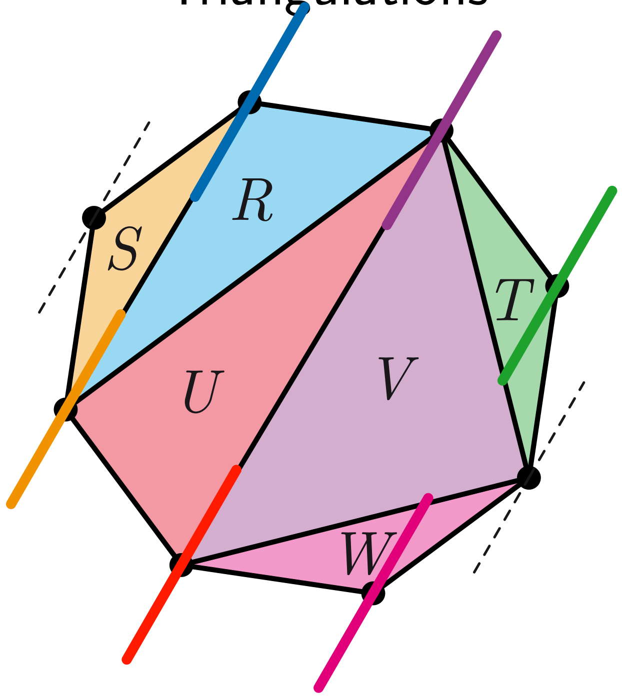


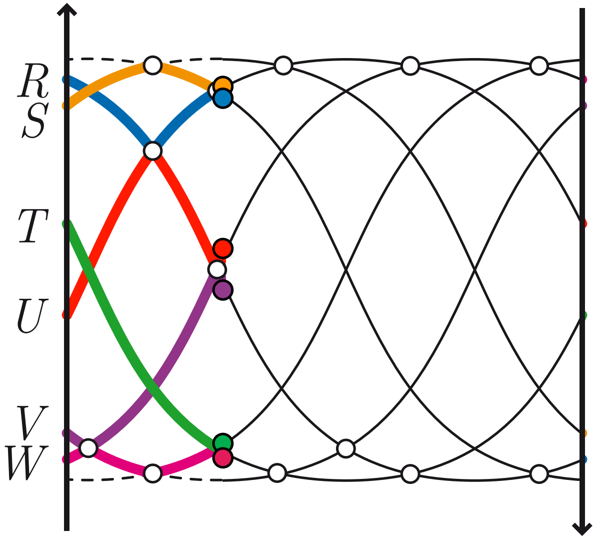
Multitriangulations



DUALITY

Triangulations





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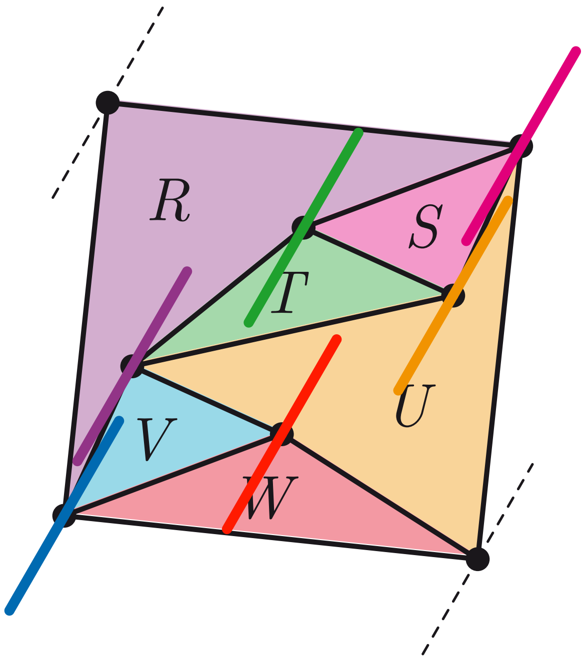

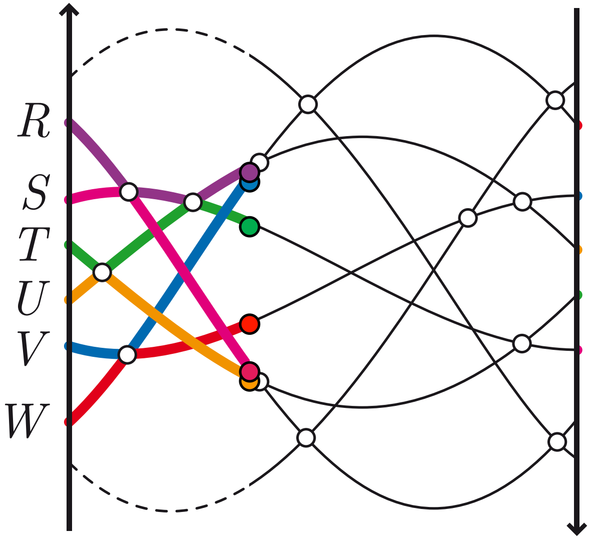

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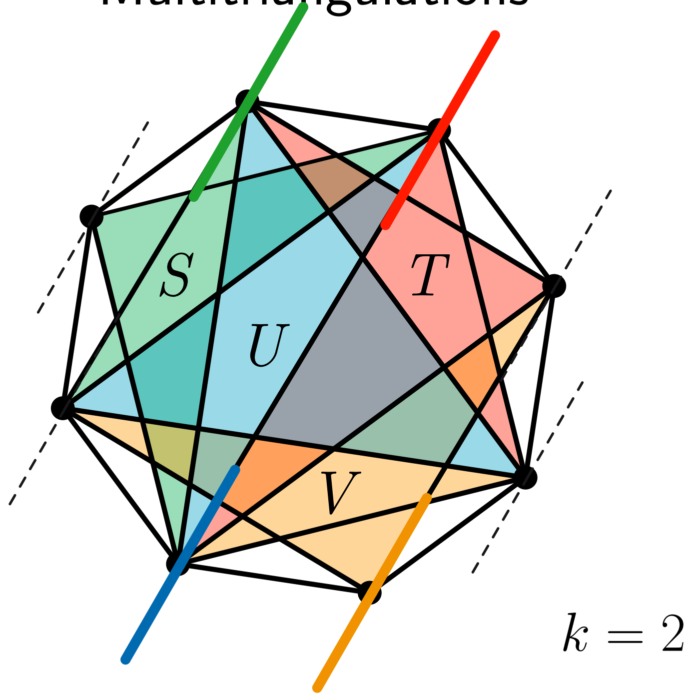
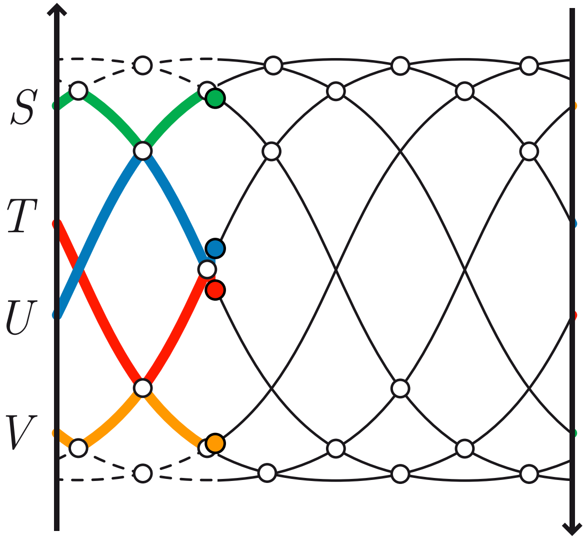


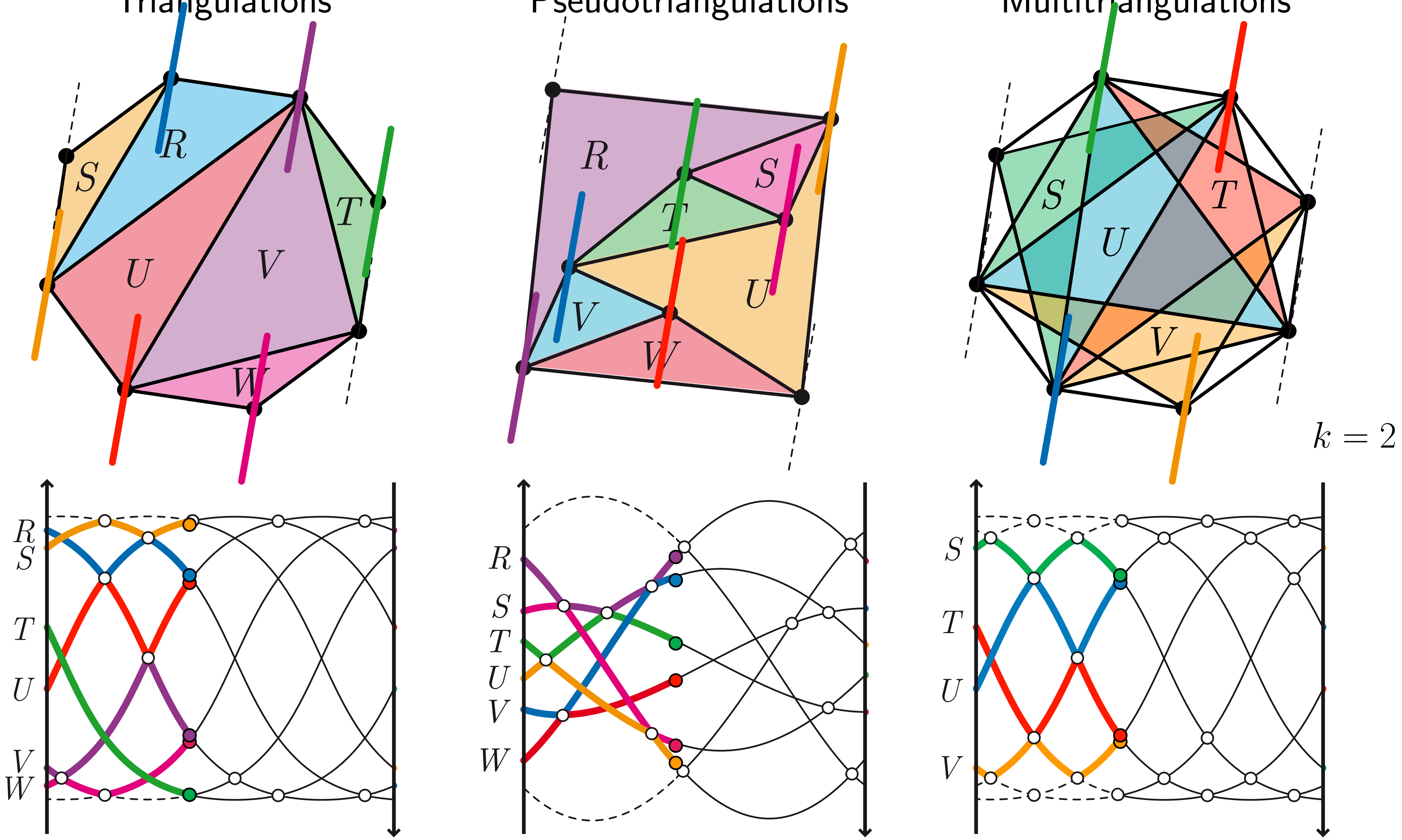


DUALITY

Triangulations

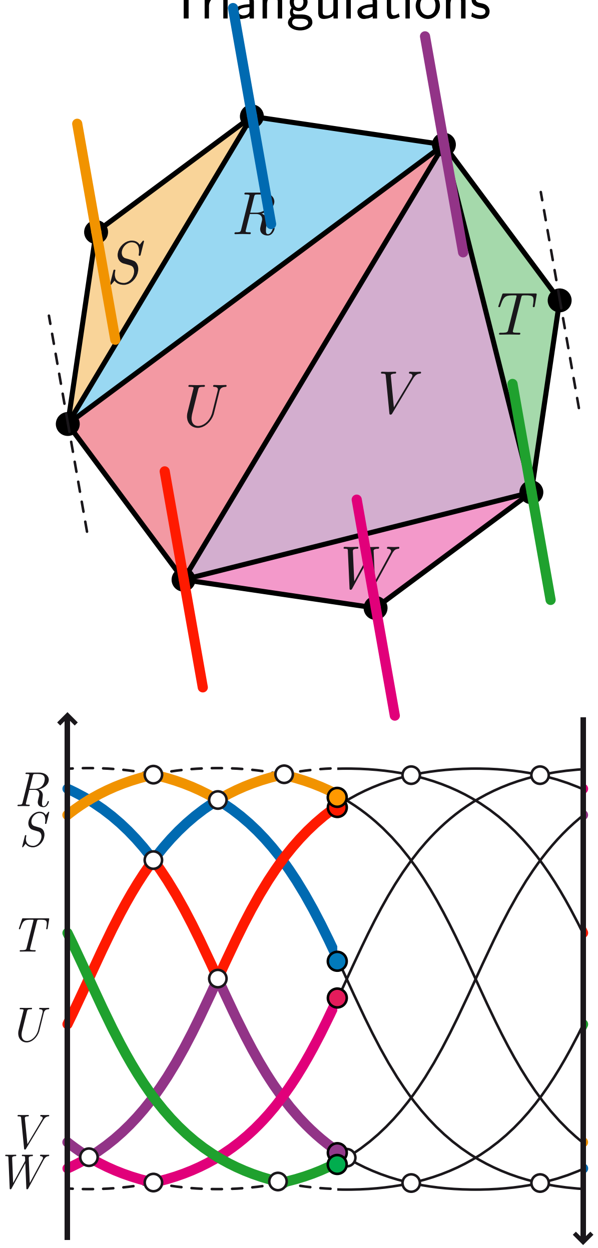
Pseudotriangulations

Multitriangulations

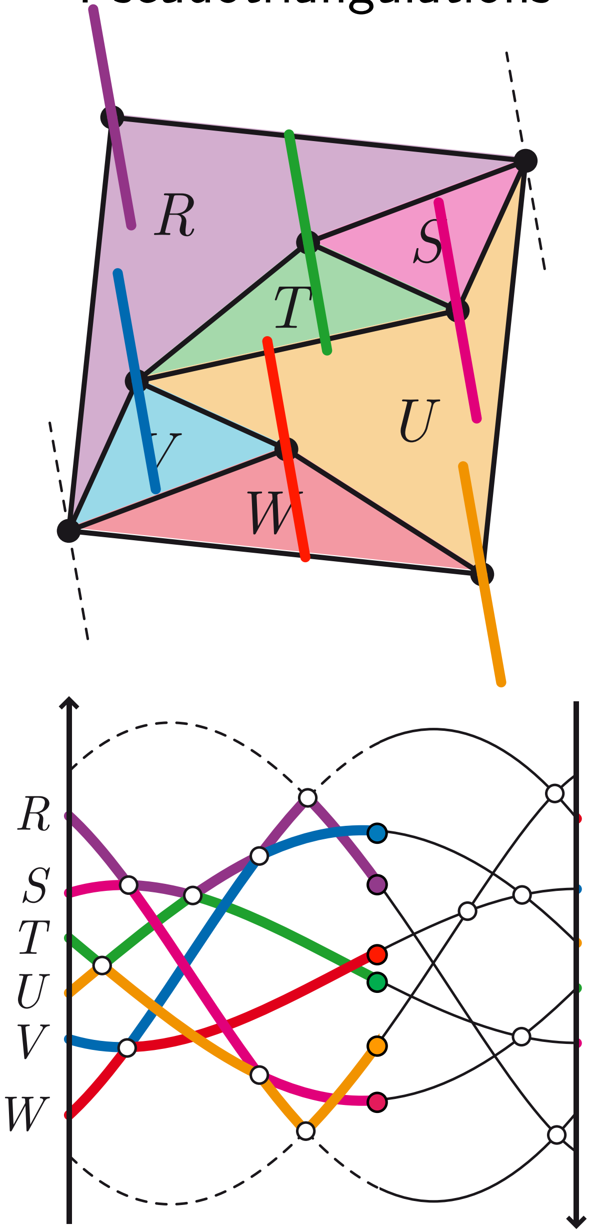


DUALITY

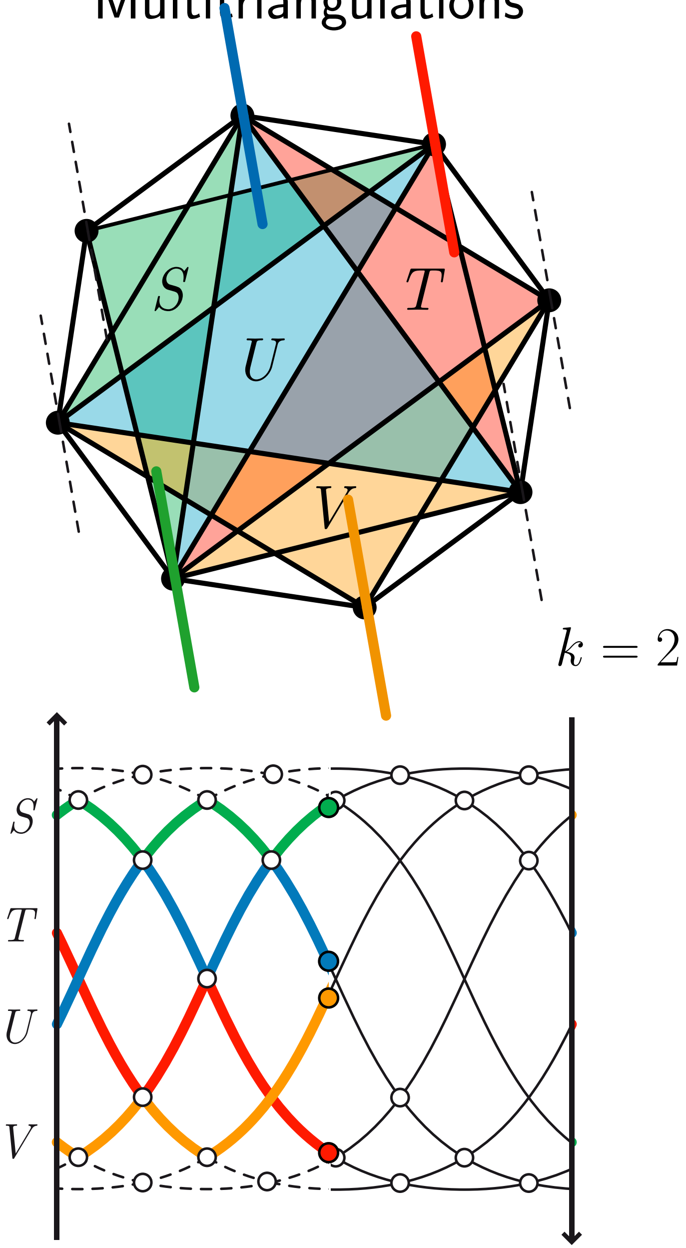
Triangulations



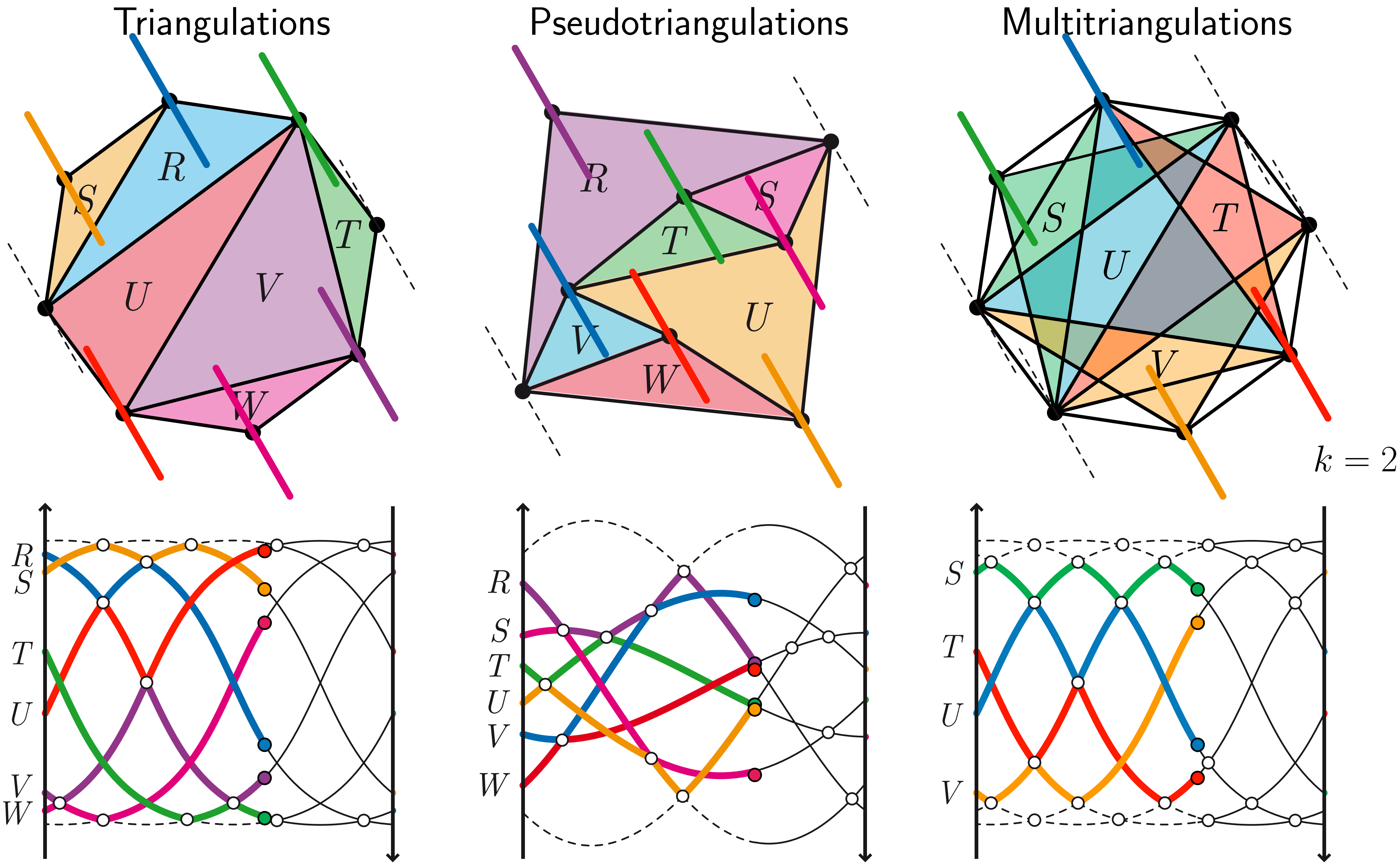
Pseudotriangulations



Multitriangulations



DUALITY


Triangulations
Pseudotriangulations
Multitriangulations
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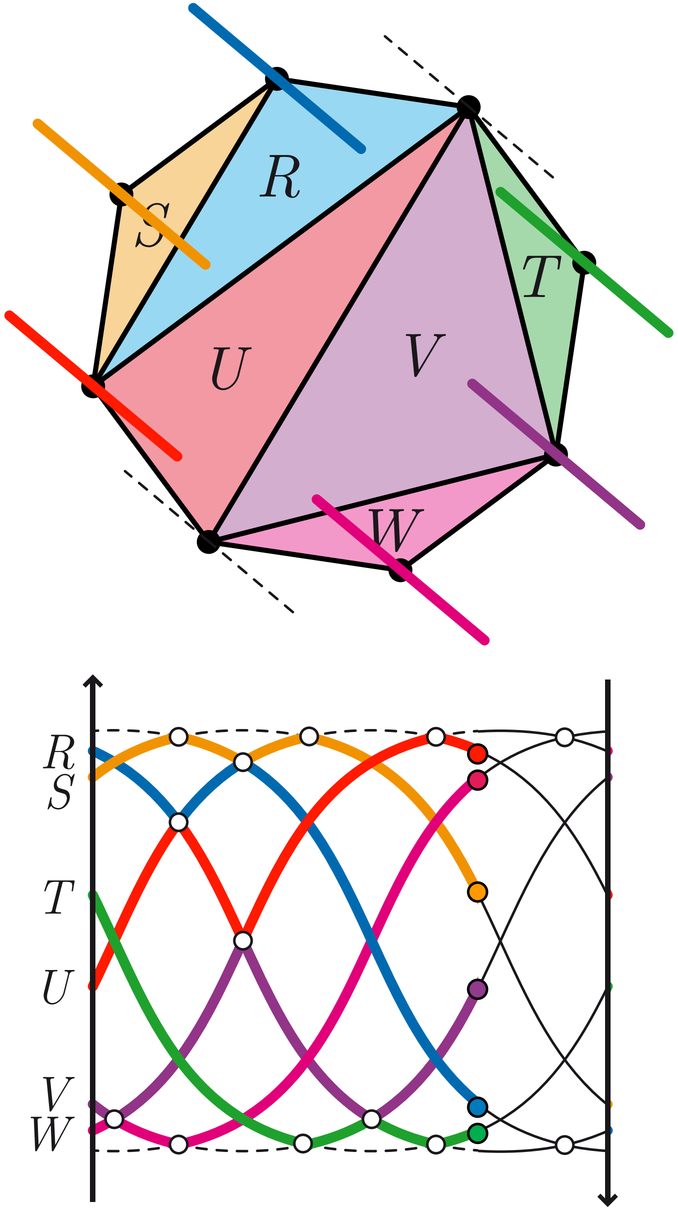
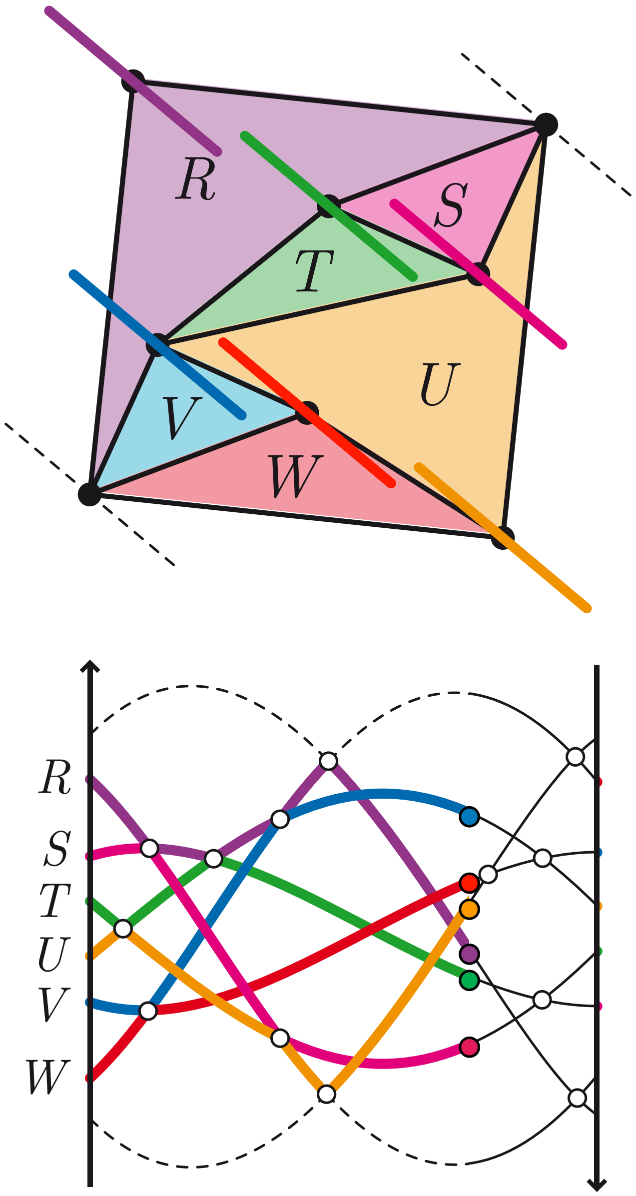
DUALITY

