UNEXPECTED DIAGONALS

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Rencontre 3DMaps
Wednesday June 21st, 2023
DIAGONALS OF POLYTOPES
$\mathcal{P}$ polytope in $\mathbb{R}^d$

\textcolor{red}{\text{diagonal of $\mathcal{P} = \delta : \mathcal{P} \to \mathcal{P} \times \mathcal{P}$}}

\textcolor{blue}{p \mapsto (p, p)}
**DIAGONALS OF POLYTOPES**

\( \mathbb{P} \) polytope in \( \mathbb{R}^d \)

**diagonal** of \( \mathbb{P} = \delta : \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P} \)

\[ \mathbb{P} \rightarrow (p, p) \]

**cellular approximation of the diagonal of \( \mathbb{P} = \delta = \text{map} \ \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P} \) s.t.**

- its image is a union of faces of \( \mathbb{P} \times \mathbb{P} \)
- it agrees with \( \delta \) on the vertices of \( \mathbb{P} \)
- it is homotopic to \( \delta \)
DIAGONALS OF POLYTOPES

\( \mathbb{P} \) polytope in \( \mathbb{R}^d \)

**diagonal** of \( \mathbb{P} = \delta : \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P} \)

\[ p \mapsto (p, p) \]

**cellular approximation** of the diagonal of \( \mathbb{P} = \text{map } \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P} \) s.t.

- its image is a union of faces of \( \mathbb{P} \times \mathbb{P} \)
- it agrees with \( \delta \) on the vertices of \( \mathbb{P} \)
- it is homotopic to \( \delta \)

Alexander – Whitney

singular homology

Serre

cubical singular homology
any vertex of the fiber polytope

\[
\sum \left( \begin{array}{c}
P \times P \\
(p, q)
\end{array} \right)
\]

\[
\downarrow \\
\frac{p+q}{2}
\]

\[P \quad P_{p+q}
\]

allows a cellular approximation of the diagonal of \(P\)

projecting back on \(P\), we see it as a polyhedral subdivision of \(P\)
the vertex of the fiber polytope selected by \((-v, v)\)

\[
\sum \left( \begin{array}{c}
P 	imes P \\ \downarrow, \downarrow \\ P \\ \frac{p+q}{2}
\end{array} \right)
\]

gives a cellular approximation of the diagonal of \(P\)
projecting back on \(P\), we see it as a polyhedral subdivision \(\Delta_{P,v}\) of \(P\)
THM.

combinatorics of the diagonal $\Delta_{P,v}$ of $\mathbb{P}$

$\simeq$

common refinement of two copies of the normal fan of $\mathbb{P}$ translated by $v$

Laplante-Anfossi '22
THM. Faces of $\Delta_{P,v} \subseteq$ pairs $(F, G)$ such that $\max_v(F) \leq \min_v(G)$

When these are exactly the facets, it is called “magical formula”
This is the case for simplices, cubes, associahedra, but not permutahedra (see later)
THM. Faces of $\Delta_{P,v} \subseteq$ pairs $(F, G)$ such that $\max_v(F) \leq \min_v(G)$

When these are exactly the facets, it is called “magical formula”
This is the case for simplices, cubes, associahedra, but not permutahedra (see later)

$$f_k(\Delta_{\text{Simplex}(n)}) = (k + 1) \binom{n + 1}{k + 2}$$

$$f_k(\Delta_{\text{Cube}(n)}) = \binom{n}{k} 2^k 3^{n-k}$$

[OEIS, A127717]  [OEIS, A038220]
weak order = permutations of $[n]$
ordered by paths of simple transpositions

Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations
**Weak order** = permutations of $[n]$ ordered by paths of simple transpositions

**Tamari lattice** = binary trees on $[n]$ ordered by paths of right rotations
**weak order** = permutations of \([n]\) ordered by paths of simple transpositions

**Tamari lattice** = binary trees on \([n]\) ordered by paths of right rotations

**sylvester congruence** = equivalence classes are sets of linear extensions of binary trees
  = equivalence classes are fibers of BST insertion
  = rewriting rule \(UacVbW \equiv_{\text{sylv}} UcaVbW\) with \(a < b < c\)

**quotient lattice** = lattice on classes with \(X \leq Y \iff \exists x \in X, y \in Y, x \leq y\)
LATTICES: WEAK ORDER & TAMARI LATTICE

weak order = permutations of \([n]\) ordered by paths of simple transpositions

Tamari lattice = binary trees on \([n]\) ordered by paths of right rotations

sylvester congruence = equivalence classes are sets of linear extensions of binary trees
= equivalence classes are fibers of BST insertion
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quotient lattice = lattice on classes with \(X \leq Y \iff \exists x \in X, y \in Y, x \leq y\)
**FANS: BRAID FAN & SYLVESTER FAN**

**Braid Fan**

\[ C(\sigma) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \]

**Sylvester Fan**

\[ C(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \to j \text{ in } T \} \]
braid fan =
\[ C(\sigma) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \]

sylvester fan =
\[ C(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \to j \text{ in } T \} \]
**FANS: BRAID FAN & SYLVESTER FAN**

\[ \text{braid fan} = \mathcal{C}(\sigma) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \]

\[ \text{sylvester fan} = \mathcal{C}(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \rightarrow j \text{ in } T \} \]

**quotient fan** = \( \mathcal{C}(T) \) is obtained by glueing \( \mathcal{C}(\sigma) \) for all linear extensions \( \sigma \) of \( T \)
**Polytopes: Permutahedron & Associahedron**

