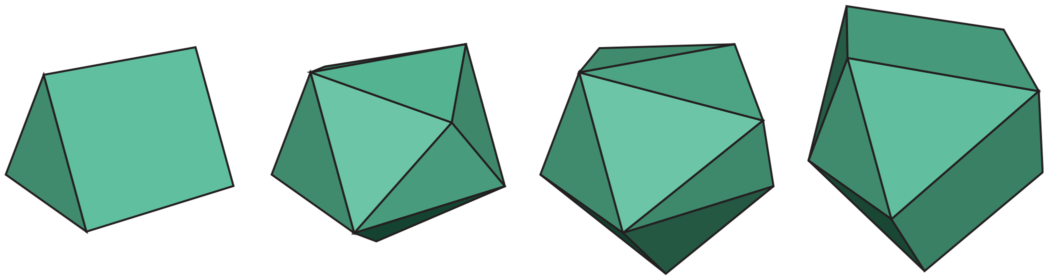
21st Canadian Conference on Computational Geometry
Vancouver, August 2009

ON SYMMETRIC REALIZATIONS OF THE SIMPLICIAL COMPLEX OF 3-CROSSING-FREE SETS OF DIAGONALS OF THE OCTAGON



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Vincent Pilaud
Ecole Normale Superieure, Paris

$k \geq 1$ and $n \geq 2k + 1$ two xed integers

ℓ-crossing = set of ℓ mutually crossing diagonals of the convex n-gon

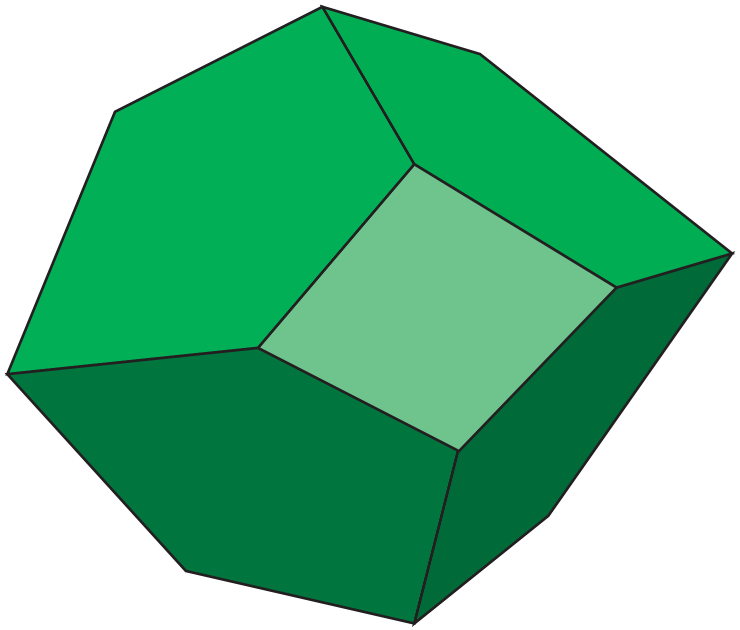
k-relevant diagonal = at least k vertices on each side
= diagonals which may appear in a $(k+1)$-crossing

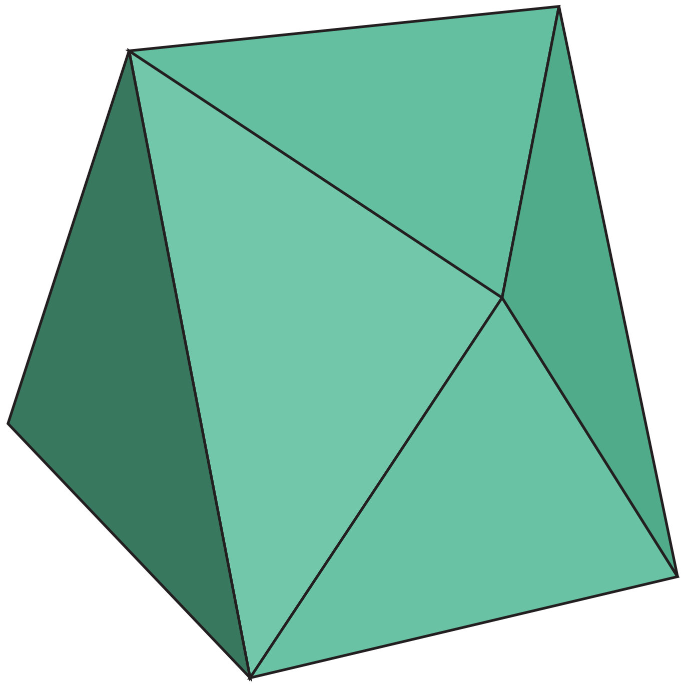
$\boxed{\ {}_{n;k} = \text{simplicial complex of } (k+1)\text{-crossing-free sets of } k\text{-relevant diagonals of the convex } n\text{-gon}}$