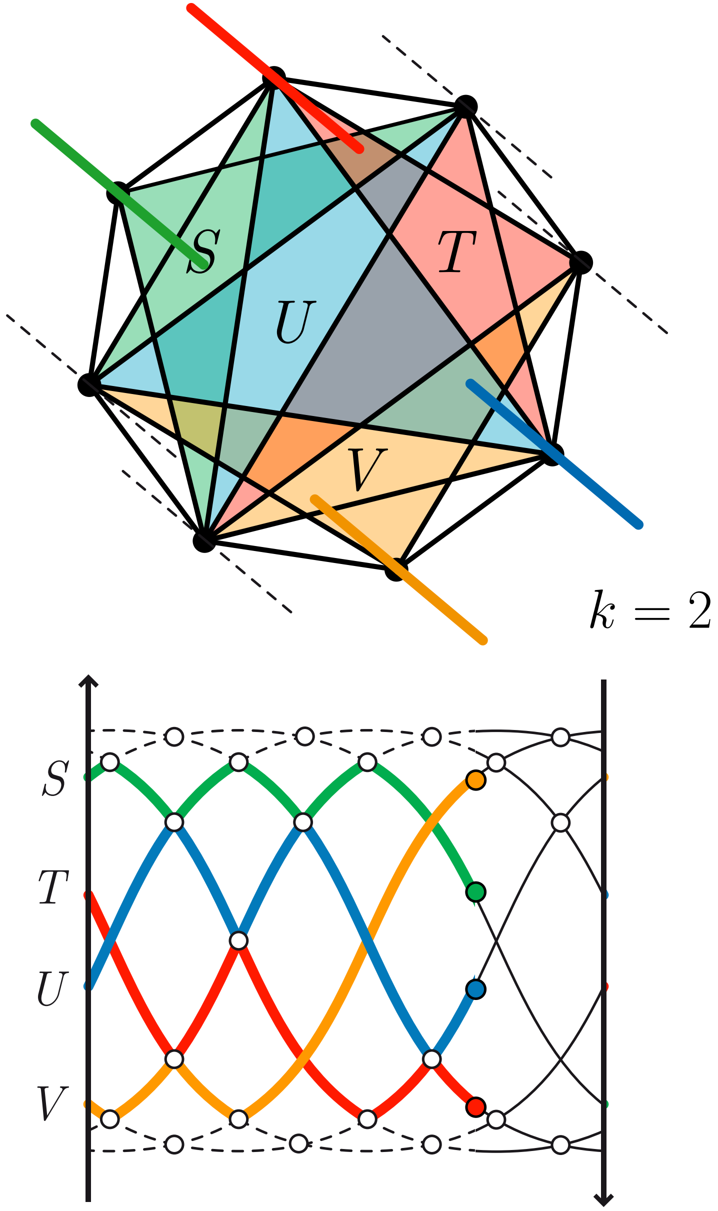
Triangulations



Pseudotriangulations

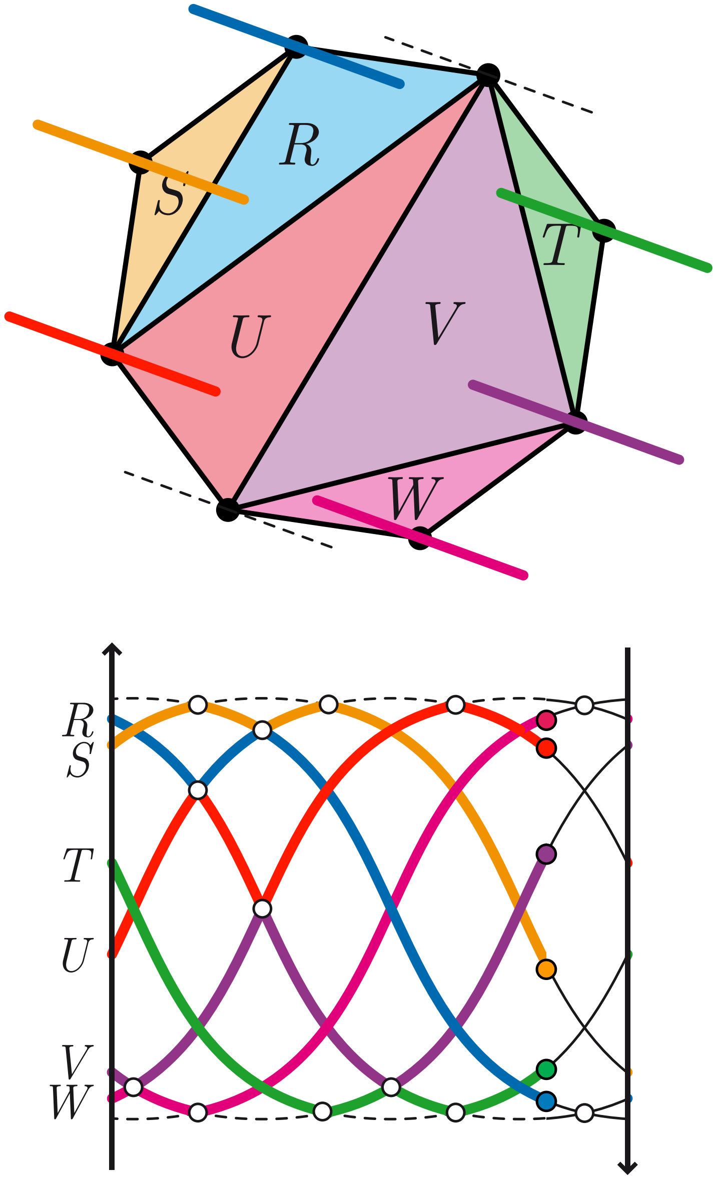


Multitriangulations

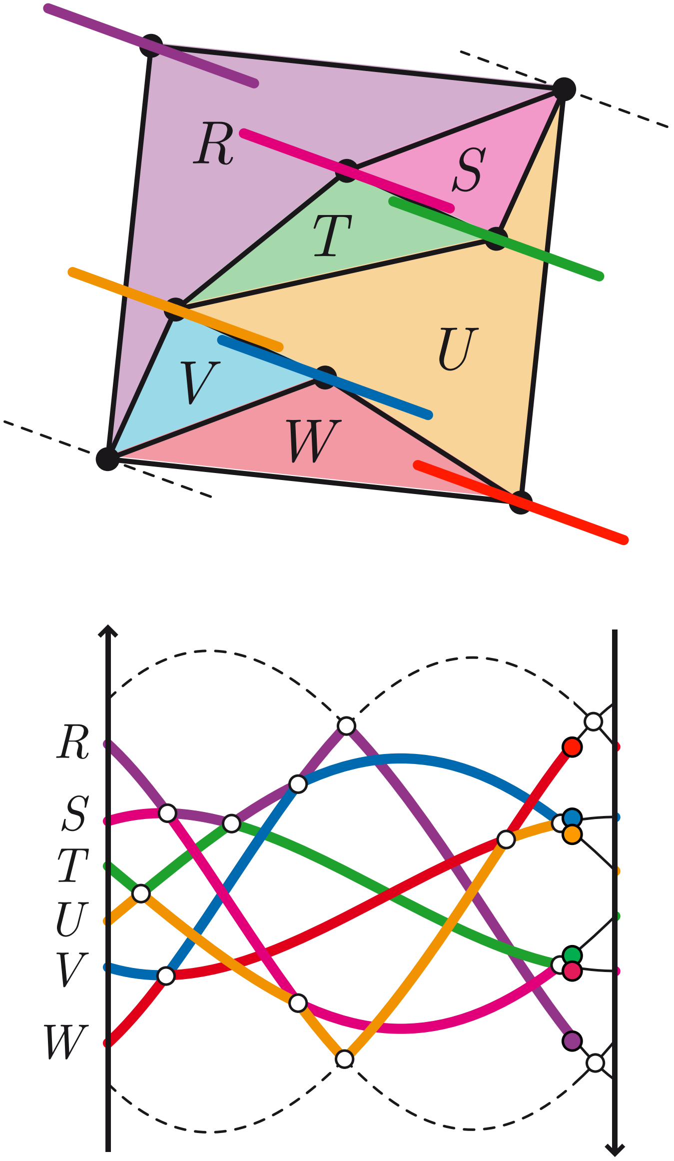


DUALITY

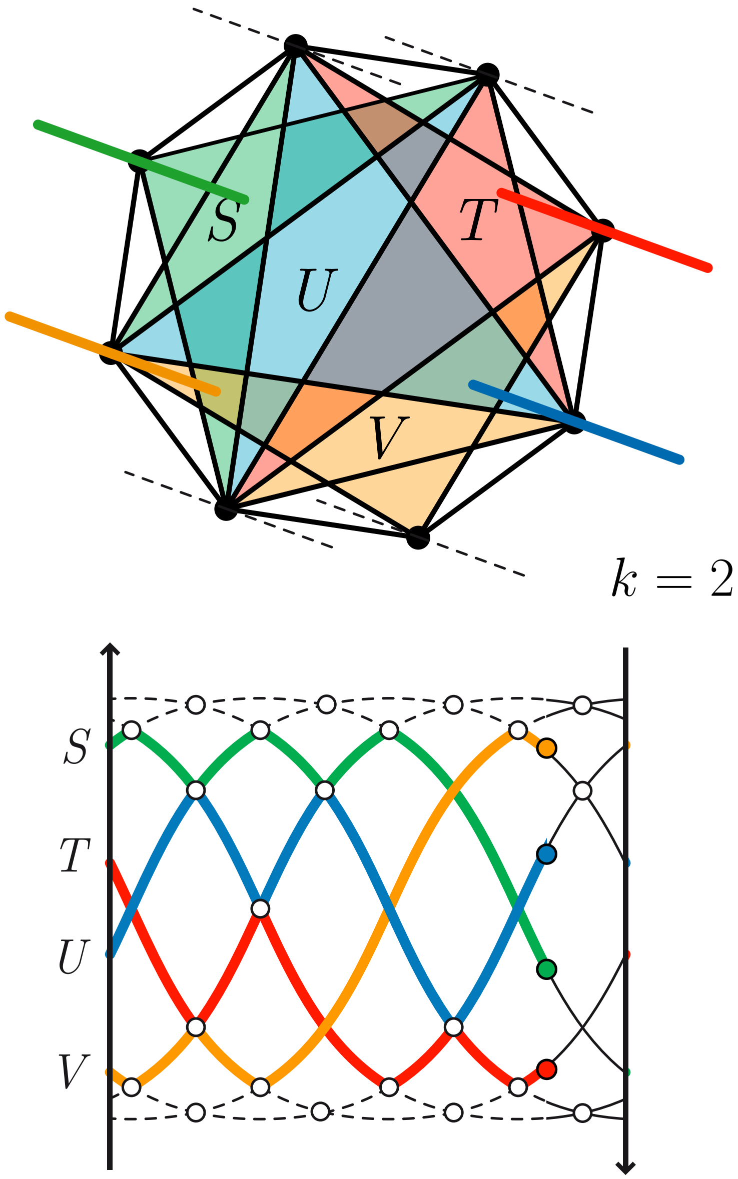
Triangulations



Pseudotriangulations



Multitriangulations



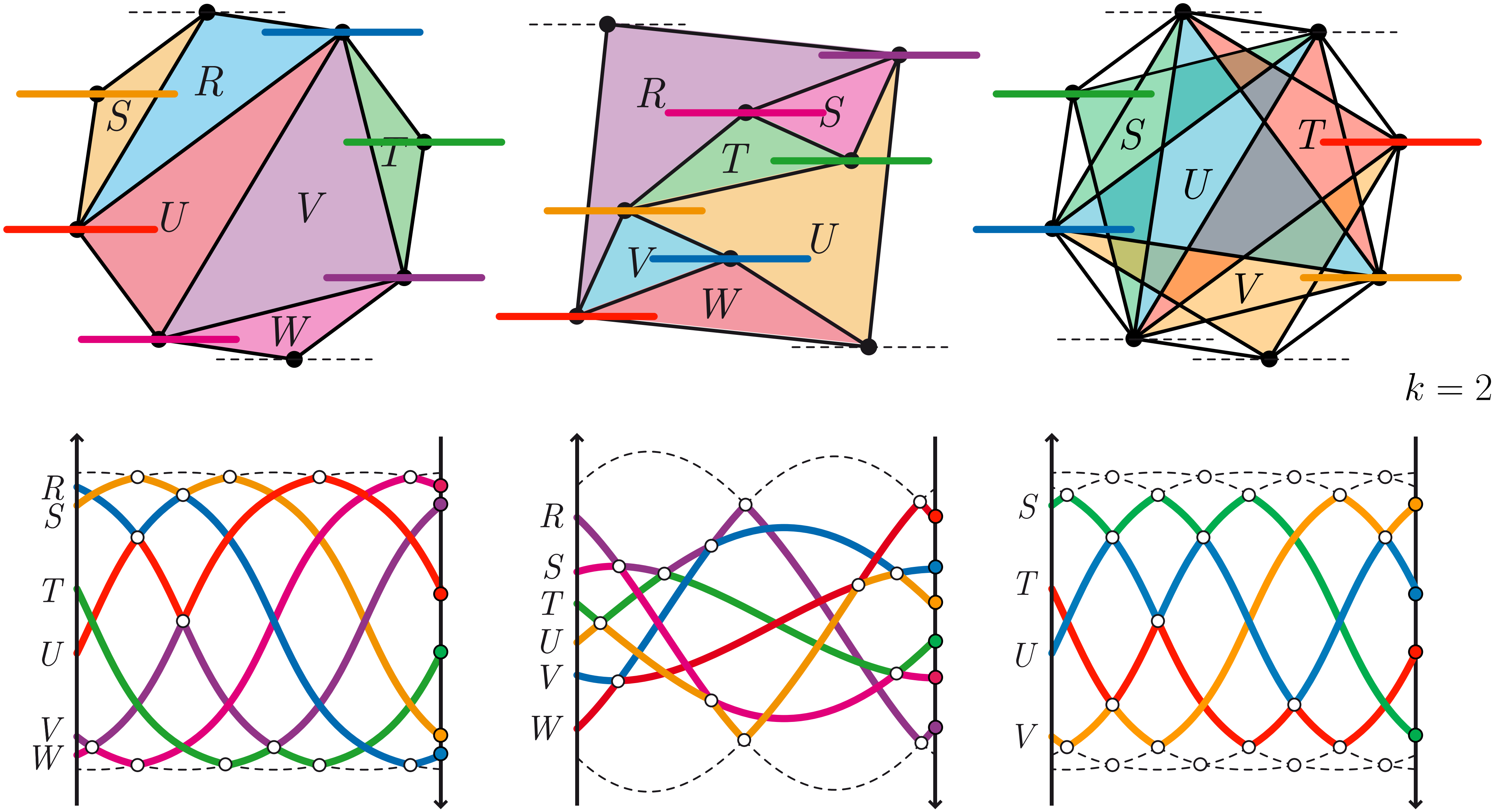
$k = 2$

DUALITY

Triangulations

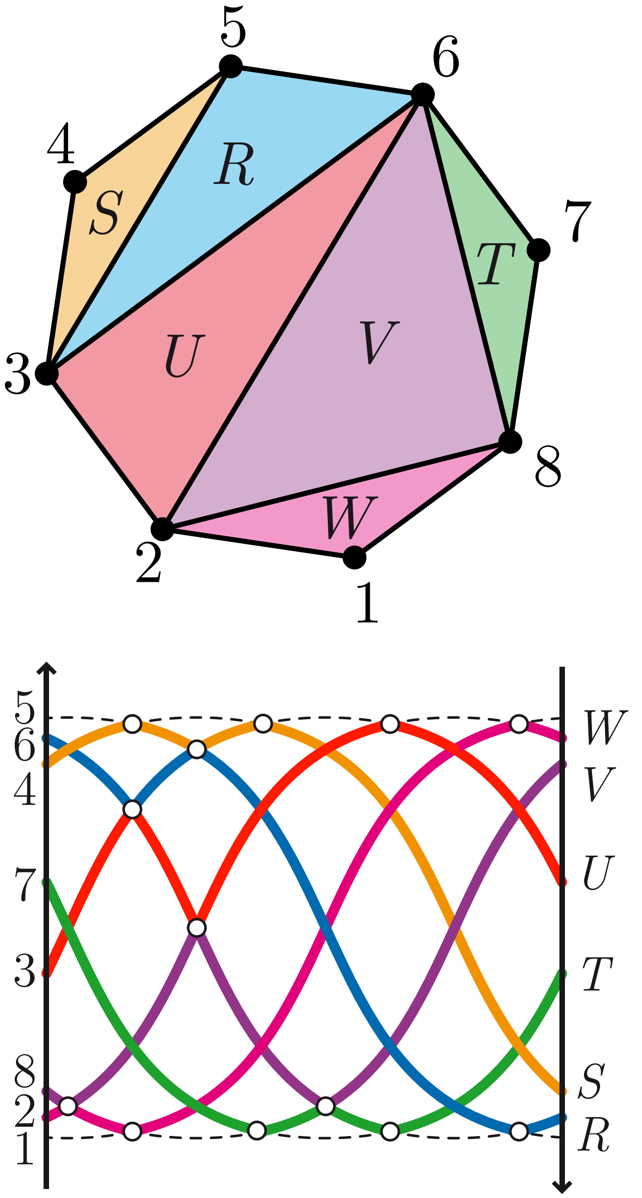
Pseudotriangulations

Multitriangulations

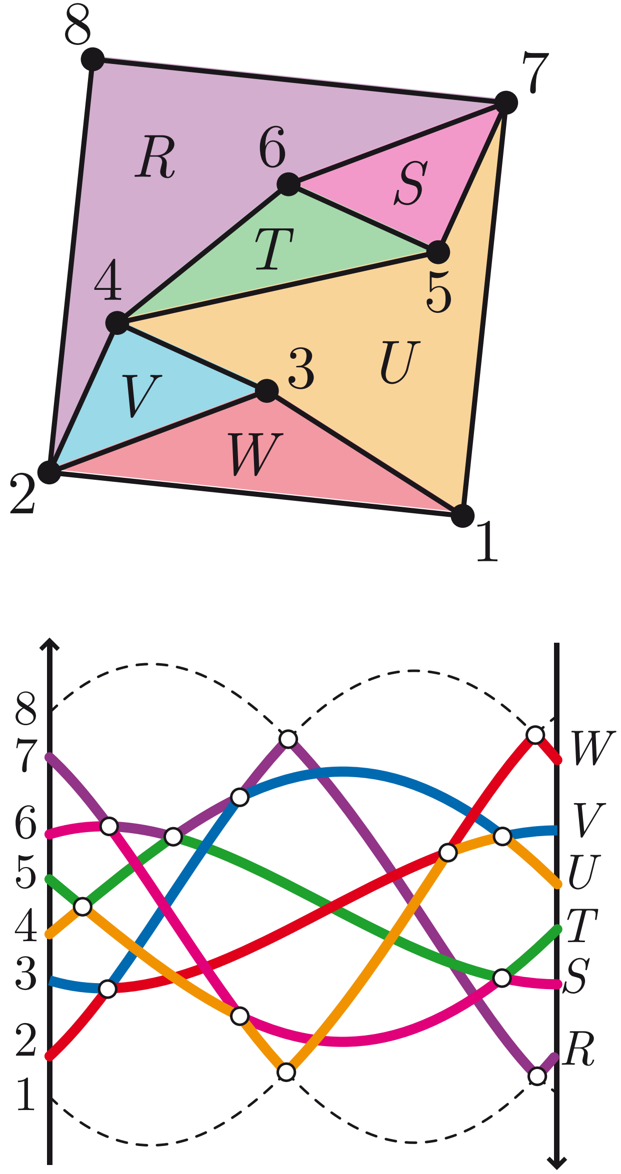


DUALITY

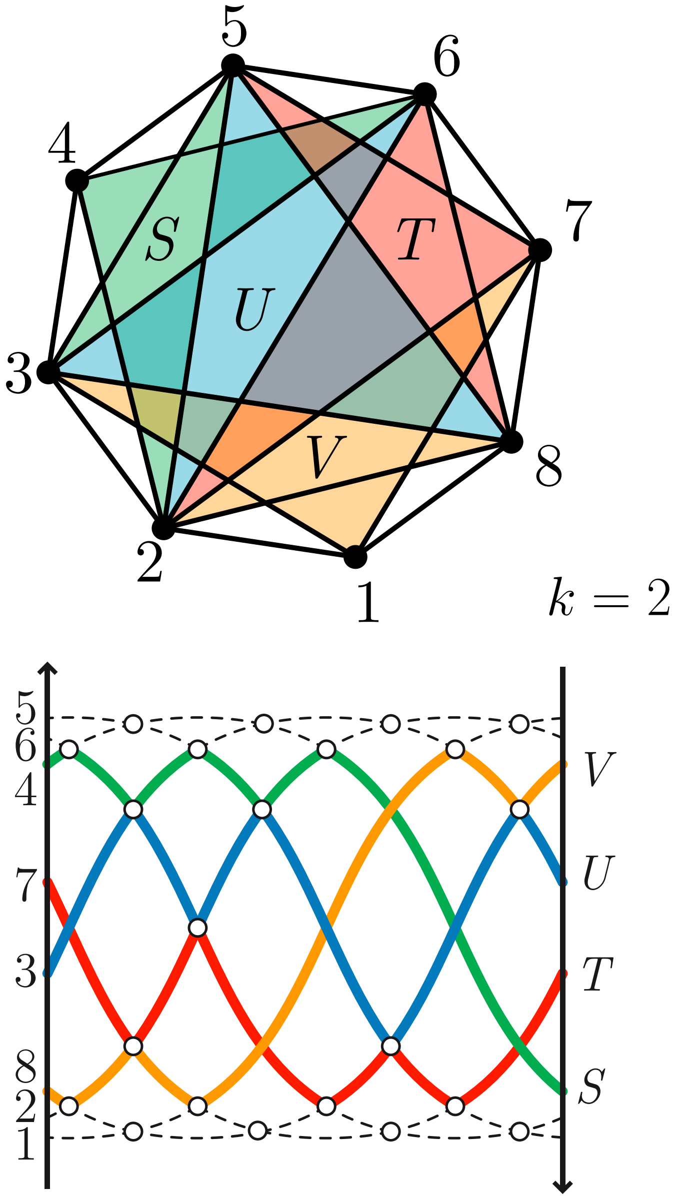
Triangulations



Pseudotriangulations

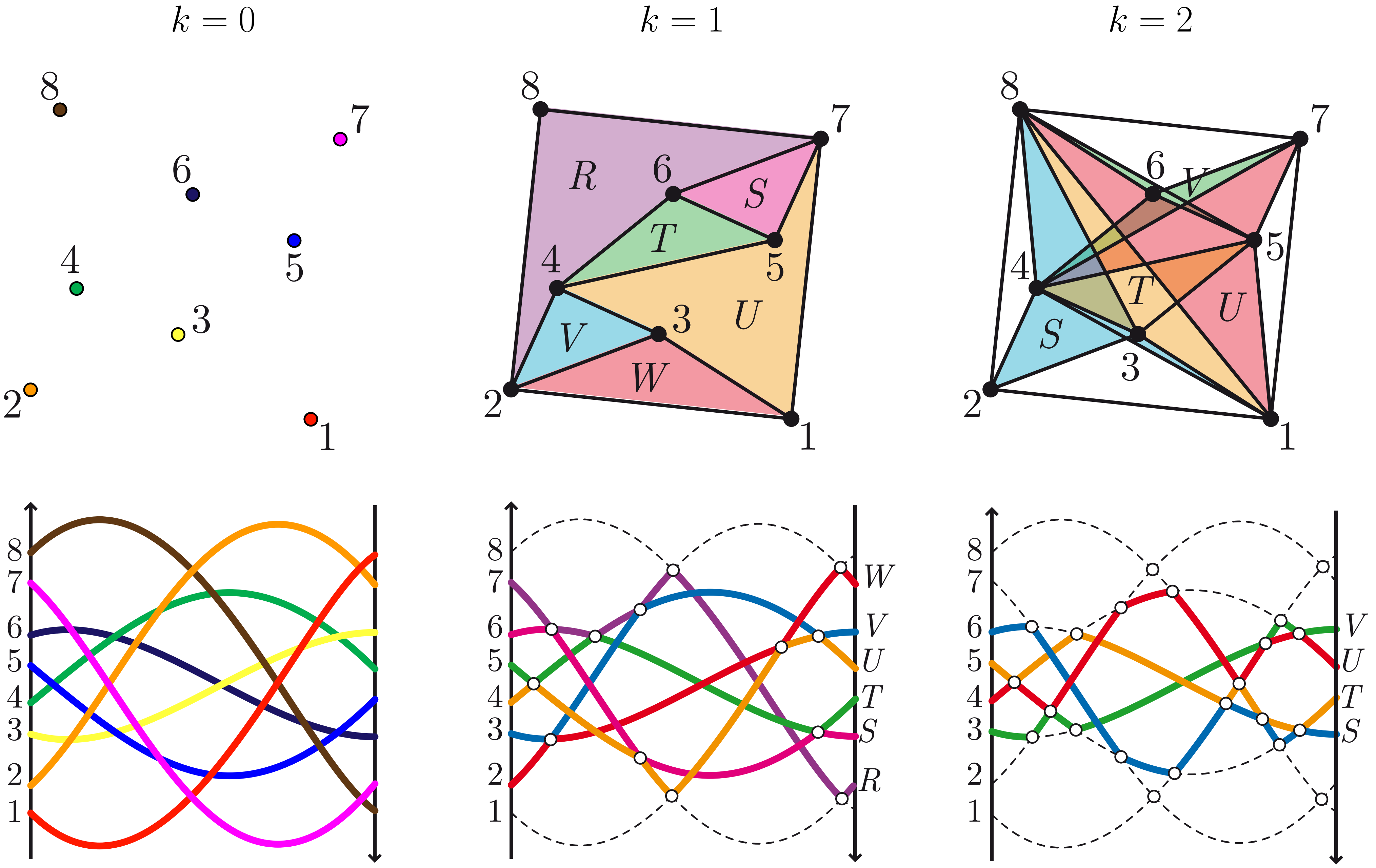


Multitriangulations



P.-Pocchiola, *Multitriangulations, pseudotriangulations and primitive sorting networks* ('12)

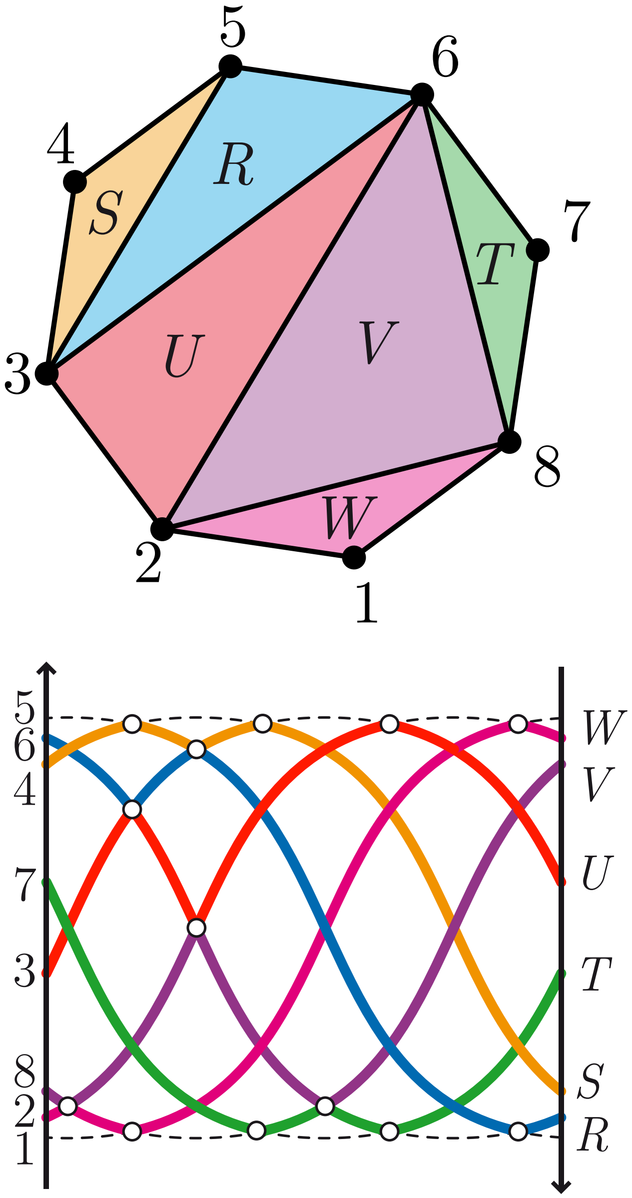
MULTIPSEUDOTRIANGULATIONS



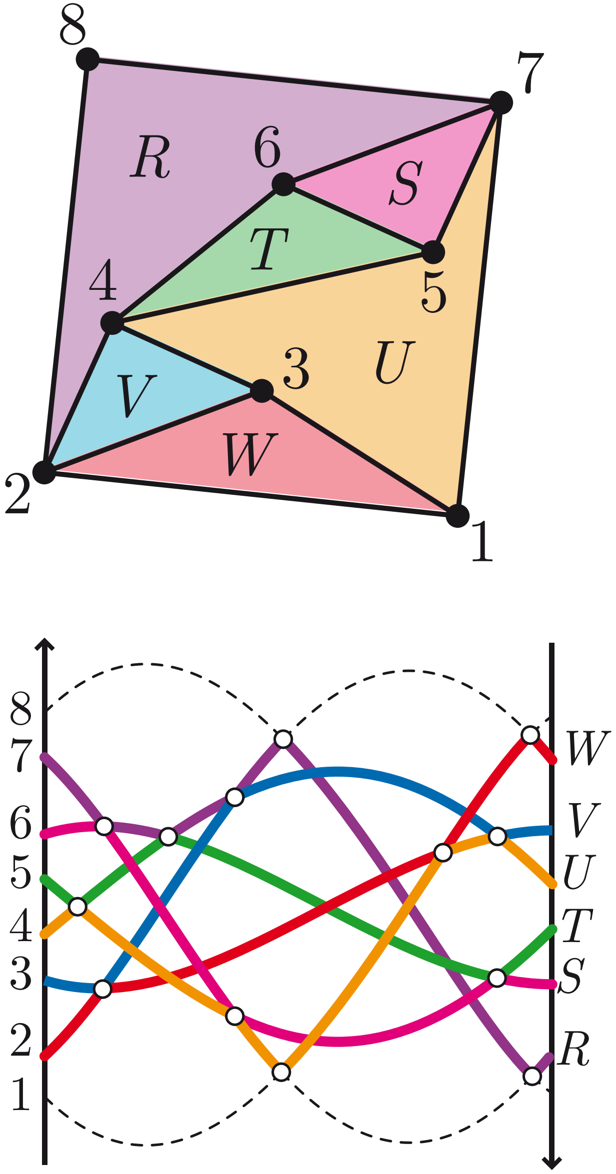
P.-Pocchiola, *Multitriangulations, pseudotriangulations and primitive sorting networks* ('12)

DUALITY

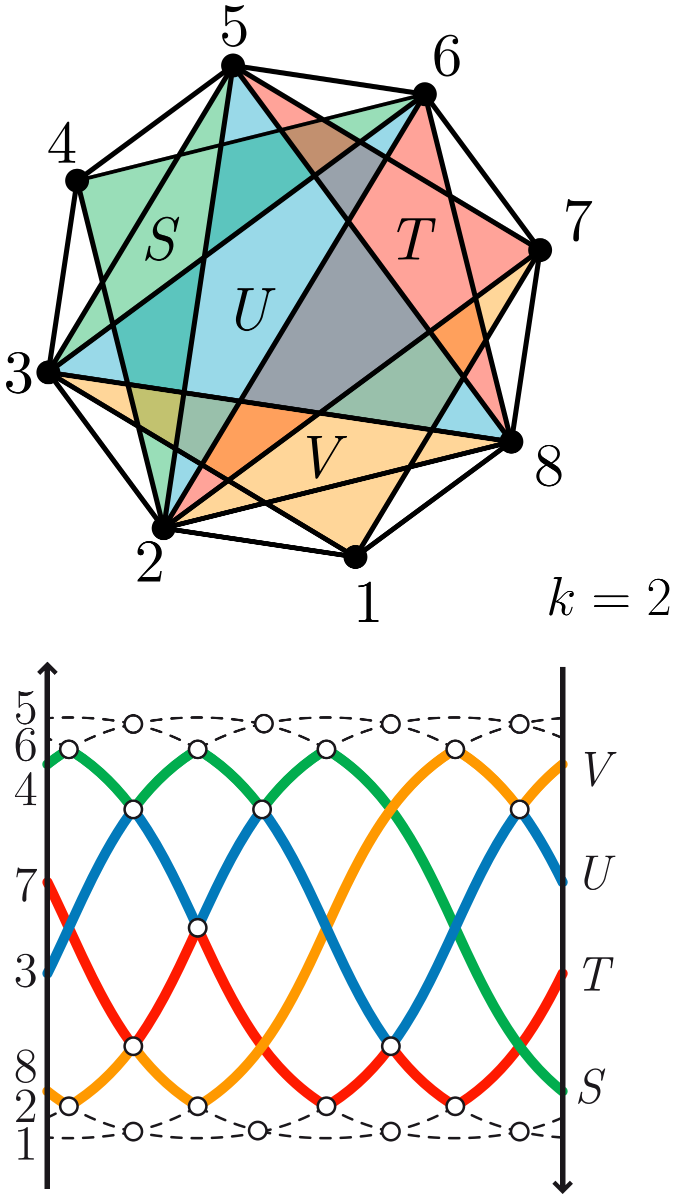
Triangulations



Pseudotriangulations



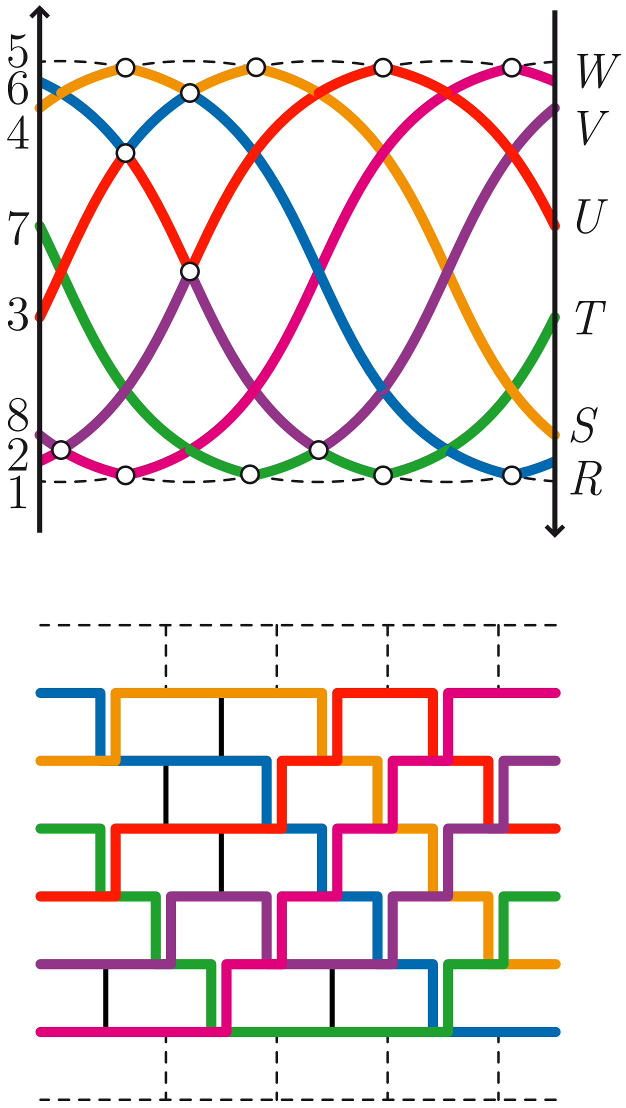
Multitriangulations



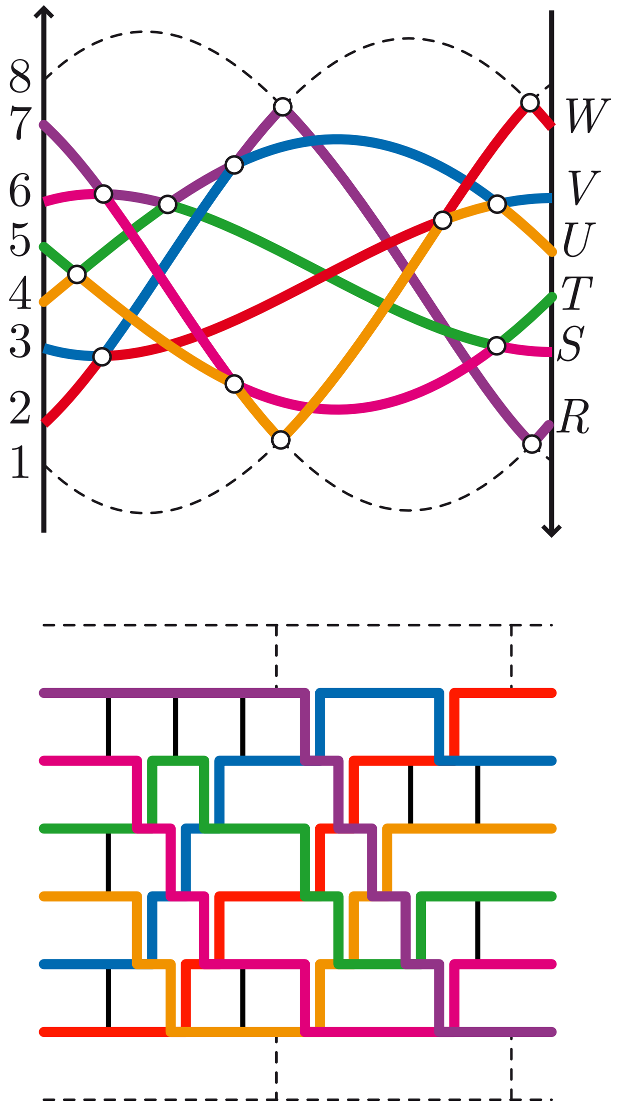
P.-Pocchiola, *Multitriangulations, pseudotriangulations and primitive sorting networks* ('12)

SORTING NETWORKS

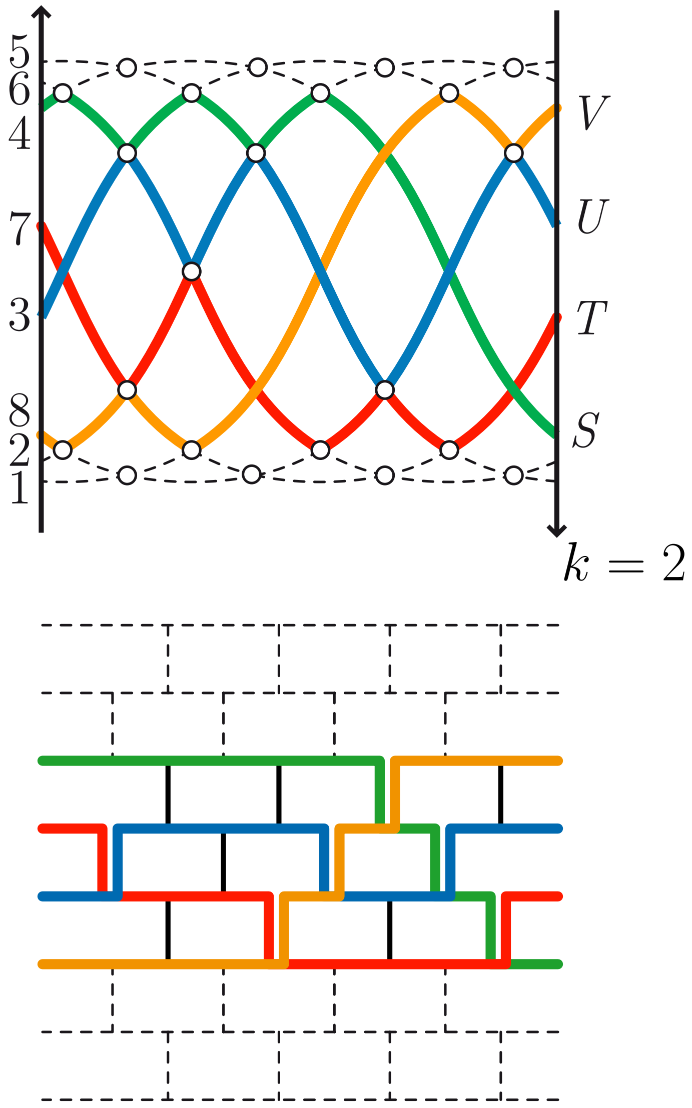
Triangulations



Pseudotriangulations

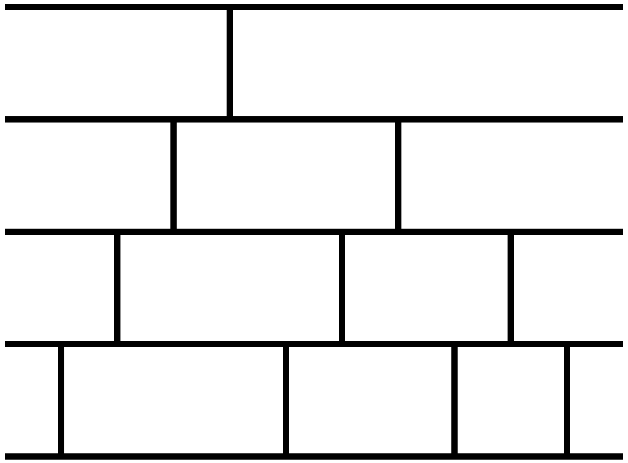


Multitriangulations

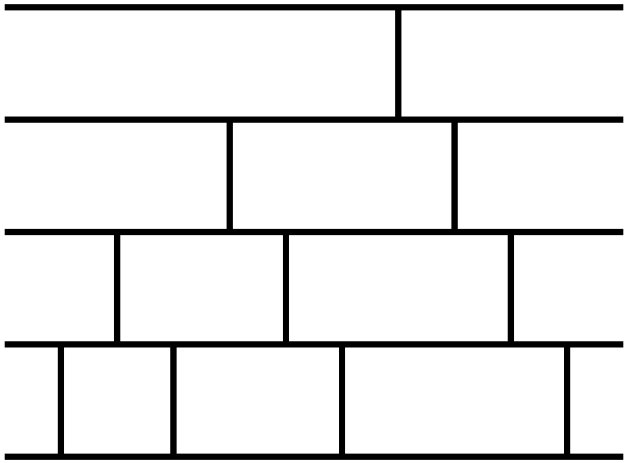


PSEUDOLINE ARRANGEMENTS ON SORTING NETWORKS

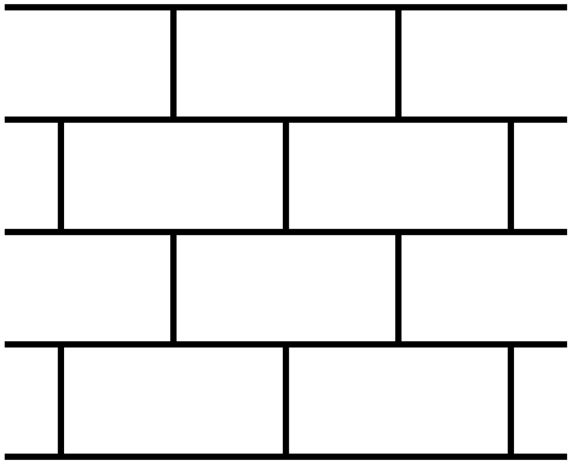
SORTING NETWORKS



Bubble sort



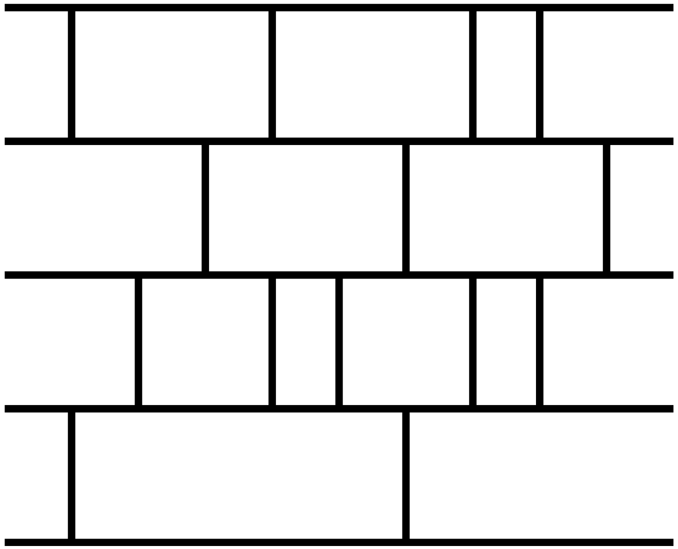
Insertion sort



Even-odd sort

Knuth, *The art of Computer Programming, Vol. 3 – Sorting and Searching* ('97)

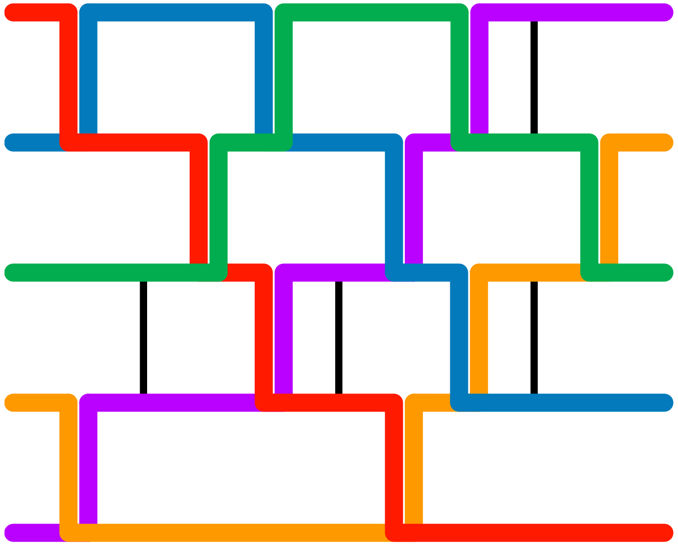
NETWORKS & PSEUDOLINE ARRANGEMENTS



network $\mathcal{N}$ $=$ n horizontal levels and m vertical commutators.

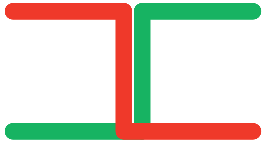
bricks of $\mathcal{N}$ $=$ bounded cells.

NETWORKS & PSEUDOLINE ARRANGEMENTS



network $\mathcal{N}$ $=$ n horizontal levels and m vertical commutators.
bricks of $\mathcal{N}$ $=$ bounded cells.

pseudoline $=$ x-monotone path which starts at a level l and ends at the level $n+1-l$.

crossing $=$  contact $=$ 

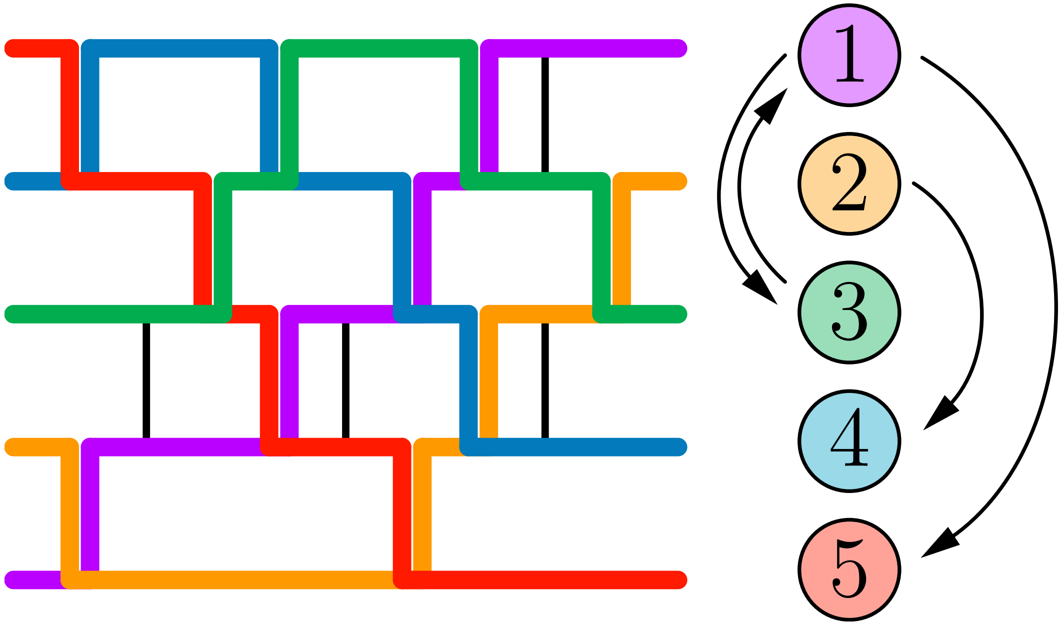
pseudoline arrangement (with contacts) $=$ n pseudolines supported by $\mathcal{N}$ which have pairwise exactly one crossing, eventually some contacts, and no other intersection.