**Permutahedron** $\text{Perm}(n)$

$$ = \text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \mid \sigma \in \mathfrak{S}_n \right\} $$

$$ = H \cap \bigcap_{\emptyset \neq J \subset [n]} H_J $$

where $H_J = \left\{ x \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \left( \frac{|J|+1}{2} \right) \right\}$

**Associahedron** $\text{Asso}(n)$

$$ = \text{conv} \left\{ [\ell(T, i) \cdot r(T, i)]_{i \in [n]} \mid T \text{ binary tree} \right\} $$

$$ = H \cap \bigcap_{1 \leq i < j \leq n} H_{[i,j]} $$

Stasheff ('63)
Shnider – Sternberg ('93)
Loday ('04)
permutahedron $\text{Perm}(n)$

\[
\text{Perm}(n) = \text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \mid \sigma \in \mathfrak{S}_n \right\}
\]

\[
= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n]} \mathbb{H}_J
\]

where $\mathbb{H}_J = \left\{ x \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \left( |J| + 1 \right) \frac{1}{2} \right\}$

associahedron $\text{Asso}(n)$

\[
\text{Asso}(n) = \text{conv} \left\{ [\ell(T, i) \cdot r(T, i)]_{i \in [n]} \mid T \text{ binary tree} \right\}
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\]

Stasheff (’63)

Shnider – Sternberg (’93)

Loday (’04)
**POLYTOPES: PERMUTAHEDRON & ASSOCIAHEDRON**

**Permutahedron** $\text{Perm}(n)$

$$= \text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \mid \sigma \in S_n \right\}$$

$$= H \cap \bigcap_{\emptyset \neq J \subseteq [n]} H_J$$

where $H_J = \left\{ x \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$

**Associahedron** $\text{Asso}(n)$

$$= \text{conv} \left\{ [\ell(T, i) \cdot r(T, i)]_{i \in [n]} \mid T \text{ binary tree} \right\}$$

$$= H \cap \bigcap_{1 \leq i < j \leq n} H_{[i,j]}$$

*Stasheff (’63)*

*Shnider – Sternberg (’93)*

*Loday (’04)*
POLYTOPES: PERMUTAHEDRON & ASSOCIAHEDRON
LATTICES – FANS – POLYTOPES

permutahedron $\Perm(n)$

$\Rightarrow$ braid fan

$\Rightarrow$ weak order on permutations

associahedron $\Asso(n)$

$\Rightarrow$ Sylvester fan

$\Rightarrow$ Tamari lattice on binary trees
$F$-VECTOR OF DIAGONALS

Saneblidze – Umble '04
Markl – Shnider '06
Loday '11

Masuda – Thomas – Tonks – Vallette '21
Laplace-Anfossi '22
\[ f_k = \sum_{F \leq G} \prod_{i \in [2]} \prod_{p \in G_i} (#F_i[p] - 1)! \]

\[ f_0 = [x^n] \exp \left( \sum_m \frac{x^m}{m(m+1)} \binom{2m}{m} \right) \]

\[ f_{n-1} = 2(n + 1)^{n-2} \]
DIAGONAL OF THE ASSOCIAHEDRON

arXiv:2303.10986

with
Alin BOSTAN (INRIA)
Frédéric CHYZAK (INRIA)
Tam$(n) = \text{Tamari lattice on binary trees with } n \text{ nodes}
**NUMBER OF TAMARI INTERVALS**

**Tam**($n$) = Tamari lattice on binary trees with $n$ nodes

**THM.** For any $n \geq 1$,

$$\# \{ S \leq T \in \text{Tam}(n) \} = \frac{2}{(3n + 1)(3n + 2)} \binom{4n + 1}{n + 1}$$

1, 3, 13, 68, 399, 2530, 16965, ... [OEIS A000260]
Tam\( (n) \) = Tamari lattice on binary trees with \( n \) nodes

\( \text{des}(T) = \) number of binary trees covered by \( T \)

\( \text{asc}(T) = \) number of binary trees covering \( T \)
Tam\( (n) \) = Tamari lattice on binary trees with \( n \) nodes

des\( (T) \) = number of binary trees covered by \( T \)

asc\( (T) \) = number of binary trees covering \( T \)

THM. For any \( n, k \geq 1 \),

\[
\# \left\{ S \leq T \in \text{Tam}(n) \mid \text{des}(S) + \text{asc}(T) = k \right\} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}
\]

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CANONICAL COMPLEX OF THE TAMARI LATTICE

\[(L, \leq, \land, \lor)\] lattice

\[
\begin{align*}
\text{join semidistributive} & \iff x \lor y = x \lor z \text{ implies } x \lor (y \land z) = x \lor y \text{ for all } x, y, z \in L \\
& \iff \text{any } x \in L \text{ admits a canonical join representation } x = \bigvee J
\end{align*}
\]

canonical join complex = simplicial complex of canonical join representations

= a simplex \( J \) for each element \( \bigvee J \) of \( L \)

\[
\begin{array}{c}
\text{Reading '15} \\
\text{Barnard '19}
\end{array}
\]
(\(L, \leq, \wedge, \vee\)) lattice

**meet** semidistributive \(\iff\) \(x \wedge y = x \wedge z\) implies \(x \wedge (y \vee z) = x \wedge y\) for all \(x, y, z \in L\)

**meet** semidistributive \(\iff\) any \(x \in L\) admits a canonical **meet** representation \(x = \bigwedge M\)

canonical **meet** complex = simplicial complex of canonical **meet** representations