EXAMPLES

$\boxed{k = 1}$ Maximal elements of $_{n;1}$ = triangulations of the n-gon

$_{n;1}$ = boundary complex of the dual of the $(n-3)$-dimensional associahedron





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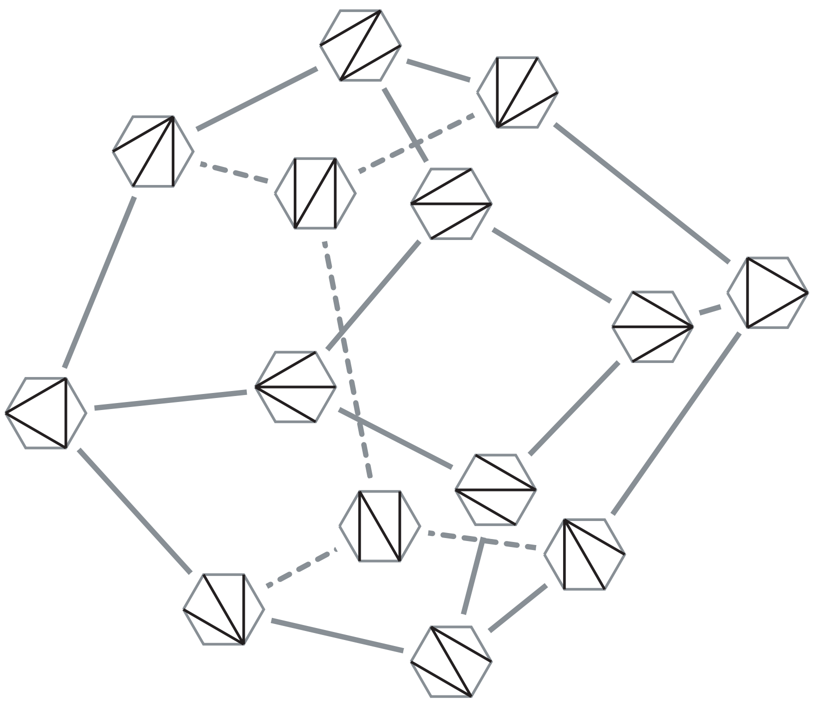