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

Contact graph $\Lambda^\#$ of a pseudoline arrangement Λ =

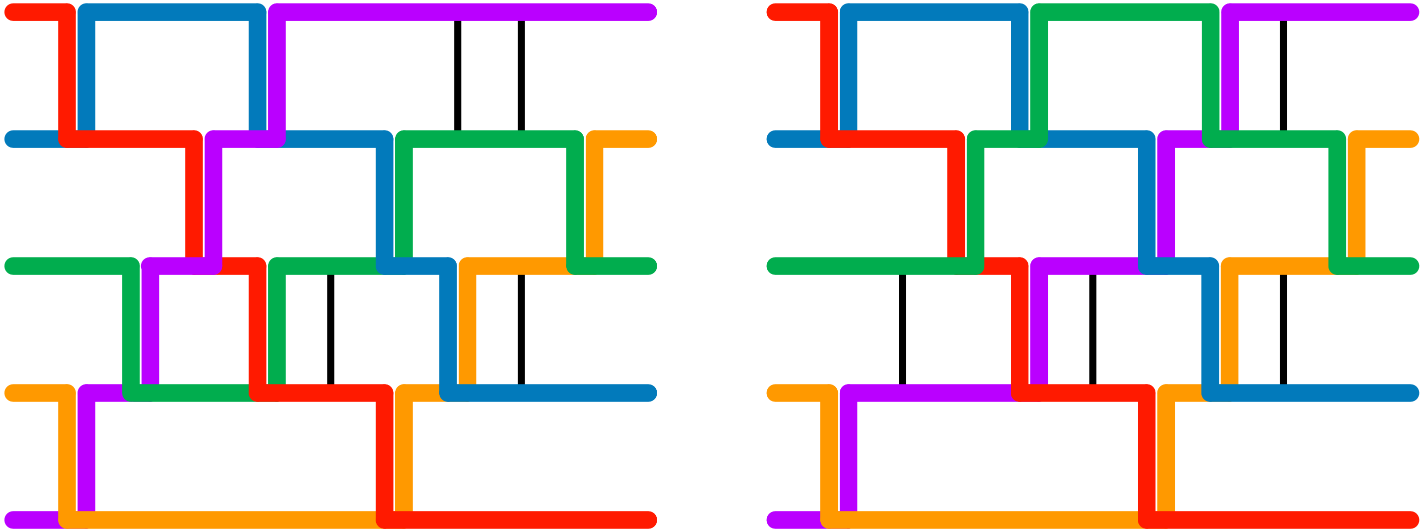
- a node for each pseudoline of Λ, and
- an arc for each contact point of Λ oriented from top to bottom.

Root configuration of $\Lambda = \mathsf{R}(\Lambda) = \left\{ e_i - e_j \;\middle|\; i \to j \in \Lambda^\# \right\}$



FLIPS

flip $=$ exchange a contact with the corresponding crossing.



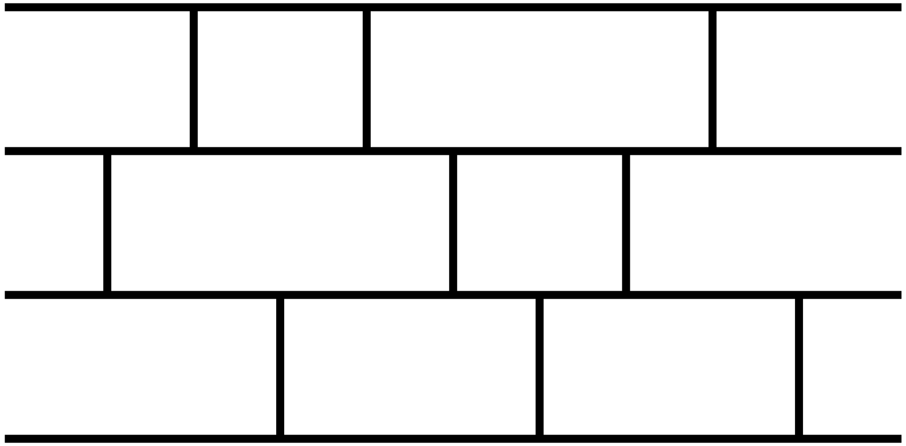
THM. Let $\mathcal{N}$ be a sorting network with n levels and m commutators. The graph of flips $G(\mathcal{N})$ is $\left(m - \binom{n}{2}\right)$-regular and connected.

QUESTION. Is $G(\mathcal{N})$ the graph of a simple $\left(m - \binom{n}{2}\right)$-dimensional polytope?

BRICK POLYTOPE

P.-Santos, *The brick polytope of a sorting network* ('12)
P.-Stump, *Brick polytopes of spherical subword complexes & gen. assoc.* ('15)

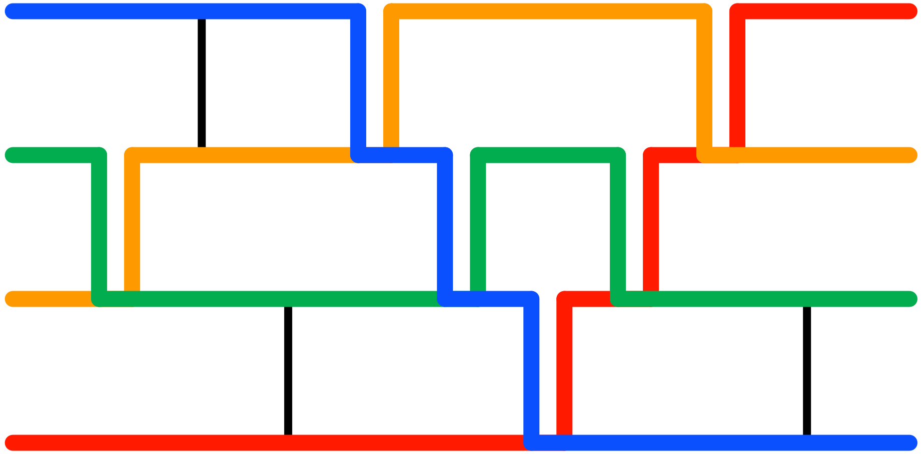
BRICK POLYTOPE



$\mathcal{N}$ a sorting network with $n+1$ levels and m commutators

P.-Santos, *The brick polytope of a sorting network* ('12)

BRICK POLYTOPE

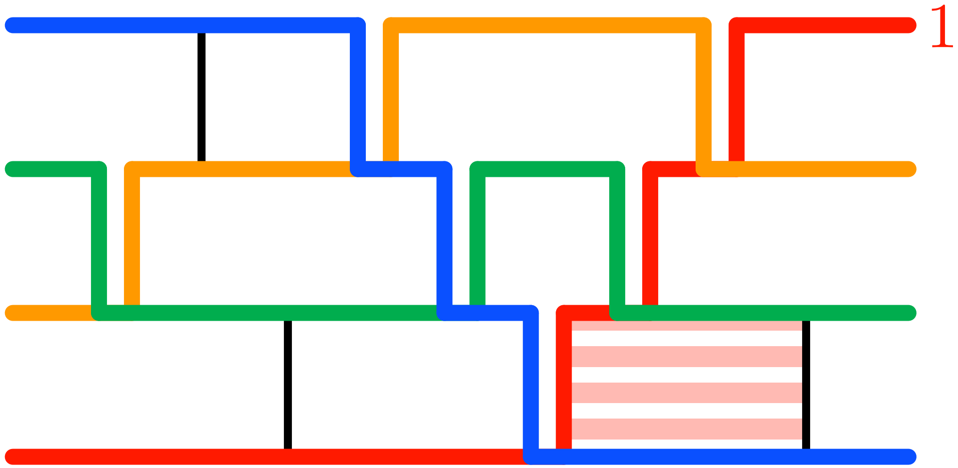


$\mathcal{N}$ a sorting network with $n+1$ levels and m commutators

Λ pseudoline arrangement supported by $\mathcal{N}$ $\quad\longmapsto\quad$ brick vector $\mathsf{B}(\Lambda) \in \mathbb{R}^{n+1}$

P.-Santos, *The brick polytope of a sorting network* ('12)

BRICK POLYTOPE



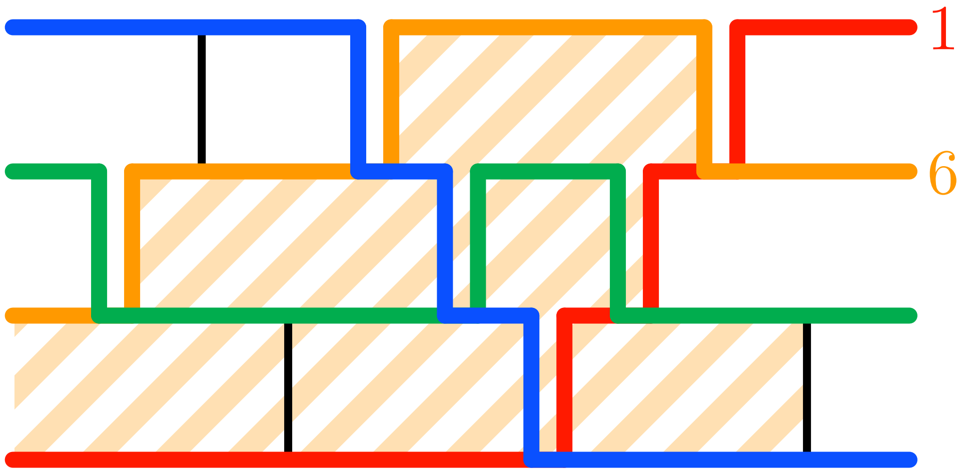
$\mathcal{N}$ a sorting network with $n+1$ levels and m commutators

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\mathsf{B}(\Lambda) \in \mathbb{R}^{n+1}$

$\mathsf{B}(\Lambda)_j$ $=$ number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

P.-Santos, *The brick polytope of a sorting network* ('12)

BRICK POLYTOPE



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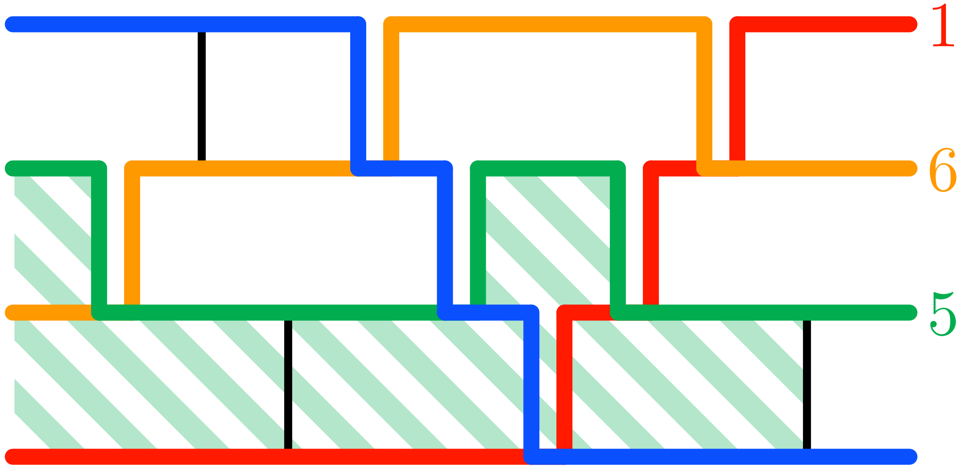

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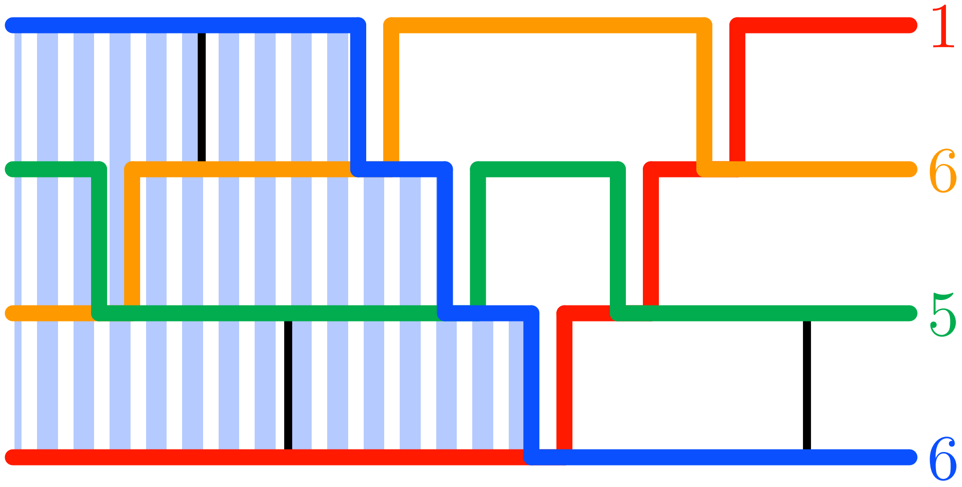
BRICK POLYTOPE



$\mathcal{N}$ a sorting network with $n+1$ levels and m commutators

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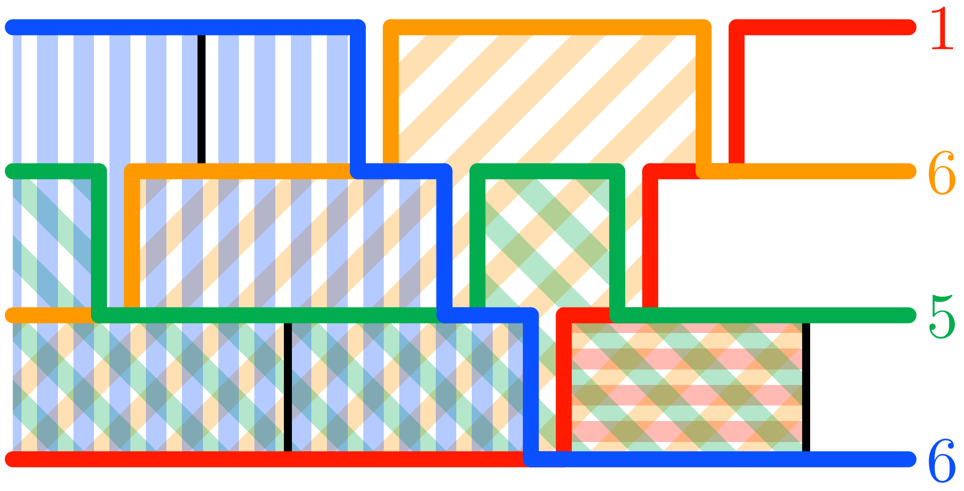
P.-Santos, *The brick polytope of a sorting network* ('12)

BRICK POLYTOPE



$\mathcal{N}$ a sorting network with $n+1$ levels and m commutators

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\mathsf{B}(\Lambda) \in \mathbb{R}^{n+1}$

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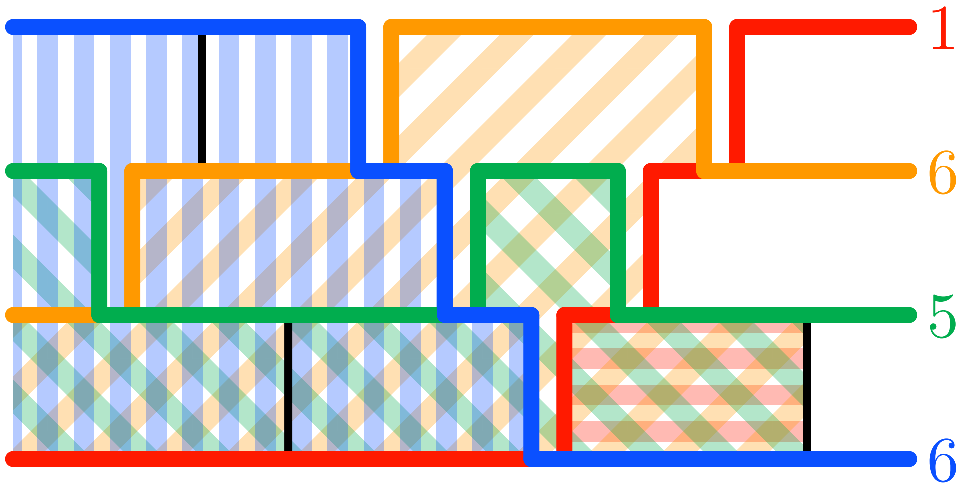
P.-Santos, *The brick polytope of a sorting network* ('12)

BRICK POLYTOPE



$\mathcal{N}$ a sorting network with $n+1$ levels and m commutators

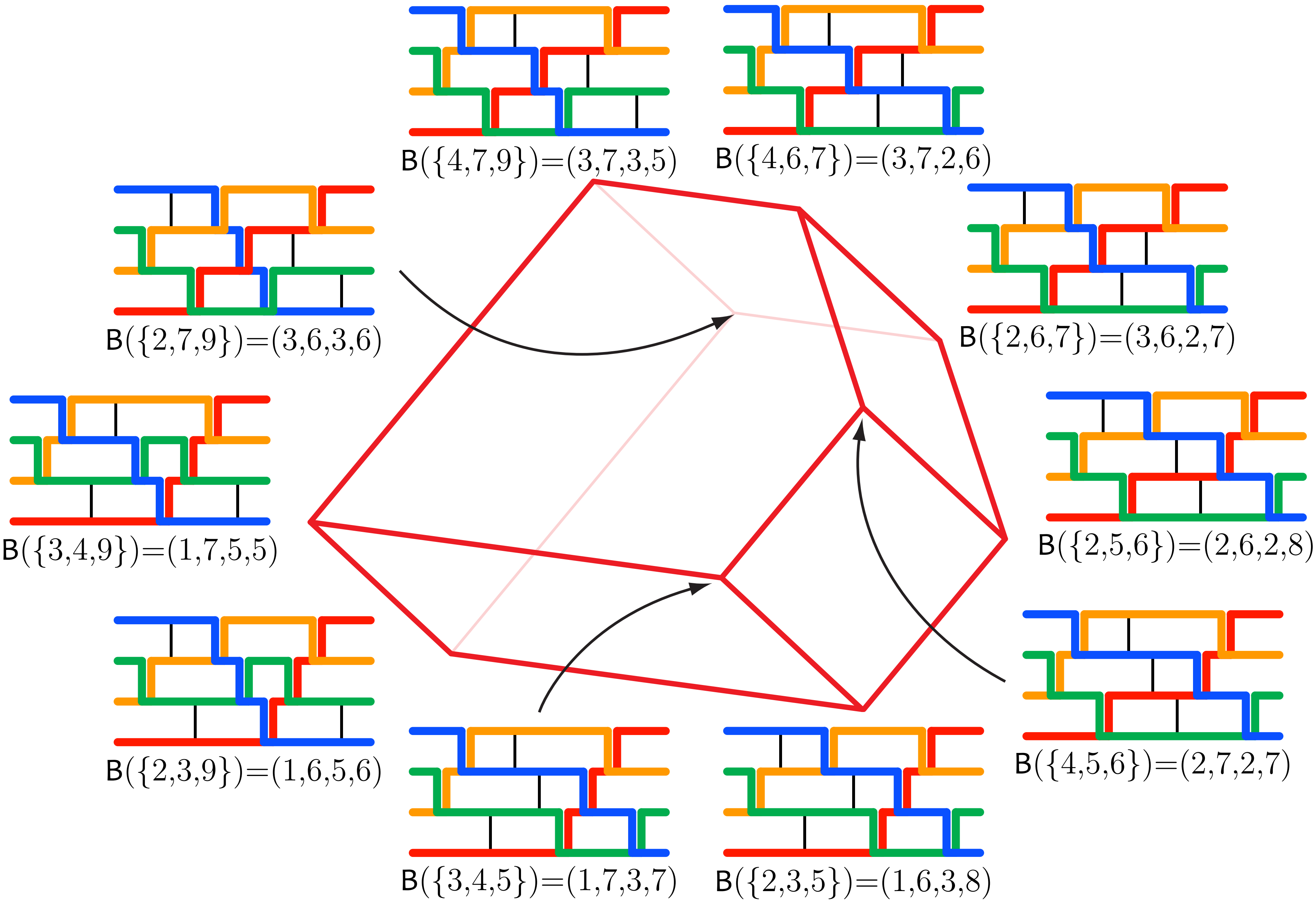
Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\mathsf{B}(\Lambda) \in \mathbb{R}^{n+1}$

$\mathsf{B}(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

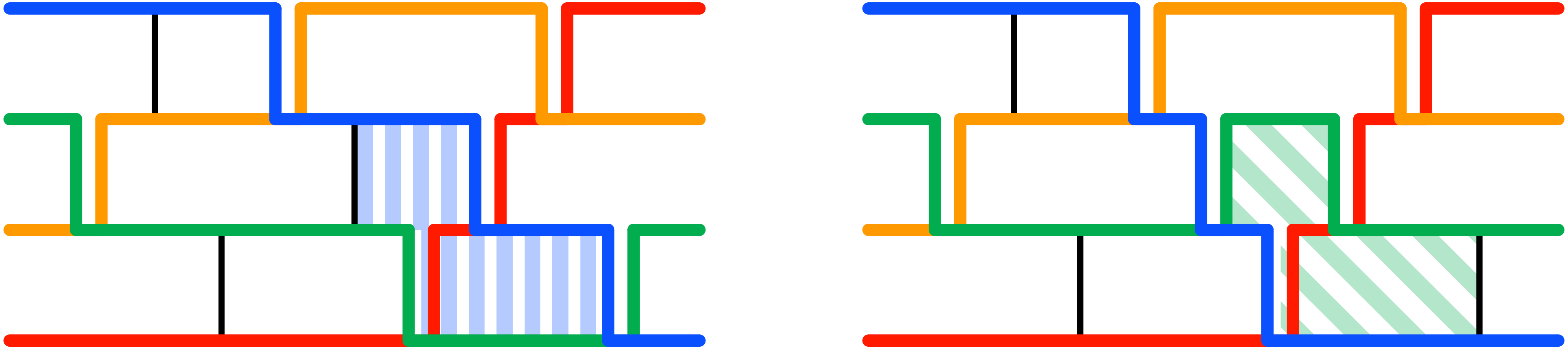
Brick polytope $\mathcal{B}(\mathcal{N}) = \mathrm{conv}\left\{\mathsf{B}(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\right\}$

P.-Santos, *The brick polytope of a sorting network* ('12)

BRICK POLYTOPE



BRICK VECTORS AND FLIPS



If Λ and Λ' are two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip between their ith and jth pseudolines, then $\mathsf{B}(\Lambda) - \mathsf{B}(\Lambda') \in \mathbb{N}_{>0}\,(e_j - e_i)$.

THM. The cone of the brick polytope $\mathcal{B}(\mathcal{N})$ at the brick vector $\mathsf{B}(\Lambda)$ is the incidence cone of the contact graph of Λ:

$$\mathrm{cone}\,\{\mathsf{B}(\Lambda') - \mathsf{B}(\Lambda) \mid \Lambda' \text{ supported by } \mathcal{N}\} = \mathrm{cone}\left\{e_j - e_i \;\middle|\; i \to j \in \Lambda^{\#}\right\}$$

COMBINATORIAL DESCRIPTION

THM. The cone of the brick polytope $\mathcal{B}(\mathcal{N})$ at the brick vector $\mathsf{B}(\Lambda)$ is the incidence cone of the contact graph of Λ:

$$\operatorname{cone}\left\{\mathsf{B}(\Lambda') - \mathsf{B}(\Lambda) \mid \Lambda' \text{ supported by } \mathcal{N}\right\} = \operatorname{cone}\left\{e_j - e_i \,\middle|\, i \to j \in \Lambda^{\#}\right\}$$

VERTICES OF $\mathcal{B}(\mathcal{N})$

The brick vector $\mathsf{B}(\Lambda)$ is a vertex of $\mathcal{B}(\mathcal{N})$ $\iff$ the contact graph $\Lambda^{\#}$ is acyclic.

GRAPH OF $\mathcal{B}(\mathcal{N})$

The graph of the brick polytope is a subgraph of the flip graph whose vertices are the pseudoline arrangements with acyclic contact graphs.

FACETS OF $\mathcal{B}(\mathcal{N})$

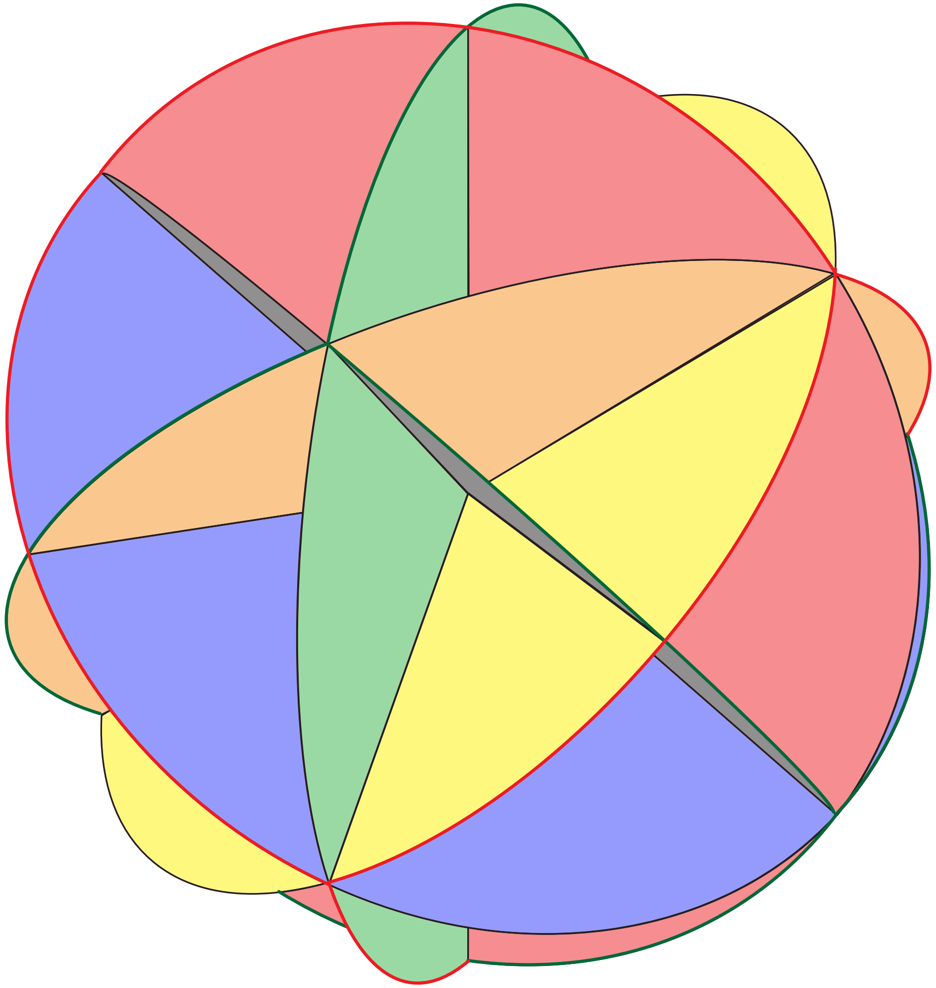
The facets of $\mathcal{B}(\mathcal{N})$ correspond to the minimal directed cuts of the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$.

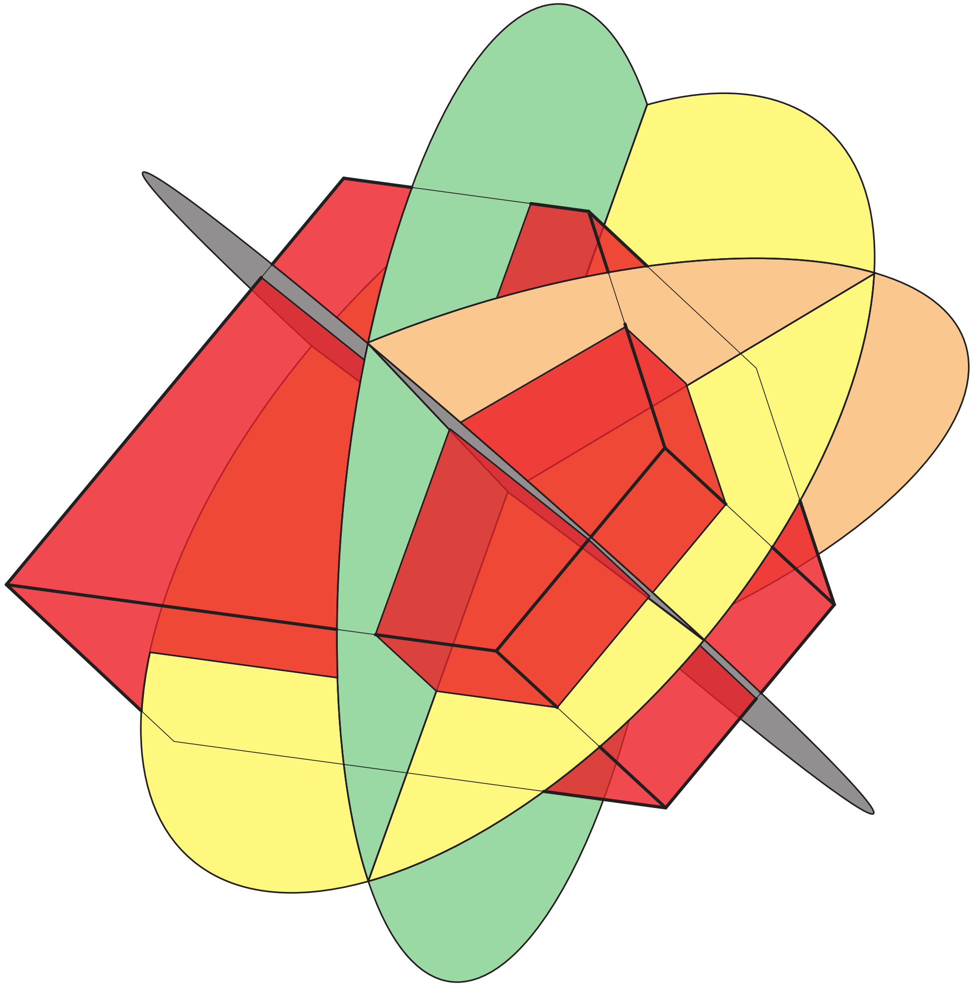
P.-Santos, *The brick polytope of a sorting network* ('12)

NORMAL FAN

THM. The Coxeter fan refines the normal fan of the brick polytope. More precisely,

$$\text{normal cone of } \mathsf{B}(\Lambda) \text{ in } \mathcal{B}(\mathcal{N}) = \bigcup_{\substack{\sigma\in\mathfrak{S}_n \\ \mathsf{R}(\Lambda)\subseteq\sigma(\Phi^+)}} \sigma(\text{ fundamental cone })$$



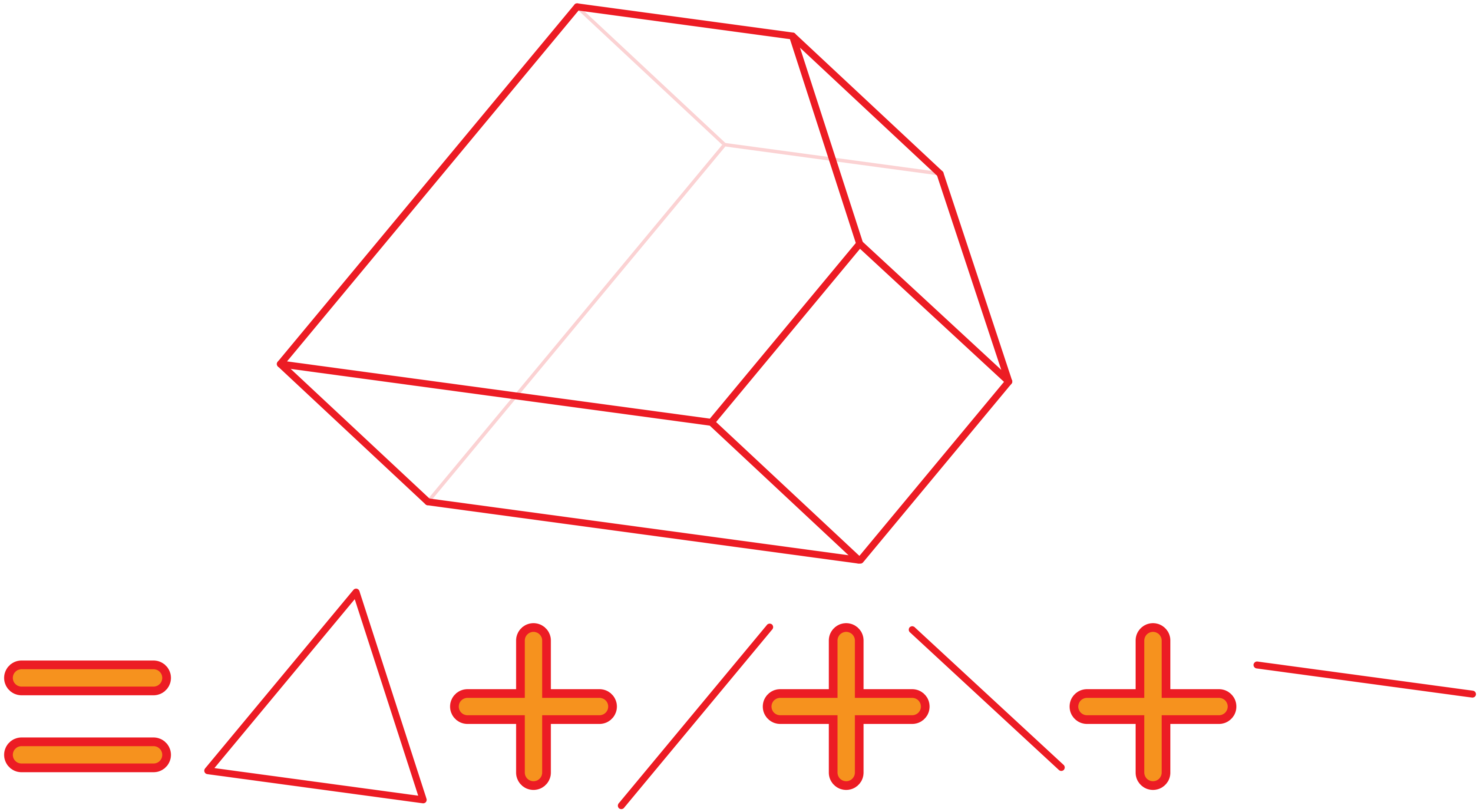


MINKOWSKI SUM

THM. $\mathsf{B}_k(\Lambda)$ = characteristic vector of pseudolines of Λ passing above the kth brick

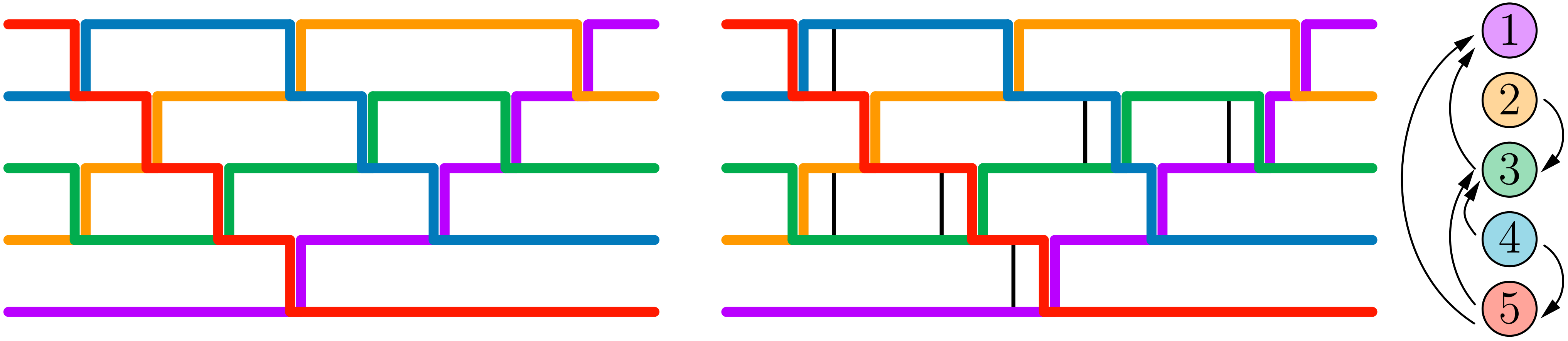
$\mathcal{B}(\mathcal{N}, k) = \operatorname{conv}\{\mathsf{B}_k(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$

$$\mathcal{B}(\mathrm{Q}) = \operatorname{conv}_\Lambda \mathsf{B}(\Lambda) = \operatorname{conv}_\Lambda \sum_k \mathsf{B}_k(\Lambda) = \sum_k \operatorname{conv}_I \mathsf{B}_k(\Lambda) = \sum_k \mathcal{B}(\mathcal{N}, k)$$



DUPLICATED NETWORKS AND ZONOTOPES

Duplicated network $\mathcal{N}_{\mathrm{G}}$ = duplicate the commutators corresp. to G in a reduced network

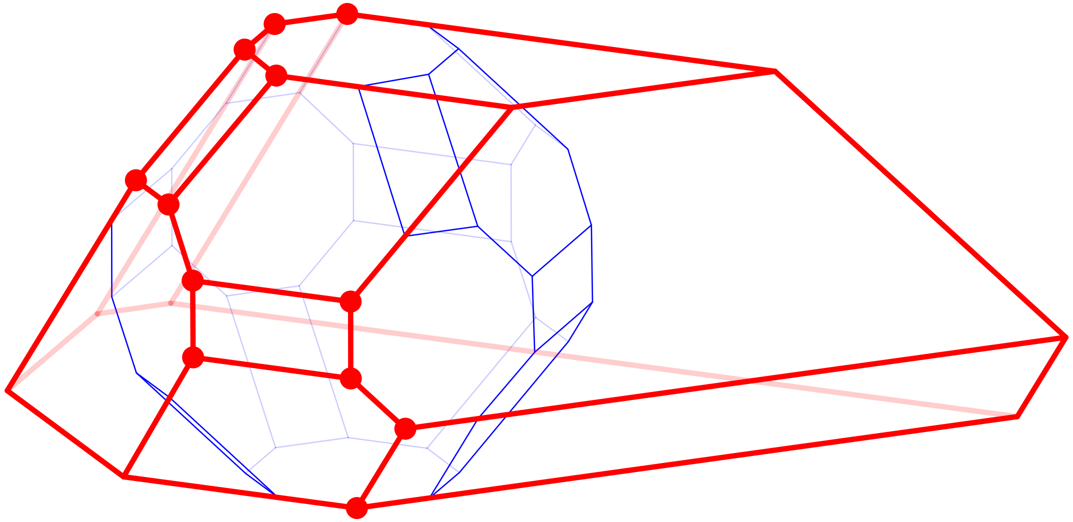


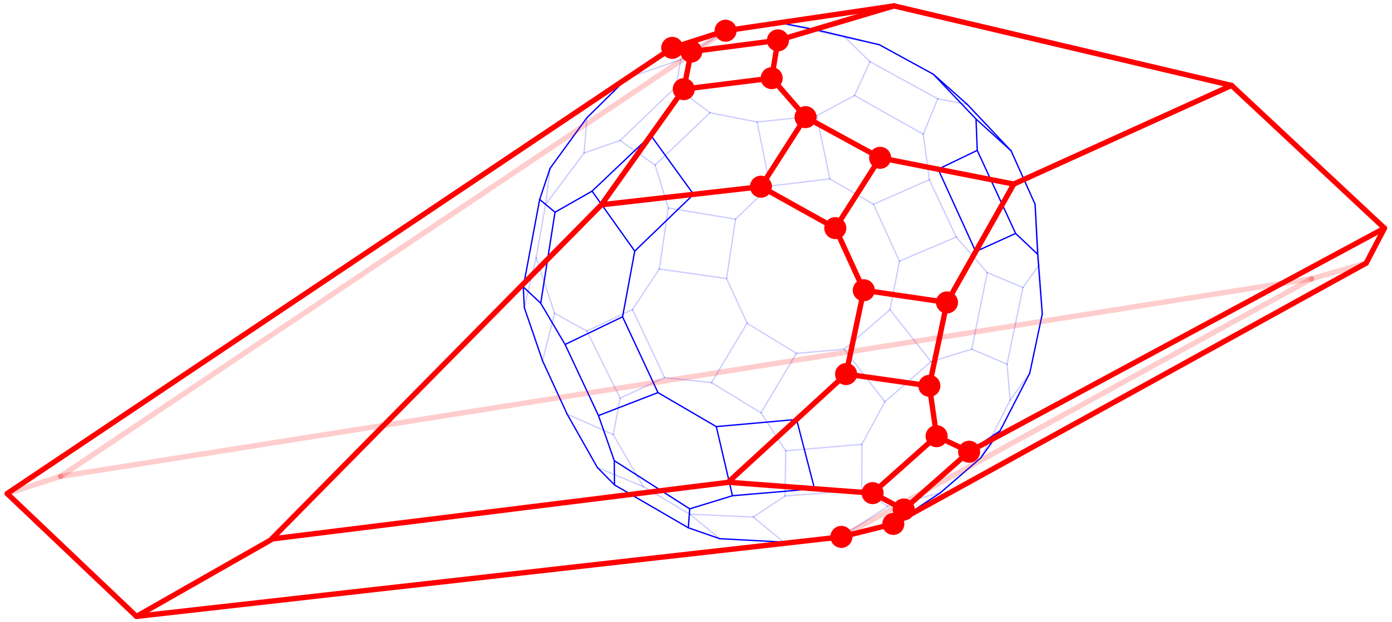
Any pseudoline arrangement supported by $\mathcal{N}_{\mathrm{G}}$ has one contact and one crossing among each pair of duplicated commutators. Therefore

pseudoline arrangements on $\mathcal{N}_{\mathrm{G}}$	$\longleftrightarrow$	orientation of G
vertices of $\mathcal{B}(\mathcal{N}_{\mathrm{G}})$	$\longleftrightarrow$	acyclic orientations of G
$\mathcal{B}(\mathcal{N}_{\mathrm{G}})$	$\longleftrightarrow$	graphical zonotope of $\mathrm{G} = \sum_{i-j \in \mathrm{G}} [\mathbf{e}_i, \mathbf{e}_j]$
normal fan of $\mathcal{N}_{\mathrm{G}}$	$\longleftrightarrow$	graphical hyperplane arrangement of G = hyp. $\{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}$ for $i - j \in \mathrm{G}$