= a simplex \(M\) for each element \(\bigwedge M\) of \(L\)
(\(L, \leq, \land, \lor\)) lattice

semidistributive \iff join semidistributive and meet semidistributive
\iff any \(x \in L\) admits canonical representations \(x = \lor J = \land M\)

canonical complex = simplicial complex of canonical representations
\(=\) a simplex \(J \sqcup M\) for each interval \(\lor J \leq \land M\) in \(L\)
CANONICAL COMPLEX OF THE TAMARI LATTICE

$(L, \leq, \land, \lor)$ lattice

semidistributive $\iff$ join semidistributive and meet semidistributive

$\iff$ any $x \in L$ admits canonical representations $x = \lor J = \land M$

canonical complex = simplicial complex of canonical representations

= a simplex $J \sqcup M$ for each interval $\lor J \leq \land M$ in $L$
THM. For any $n, k \geq 1$, 

$$f_k(\mathcal{C}C_n) = \# \{ S \leq T \in \text{Tam}(n) \mid \text{des}(S) + \text{asc}(T) = k \} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}$$

Bostan – Chyzak – P. ’23+

Reading ’15
Albertin – P. ’22
1 + 12 + 33 + 22 = 68
SECOND REFINED FORMULA ON TAMARI INTERVALS

Tam(n) = Tamari lattice on binary trees with n nodes

\( \text{des}(T) = \) number of binary trees covered by T

\( \text{asc}(T) = \) number of binary trees covering T

**THM.** For any \( n, k \geq 1 \),

\[
\sum_{S \leq T \in \text{Tam}(n)} \binom{\text{des}(S) + \text{asc}(T)}{k} = \frac{2}{(3n + 1)(3n + 2)} \binom{n - 1}{k} \binom{4n + 1 - k}{n + 1}
\]

\[
\begin{array}{c|cccccccc}
 n \backslash k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 1 \\
2 & 3 & 2 \\
3 & 13 & 18 & 6 \\
4 & 68 & 144 & 99 & 22 \\
5 & 399 & 1140 & 1197 & 546 & 91 \\
6 & 2530 & 9108 & 12903 & 8976 & 3060 & 408 \\
7 & 16965 & 73710 & 131625 & 123500 & 64125 & 17442 & 1938 \\
8 & 118668 & 604128 & 1302651 & 1540770 & 1078539 & 446292 & 100947 & 9614 \\
9 & 857956 & 5008608 & 12660648 & 18086640 & 15958800 & 8898240 & 3058770 & 592020 & 49335 \\
\end{array}
\]
$\Delta_{\text{Asso}(n)} = \text{diagonal of } (n - 1)\text{-dimensional associahedron}$
\[ \Delta_{A_{\text{asso}}(n)} = \text{diagonal of } (n - 1)\text{-dimensional associahedron} \]
Δ_{Asso}(n) = diagonal of \((n - 1)\)-dimensional associahedron

**THM. (Magical formula)**

Masuda – Thomas – Tonks – Vallette ’21

\(k\)-faces of \(\Delta_{Asso}(n)\) \(\leftrightarrow\) \((F, G)\) faces of \(\tilde{A}_{sso}(n)\) with

\[
\dim(F) + \dim(G) = k \text{ and } \max(F) \leq \min(G)
\]
$\Delta_{\text{Asso}(n)} = \text{diagonal of } (n - 1)\text{-dimensional associahedron}$

**THM.** For any $n, k \geq 1$,

$$f_k(\Delta_{\text{Asso}(n)}) = \sum_{S \leq T \in \text{Tam}(n)} \binom{\text{des}(S) + \text{asc}(T)}{k} = \frac{2}{(3n + 1)(3n + 2)} \binom{n - 1}{k} \binom{4n + 1 - k}{n + 1}$$
**THM.** For any $n, k \geq 1$,

\[
f_k(\mathcal{C}C_n) = \# \{ S \leq T \in \text{Tam}(n) \mid \text{des}(S) + \text{asc}(T) = k \} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}
\]

**THM.** For any $n, k \geq 1$,

\[
f_k(\Delta_{\text{asso}}(n)) = \sum_{S \leq T \in \text{Tam}(n)} \binom{\text{des}(S) + \text{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1}
\]
CONNECTION BETWEEN THE TWO FORMULAS

**THM.** For any $n, k \geq 1$,

\[ f_k(CC_n) = \# \{ S \leq T \in \text{Tam}(n) \mid \text{des}(S) + \text{asc}(T) = k \} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k} \]

**THM.** For any $n, k \geq 1$,

\[ f_k(\Delta_{asso}(n)) = \sum_{S \leq T \in \text{Tam}(n)} \binom{\text{des}(S) + \text{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1} \]

Second formula follows from the first since ...

**THM.** For any $n, k, r \in \mathbb{N}$,

\[ \sum_{\ell=k}^{n-1} \binom{n+1}{\ell+2} \binom{r}{\ell} \binom{\ell}{k} = \frac{n(n+1)}{(r+1)(r+2)} \binom{n-1}{k} \binom{r+n+1-k}{n+1} \]

... by application of Chu – Vandermonde equality
\[ n(T) = \text{number of nodes of } T \]
\[ \ell(T) = \text{number of bounded edges on the left branch of } T \]

\[
\Delta(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)}
\]
\[ n(T) = \text{number of nodes of } T \]
\[ \ell(T) = \text{number of bounded edges on the left branch of } T \]

\[
\mathbb{A}(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\text{des}(S)+\text{asc}(T)}
\]

We want to compute

\[
A := A(t, z) := \sum_{S \leq T} t^{n(S)} z^{\text{des}(S)+\text{asc}(T)} = \mathbb{A}(1, 1, t, z)
\]

we will use \( u \) and \( v \) as catalytic variables ...
QUADRATIC EQUATION

\[ n(T) = \text{number of nodes of } T \]
\[ \ell(T) = \text{number of bounded edges on the left branch of } T \]

\[
A_u(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)}
\]

We want to compute

\[
A := A(t, z) := \sum_{S \leq T} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)} = A(1, 1, t, z)
\]

we will use \( u \) and \( v \) as catalytic variables ...