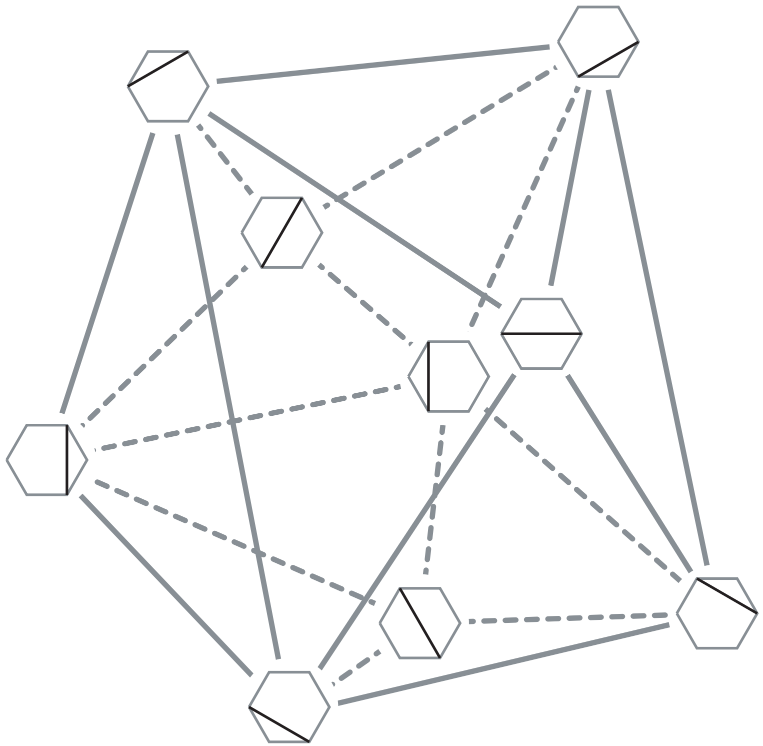

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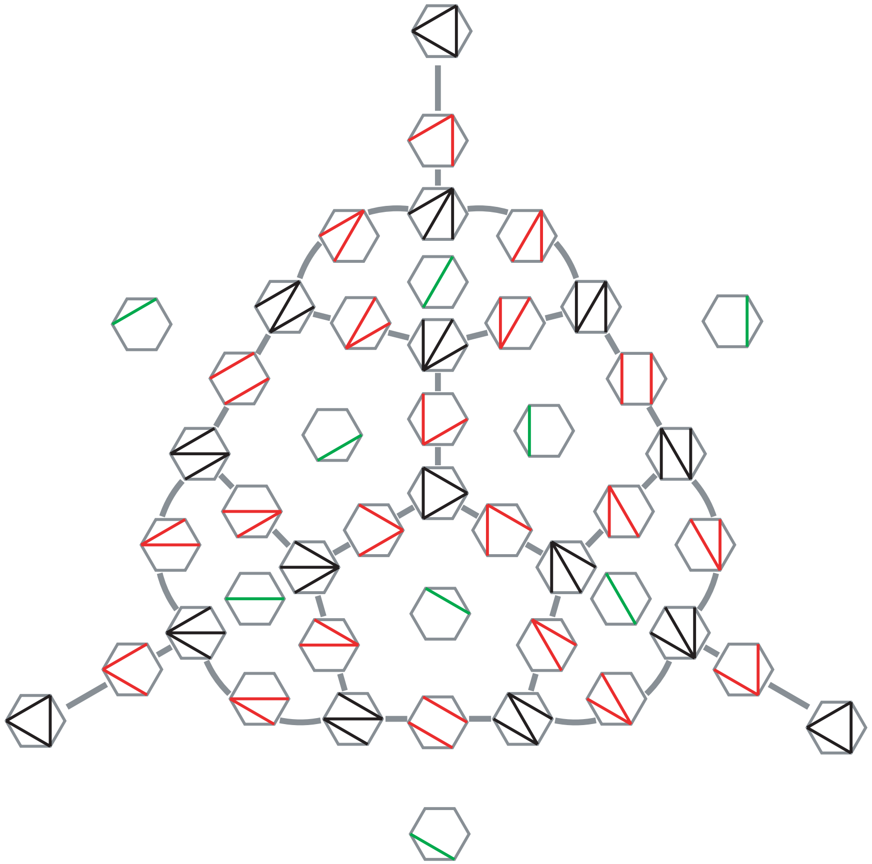