COXETER BRICK POLYTOPES

All this story extends to arbitrary finite Coxeter groups

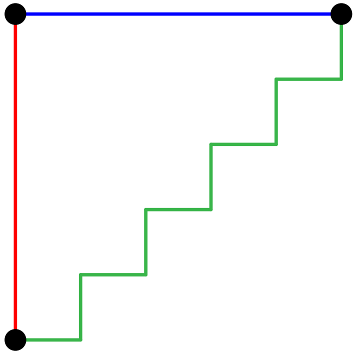




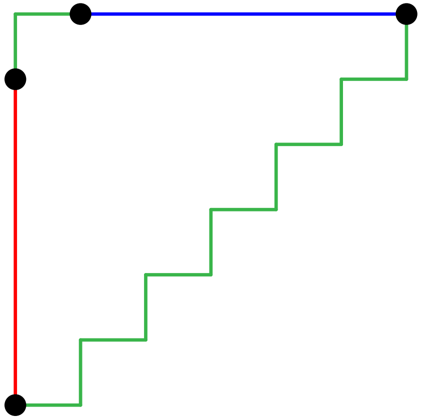
P.-Stump, *Brick polytopes of spherical subword complexes & gen. assoc.* ('15)

TWISTS

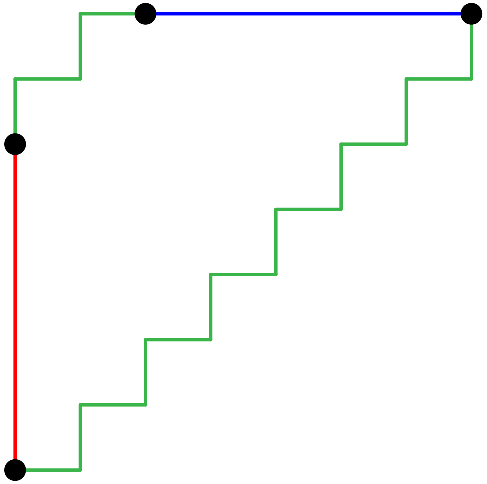
k-TWISTS



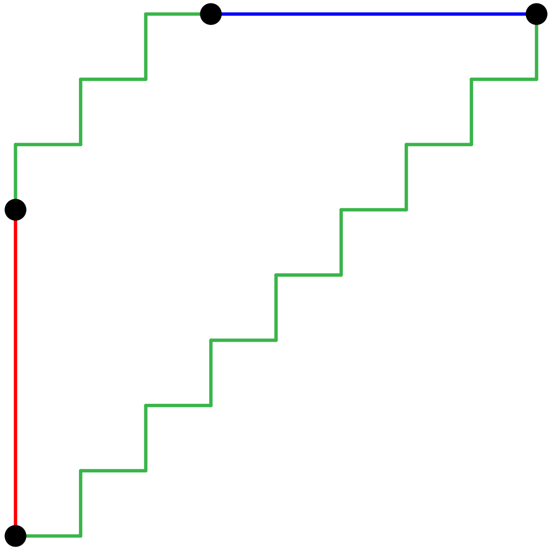
$k = 0$



$k = 1$



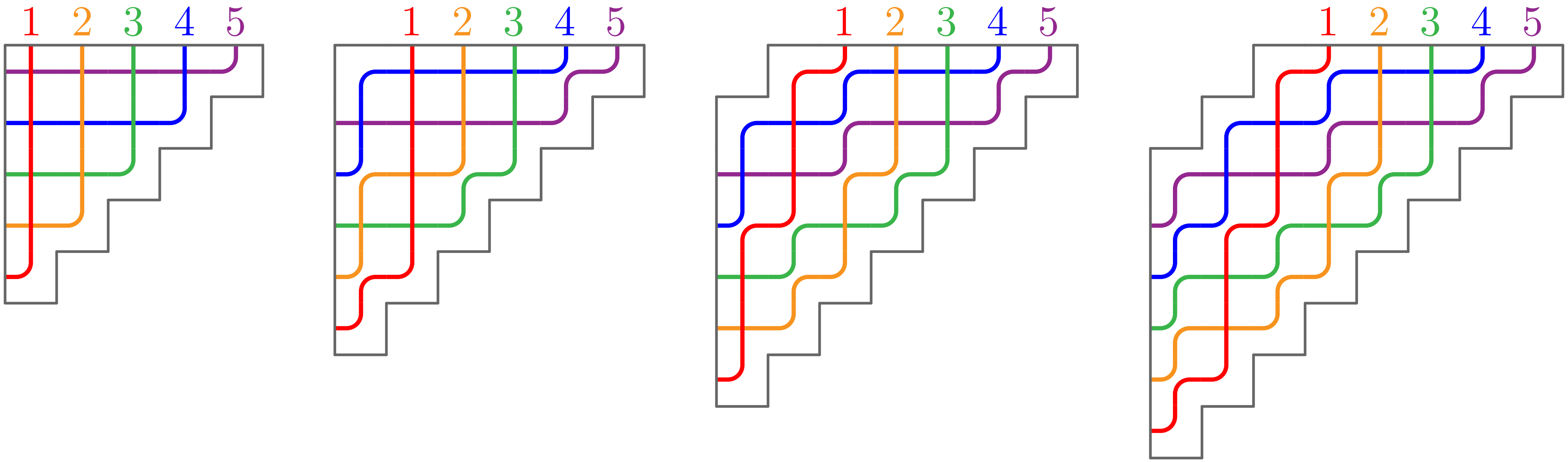
$k = 2$



$k = 3$

trapezoidal shape of height n and width k

k-TWISTS



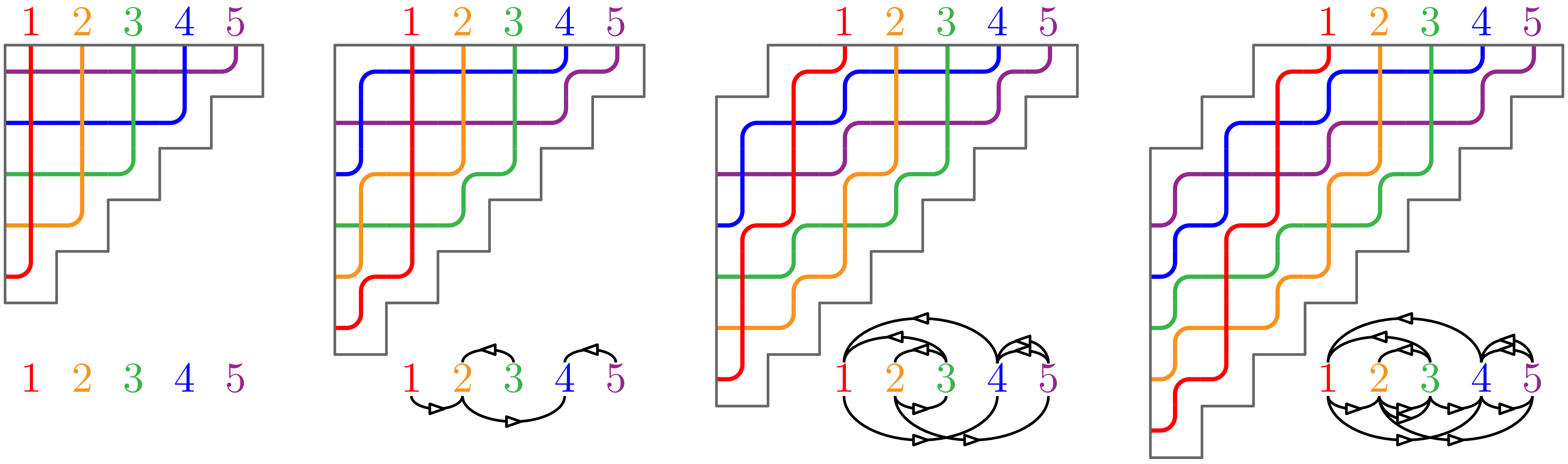
$k = 0$ $k = 1$ $k = 2$ $k = 3$

(k, n)-twist $=$ pipe dream in the trapezoidal shape of height n and width k

Bergeron-Billey. RC-graphs and Schubert polynomials. 1993

Knutson-Miller. Gröbner geometry of Schubert polynomials. 2005

k-TWISTS



$k = 0$ $k = 1$ $k = 2$ $k = 3$

(k, n)-twist $=$ pipe dream in the trapezoidal shape of height n and width k

contact graph of a twist T $=$ vertices are pipes of T and arcs are elbows of T

1-TWISTS AND TRIANGULATIONS

Correspondence

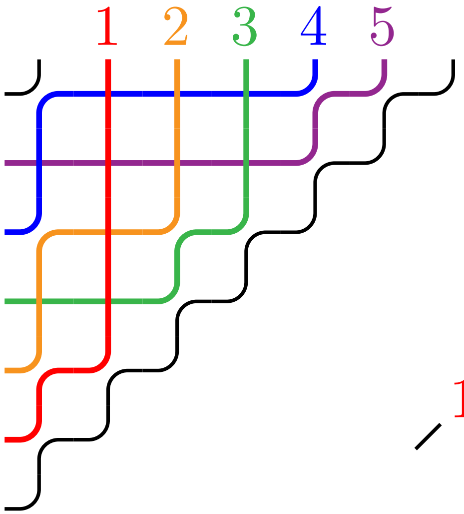
elbow in row i and column j $\longleftrightarrow$ diagonal $[i, j]$ of the $(n+2)$-gon

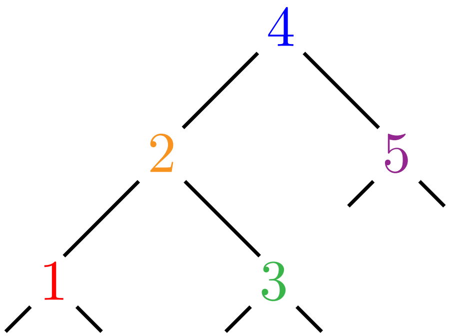
$(1, n)$-twist T $\longleftrightarrow$ triangulation $\mathrm{T}^\star$ of the $(n+2)$-gon

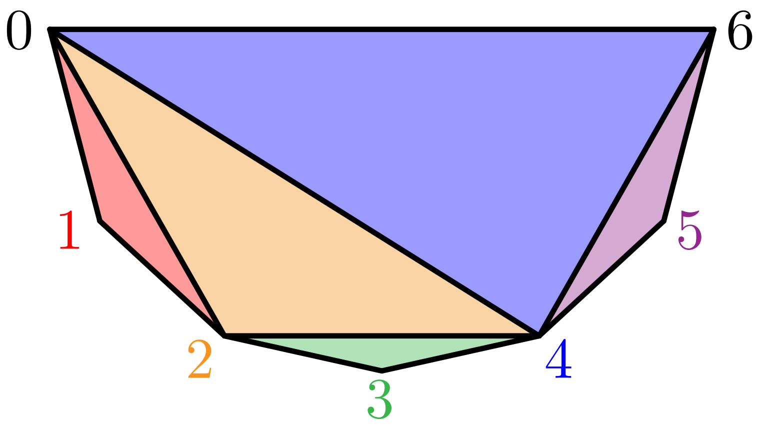
pth relevant pipe of T $\longleftrightarrow$ pth triangle of $\mathrm{T}^\star$

contact graph of T $\longleftrightarrow$ dual binary tree of $\mathrm{T}^\star$

elbow flips in T $\longleftrightarrow$ diagonal flips in $\mathrm{T}^\star$







Woo. Catalan numbers and Schubert Polynomials for $w = 1(n+1)\ldots 2$. Unpub 2004

P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012

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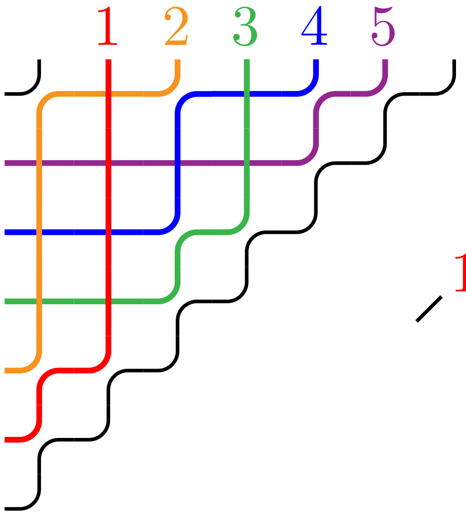
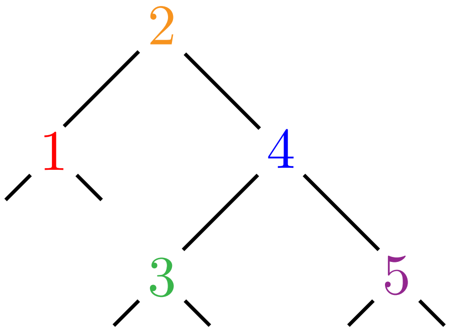
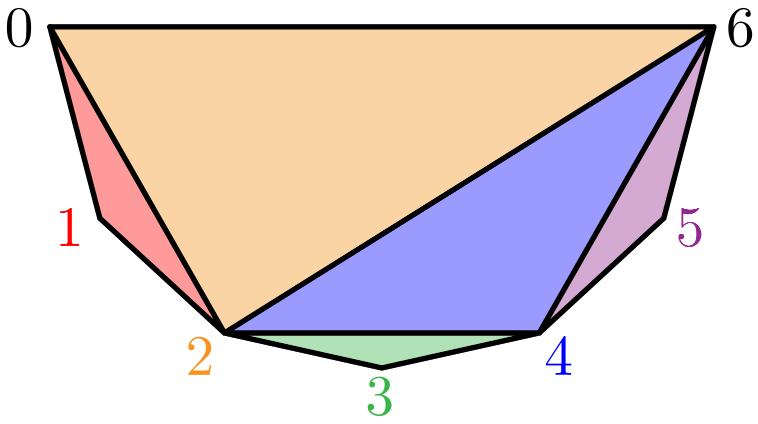






Woo. Catalan numbers and Schubert Polynomials for $w = 1(n+1)\dots 2$. Unpub 2004

P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012

k-TWISTS AND k-TRIANGULATIONS

Correspondence

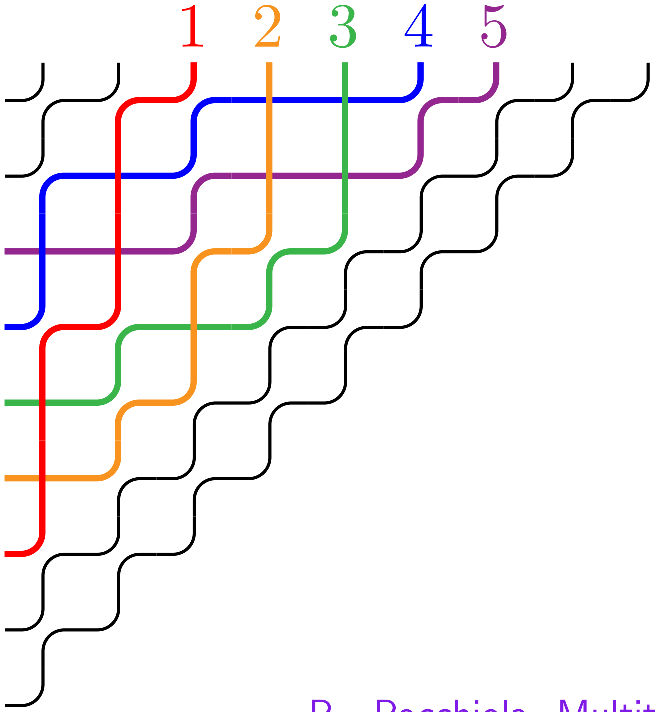
elbow in row i and column j $\longleftrightarrow$ diagonal $[i, j]$ of the $(n + 2k)$-gon

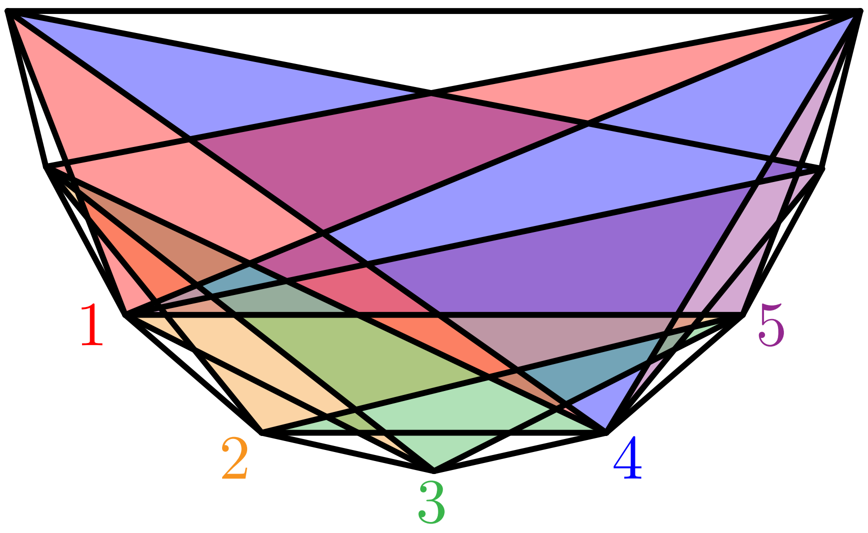
(k, n)-twist T $\longleftrightarrow$ k-triangulation $\mathrm{T}^\star$ of the $(n + 2k)$-gon

pth relevant pipe of T $\longleftrightarrow$ pth k-star of $\mathrm{T}^\star$

contact graph of T $\longleftrightarrow$ dual graph of $\mathrm{T}^\star$

elbow flips in T $\longleftrightarrow$ diagonal flips in $\mathrm{T}^\star$

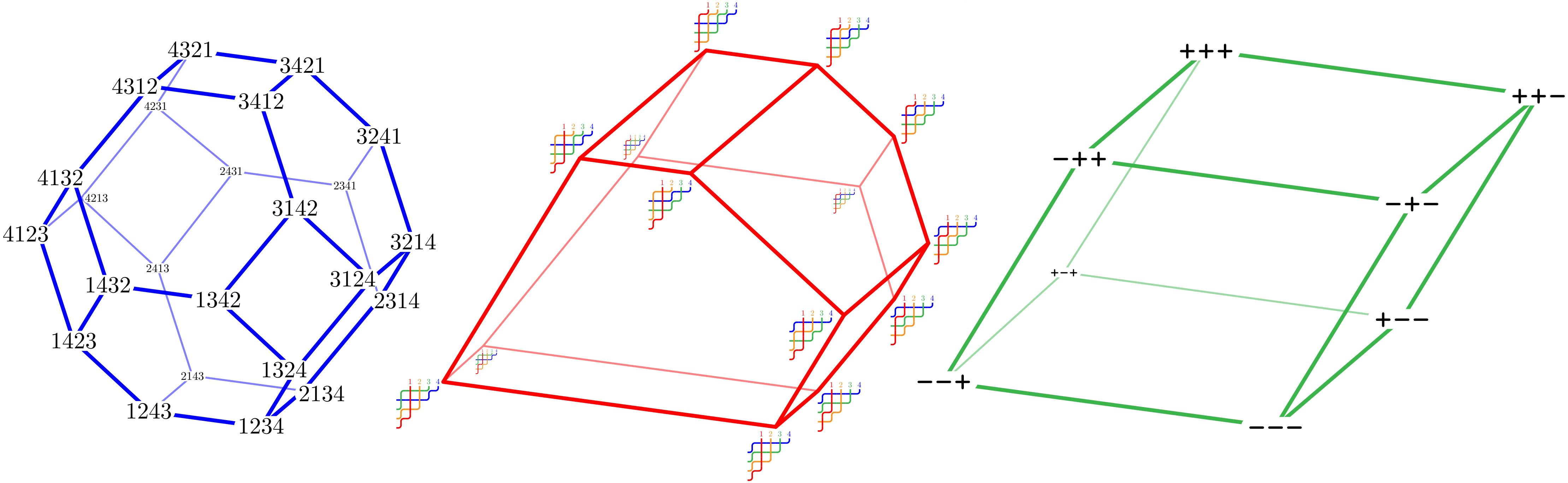




P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012

BRICK POLYTOPES OF TWISTS

$k = 1$



permutahedron Perm(n)
$\mathrm{conv}(\mathfrak{S}_{n+1})$

brick polytope Brick$^k(n)$
$\mathrm{conv}\,\{\mathsf{B}(\Lambda) \mid \Lambda\ (k,n)\text{-twist}\}$

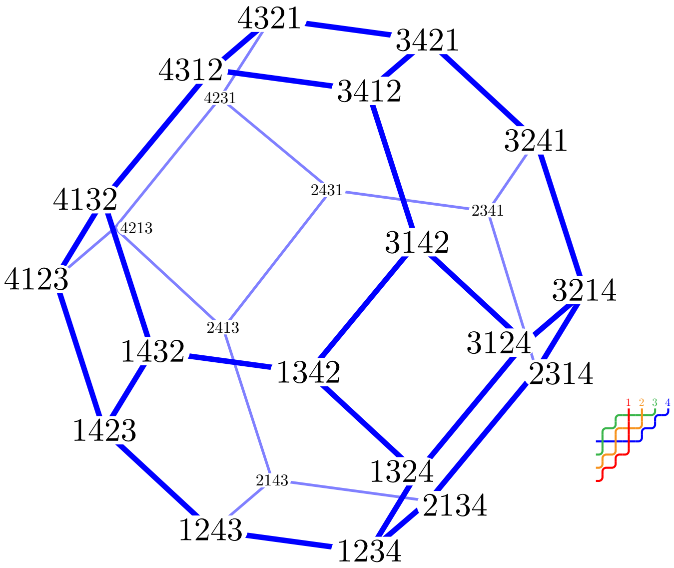
zonotope Zono$^k(n)$
$\sum_{|i-j| \le k} [\mathbf{e}_i, \mathbf{e}_j]$

$k = 1$ gives Loday's associahedron!!

…

BRICK POLYTOPES OF TWISTS

$k = 2$



permutahedron Perm(n)

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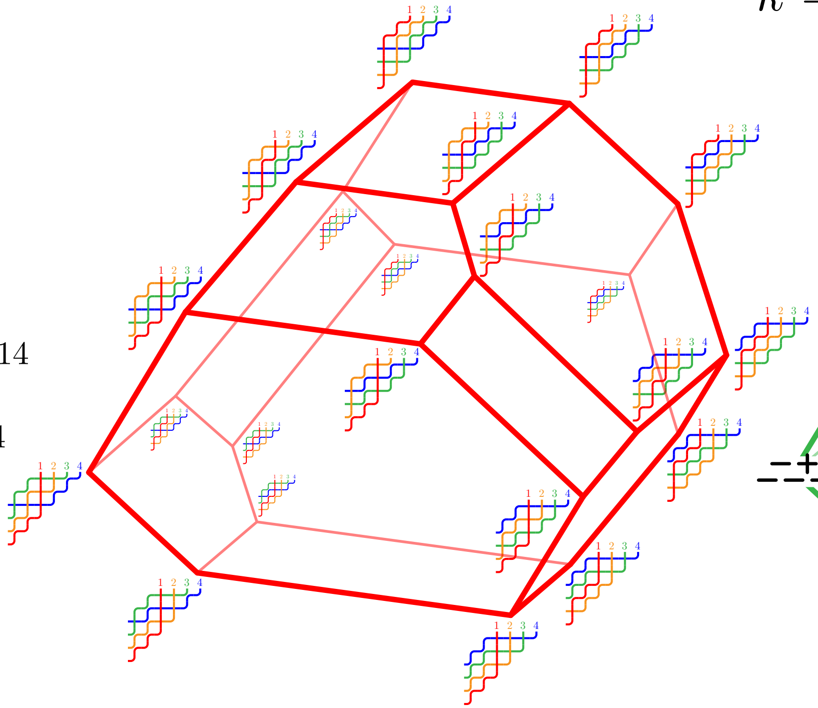

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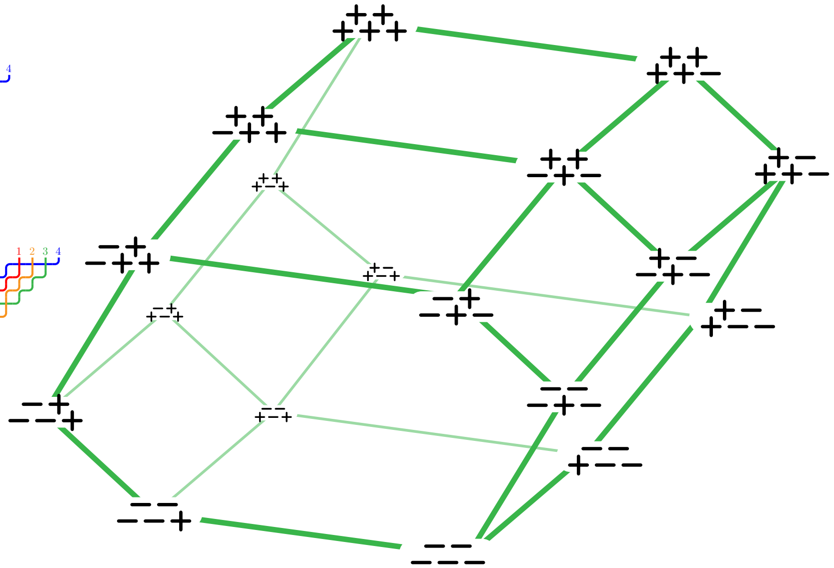


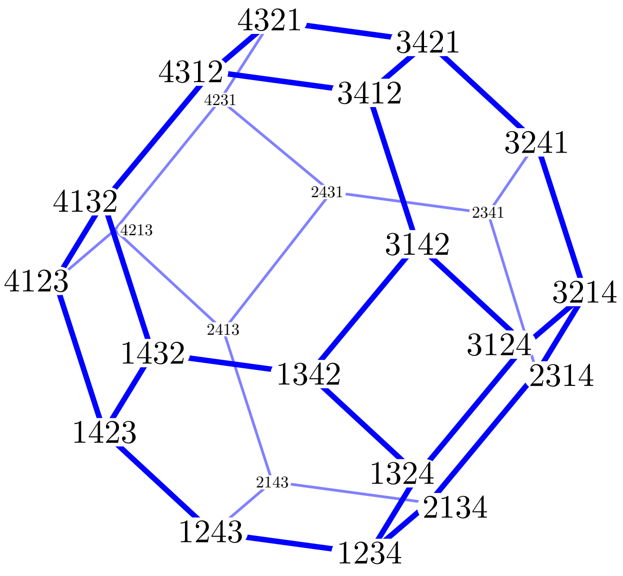
zonotope Zono$^k(n)$

$$\sum_{|i-j|\le k} [\mathbf{e}_i, \mathbf{e}_j]$$

…

BRICK POLYTOPES OF TWISTS

$k = 3$



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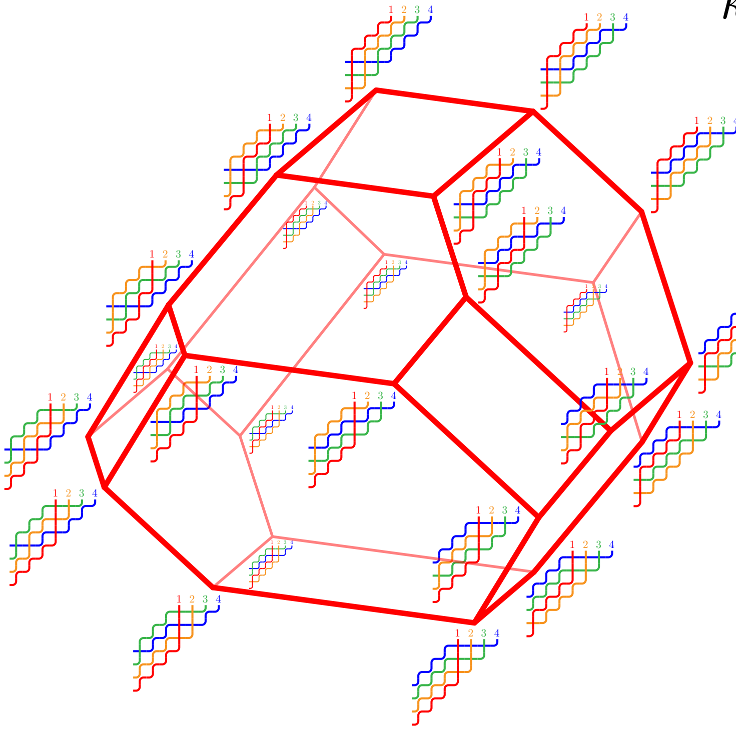

brick polytope Brick$^k(n)$

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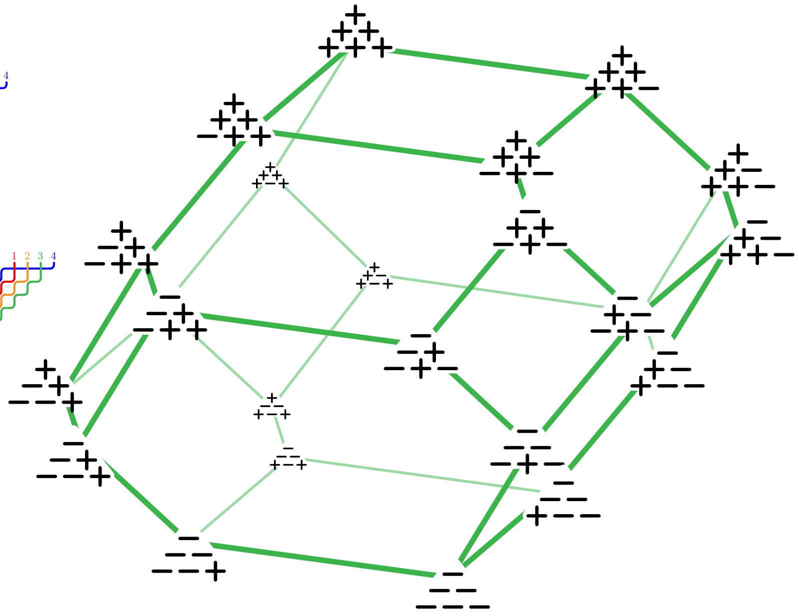

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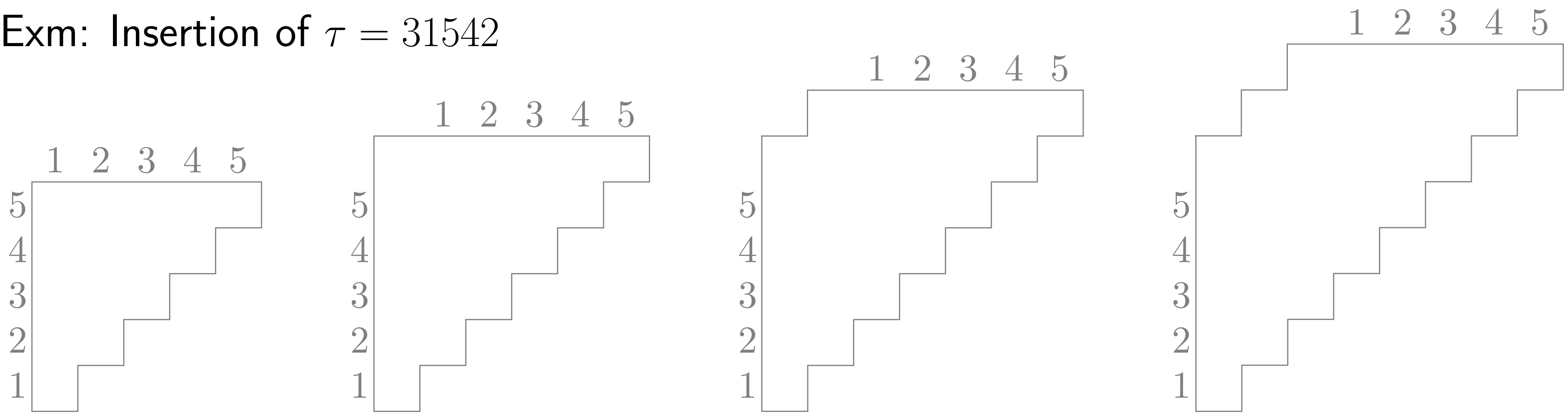
k-TWIST INSERTION

Input: a permutation $\tau = \tau_1 \cdots \tau_n$

Algo: Insert pipes one by one (from right to left) as northwest as possible

Output: an acyclic (k, n)-twist $\text{ins}^k(\tau)$

Exm: Insertion of $\tau = 31542$



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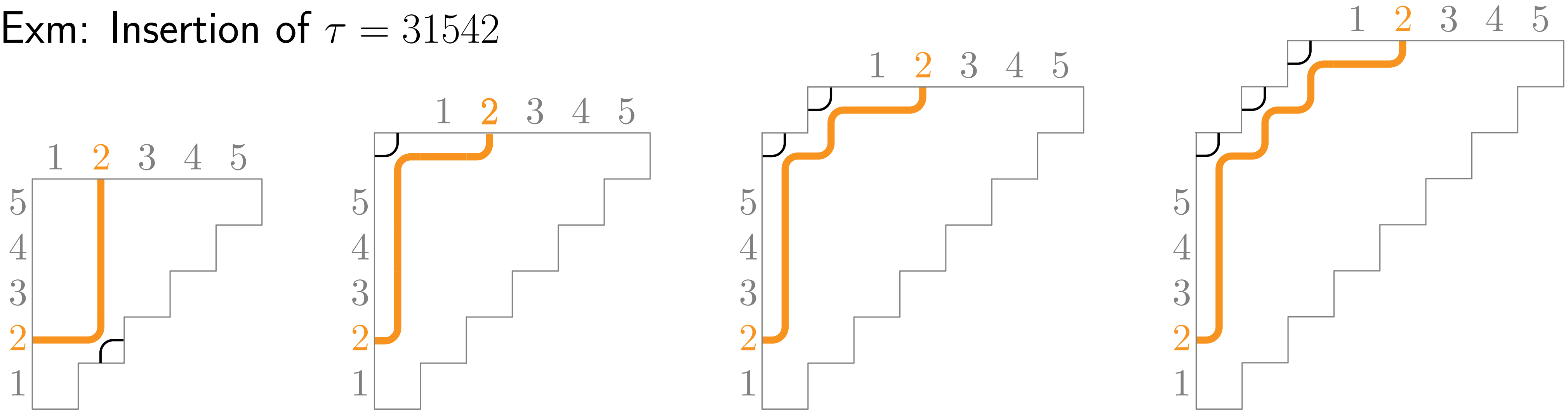

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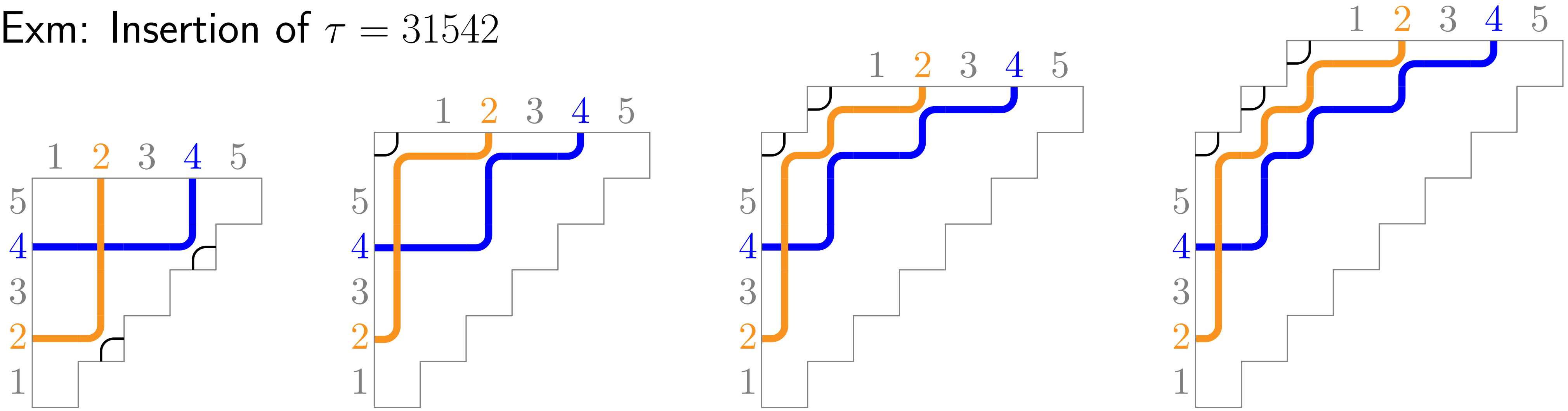