PROP. The generating functions \( A_u := A(u, 1, t, z) \) and \( A_1 := A(1, 1, t, z) \) satisfy the quadratic functional equation

\[
(u - 1)A_u = t(u - 1 + u(u + z - 1)A_u - zA_1)(1 + uzA_u)
\]
GRAFTING DECOMPOSITIONS

\[ S \setminus T = \text{binary tree obtained by grafting } S \text{ on the leftmost leaf of } T \]
\[ S = S_0 \setminus S_1 \ldots \setminus S_k \text{ grafting decomposition} \]

\[
\begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{array}
\]

\[
\begin{array}{ccc}
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\text{ } & \text{ } & \text{ } \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{array}
\]

LEM. If \( S = S_0 \setminus S_1 \ldots \setminus S_k \) and \( T = T_0 \setminus T_1 \ldots \setminus T_k \) are s.t. \( n(S_i) = n(T_i) \) for all \( i \in [k] \), then \( S \leq T \iff S_i \leq T_i \) for all \( i \in [k] \)  
Chapoton ’07

\[
\begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{array}
\]

\[
\begin{array}{ccc}
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\]

\[
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\end{array}
\]

\[
\begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{array}
\]

LEM. If \( S \leq T \), then we can write \( S = S_0 \setminus S_1 \ldots \setminus S_\ell \) and \( T = T_0 \setminus T_1 \ldots \setminus T_\ell \) where \( \ell = \ell(T) \) and \( n(S_i) = n(T_i) \) for all \( i \in [\ell] \)  
Chapoton ’07

\( \ell(T) = \text{number of bounded edges on the left branch of } T \)
\( n(T) \) = number of nodes of \( T \)
\( \ell(T) \) = number of bounded edges on the left branch of \( T \)

\[
A(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)}
\]

Consider

\[
A_u(t, z) := A(u, 1, t, z) \quad \text{and} \quad A_u^o(t, z) := A(u, 0, t, z)
\]

= all Tamari intervals \hspace{1cm} = \text{indecomposable intervals}
$A_u = A_u(t, z) = \text{all Tamari intervals}$

$A^o_u = A^o_u(t, z) = \text{indecomposable intervals}$

$$\sum_{S \leq T} u^{\ell(S)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)}$$
QUADRATIC EQUATION

$$A_u = A_u(t, z) = \text{all Tamari intervals}$$

$$A^\circ_u = A^\circ_u(t, z) = \text{indecomposable intervals}$$

1. all intervals = indecomposable intervals \ (ε + all intervals)
   $$A_u = A^\circ_u (1 + uzA_u)$$

$$\sum_{S \leq T} u^{\ell(S)} t^{n(S)} z^{\text{des}(S)+\text{asc}(T)}$$

Chapoton '07
Quadratic Equation

$A_u = A_u(t, z) = \text{all Tamari intervals}$

$A^\circ_u = A^\circ_u(t, z) = \text{indecomposable intervals}$

1. all intervals = indecomposable intervals \( \setminus (\varepsilon + \text{all intervals}) \)
   \[ A_u = A^\circ_u \left( 1 + u z A_u \right) \]

2. from any Tamari interval \((S, T)\) where \(S = S_0/S_1/\ldots/S_{\ell(S)}\), we can construct \(\ell(S) + 2\) indecomposable Tamari intervals \((S'_k, T')\) for \(0 \leq k \leq \ell(S) + 1\), where
   \[ S'_k = \left( S_0/\ldots/S_{k-1} \right)/Y \setminus \left( S_k/\ldots/S_{\ell(S)} \right) \quad \text{and} \quad T' = Y \setminus T \]

\[ S'_0 = Y/(S_0/S_1/S_2) \quad S'_1 = S_0/Y \setminus (S_1/S_2) \quad S'_2 = (S_0/S_1)/Y \setminus S_2 \quad S'_3 = (S_0/S_1/S_2)/Y \]

... and all indecomposable intervals are obtained this way

\[ A^\circ_u = t \left( 1 + z \frac{u A_u - A_1}{u - 1} + u A_u \right) \]

Chapoton '07
QUADRATIC EQUATION

\[ A_u = A_u(t, z) = \text{all Tamari intervals} \]
\[ A_u^\circ = A_u^\circ(t, z) = \text{indecomposable intervals} \]

\[ \sum_{S \leq T} u^{\ell(S)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)} \]

1. \[ A_u = A_u^\circ(1 + uzA_u) \]

2. \[ A_u^\circ = t \left( 1 + z \frac{uA_u - A_1}{u - 1} + uA_u \right) \]

PROP. The generating functions \( A_u := \mathbb{A}(u, 1, t, z) \) and \( A_1 := \mathbb{A}(1, 1, t, z) \) satisfy the quadratic functional equation

\[ (u - 1)A_u = t \left( u - 1 + u(u + z - 1)A_u - zA_1 \right) \left( 1 + uzA_u \right) \]
**QUADRATIC METHOD**

**PROP.** The generating functions $A_u := \mathbb{A}(u, 1, t, z)$ and $A_1 := \mathbb{A}(1, 1, t, z)$ satisfy the quadratic functional equation

$$(u - 1)A_u = t\left(u - 1 + u(u + z - 1)A_u - zA_1\right)\left(1 + uzA_u\right)$$

