ENUMERATING DOUBLE PSEUDOLINE ARRANGEMENTS

Julien Ferté, Vincent Pilaud, and Michel Pocchiola

École Normale Supérieure, Paris
INTRODUCTION
**INTRODUCTION**

Pseudoline Arrangements

Projective plane $\mathcal{P}$ = disk with antipodal boundary points identified

A simple closed curve of $\mathcal{P}$ is a pseudoline if it is not contractible

The complement of a pseudoline is a topological disk
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Pseudoline arrangements correspond via duality to points configurations in geometric projective planes

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A simple closed curve of $\mathcal{P}$ is a **double pseudoline** if it is contractible

The complement of a double pseudoline $\ell$ has two connected components: a Möbius strip $M_\ell$ and a topological disk $D_\ell$
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Double Pseudoline Arrangements

**Projective plane** $\mathcal{P} =$ disk with antipodal boundary points identified

A simple closed curve of $\mathcal{P}$ is a **double pseudoline** if it is contractible

A **double pseudoline arrangement** is a finite set of double pseudolines such that any two of them

(i) have exactly four intersection points (and cross transversally at these points), and

(ii) induce a cell decomposition of $\mathcal{P}$

Double pseudoline arrangements correspond via duality to configurations of disjoint convex bodies in geometric projective planes


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Two arrangements are isomorphic if and only if their face lattices are isomorphic.
Two arrangements $A$ and $B$ are isomorphic if there is a homeomorphism of $\mathcal{P}$ that sends $A$ on $B$. The number of isomorphism classes of arrangements of $n$ pseudolines is

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<th>$n$</th>
<th>1</th>
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<tr>
<td>$a_n$</td>
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On-line Encyclopedia of Integer Sequences Identification A006248


O. Aichholzer, F. Aurenhammer, & H. Krasser, Enumerating order types for small point sets with applications (2002)

The value $a_{11}$ is due to F. Aurenhammer (2002)
Isomorphism

Two arrangements $A$ and $B$ are isomorphic if there is a homeomorphism of $\mathcal{P}$ that sends $A$ on $B$.

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Our result: The number of isomorphism classes of arrangements of $n$ double pseudolines is:

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Comments on the computation of $A_5$ :

1. **RUNNING TIME** : $\simeq 3$ weeks on 4 processors of 2 GHz
2. **RESULT SIZE** : complete data base represents $\simeq 15$ Go

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Example

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Enumerating double pseudoline arrangements
MUTATIONS
A mutation is a homotopy of arrangements in which only one curve \( \ell \) moves, sweeping a single vertex of the remaining arrangement \( L \setminus \{\ell\} \)
**MUTATIONS**

Connectivity

**THEOREM.** Any two double pseudoline arrangements (with the same number of double pseudolines) are homotopic via a finite sequence of mutations, followed by a homeomorphism

**L. Habert & M. Pocchiola,** *Arrangements of double pseudolines* (2006)
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⇒ first enumeration algorithm: exploring the graph of mutations
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Fails for arrangements of five double pseudolines (*RAM memory limitation*)

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**THEOREM.** Any two double pseudoline arrangements containing a subarrangement $L$ (and with the same number of double pseudolines) are homotopic via a finite sequence of mutations where $L$ remains fixed, followed by a homeomorphism

**V. PILAUD & M. POCCHIOLA,** *A relative homotopy theorem for arrangements of double pseudolines*
INCREMENTAL ALGORITHM
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Notations

\( \mathcal{A}_n \) = the set of isomorphism classes of arrangements of \( n \) double pseudolines

pointed arrangement \( \mathcal{A}^\bullet \) = arrangement \( A \) with a distinguished double pseudoline

\( \mathcal{A}^\bullet_n \) = the set of isomorphism classes of pointed arrangements
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An isomorphism between two pointed arrangements \( A^\bullet \) and \( B^\bullet \) is a homeomorphism of \( \mathcal{P} \) that sends \( A^\bullet \) on \( B^\bullet \) respecting the distinguished double pseudoline
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Incremental method

Given the set $A_n = \{a_1, \ldots, a_p\}$, our algorithm enumerates $A_{n+1}$ by mutation of an added double pseudoline.
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1. add a double pseudoline $\alpha$ to the arrangement $a_i$
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3. select in $S_i$ the set $R_i$ with no subarrangements in $\{a_1, \ldots, a_{i-1}\}$
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3. select in $S_i$ the set $R_i$ with no subarrangements in $\{a_1, \ldots, a_{i-1}\}$

$R_i$ is the set of arrangements of $A_{n+1}$ whose first subarrangement among $\{a_1, \ldots, a_p\}$ is $a_i$.

\[ A_{n+1} = \bigsqcup_{i=1}^{p} R_i \]
How can we add a double pseudoline to an arrangement \( A \)?

1. choose an arbitrary double pseudoline and duplicate it
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**INCREMENTAL ALGORITHM**

Adding a double pseudoline

How can we add a double pseudoline to an arrangement $A$?

1. choose an arbitrary double pseudoline and **duplicate** it
2. **pump** the added double pseudoline $\ell$ until no vertex of $A$ lies in $M_\ell$
How can we add a double pseudoline to an arrangement $A$?

1. choose an arbitrary double pseudoline and duplicate it
2. pump the added double pseudoline $\ell$ until no vertex of $A$ lies in $M_\ell$
3. add four crossings
TWO OPEN PROBLEMS
TWO OPEN PROBLEMS

Axiomatization

Pseudoline arrangements admit simple axiomatizations:
(i) few axioms
(ii) dealing with configurations of at most five pseudolines

Enumeration = complete list of arrangements of at most five double pseudolines
= axiomatization
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Well, we have about 200,000,000 axioms

Is it possible to algorithmically reduce our axiomatization?
Certain pseudoline arrangements are not realizable in the Euclidean plane.

Inflating pseudolines into thin double pseudolines in such an arrangement give rise to non-realizable double pseudoline arrangement.

Are there smaller examples?
Thank you.