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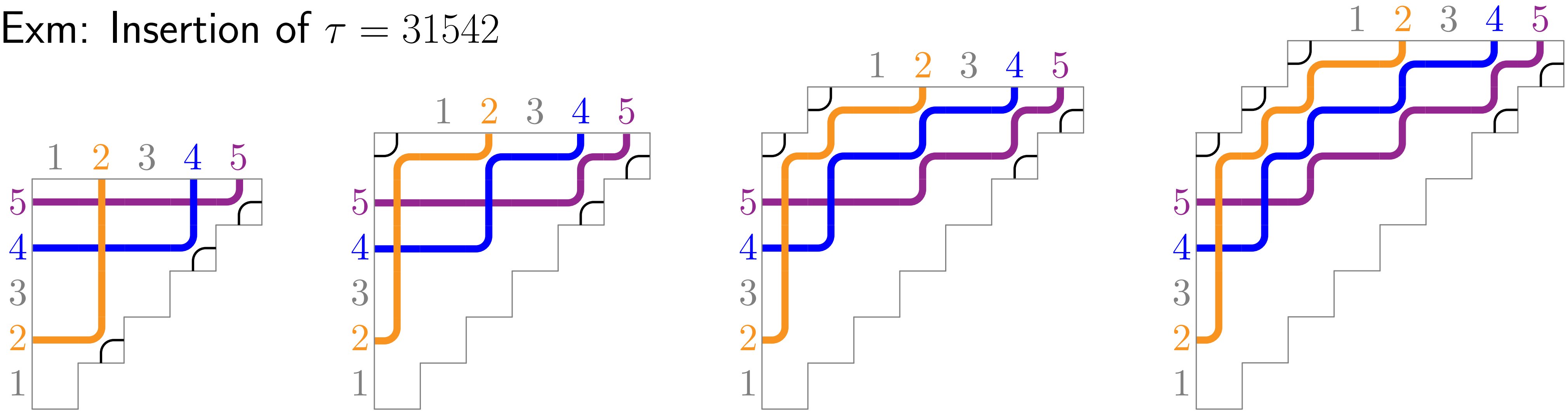

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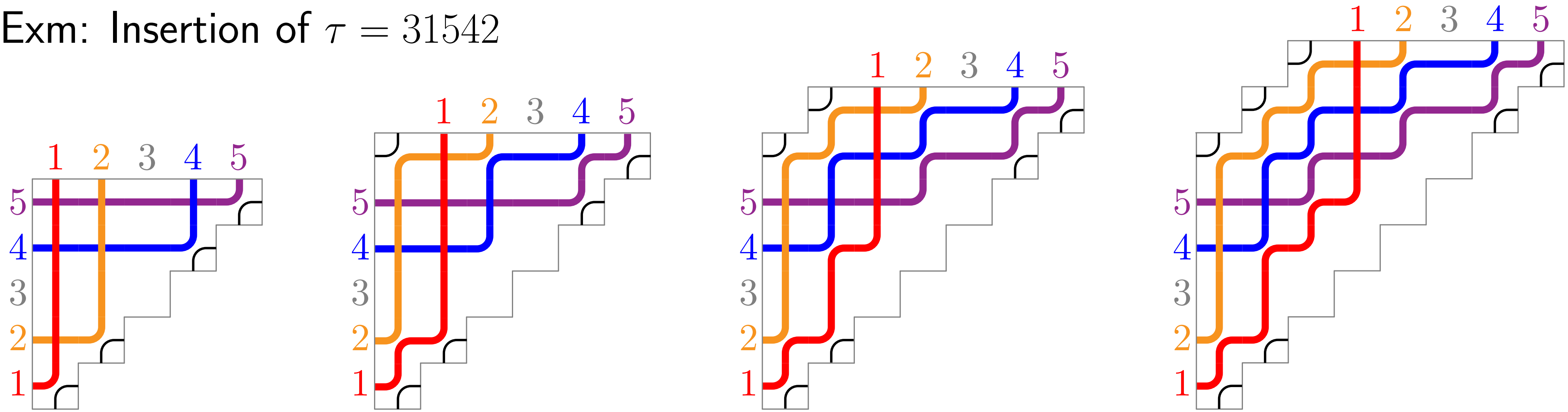

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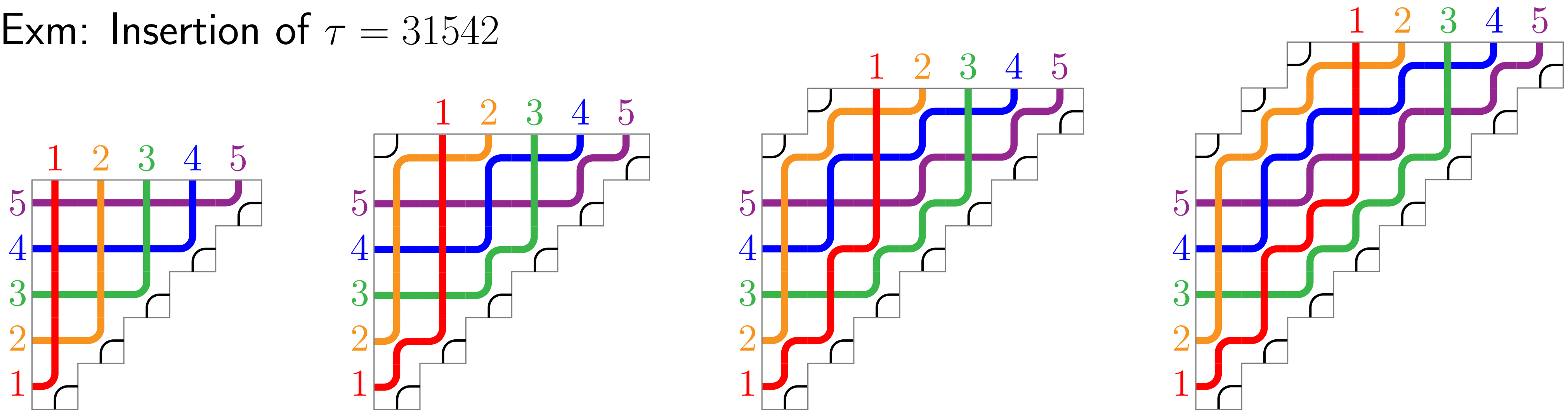

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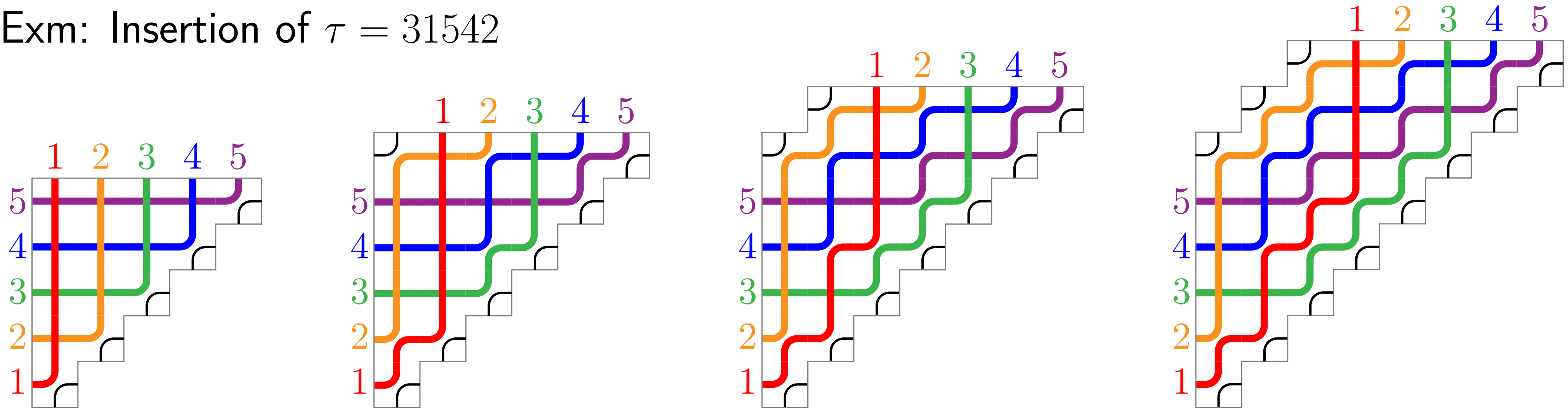


THM. ins^k is a surjection from permutations of $[n]$ to acyclic (k, n)-twists.
fiber of a (k, n)-twist T = linear extensions of its contact graph $\mathrm{T}^{\#}$.

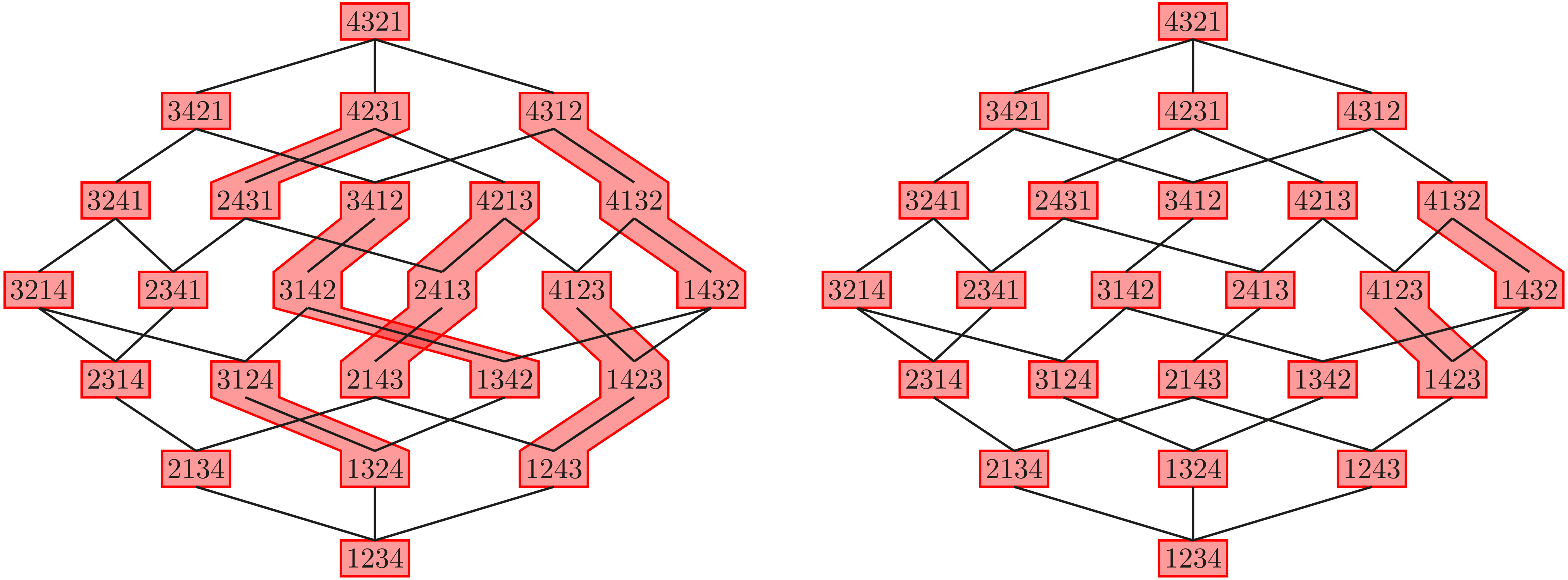
Exm: insertion in binary search trees

...

k-TWIST CONGRUENCE

k-twist congruence = equivalence relation $\equiv^k$ on $\mathfrak{S}_n$ defined as the transitive closure of the rewriting rule

$$UacV_1b_1V_2b_2\cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2\cdots V_kb_kW \qquad \text{if } a < b_i < c \text{ for all } i \in [k].$$



PROP. The k-twist congruence is a lattice congruence of the weak order.

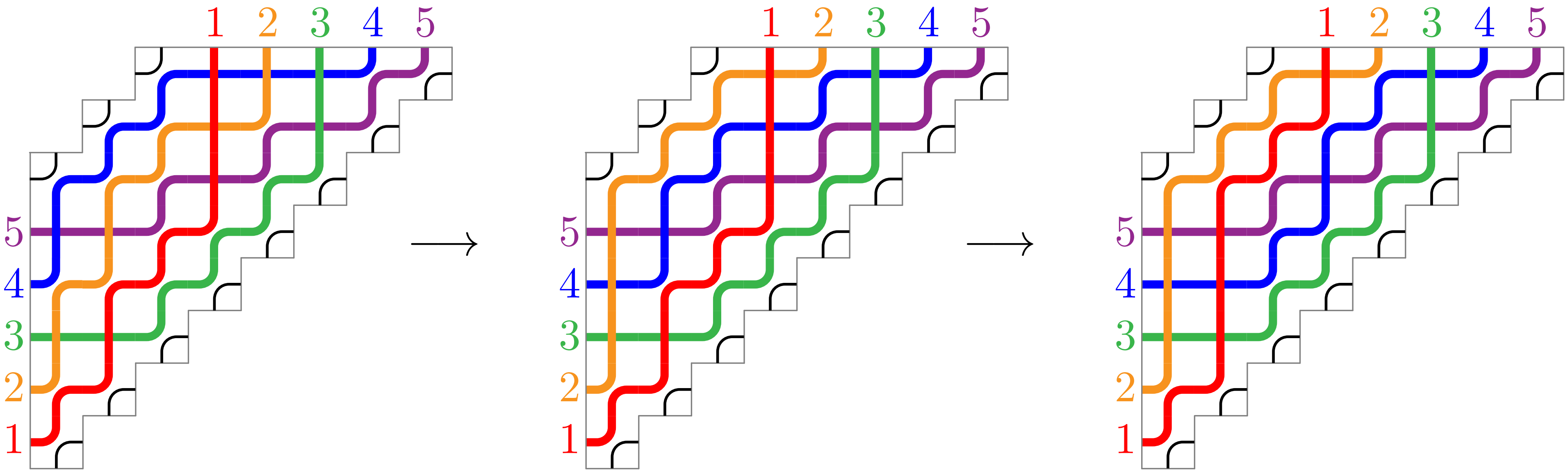
PROP. For any $\tau, \tau' \in \mathfrak{S}_n$, we have $\tau \equiv^k \tau' \iff \mathrm{ins}^k(\tau) = \mathrm{ins}^k(\tau')$.

INCREASING FLIP LATTICE

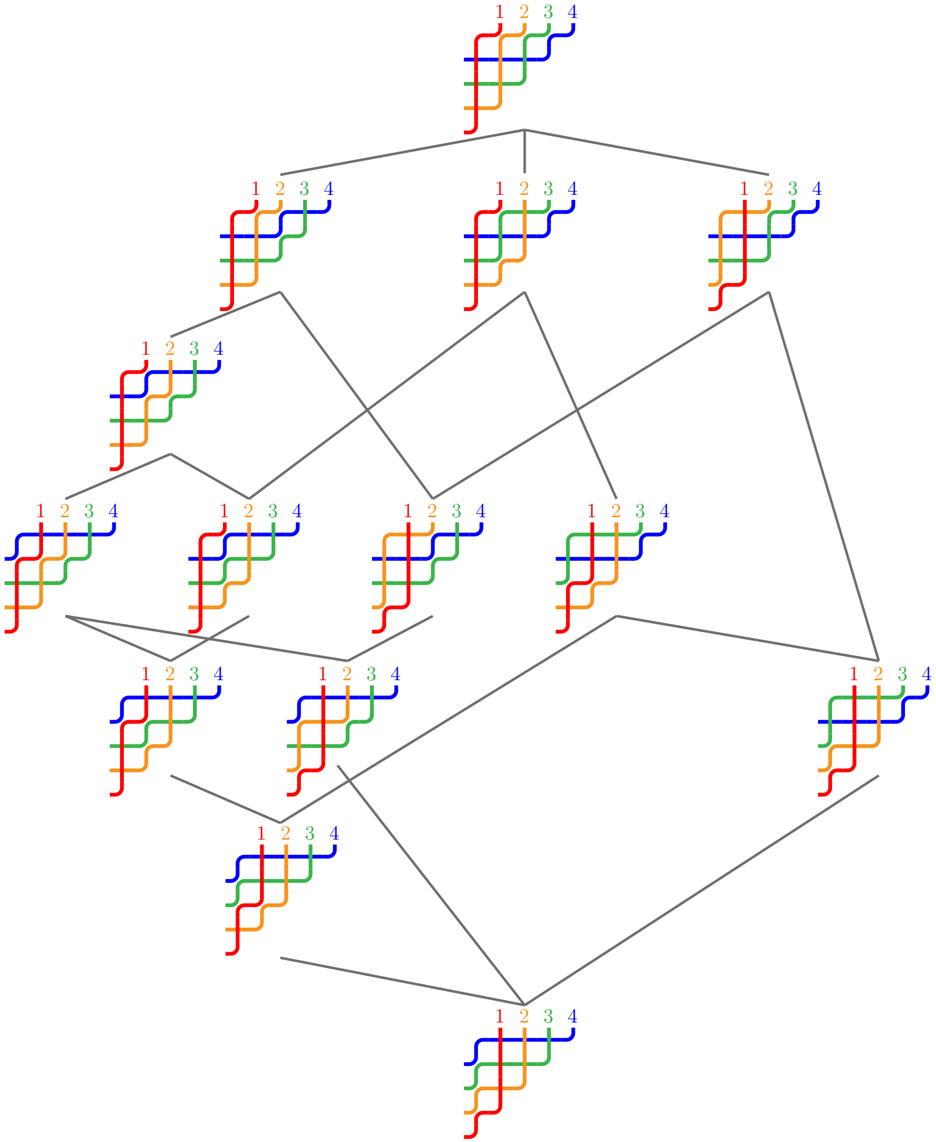
flip in a k-twist $=$ exchange an elbow with the unique crossing between its two pipes

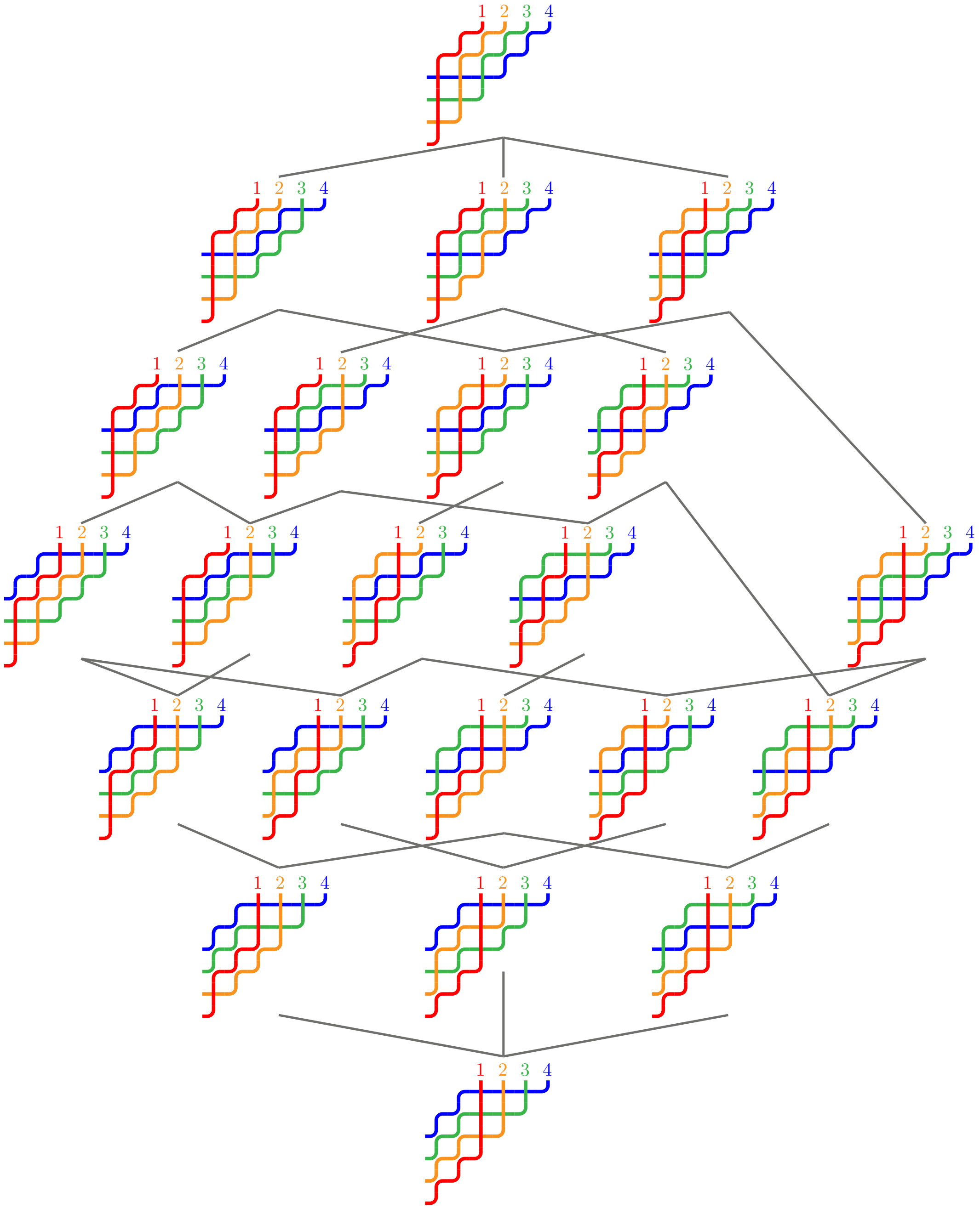
increasing flip $=$ the elbow is southwest of the crossing

increasing flip order $=$ transitive closure of the increasing flip graph



INCREASING FLIP LATTICE





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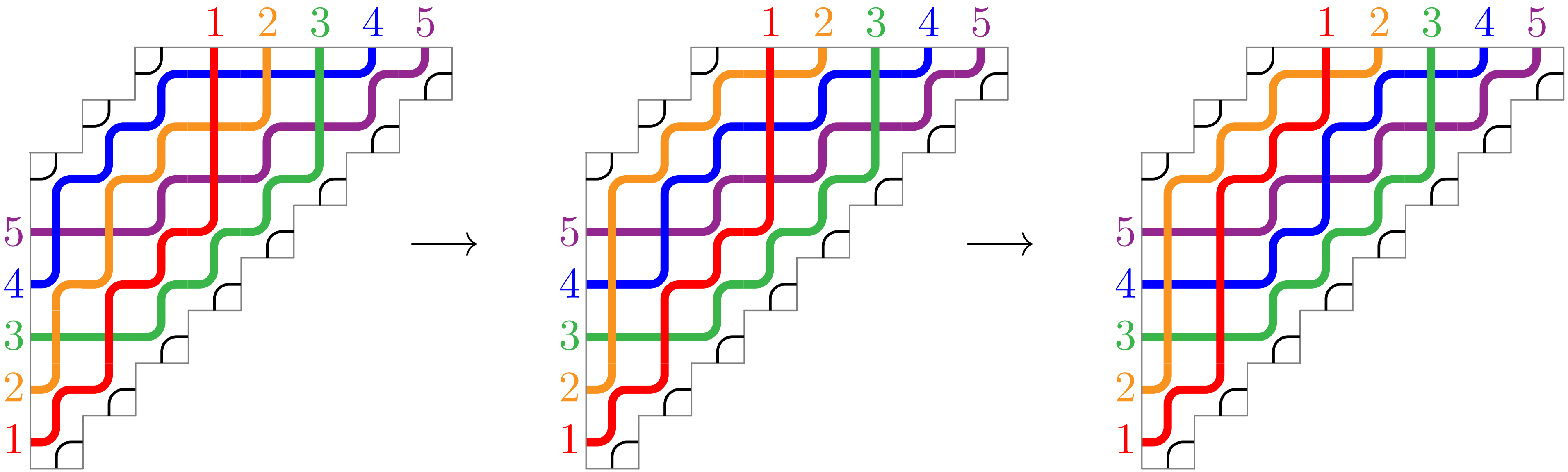


PROP. The increasing flip order on acyclic k-twists is isomorphic to:

- the quotient lattice of the weak order by the k-twist congruence $\equiv^k$,
- the subposet of the weak order induced by the permutations of $\mathfrak{S}_n$ avoiding the pattern $1(k+2)-(\sigma_1+1)-\cdots-(\sigma_k+1)$ for all $\sigma \in \mathfrak{S}_k$.

TWIST ALGEBRA

SHUFFLE AND CONVOLUTION

For $n, n' \in \mathbb{N}$, consider the set of perms of $\mathfrak{S}_{n+n'}$ with at most one descent, at position n:

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n')\}$$

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

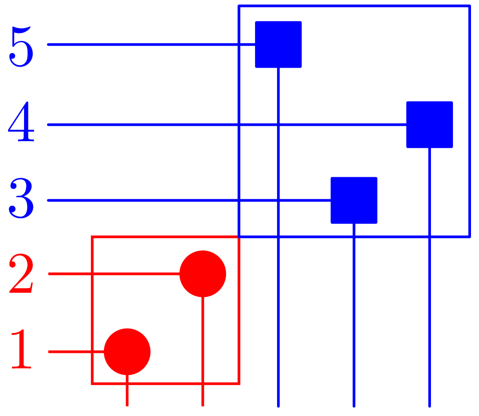
shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1)+n, \dots, \tau'(n')+n] \in \mathfrak{S}_{n+n'}$

shifted shuffle product $\tau \mathbin{\bar{\sqcup\!\sqcup}} \tau' = \left\{(\tau\bar{\tau}') \circ \pi^{-1} \;\middle|\; \pi \in \mathfrak{S}^{(n,n')}\right\} \subset \mathfrak{S}_{n+n'}$

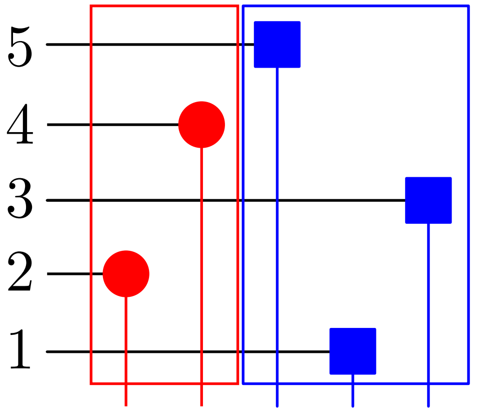
convolution product $\tau \star \tau' = \left\{\pi \circ (\tau\bar{\tau}') \;\middle|\; \pi \in \mathfrak{S}^{(n,n')}\right\} \subset \mathfrak{S}_{n+n'}$

Exm: $12 \mathbin{\bar{\sqcup\!\sqcup}} 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

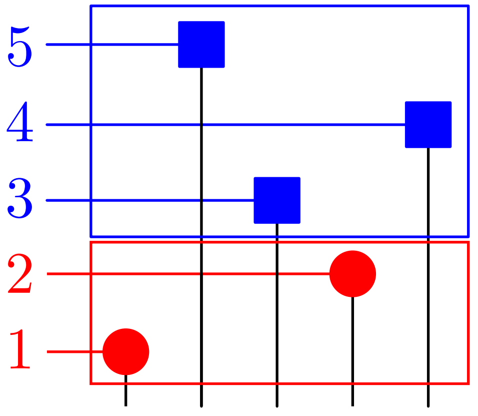
$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



concatenation



shuffle



convolution

MALVENUTO & REUTENAUER'S HOPF ALGEBRA ON PERMUTATIONS

DEF. Combinatorial Hopf Algebra $=$ combinatorial vector space $\mathcal{B}$ endowed with

$$\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$$

$$\text{coproduct } \triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

which are "compatible", ie.

$$\begin{tikzcd}
\mathcal{B} \otimes \mathcal{B} \arrow[r, "\cdot"] \arrow[d, "\triangle \otimes \triangle"'] & \mathcal{B} \arrow[r, "\triangle"] & \mathcal{B} \otimes \mathcal{B} \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[rr, "I \otimes \mathrm{swap} \otimes I"'] & & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[u, "\cdot \otimes \cdot"']
\end{tikzcd}$$

Malvenuto-Reutenauer algebra $=$ Hopf algebra FQSym with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ and where

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

Malvenuto-Reutenauer. *Duality between quasi-symmetric functions and the Solomon descent algebra*. 1995

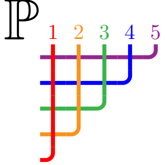
HOPF SUBALGEBRA

k-Twist algebra $=$ vector subspace Twist^k of FQSym generated by

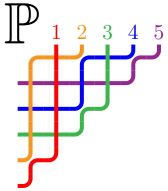
$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau\in\mathfrak{S}\\ \mathrm{ins}^k(\tau)=\mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau\in\mathcal{L}(\mathrm{T}^\#)} \mathbb{F}_\tau,$$

for all acyclic k-twists T.

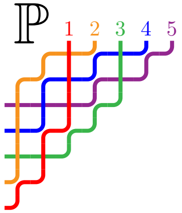
Exm:

$\mathbb{P}$


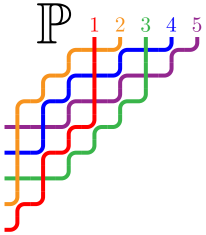
$= \sum_{\tau\in\mathfrak{S}_5} \mathbb{F}_\tau$

$\mathbb{P}$


$= \mathbb{F}_{13542} + \mathbb{F}_{15342} + \mathbb{F}_{31542} + \mathbb{F}_{51342} + \mathbb{F}_{35142} + \mathbb{F}_{53142} + \mathbb{F}_{35412} + \mathbb{F}_{53412}$

$\mathbb{P}$


$= \mathbb{F}_{31542} + \mathbb{F}_{35142}$

$\mathbb{P}$


$= \mathbb{F}_{31542}.$

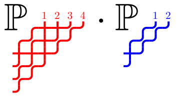
THEO. Twist^k is a subalgebra of FQSym

Loday-Ronco. *Hopf algebra of the planar binary trees.* 1998

Hivert-Novelli-Thibon. *The algebra of binary search trees.* 2005

GAME: Explain the product and coproduct directly on the k-twists...

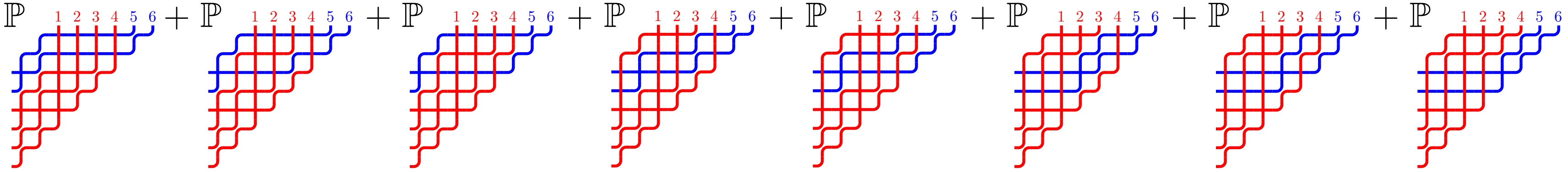
PRODUCT



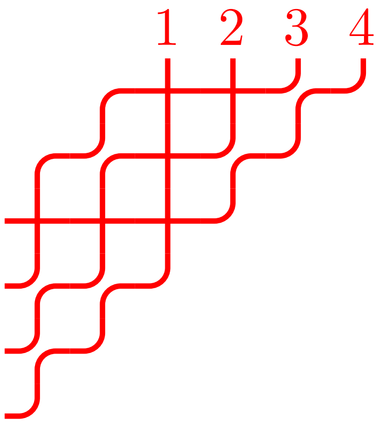
$$\mathbb{P} \cdot \mathbb{P} = (\mathbb{F}_{1423} + \mathbb{F}_{4123}) \cdot \mathbb{F}_{21}$$

$$= \begin{bmatrix} \mathbb{F}_{142365} \\ + \mathbb{F}_{412365} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{142635} \\ + \mathbb{F}_{146235} \\ + \mathbb{F}_{412635} \\ + \mathbb{F}_{416235} \\ + \mathbb{F}_{461235} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164235} \\ + \mathbb{F}_{614235} \\ + \mathbb{F}_{641235} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{142653} \\ + \mathbb{F}_{146253} \\ + \mathbb{F}_{412653} \\ + \mathbb{F}_{416253} \\ + \mathbb{F}_{461253} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164253} \\ + \mathbb{F}_{614253} \\ + \mathbb{F}_{641253} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{146523} \\ + \mathbb{F}_{416523} \\ + \mathbb{F}_{461523} \\ + \mathbb{F}_{465123} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164523} \\ + \mathbb{F}_{614523} \\ + \mathbb{F}_{641523} \\ + \mathbb{F}_{645123} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{165423} \\ + \mathbb{F}_{615423} \\ + \mathbb{F}_{651423} \\ + \mathbb{F}_{654123} \end{bmatrix}$$

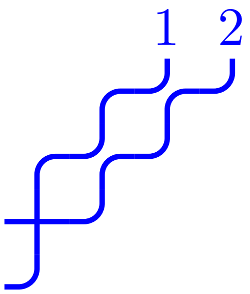
$$= \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P}$$



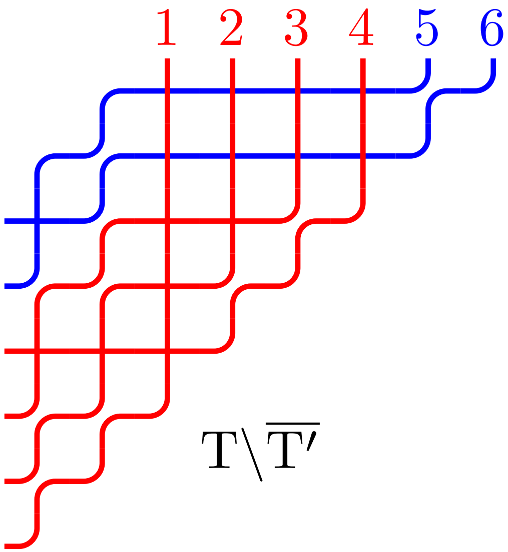
PROP. For $\mathrm{T} \in \mathcal{AT}^k(n)$ and $\mathrm{T}' \in \mathcal{AT}^k(n')$ acyclic k-twists, $\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}}$, where S runs over the interval between $\mathrm{T}\backslash\mathrm{T}'$ and T/T' in the $(k, n+n')$-twist lattice.