Quadratic equation with a catalytic variable... quadratic method

The discriminant of this quadratic polynomial must have multiple roots, hence, its own discriminant vanishes

**CORO.** The generating function $A = A(t, z)$ is a root of the polynomial

$$
t^3z^6X^4 + t^2z^4(tz^2 + 6tz - 3t + 3)X^3 + tz^2(6t^2z^3 + 9t^2z^2 - 12t^2z + 2tz^2 + 3t^2 - 6tz + 21t + 3)X^2 + (12t^3z^4 - 4t^3z^3 - 9t^3z^2 - 10t^2z^3 + 6t^3z + 26t^2z^2 - t^3 + 6t^2z + tz^2 + 3t^2 - 12tz - 3t + 1)X + t(8t^2z^3 - 12t^2z^2 + 6t^2z - tz^2 - t^2 + 10tz + 2t - 1)$$
The generating function $A = A(t, z)$ is a root of the polynomial

\[
t^3z^6X^4 + t^2z^4(tz^2 + 6tz - 3t + 3)X^3 + tz^2(6t^2z^3 + 9t^2z^2 - 12t^2z + 2t^2z + 3t^2 - 6tz + 21t + 3)X^2 \\
+ (12t^3z^4 - 4t^3z^3 - 9t^3z^2 - 10t^2z^3 + 6t^3z + 26t^2z^2 - t^3 + 6t^2z + tz^2 + 3t^2 - 12tz - 3t + 1)X \\
+ t(8t^2z^3 - 12t^2z^2 + 6t^2z - tz^2 - t^2 + 10tz + 2t - 1)
\]

Reparametrize by

\[
t = \frac{s}{(s + 1)(sz + 1)^3} \quad X = s - zs^2 - zs^3
\]

The generating function $A = A(t, z)$ can be written

\[
A = S - zS^2 - zS^3 \quad \text{where} \quad t = \frac{S}{(S + 1)(Sz + 1)^3}
\]
CORO. The generating function $A = A(t, z)$ can be written

$$A = S - zS^2 - zS^3 \quad \text{where} \quad t = \frac{S}{(S + 1)(Sz + 1)^3}$$

THM. (Lagrange inversion) If $S = t\psi(S)$, then $[t^n S^r] = \frac{r}{n} [s^{n-r}] \phi(s)^n$ for any $r \geq 1$

Here $\phi(s) := (s + 1)(sz + 1)^3$

Hence $[s^a] \phi(s)^n = [s^a](s + 1)^n(sz + 1)^{3n} = \sum_{i+j=a} \binom{n}{i} \binom{3n}{j} z^j$

Hence $[t^n z^k] S^r = \frac{r}{n} [s^{n-r} z^k] \phi(s)^n = \frac{r}{n} \binom{n}{n-r-k} \binom{3n}{k} = \frac{r}{n} \binom{n}{k+r} \binom{3n}{k}$

Finally,

$$[t^n z^k] A = [t^n z^k] S - [t^n z^{k-1}] S^2 - [t^n z^{k-1}] S^3 = \frac{2}{n(n+1)} \binom{3n}{k} \binom{n+1}{k+2}$$
Tam\( (n) \) = Tamari lattice on binary trees with \( n \) nodes

**THM.** For any \( n \geq 1 \),

\[
\# \{ S \leq T \in \text{Tam}(n) \} = \frac{2}{(3n+1)(3n+2)} \binom{4n+1}{n+1}
\]

Chapoton '07

Also counts rooted 3-connected planar triangulations with \( 2n + 2 \) faces  

Tutte
Tam\((n)\) = Tamari lattice on binary trees with \(n\) nodes

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Chapoton '07

Also counts rooted 3-connected planar triangulations with \(2n + 2\) faces

Tutte

\[\text{Diagram}\]
Tam\( (n) \) = Tamari lattice on binary trees with \( n \) nodes

**THM.** For any \( n \geq 1 \),

\[
\#\{S \leq T \in \text{Tam}(n)\} = \frac{2}{(3n + 1)(3n + 2)} \binom{4n + 1}{n + 1}
\]

Chapoton '07

Also counts rooted 3-connected planar triangulations with \( 2n + 2 \) faces

Tutte

Bernardi – Bonichon, '09
A planar triangulation with external vertices $v_0, v_1, v_3$
$n$ internal nodes, $3n$ internal edges, $2n + 1$ internal triangles

Schnyder wood = color (with 0, 1, 2) and orient the internal edges s.t.
- the edges colored $i$ form a spanning tree oriented towards $v_i$
- each vertex satisfies the vertex rule:

Schnyder '89
$M$ planar triangulation with external vertices $v_0, v_1, v_3$
$n$ internal nodes, $3n$ internal edges, $2n + 1$ internal triangles

Schnyder wood = color (with $0, 1, 2$) and orient the internal edges s.t.
- the edges colored $i$ form a spanning tree oriented towards $v_i$
- each vertex satisfies the vertex rule:

Used for graph drawing and representations:
A planar triangulation with external vertices \( v_0, v_1, v_3 \)

- \( n \) internal nodes, \( 3n \) internal edges, \( 2n + 1 \) internal triangles

**Schnyder wood** = color (with 0, 1, 2) and orient the internal edges s.t.
- the edges colored \( i \) form a spanning tree oriented towards \( v_i \)
- each vertex satisfies the vertex rule:

**THM.** The Schnyder woods of a planar triangulation form a lattice structure under reorientations of clockwise essential cycles