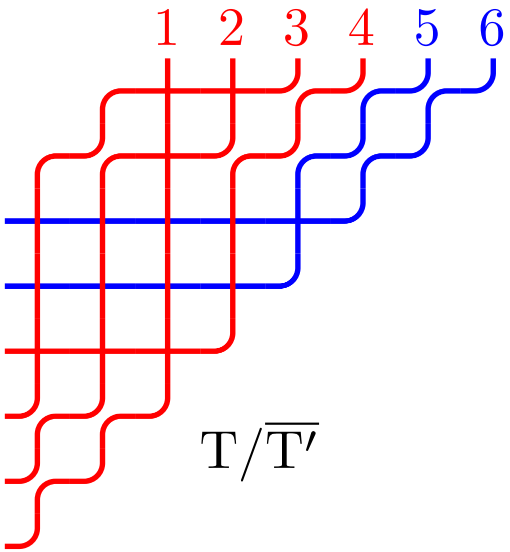
T



T'



$\mathrm{T}\backslash\overline{\mathrm{T}'}$



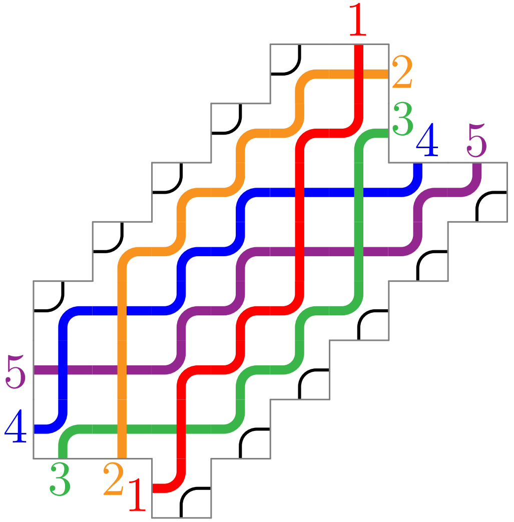
$\mathrm{T}/\overline{\mathrm{T}'}$

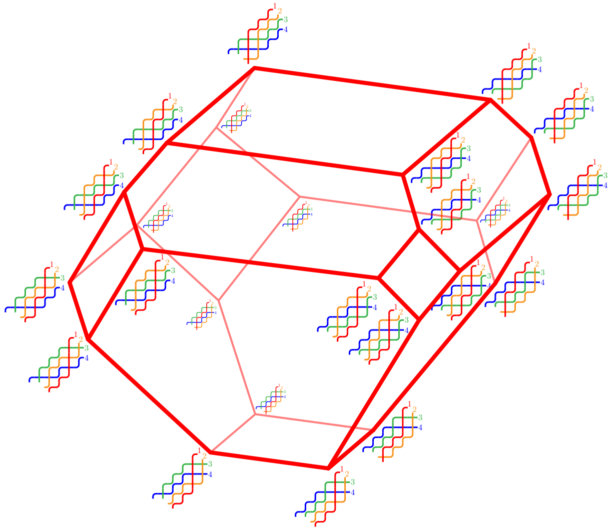
…

THREE EXTENSIONS

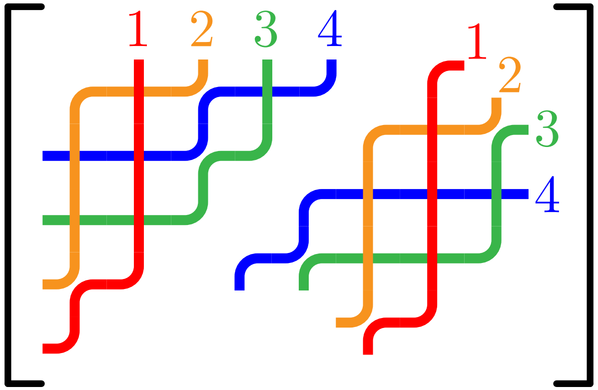
THREE EXTENSIONS

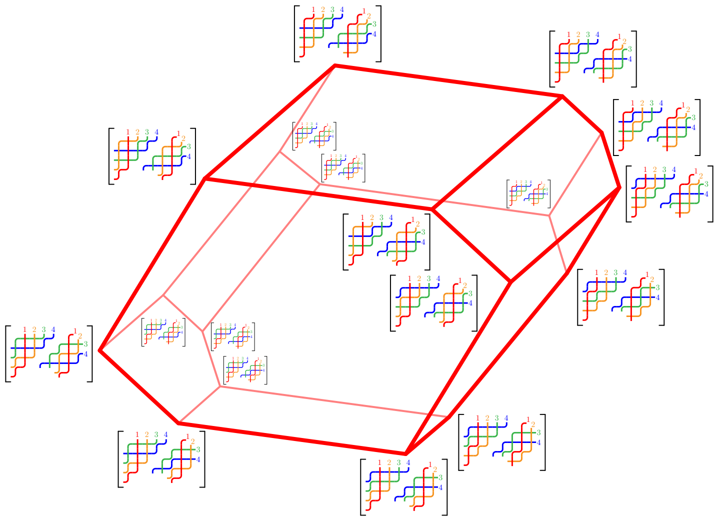
CAMBRIANIZATION



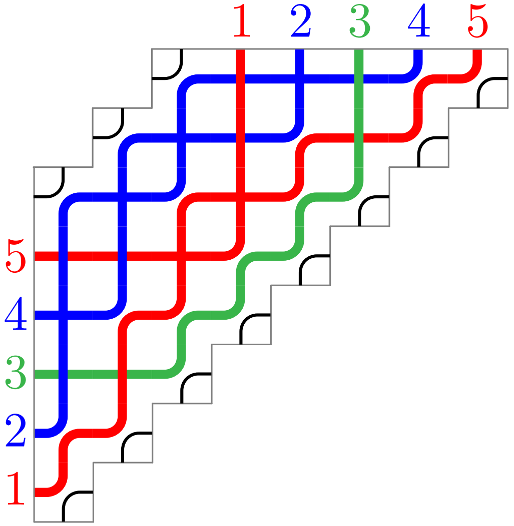


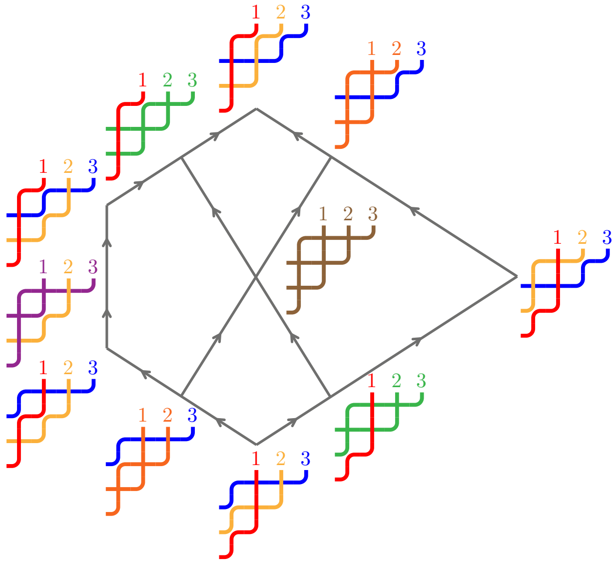
TUPLIZATION





SCHRÖDERIZATION





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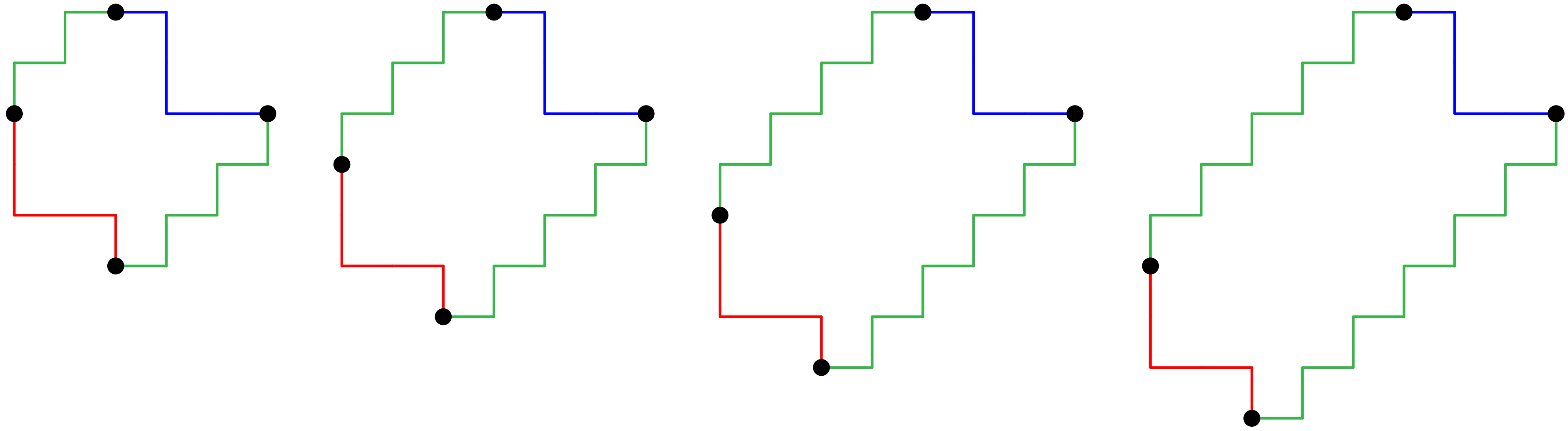
CAMBRIANIZATION



$k = 0$ $\quad$ $k = 1$ $\quad$ $k = 2$ $\quad$ $k = 3$

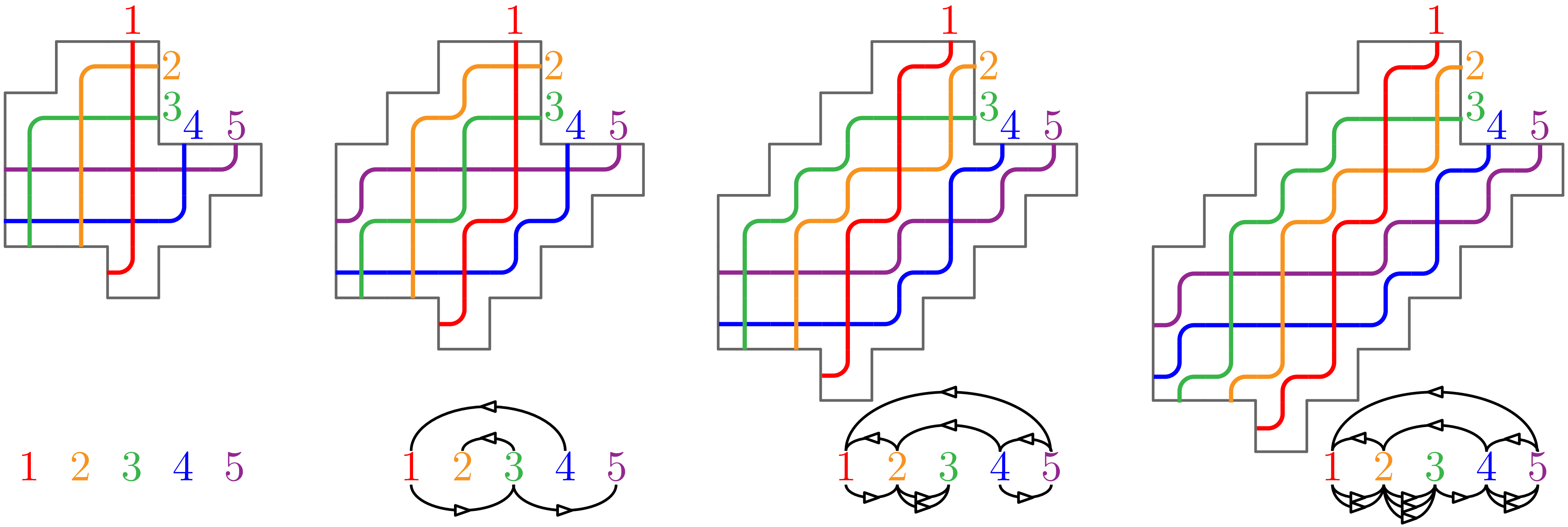
$k \in \mathbb{N}$ and $\varepsilon \in \pm^n$, define a shape $\mathrm{Sh}_\varepsilon^k$ formed by four monotone lattices paths:

(i) enter path: from $(|\varepsilon|_+, 0)$ to $(0, |\varepsilon|_-)$ with pth step north if $\varepsilon_p = -$ and west if $\varepsilon_p = +$,

(ii) exit path: from $(|\varepsilon|_+ + k, n + k)$ to $(n + k, |\varepsilon|_- + k)$ with pth step east if $\varepsilon_p = -$ and south if $\varepsilon_p = +$,

(iii) accordion paths: the path $(NE)^{|\varepsilon|_+ + k}$ from $(0, |\varepsilon|_-)$ to $(|\varepsilon|_+ + k, n + k)$ and the path $(EN)^{|\varepsilon|_- + k}$ from $(|\varepsilon|_+, 0)$ to $(n + k, |\varepsilon|_- + k)$.

CAMBRIANIZATION



$k = 0$ $k = 1$ $k = 2$ $k = 3$

Cambrian (k, ε)-twist = pipe dream in $\mathsf{Sh}_{\varepsilon}^{k}$

contact graph of a twist T = vertices are pipes of T and arcs are elbows of T

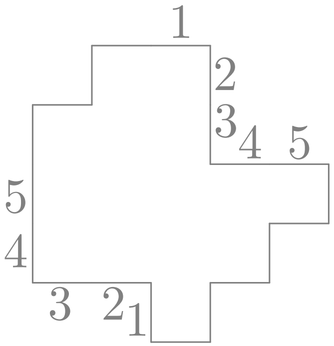
CAMBRIANIZATION

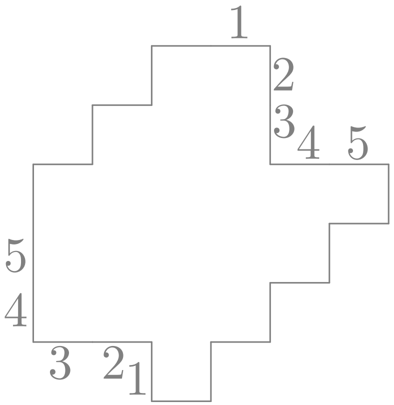
Input: a signed permutation $\tau = \tau_1 \cdots \tau_n$

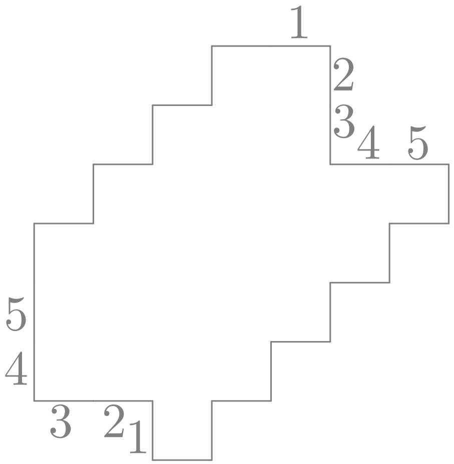
Algo: Insert pipes one by one (from right to left) as northwest as possible

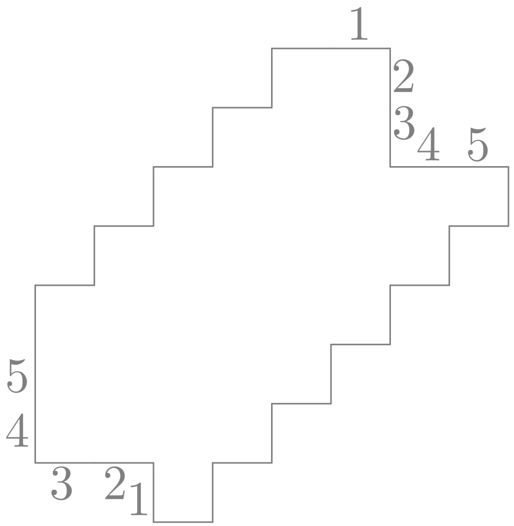
Output: an acyclic Cambrian (k, ε)-twist $\mathrm{ins}^k(\tau)$

Exm: Insertion of $\tau = \overline{3}\underline{154}\overline{2}$









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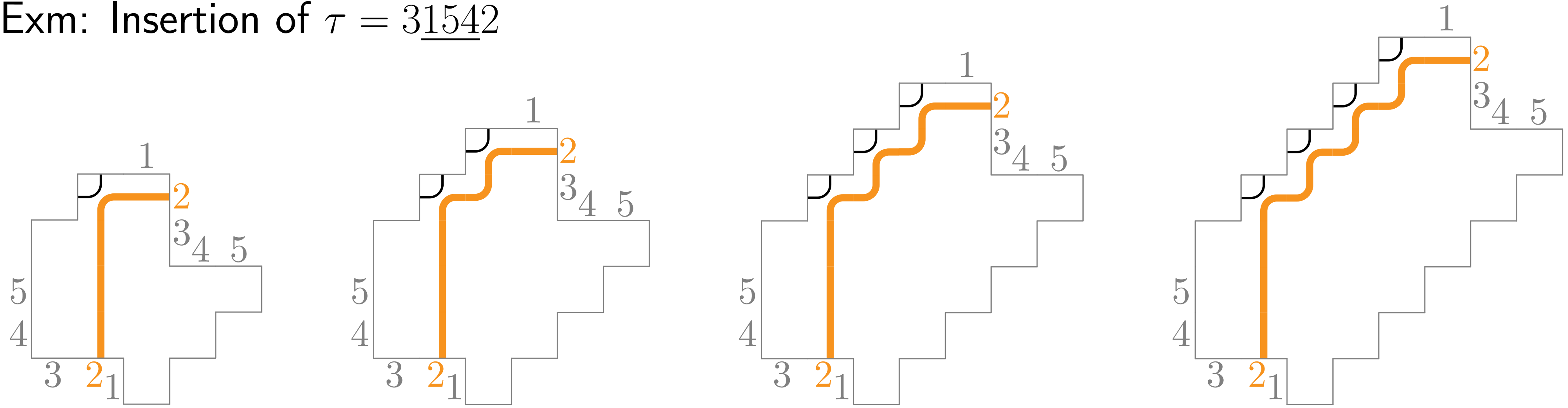

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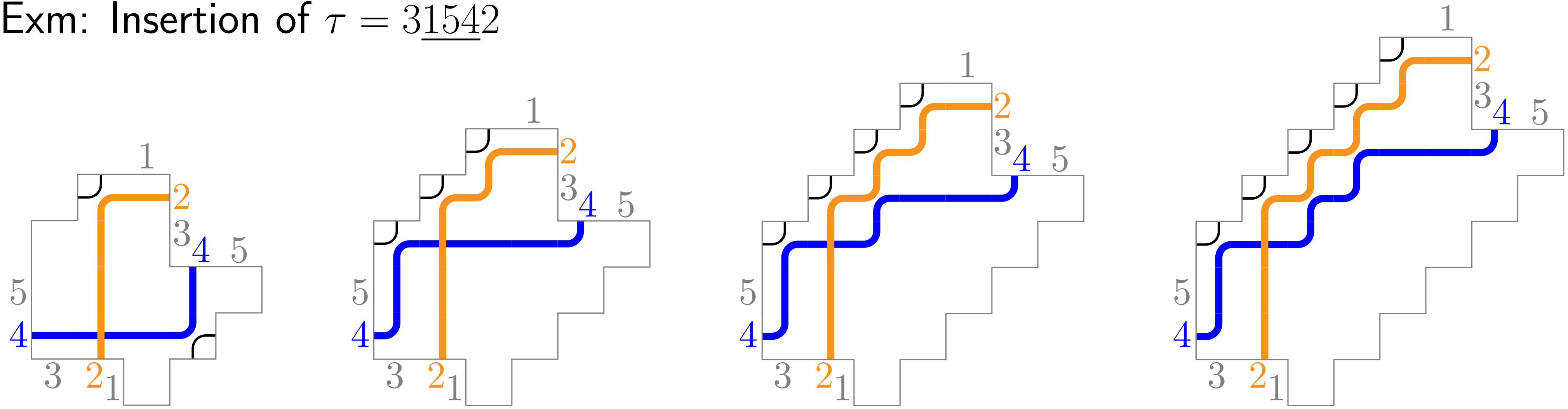

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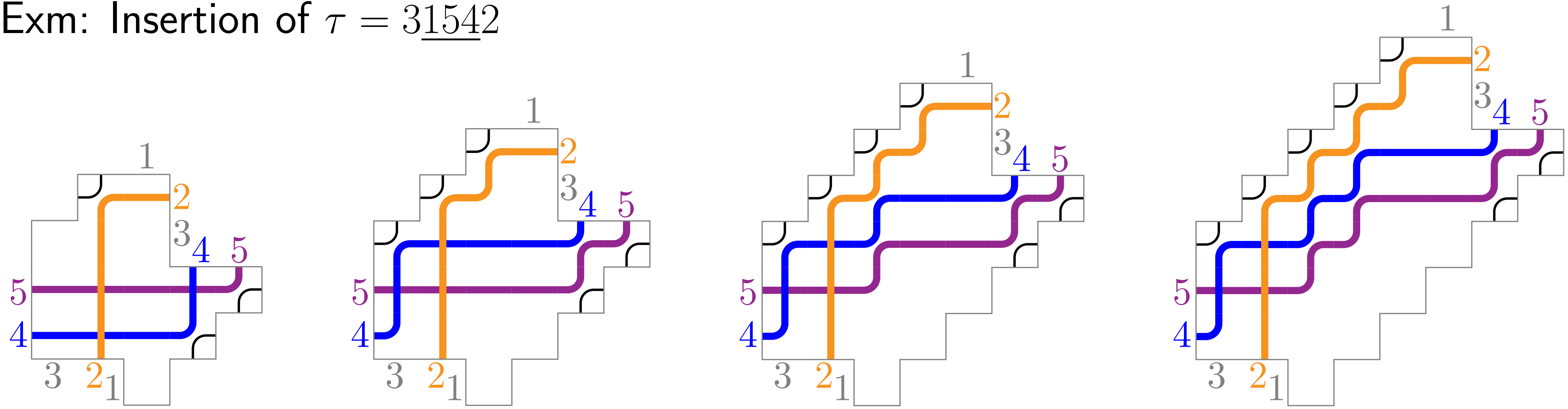

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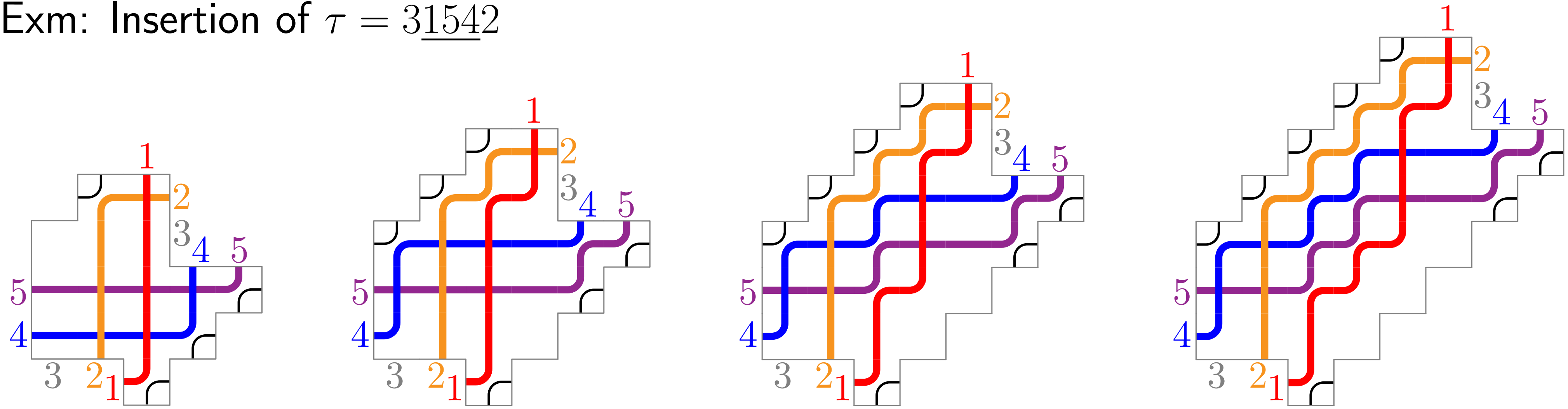

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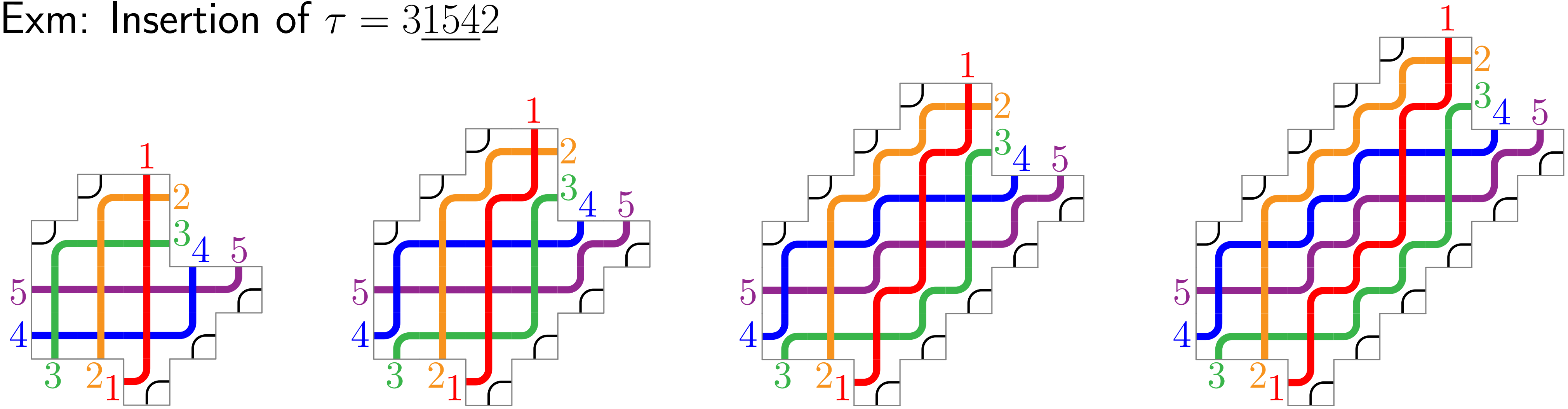

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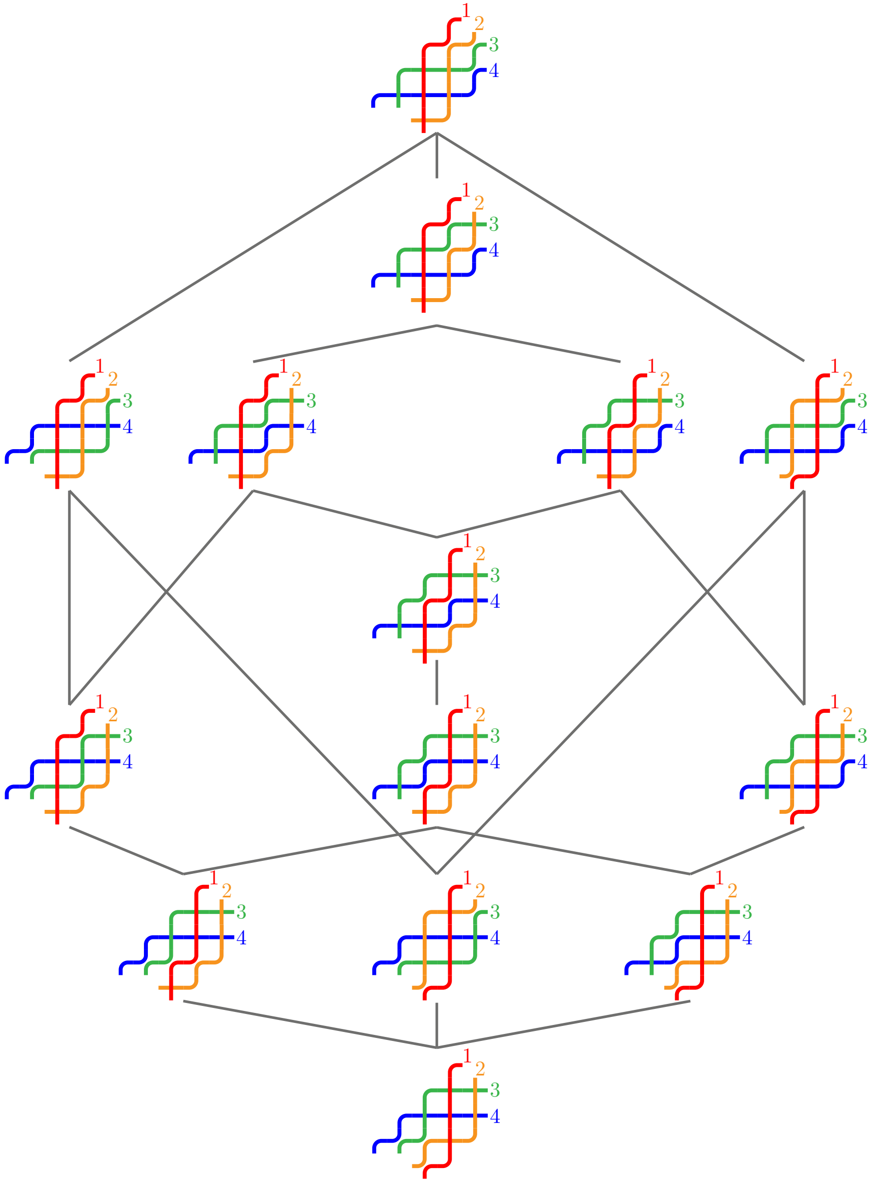
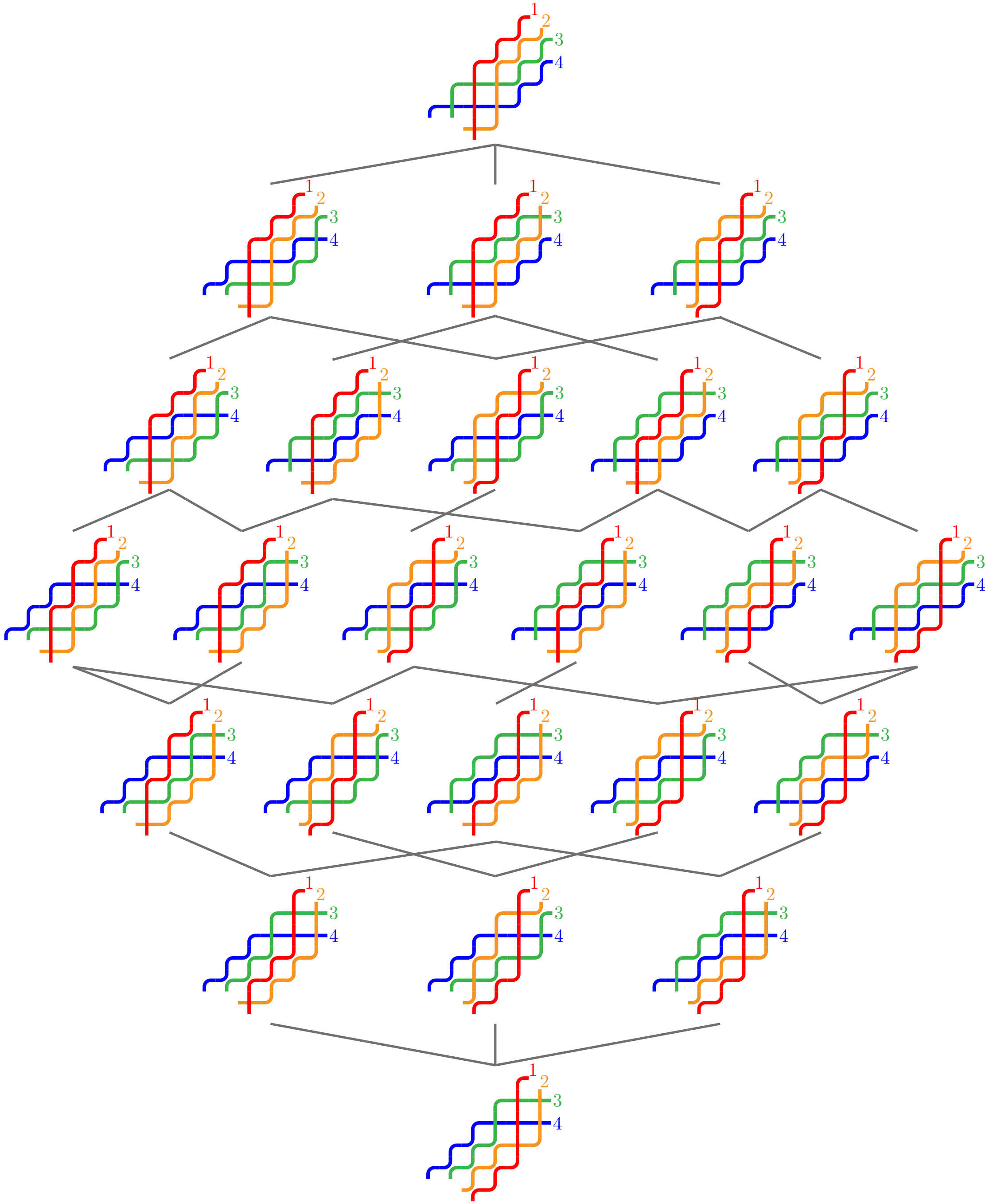


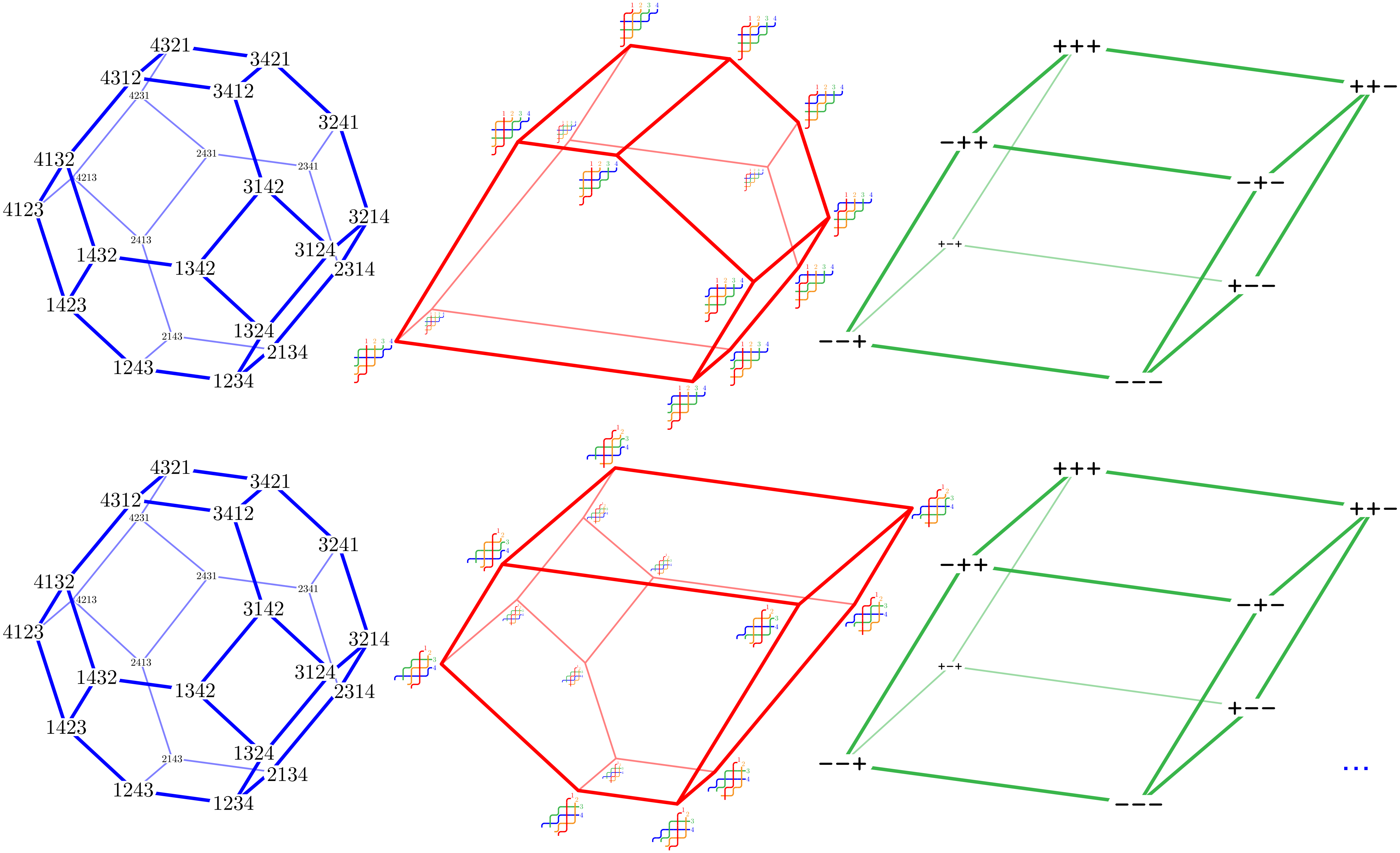


CAMBRIANIZATION

permutahedron Perm

brick polytope Brick$^k(\varepsilon)$

zonotope Zono$^k(n)$



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permutahedron Perm

brick polytope $\mathrm{Brick}^k(\varepsilon)$

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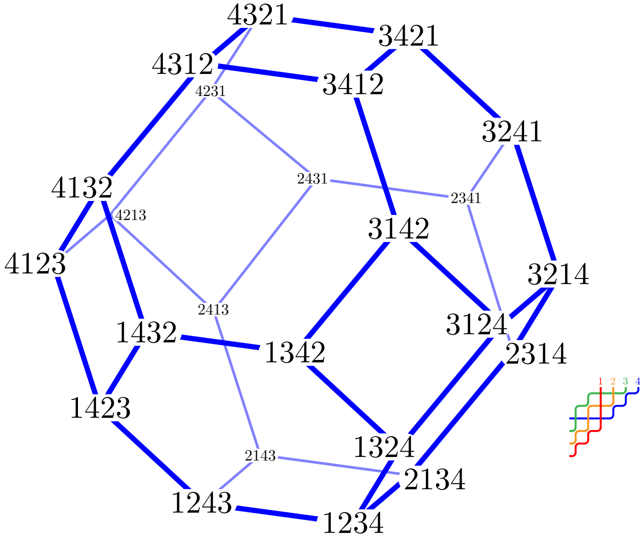