**CORO.** Any planar triangulation admits a unique Schnyder wood with no clockwise cycle

Ossona de Mendez '94
Propp '97
Felsner '04
binary trees $S \leq T$
with $n$ nodes

Dyck paths $\mu \leq \nu$
with semilength $n$

planar triangulations
with $n$ internal vertices

contour of $T$
transform $\downarrow$ to $\diagup$
and $\leftarrow$ to $\downarrow$

contour of $T_0$
transform $\leftarrow \bullet$ to $\diagup$
and $\rightarrow \bullet$ to $\downarrow$

Bernardi – Bonichon, ’09
binary trees $S \leq T$ with $n$ nodes

descents of $S$ ascents of $T$

Dyck paths $\mu \leq \nu$ double falls of $\mu$ valleys of $\nu$

with semilength $n$

planar triangulations intermediate intermediate red vertices blue vertices

with $n$ internal vertices
THM. The generating function \( F := F(u, v, w) := \sum_{S \leq T} u^\uparrow v^\downarrow w^\downarrow \) is given by

\[
wv F = uU + vV + wUV - \frac{UV}{(1 + U)(1 + V)}
\]

where the series \( U := U(u, v, w) \) and \( V := V(u, v, w) \) satisfy the system

\[
U = (v + wU)(1 + U)(1 + V)^2
\]
\[
V = (u + wV)(1 + V)(1 + U)^2
\]
The function $A := A(t, z) := \sum_{S \leq T} t^n(S) z^{\text{des}(S) + \text{asc}(T)} = tF(tz, tz, t)$ is given by

$$tz^2 A = 2t z S + t S^2 - \frac{S^2}{(1 + S)^2}$$

where the series $S := S(t, z)$ satisfies

$$S = t(z + S)(1 + S)^3$$

... and Lagrange inversion again (thanks to Éric Fusy)
$T$ binary tree with $n$ nodes, labeled in inorder and oriented towards its root.

**Canopy of** $T = \text{vector } \text{can}(T) \in \{-, +\}^{n-1}$ with $\text{can}(T)_i = -$  
$\iff (j + 1)$st leaf of $T$ is a right leaf  
$\iff$ there is an oriented path joining its $j$th node to its $(j + 1)$st node  
$\iff$ the $j$th node of $T$ has an empty right subtree  
$\iff$ the $(j + 1)$st node of $T$ has a non-empty left subtree  
$\iff$ the cone corresponding to $T$ is located in the halfspace $x_j \leq x_{j+1}$
$T$ binary tree with $n$ nodes, labeled in inorder and oriented towards its root.

**canopy of** $T = \text{vector} \ can(T) \in \{-, +\}^{n-1}$ with $\text{can}(T)_i = -$ 

$\iff$ the $j$th node of $T$ has an empty right subtree

$\iff$ the $(j + 1)$st node of $T$ has a non-empty left subtree

**LEM.** $\text{asc}(T) = \# \{i \mid \text{can}(T)_i = -\}$ and $\text{des}(T) = \# \{i \mid \text{can}(T)_i = +\}$

**LEM.** If $S \leq T$, then

- $\text{can}(S) \leq \text{can}(T)$ componentwise
- $\text{des}(S) = \# \{i \mid \text{can}(S)_i = \text{can}(T)_i = +\}$ and $\text{asc}(S) = \# \{i \mid \text{can}(S)_i = \text{can}(T)_i = -\}$

**CORO.** $\text{des}(S) + \text{asc}(T) = \# \text{canopy agreements between } S \text{ and } T$
\[ \sum \text{meandres} \]
\[ \sum_{\text{meandres}} \left( \begin{array}{ccc} + & + & -1 \end{array} \right) \]
\[ \sum_{\text{meandres}} (1 + 1 + 1 - 1) uvw = \sum_{\text{cyan half-meanders}} uvw \cdot \sum_{\text{orange half-meanders}} uvw \]

Fang – Fusy – Nadeau '23
\[
\sum_{\text{meandres}} (u + v + w - 1) uvw = \text{CHM}(u, v, w) \cdot \text{OHM}(u, v, w)
\]
\[ \sum_{\text{meandres}} \left( u + v + w - 1 \right) u v w = \text{CHM}(u, v, w) \cdot \text{OHM}(u, v, w) \]

\[ \text{CHM} = \frac{1}{(1 - \text{CHM})^2} \left( u + \frac{w}{1 - \text{OHM}} \right) \]
\[
\sum_{\text{meandres}} \left( u^2 + v^2 + w^2 - 1 \right) = \text{CHM}(u, v, w) \cdot \text{OHM}(u, v, w)
\]

\[
\text{CHM} = \frac{1}{(1 - \text{CHM})^2} \left( u + \frac{w \text{OHM}}{1 - \text{OHM}} \right) \quad \text{and} \quad \text{OHM} = \frac{1}{(1 - \text{OHM})^2} \left( v + \frac{w \text{CHM}}{1 - \text{CHM}} \right)
\]
\[ \sum_{\text{meandres}} \left( \frac{1}{t} + \frac{1}{z} + \frac{1}{t} - 1 \right) (tz)^\frac{1}{2} (tz)^\frac{1}{2} t = HM(t, z)^2 \]

where

\[ HM = \frac{t}{(1 - HM)^2} \left( z + \frac{HM}{1 - HM} \right) \]
\[
\sum_{\text{meandres}} (\frac{1}{3} + \frac{1}{5} + \frac{1}{7} - 1) (tz)^{\frac{1}{3}} (tz)^{\frac{1}{5}} t^\frac{1}{7} = \text{HM}(t, z)^2
\]

where
\[
\text{HM} = \frac{t}{(1 - \text{HM})^2} \left( z + \frac{\text{HM}}{1 - \text{HM}} \right)
\]