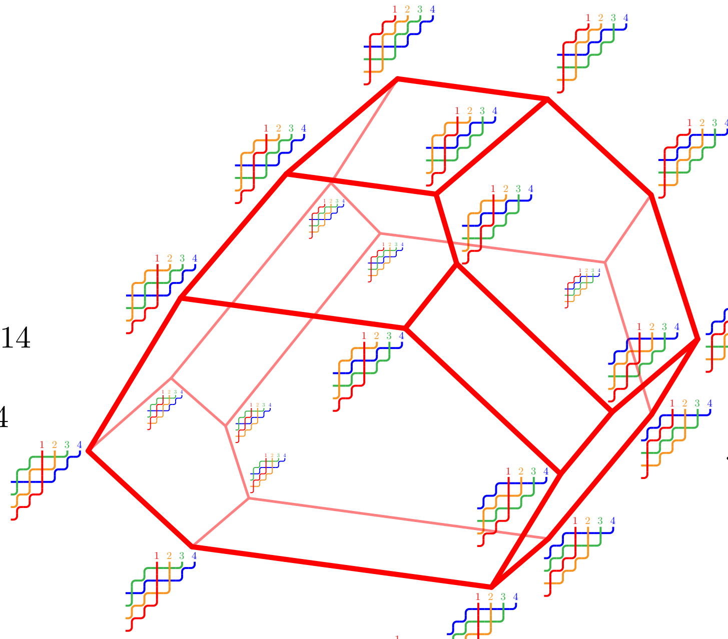

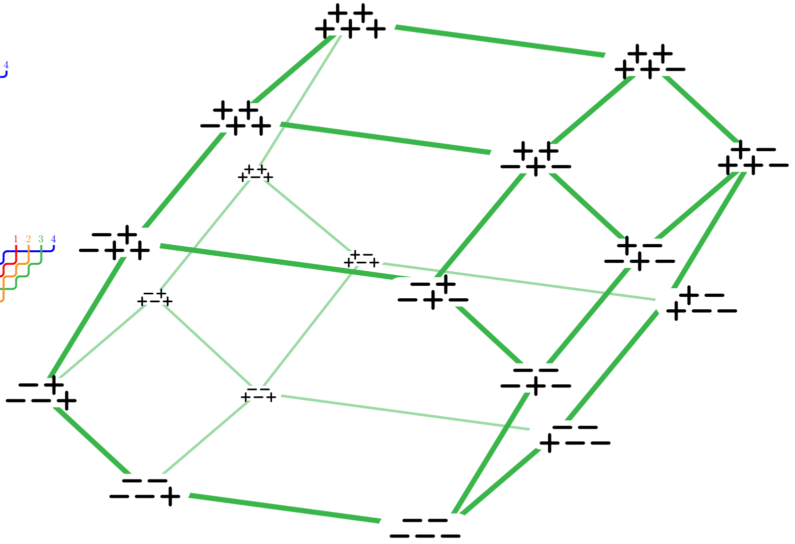

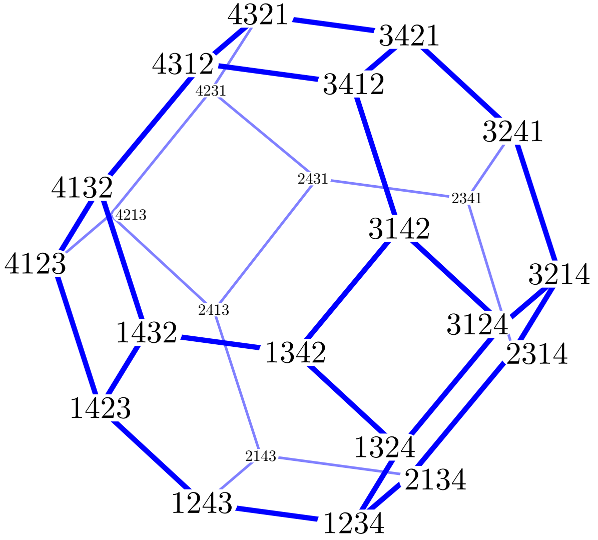

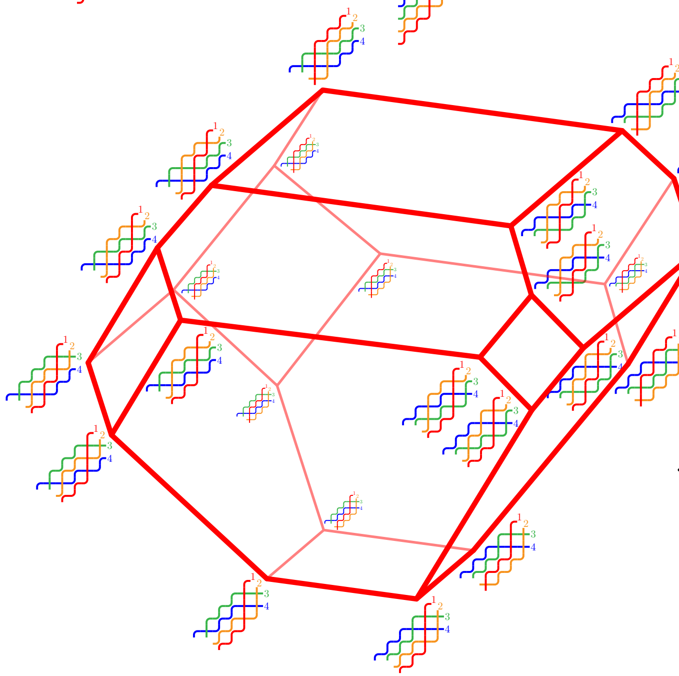

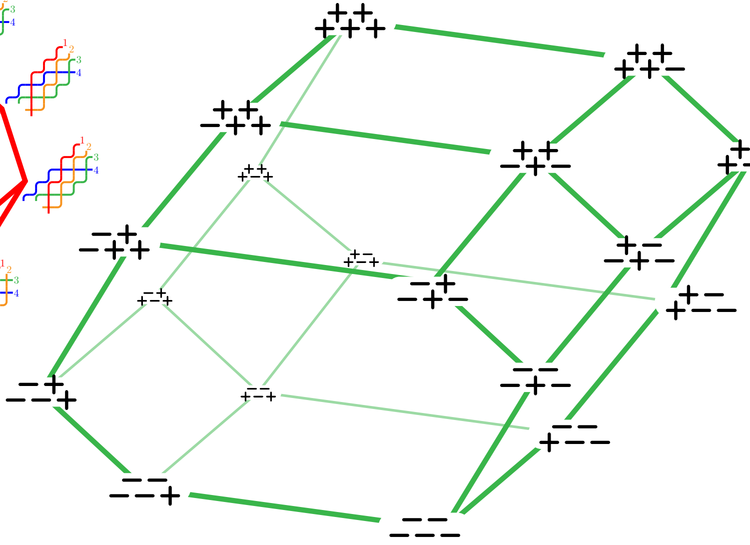

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CAMBRIANIZATION

permutahedron Perm

brick polytope $\mathrm{Brick}^k(\varepsilon)$

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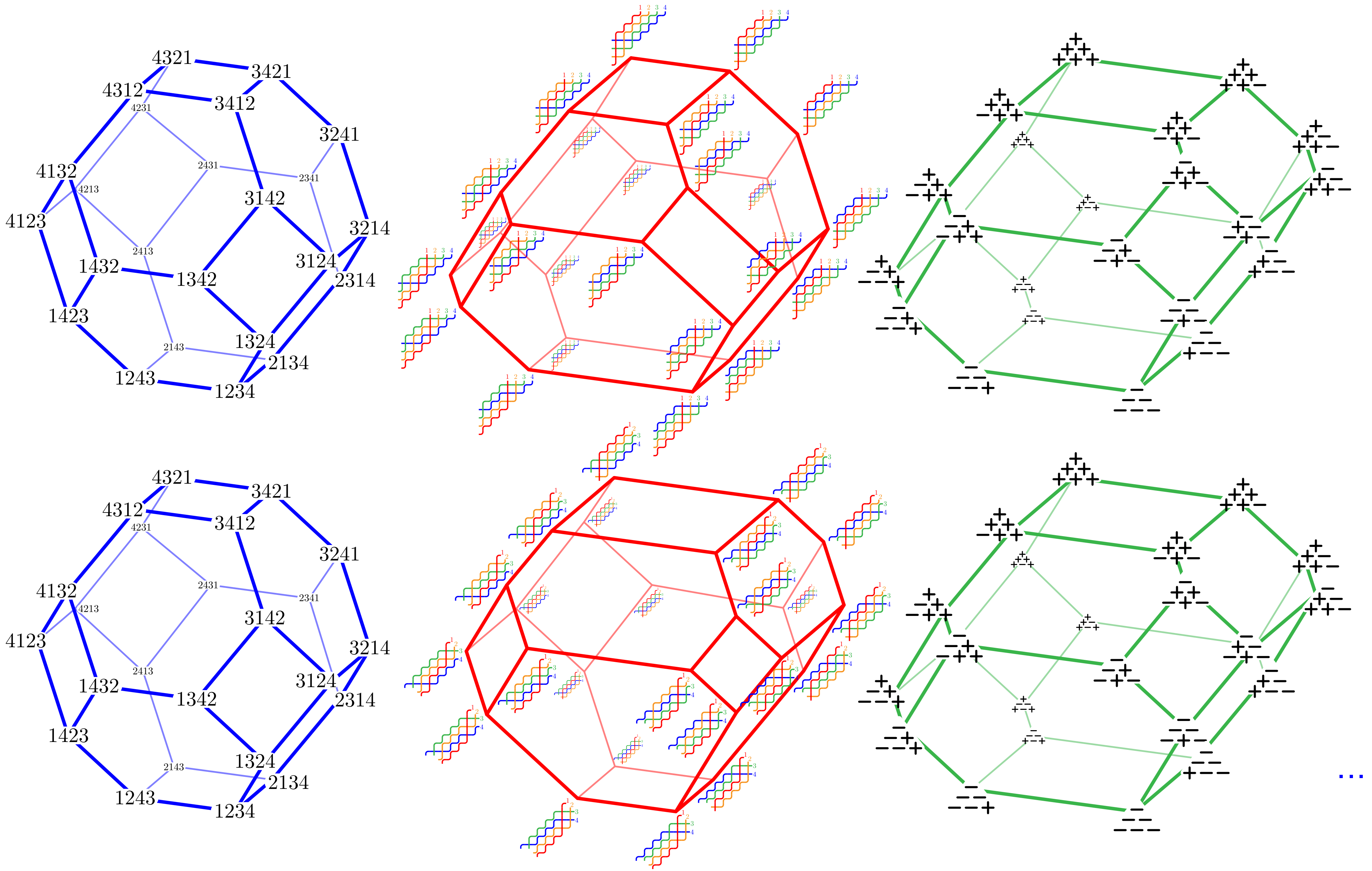
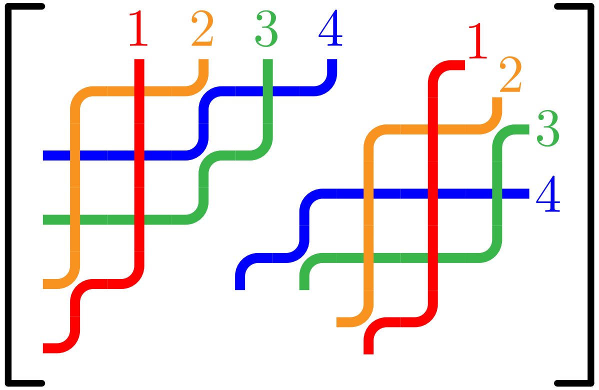


TUPLIZATION

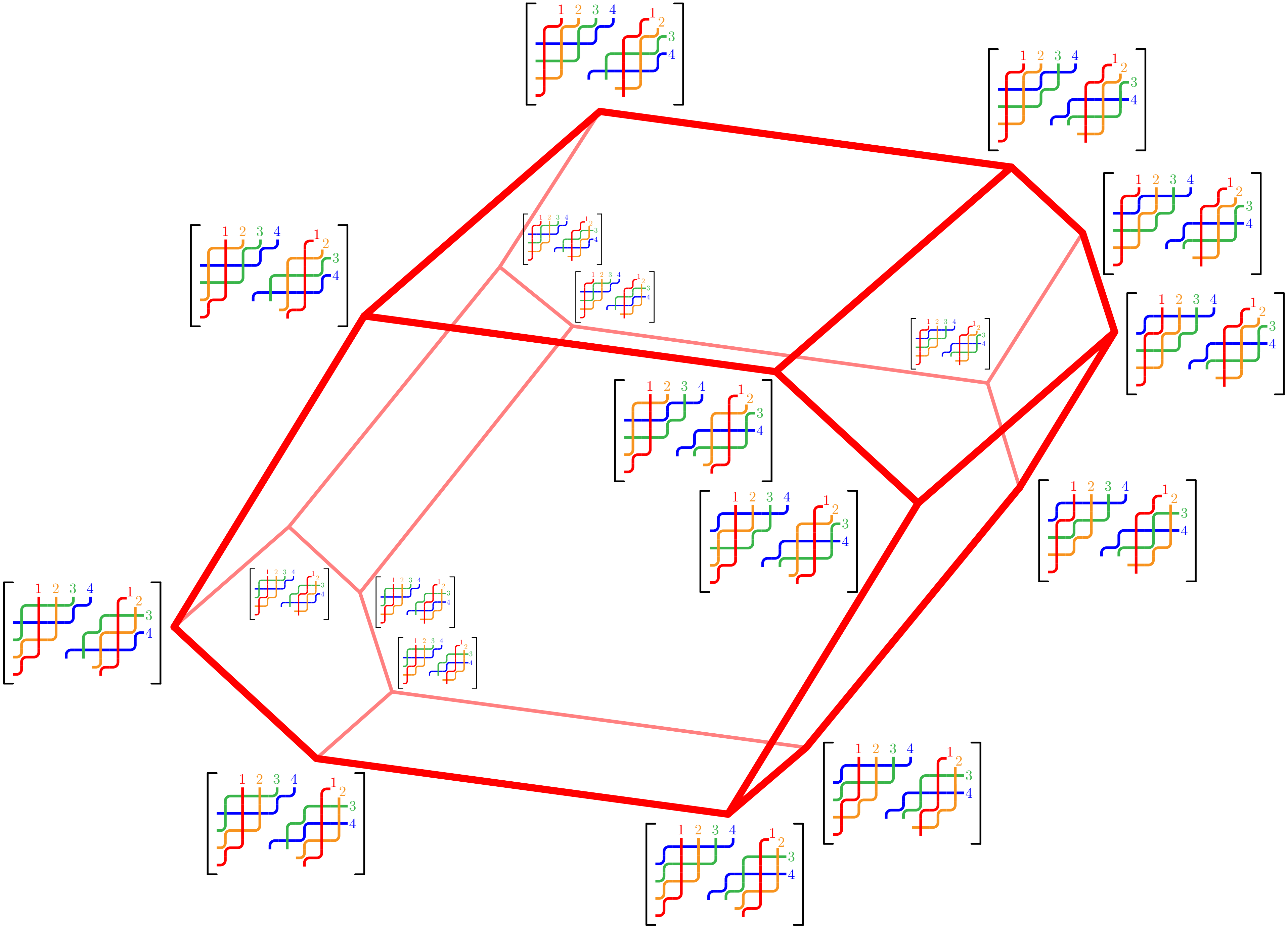
$\mathcal{E} = [\varepsilon_1, \dots, \varepsilon_\ell]$ an ℓ-tuple of signatures

$(k, \mathcal{E})$-twist tuple $=$ an ℓ-tuple $[\mathrm{T}_1, \dots, \mathrm{T}_\ell]$ where

- T_i is a (k, ε_i)-twist
- the union of the contact graphs $\mathrm{T}_1^\# \cup \dots \cup \mathrm{T}_\ell^\#$ is acyclic



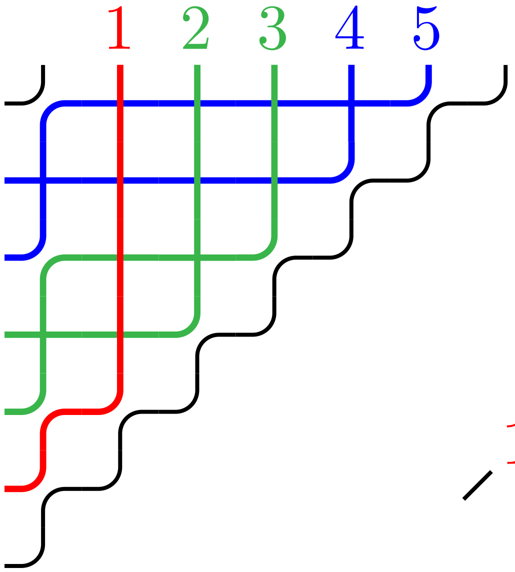
TUPLIZATION

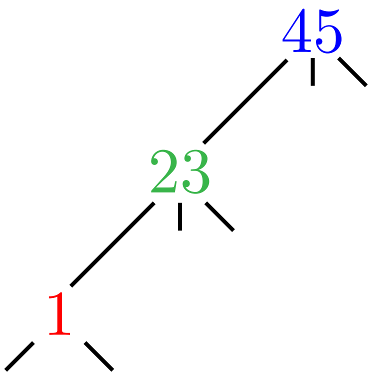


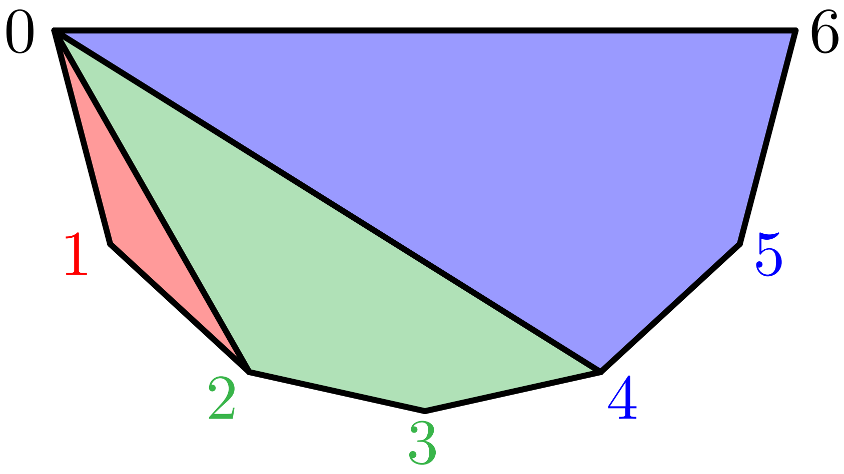
SCHRODERIZATION

hyperpipe = union of pipes whose common elbows are changed to crossings

(k, n)-hypertwist = collection of hyperpipes obtained from a (k, n)-twist T by merging subsets of pipes inducing connected subgraphs of $\mathrm{T}^{\#}$







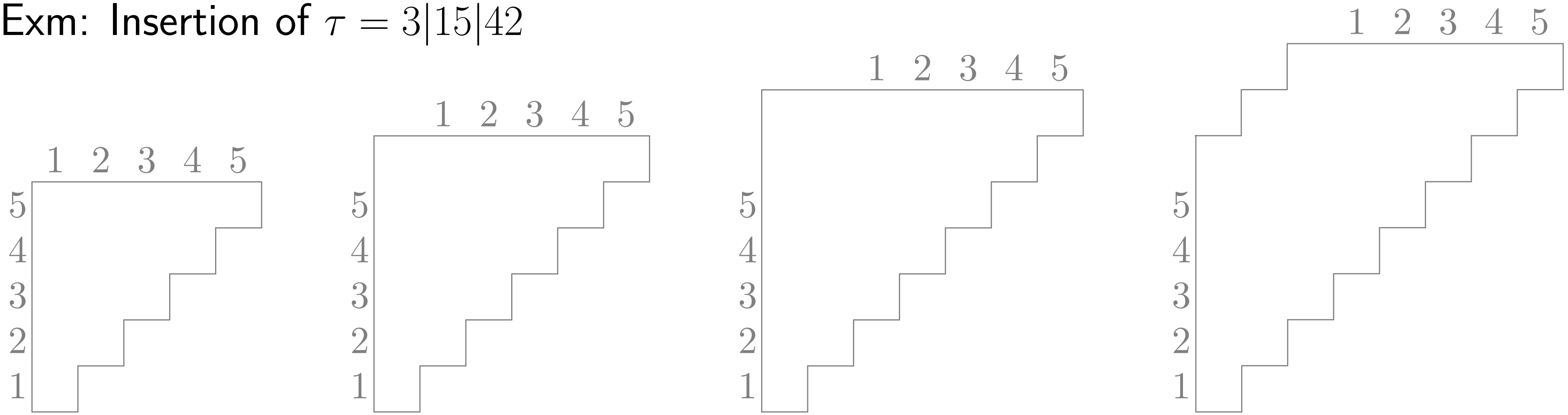
SCHRODERIZATION

Input: an ordered partition $\lambda = \lambda_1 \cdots \lambda_n$

Algo: Insert hyperpipes one by one (from right to left) as northwest as possible

Output: an acyclic (k, n)-hypertwist ins$^k(\lambda)$

Exm: Insertion of $\tau = 3|15|42$



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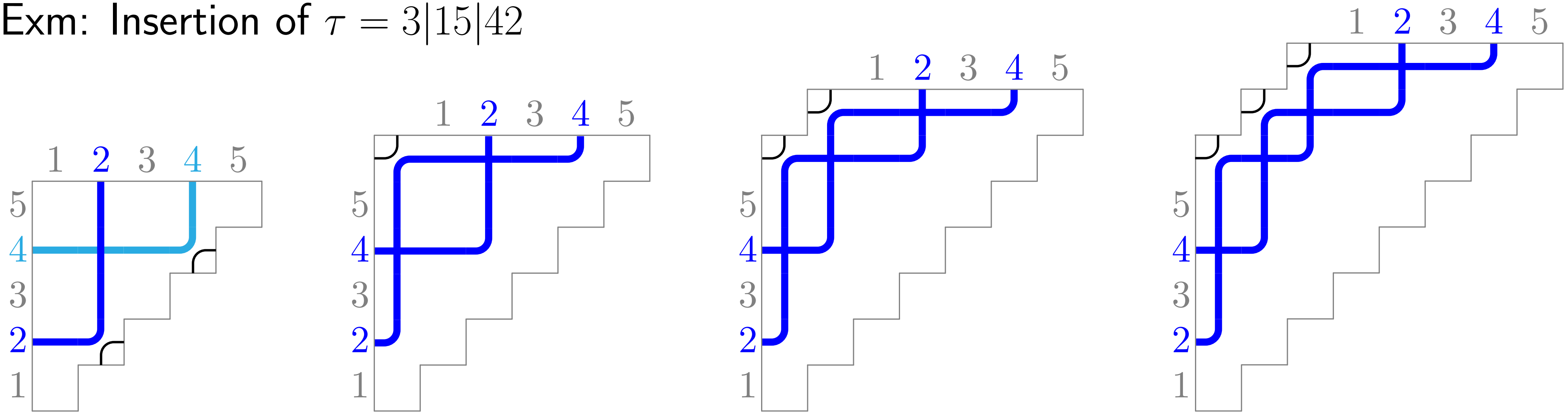

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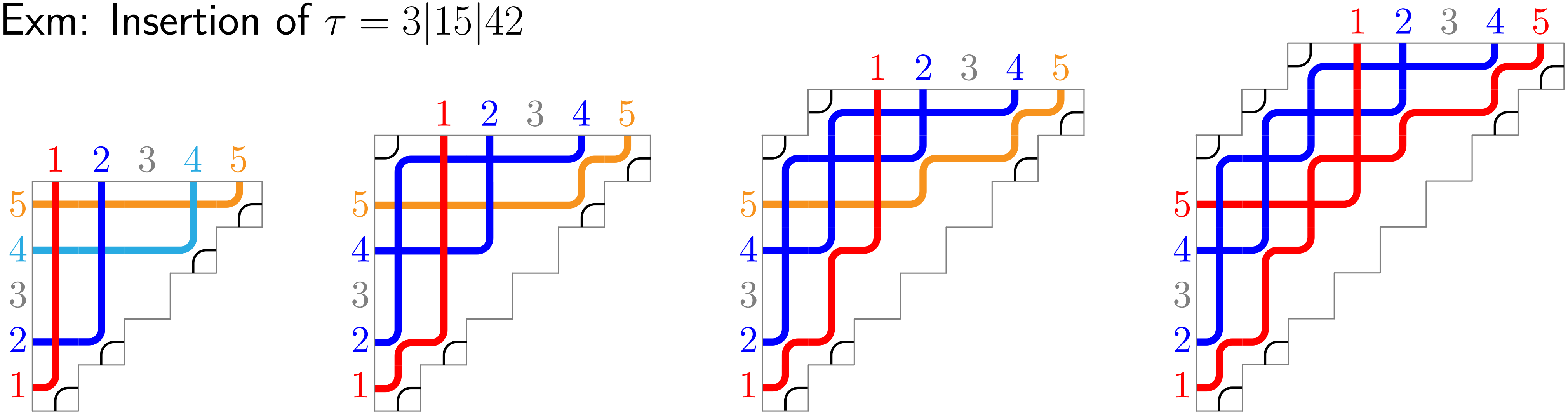

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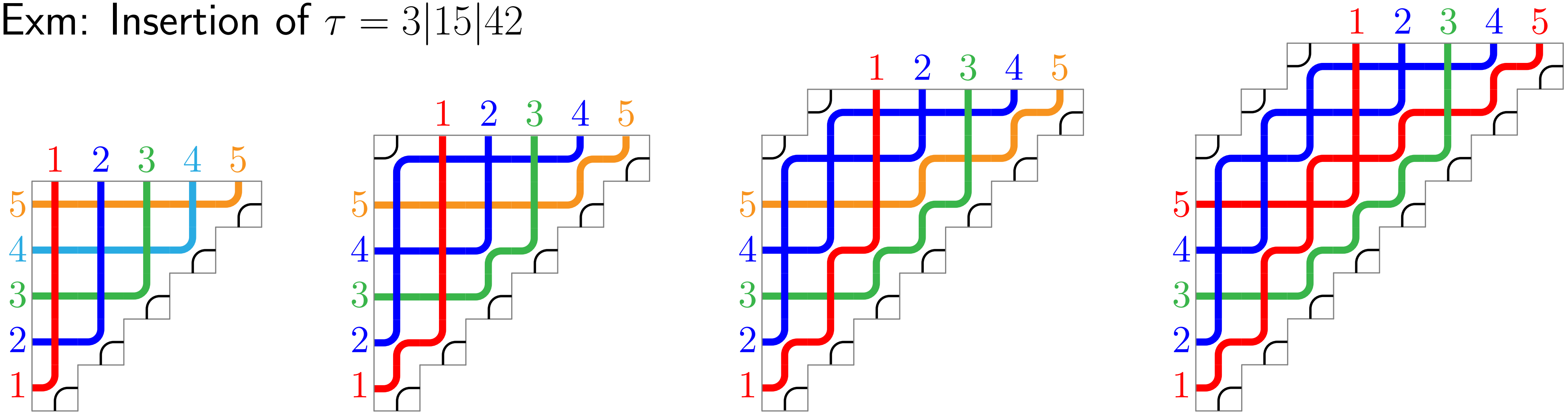
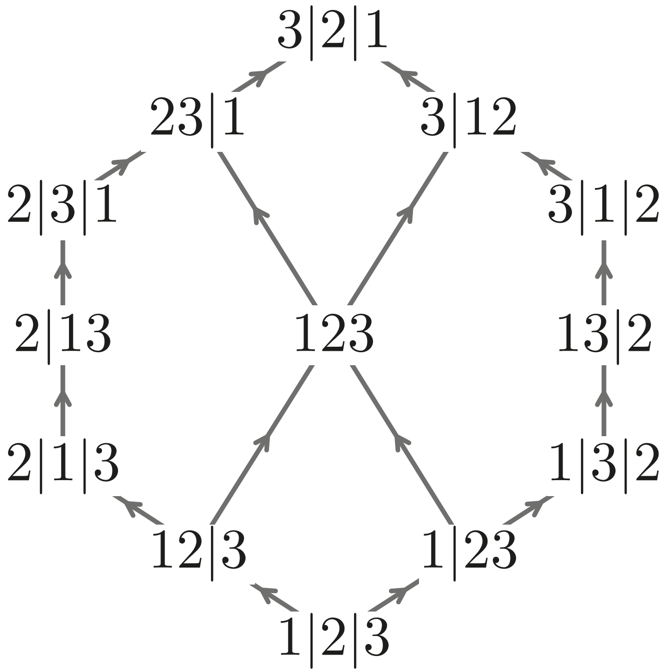
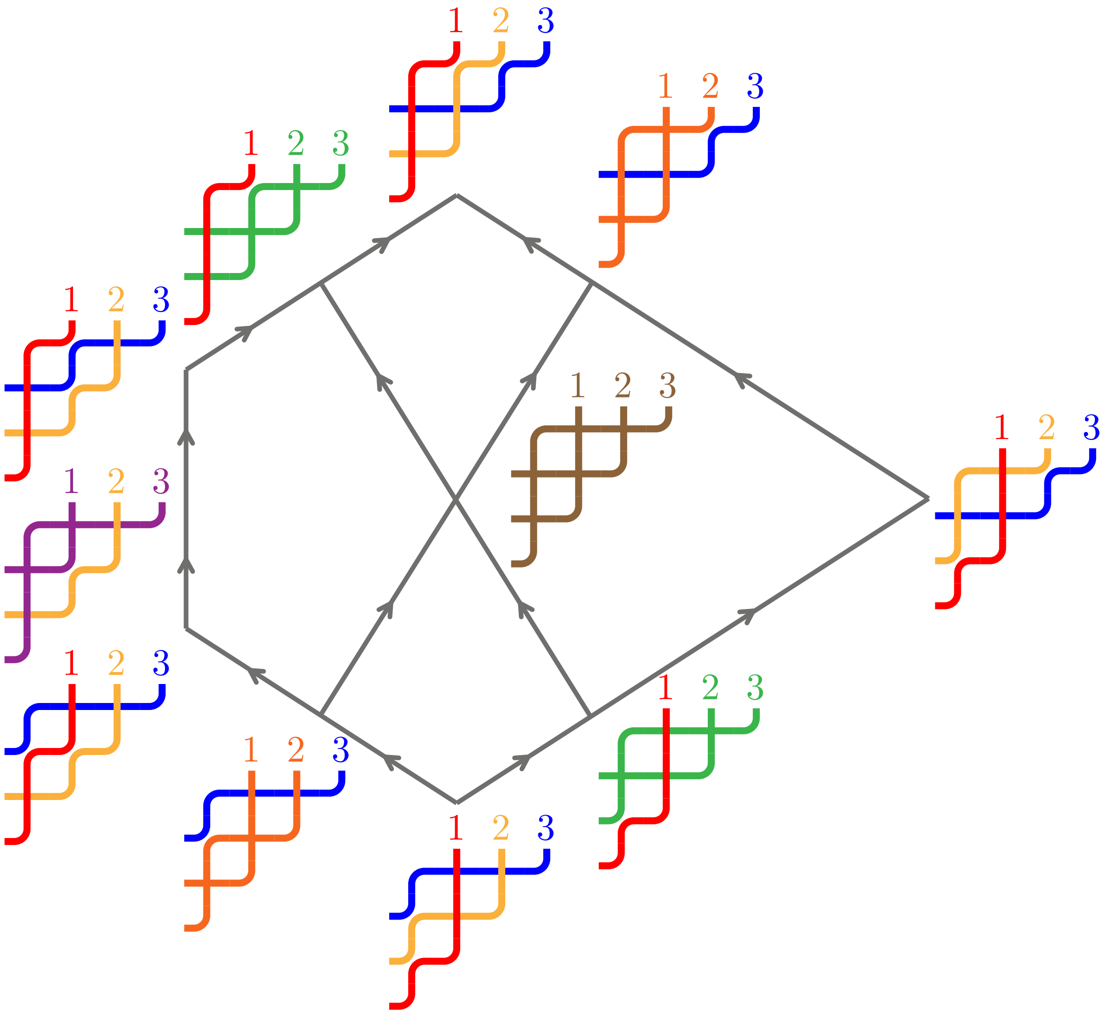
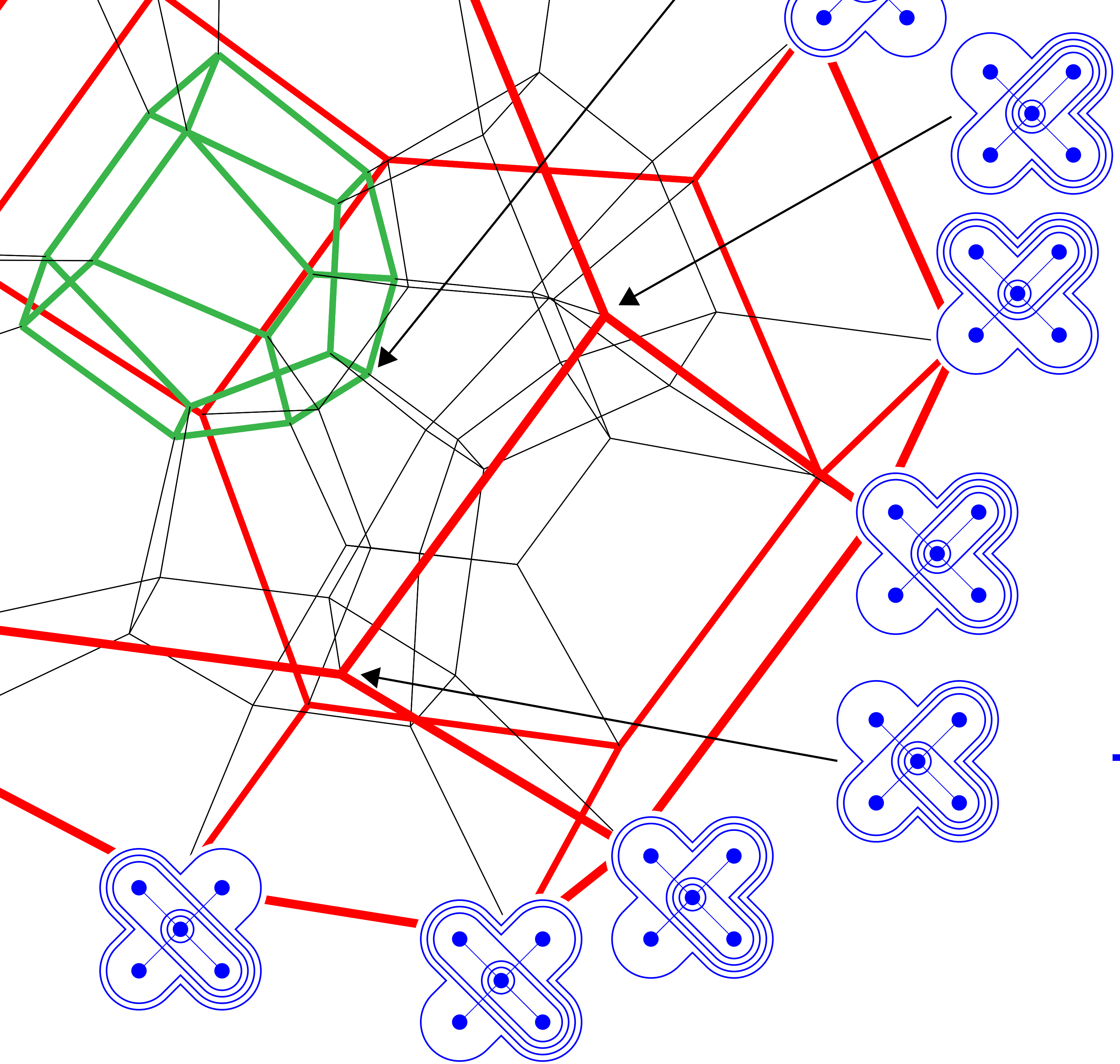


SCHRODERIZATION







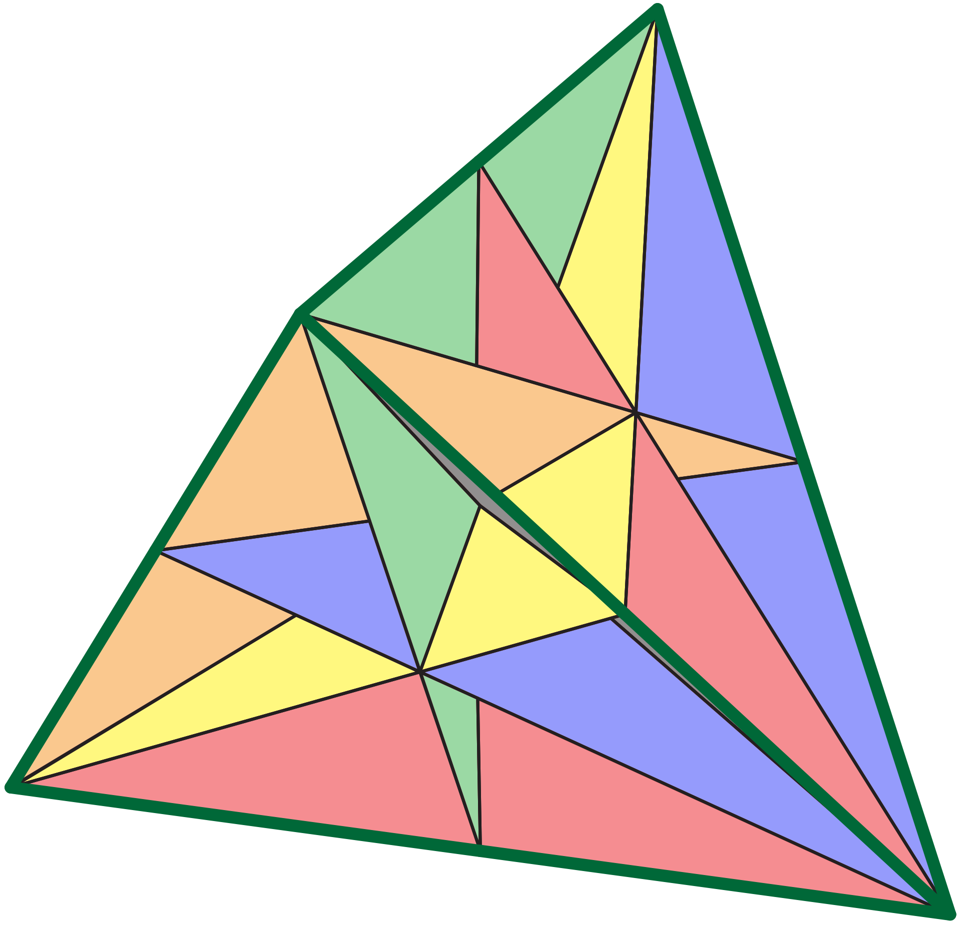
THANKS

FINITE COXETER GROUPS

Humphreys, *Reflection groups and Coxeter groups* ('90)
Bjorner-Brenti, *Combinatorics of Coxeter groups* ('05)

FINITE COXETER GROUPS

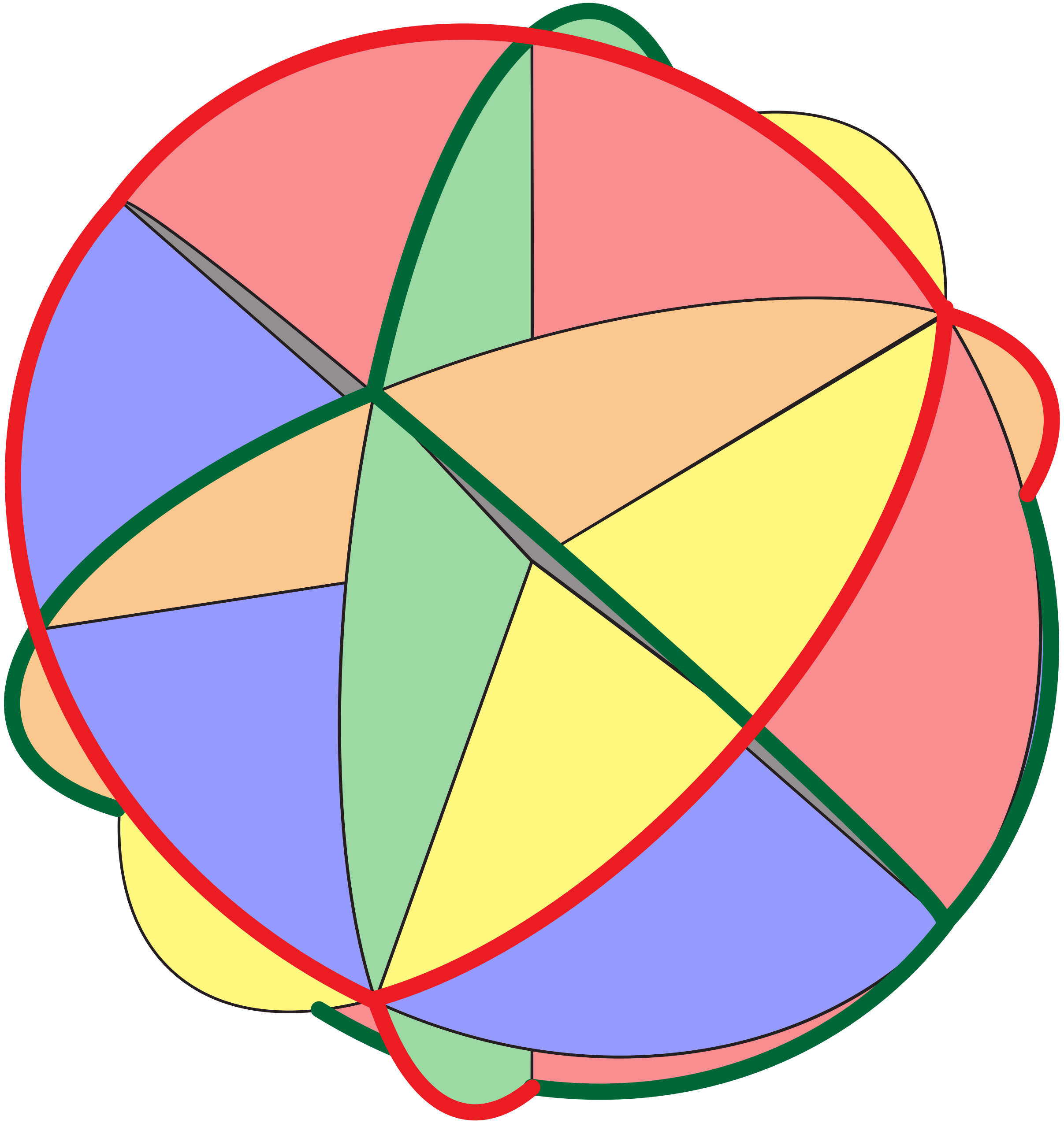
$W =$ finite Coxeter group



FINITE COXETER GROUPS

W = finite Coxeter group

Coxeter fan

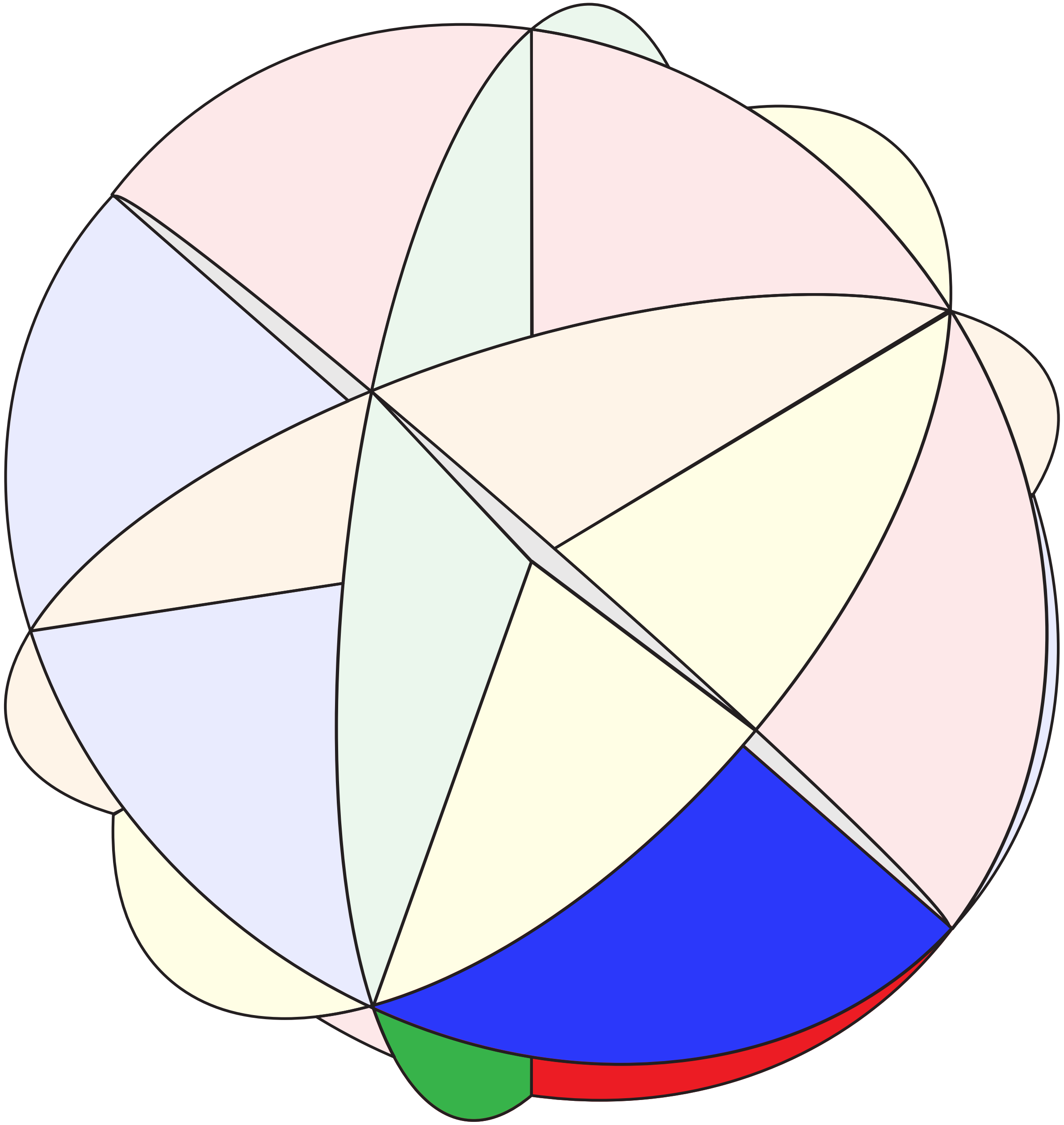


FINITE COXETER GROUPS

W = finite Coxeter group

Coxeter fan

fundamental chamber



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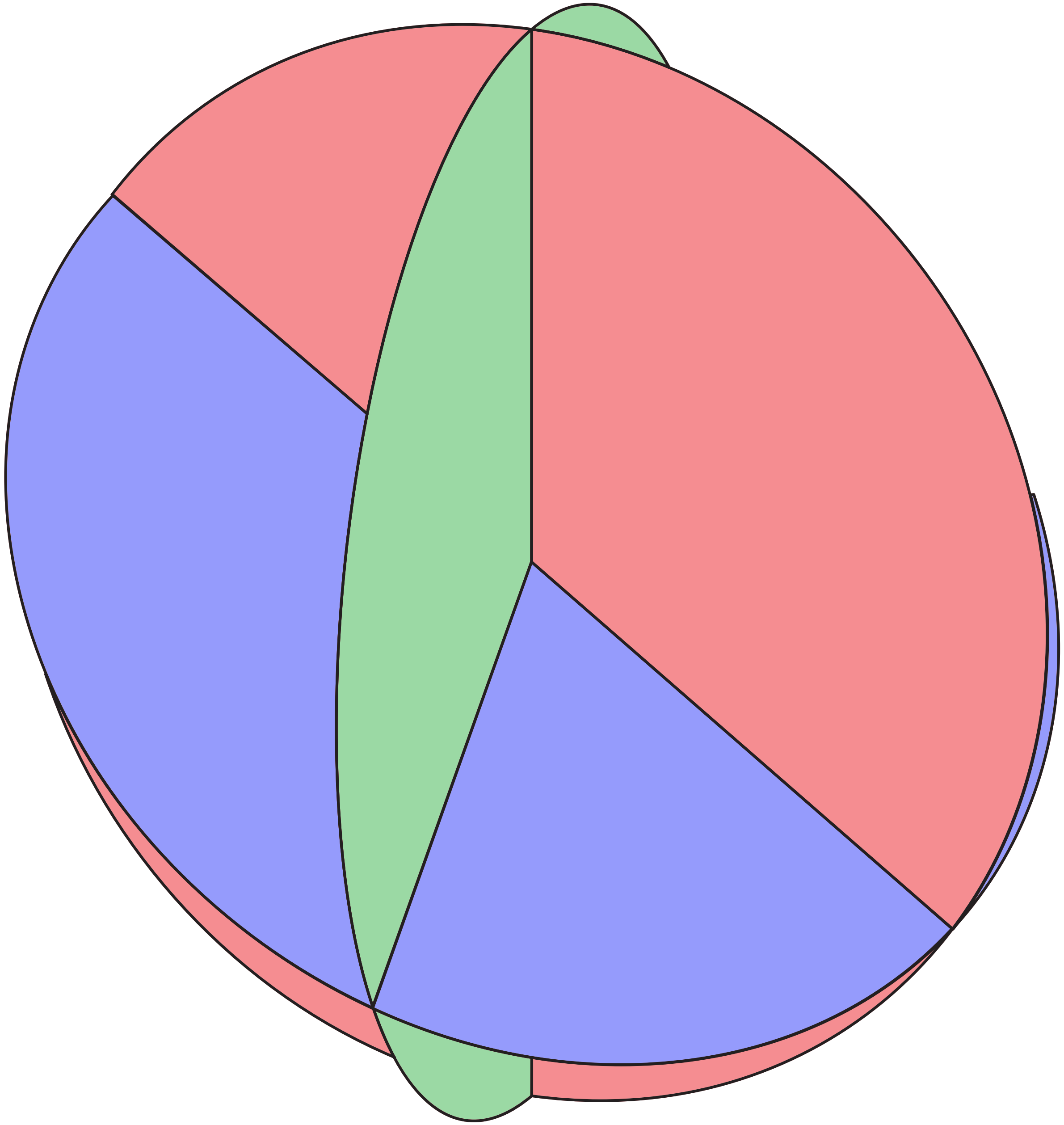


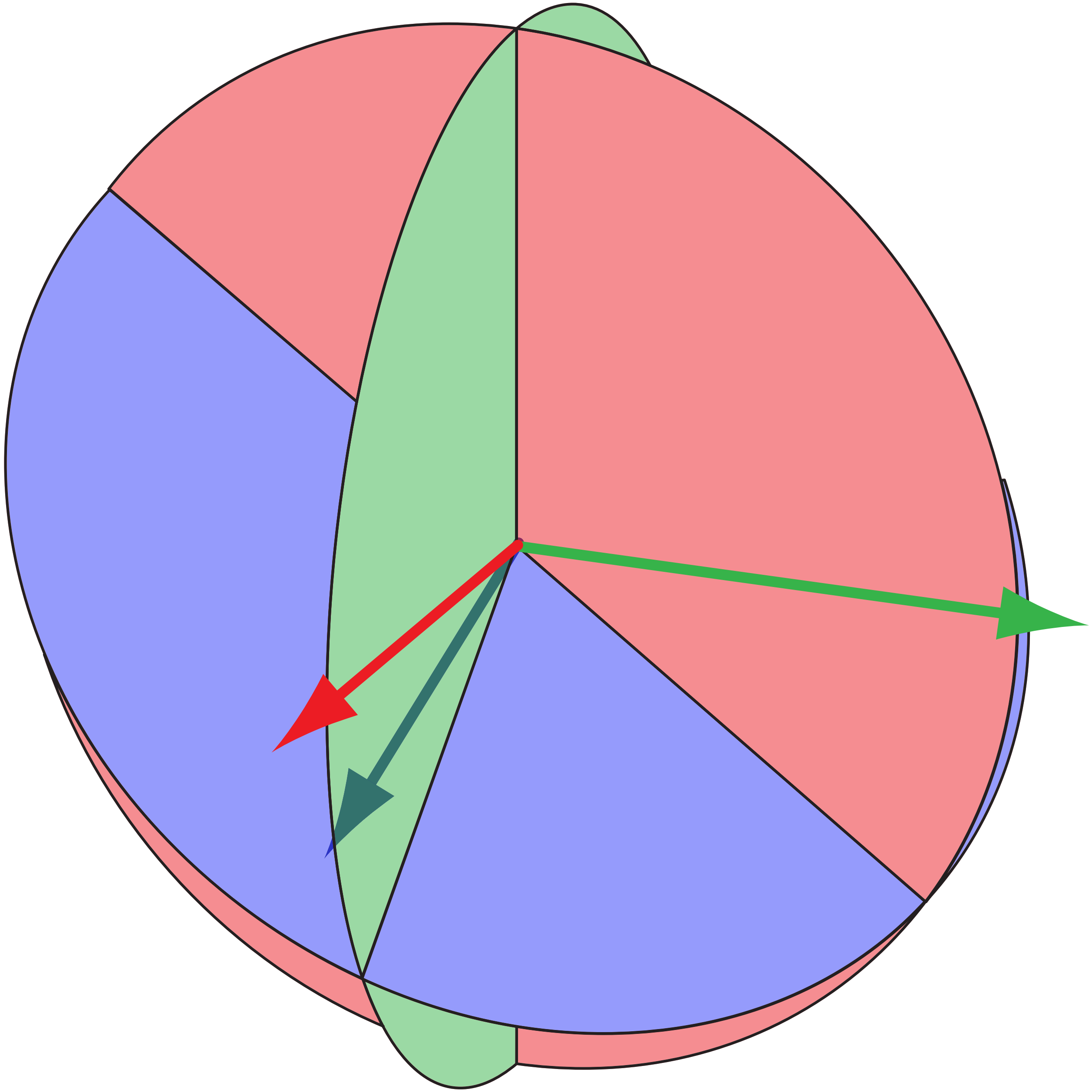
W = finite Coxeter group

Coxeter fan

fundamental chamber

S = simple reflections

FINITE COXETER GROUPS



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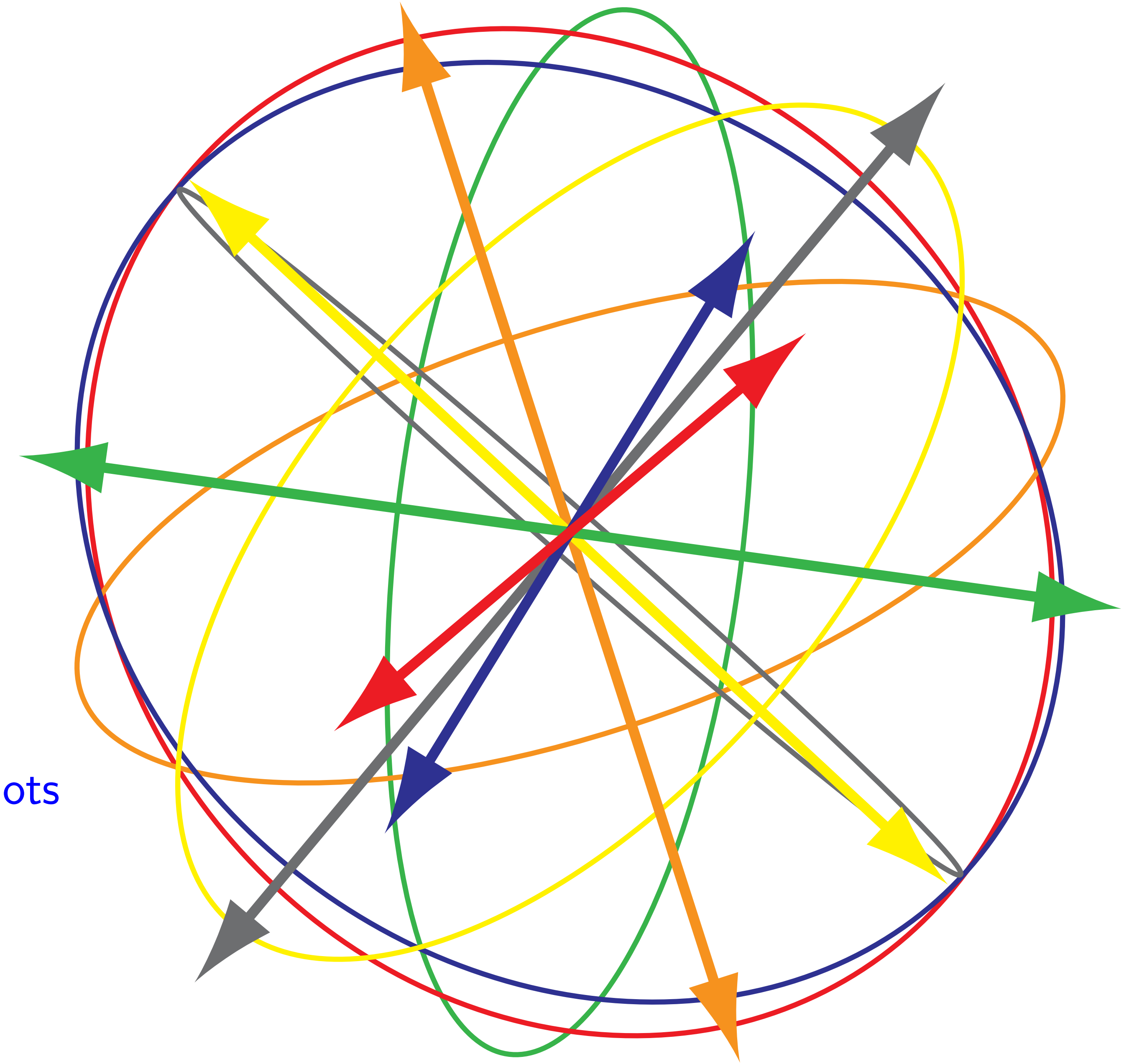
$\Delta = \{\alpha_s \mid s \in S\}$ = simple roots

FINITE COXETER GROUPS



W = finite Coxeter group

Coxeter fan

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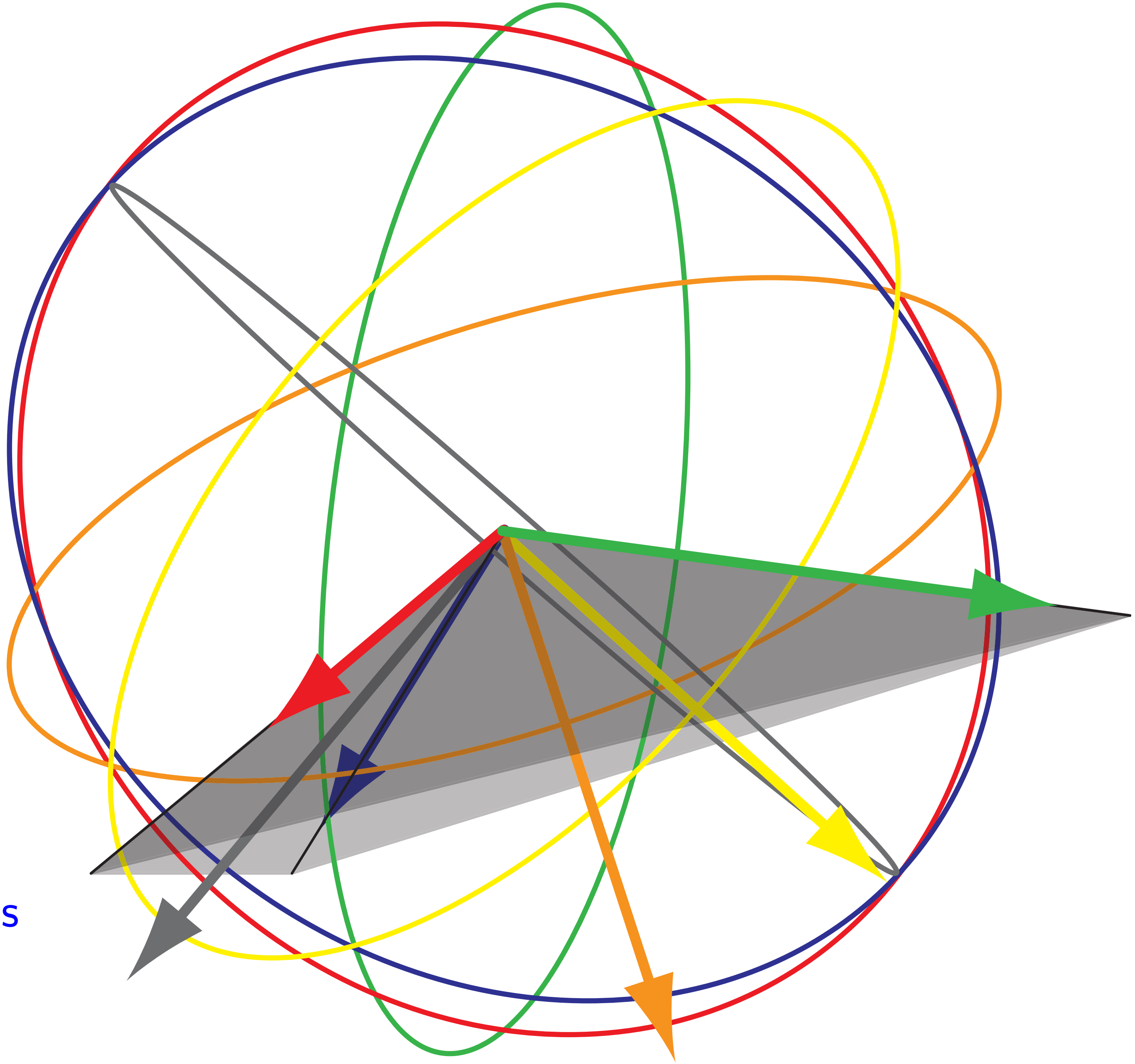
S = simple reflections

$\Delta = \{\alpha_s \mid s \in S\}$ = simple roots

$\Phi = W(\Delta)$ = root system

FINITE COXETER GROUPS



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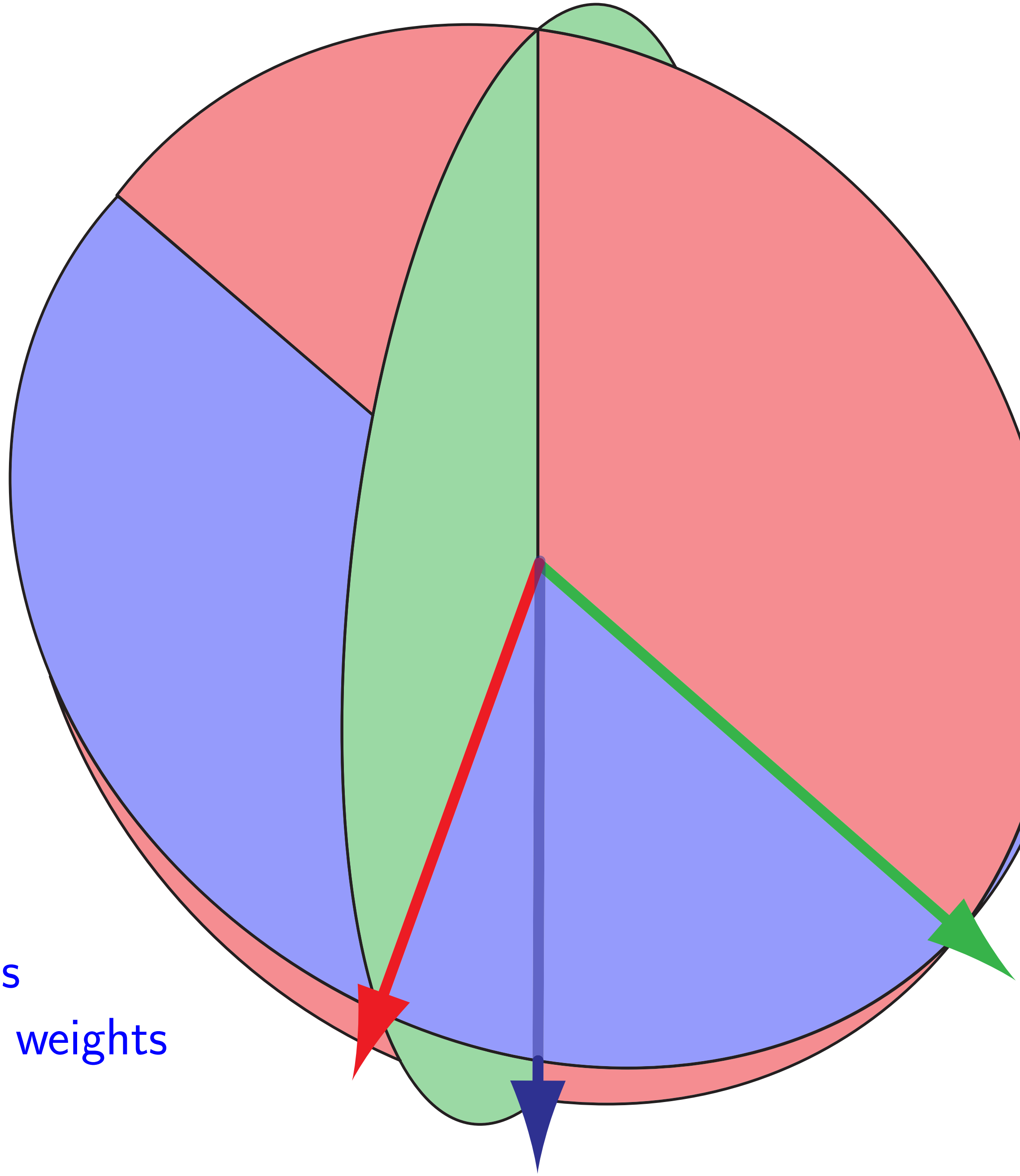
$\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta]$ = positive roots

FINITE COXETER GROUPS



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fundamental chamber

S = simple reflections

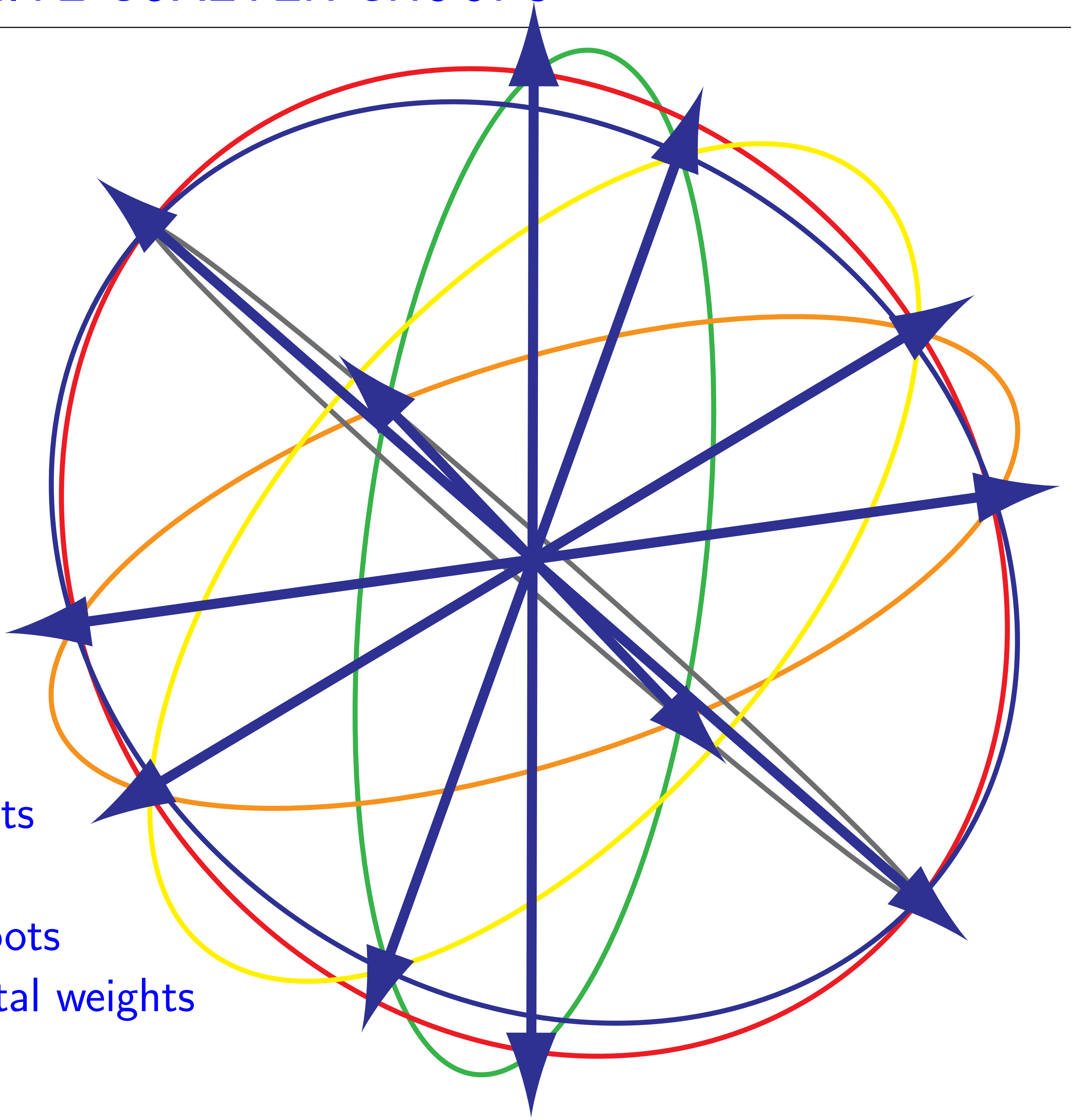
$\Delta = \{\alpha_s \mid s \in S\}$ = simple roots

$\Phi = W(\Delta)$ = root system

$\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta]$ = positive roots

$\nabla = \{\omega_s \mid s \in S\}$ = fundamental weights

FINITE COXETER GROUPS



W = finite Coxeter group

Coxeter fan

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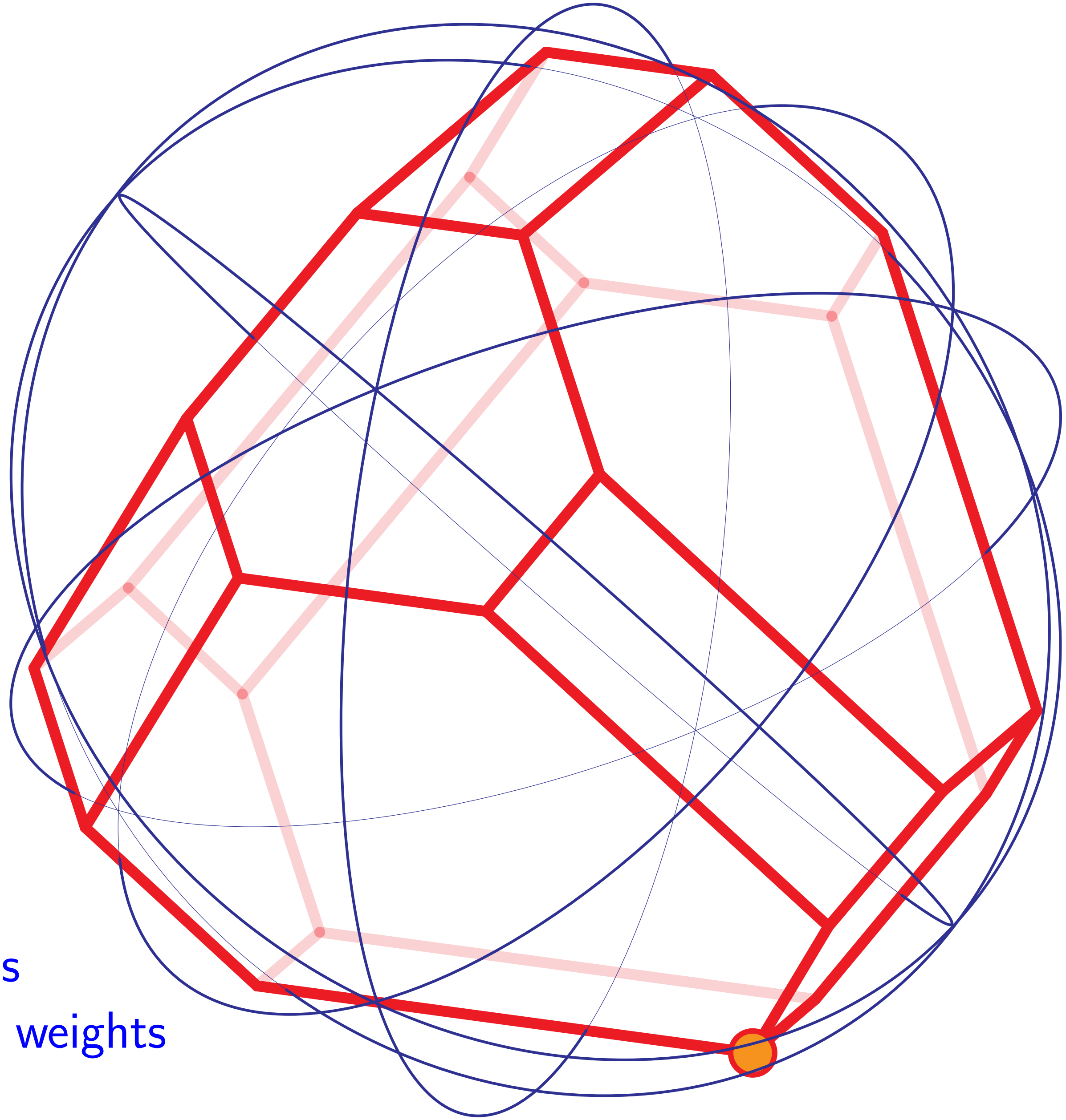

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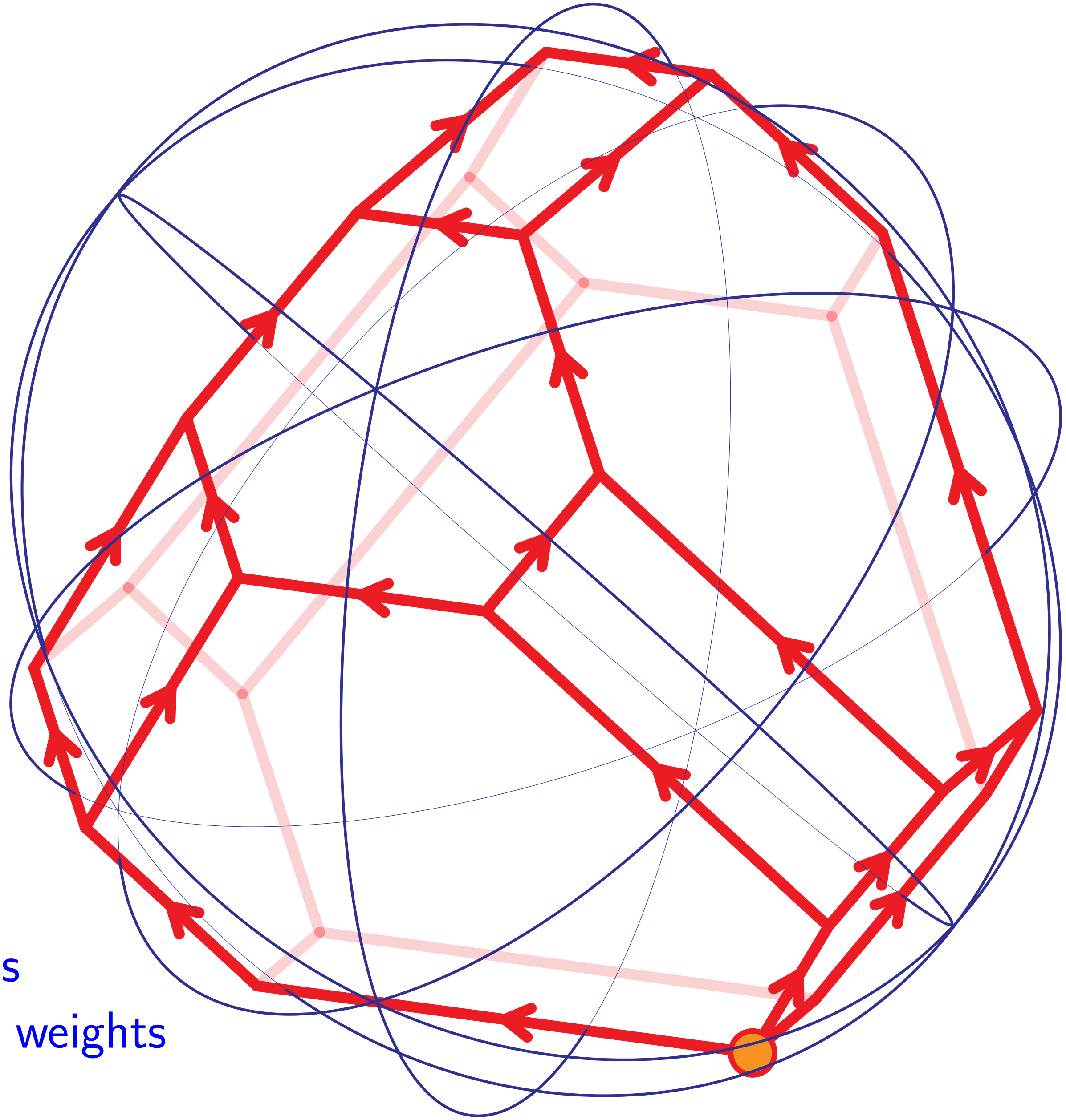

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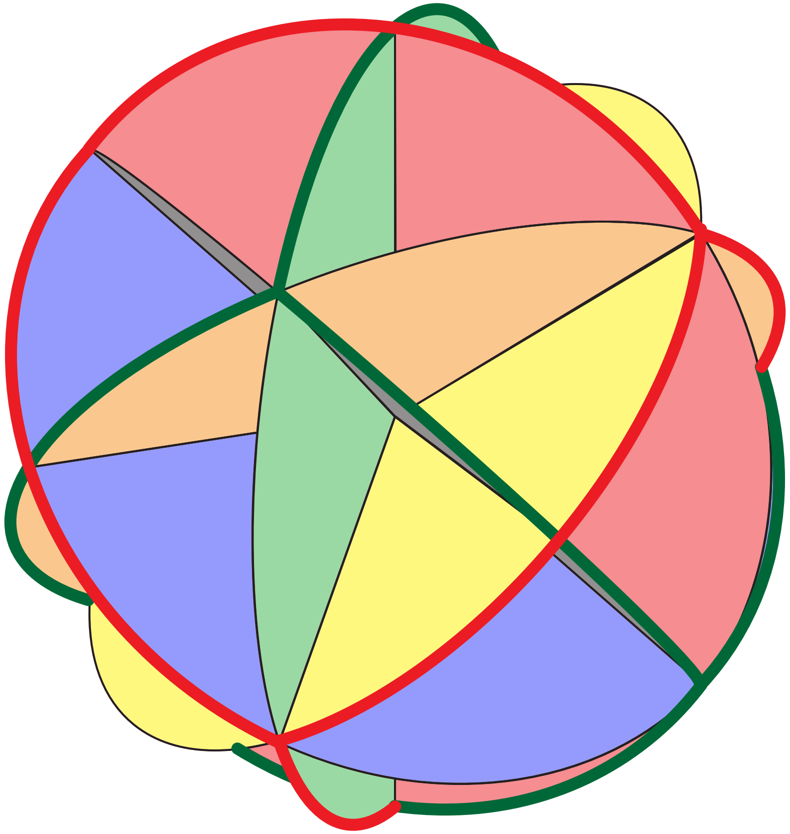
permutahedron

weak order = $u \leq w \iff \exists v \in W,\ uv = w$ and $\ell(u) + \ell(v) = \ell(w)$

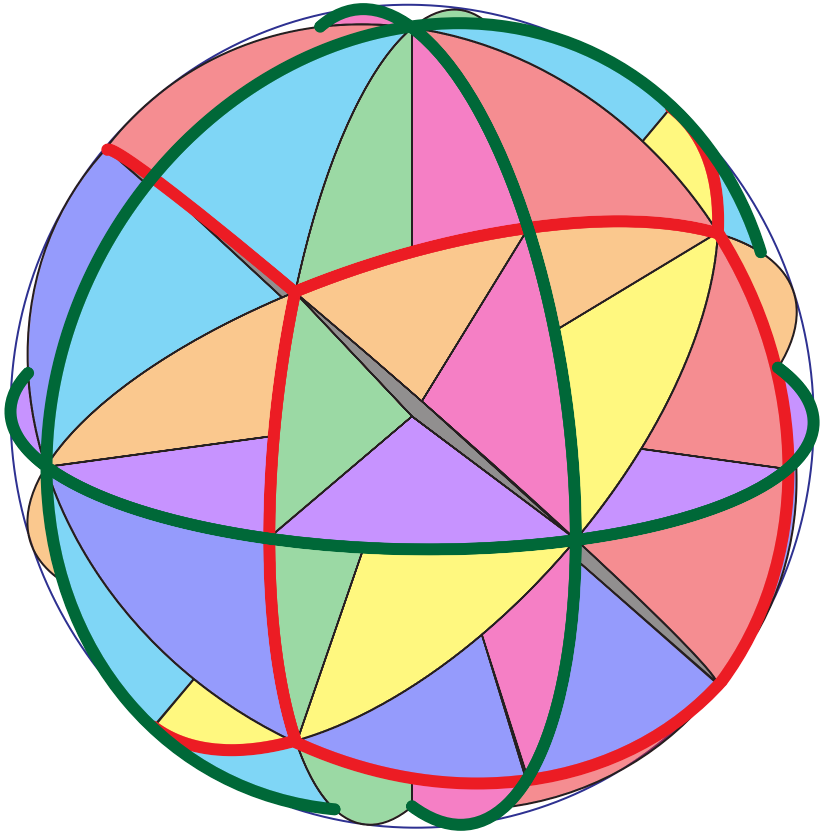
EXAMPLES: TYPE A AND B

TYPE A_n $=$ symmetric group $\mathfrak{S}_{n+1}$



$$S = \{(i, i+1) \mid i \in [n]\}$$
$$\Delta = \{e_{i+1} - e_i \mid i \in [n]\}$$
$$\text{roots} = \{e_i - e_j \mid i, j \in [n+1]\}$$
$$\nabla = \Big\{\sum_{j>i} e_j \;\Big|\; i \in [n]\Big\}$$

TYPE B_n $=$ semidirect product $\mathfrak{S}_n \rtimes (\mathbb{Z}_2)^n$



$$S = \{(i, i+1) \mid i \in [n-1]\} \cup \{\chi\}$$
$$\Delta = \{e_{i+1} - e_i \mid i \in [n-1]\} \cup \{e_1\}$$
$$\text{roots} = \{\pm e_i \pm e_j \mid i, j \in [n]\} \cup \{\pm e_i \mid i \in [n]\}$$
$$\nabla = \Big\{\sum_{j\geq i} e_j \;\Big|\; i \in [n]\Big\}$$

back to cluster algebras