Lagrange inversion again:
\[
[t^n z^k] \text{HM}^2 = \frac{2}{n} [s^{n-2} z^k] \frac{1}{(1 - s)^{2n}} \left( z + \frac{s}{1 - s} \right)^n = \frac{2}{n} \binom{n}{k} [s^{n-2}] \frac{s^{n-k}}{(1 - s)^{3n-k}}
\]
\[
= \frac{2}{n} \binom{n}{k} [s^{k-2}] \frac{1}{(1 - s)^{3n-k}} = \frac{2}{n} \binom{n}{k} \binom{3n - 3}{k - 2}
\]
\[ \sum_{\text{meandres}} \left( \frac{1}{t} + \frac{1}{z} + \frac{1}{t} - 1 \right) (tz)^{\frac{1}{t}} (tz)^{\frac{1}{z}} t^{z} = \text{HM}(t, z)^{2} \]

where \[ \text{HM} = \frac{t}{(1 - \text{HM})^{2}} \left( z + \frac{\text{HM}}{1 - \text{HM}} \right) \]

Lagrange inversion again:

\[ [t^{n} z^{k}] \text{HM}^{2} = \frac{2}{n} [s^{n-2} z^{k}] \frac{1}{(1 - s)^{2n}} \left( z + \frac{s}{1 - s} \right)^{n} = \frac{2}{n} \binom{n}{k} [s^{n-2}] \frac{s^{n-k}}{(1 - s)^{3n-k}} \]

\[ = \frac{2}{n} \binom{n}{k} [s^{k-2}] \frac{1}{(1 - s)^{3n-k}} = \frac{2}{n} \binom{n}{k} \left( \frac{3n - 3}{k - 2} \right) \]

Hence

\[ [t^{n} z^{k}] A(t, z) = \frac{1}{n + 1} [t^{n+1} z^{k+2}] \text{HM}^{2} = \frac{2}{n(n + 1)} \binom{n + 1}{k + 2} \binom{3n}{k} \]
DIAGONAL OF THE PERMUTAHEDRON

with
Bérénice DELCROIX-OGER (Univ. Montpellier)
Matthieu JOSUAT-VERGES (CNRS & Univ. Paris Cité)
Guillaume LAPLANTE-ANFOSSI (Univ. Melbourne)
Kurt STOECKL (Univ. Melbourne)
\[ \Delta_{\text{Perm}(n)} = \text{diagonal of } (n - 1)-\text{dimensional permutahedron} \]

**THM.** $k$-faces of $\Delta_{\text{Perm}(n)} \iff (\mu, \nu)$ ordered partitions of $[n]$ such that

\[ \forall (I, J) \in D(n), \exists k \in [n], \#\mu[k] \cap I > \#\mu[k] \cap J \]

or \[ \exists \ell \in [n], \#\mu[\ell] \cap I < \#\mu[\ell] \cap J \]

where $D(n) := \{(I, J) \mid I, J \subseteq [n], \#I = \#J, I \cap J = \emptyset, \min(I \cup J) \in I\}$

Laplante-Anfossi '22
\[ \Delta_{\text{Perm}(n)} = \text{diagonal of } (n - 1)\text{-dimensional permutahedron} \]

**PROP.** \( B_n^2 \) = two generically translated copies of the braid arrangement

\[ f_k\left( \Delta_{\text{Perm}(n)} \right) = f_{n-k-1}\left( B_n^2 \right) \]
flat poset $Fl(\mathcal{A})$ of an hyperplane arrangement $\mathcal{A} =$ reverse inclusion poset on nonempty intersections of hyperplanes of $\mathcal{A}$
flat poset $\text{Fl}(\mathcal{A})$ of an hyperplane arrangement $\mathcal{A} =$ reverse inclusion poset on nonempty intersections of hyperplanes of $\mathcal{A}$

**EXM.** flat poset of braid arrangement $\mathcal{B}_n$

$$\{ x \in \mathbb{R}^n \mid x_i = x_j \text{ for all } i, j \text{ in the same part of } \pi \}$$

refinement poset on partitions of $[n]$$$

$$\pi$$
flat poset $Fl(\mathcal{A})$ of an hyperplane arrangement $\mathcal{A} =$ reverse inclusion poset on nonempty intersections of hyperplanes of $\mathcal{A}$

Möbius function $\mu$ of a poset: $\mu(x, x) = 1$ and $\sum_{x \leq y \leq z} \mu(x, y) = 0$ for all $x < z$
flat poset $\text{Fl}(\mathcal{A})$ of an hyperplane arrangement $\mathcal{A} =$ reverse inclusion poset on nonempty intersections of hyperplanes of $\mathcal{A}$

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Möbius function $\mu$ of a poset: $\mu(x, x) = 1$ and $\sum_{x \leq y \leq z} \mu(x, y) = 0$ for all $x < z$

Möbius polynomial $\mu_{\mathcal{A}}(x, y) = \sum_{F \leq G} \mu(F, G) x^{\dim(F)} y^{\dim(G)}$

**THM.** $f_{\mathcal{A}}(x) = \mu_{\mathcal{A}}(-x, -1)$ and $b_{\mathcal{A}}(x) = \mu_{\mathcal{A}}(-x, 1)$

Zaslavsky '75
$\mathcal{B}_n^\ell = \text{union of } \ell \text{ generically translated copies of the braid arrangement}$
$B_n^\ell = \text{union of } \ell \text{ generically translated copies of the braid arrangement}$

$(\ell, n)$ partition forest $=$

$\ell$-tuple of partitions of $[n]$ whose intersection hypergraph is a forest

PROP. Intersection poset of $B_n^\ell \leftrightarrow$ refinement poset on $(\ell, n)$ partition forests
$\mathcal{B}_n^\ell =$ union of $\ell$ generically translated copies of the braid arrangement

$(\ell, n)$ partition forest =
$\ell$-tuple of partitions of $[n]$ whose intersection hypergraph is a forest

**PROP.** Intersection poset of $\mathcal{B}_n^\ell \leftrightarrow$ refinement poset on $(\ell, n)$ partition forests
\( \mathbb{P}_p = \text{refinement poset on partitions of } [p] \)

\( \mathbb{PF}^\ell_n = \text{refinement poset on } (\ell, n) \text{ partition forests} \)

**FACT 1.** The Möbius function of \( \mathbb{P}_p \) is \( \mu(\hat{0}, \hat{1}) = (-1)^{p-1}(p - 1)! \)

**FACT 2.** In \( \mathbb{P}_p \), \( [F, G] \simeq \prod_{p \in G} \mathbb{P}_{\#F[p]} \) where \( F[p] = \text{restriction of } F \text{ to } p \)

**FACT 2.** \( [F, G] \simeq \prod_{i \in [\ell]} [F_i, G_i] \) for \( F = (F_1, \ldots, F_\ell) \) and \( G = (G_1, \ldots, G_\ell) \) in \( \mathbb{PF}^\ell_n \)

**FACT 4.** Möbius is multiplicative \( \mu_{P \times Q}((p, q), (p', q')) = \mu_P(p, p') \cdot \mu_Q(q, q') \)
MÖBIUS POLYNOMIAL

\[ \mathbb{P}_p = \text{refinement poset on partitions of } [p] \]
\[ \mathbb{PF}_n^\ell = \text{refinement poset on } (\ell, n) \text{ partition forests} \]

**FACT 1.** The Möbius function of \( \mathbb{P}_p \) is \( \mu(\hat{0}, \hat{1}) = (-1)^{p-1}(p - 1)! \)

**FACT 2.** In \( \mathbb{P}_p \),
\[ [F, G] \simeq \prod_{p \in G} \mathbb{P}^{\#F[p]} \]
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**FACT 2.**
\[ [F, G] \simeq \prod_{i \in [\ell]} [F_i, G_i] \]
for \( F = (F_1, \ldots, F_\ell) \) and \( G = (G_1, \ldots, G_\ell) \) in \( \mathbb{PF}_n^\ell \)

**FACT 4.** Möbius is multiplicative
\[ \mu_{P \times Q}((p, q), (p', q')) = \mu_P(p, p') \cdot \mu_Q(q, q') \]

**THM.**
\[ \mu_{\mathcal{B}_n^\ell} = x^{n-1-\ell n} y^{n-1-\ell n} \sum_{F \leq G} \prod_{i \in [\ell]} x^{\#F_i} y^{\#G_i} \prod_{p \in G_i} (-1)^{\#F[p]-1}(\#F[p] - 1)! \]

### Theorem

\[
f_{B_n^\ell}(x) = x^{n-1-\ell n} \sum_{F \leq G} \prod_{i \in [\ell]} x^{\# F_i} \prod_{p \in G_i} (\# F_i[p] - 1)!
\]

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\(\ell = 1\)

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<td>34</td>
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<td>924</td>
<td>2436</td>
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\(\ell = 3\)
THM. \[ b_{B_n^\ell}(x) = (-1)^\ell x^{n-1-\ell n} \sum_{F \leq G} \prod_{i \in [\ell]} x^{\#F_i} \prod_{p \in G_i} -\left(\#F_i[p] - 1\right)! \]
**THM.** \( f_0(\mathcal{B}_n^\ell) = \# \{(\ell, n) \text{ partition trees}\} = \ell (n(\ell - 1) + 1)^{n-2} \)

<table>
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<tr>
<th>(n) (\ell)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</tr>
</thead>
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<td>676</td>
<td>1445</td>
<td>2646</td>
</tr>
</tbody>
</table>

1 ↦ \(2(n+1)^{n-2}\) [OEIS, A007334]
**THM.** \( f_0(\mathcal{B}^2_n) = \# \{ (2, n) \text{ partition trees} \} = \# \text{spanning trees of } K_{n+1} \text{ with } 01 \)


1, 2, 8, 50, 432, 4802, 65536, 1062882, 20000000, 428717762, \ldots

[OEIS, A007334]
THM. \( f_0(\mathcal{B}_n^2) = \# \{(2, n) \text{ partition trees} \} = \# \text{spanning trees of } K_{n+1} \cdot \frac{2}{n+1} = 2(n + 1)^{n-2} \)

[OEIS, A007334]

1, 2, 8, 50, 432, 4802, 65536, 1062882, 20000000, 428717762, \ldots
THM. \( f_{n-1}(B^\ell_n) = n! \left[ z^n \right] \exp \left( \sum_{m \geq 1} \frac{F_{\ell,m} z^m}{m} \right) \) where \( F_{\ell,m} = \frac{1}{(\ell - 1)m + 1} \binom{\ell m}{m} \)


<table>
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<tr>
<th>( n \backslash \ell )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>3589</td>
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</tbody>
</table>

\( n! \leftarrow \) [OEIS, A213507]
THM.  \( b_{n-1}(B^\ell_n) = (n - 1)! [z^{n-1}] \exp \left( (\ell - 1) \sum_{m \geq 1} F_{\ell,m} z^m \right) \)