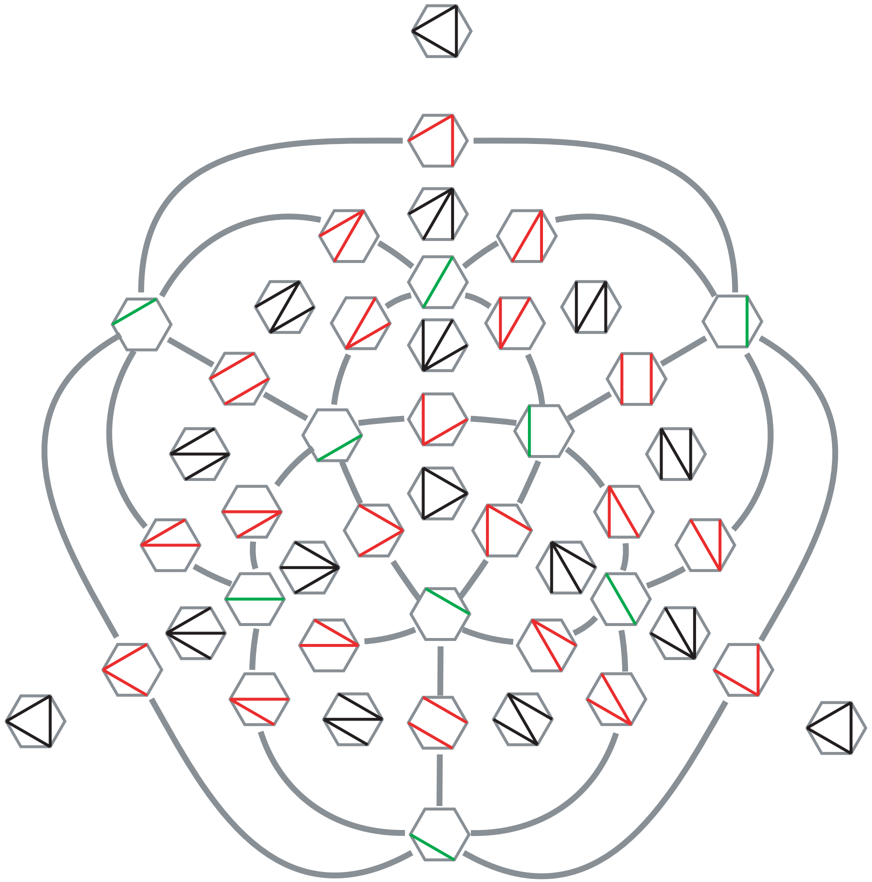

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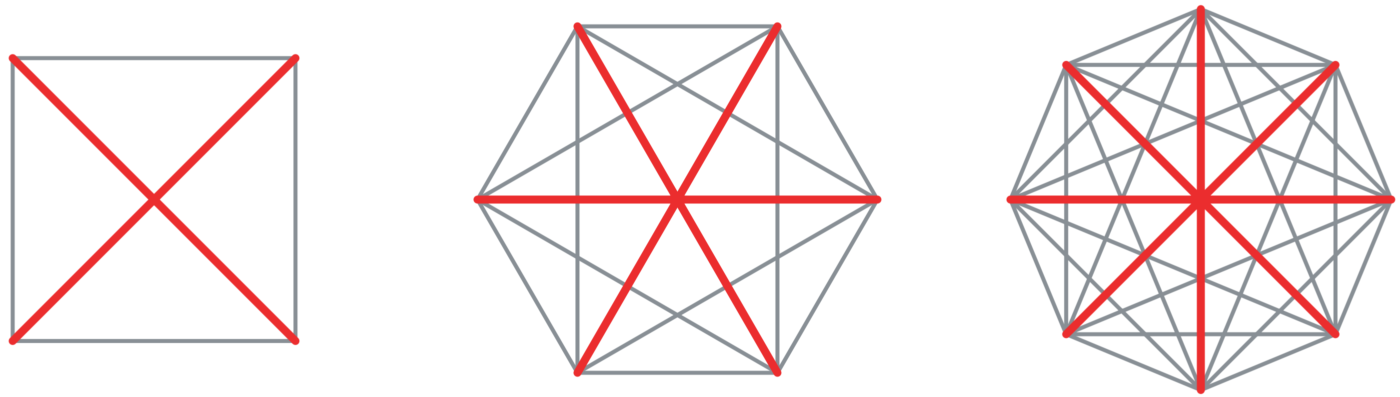
$\boxed{n = 2k+2}$ $_{2k+1;k}$ = boundary complex of the k-simplex



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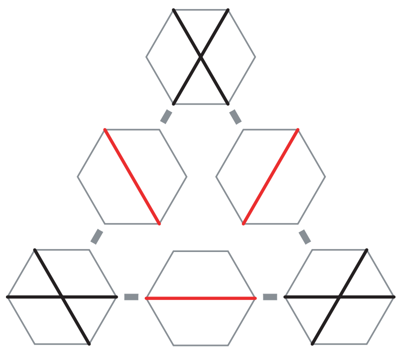

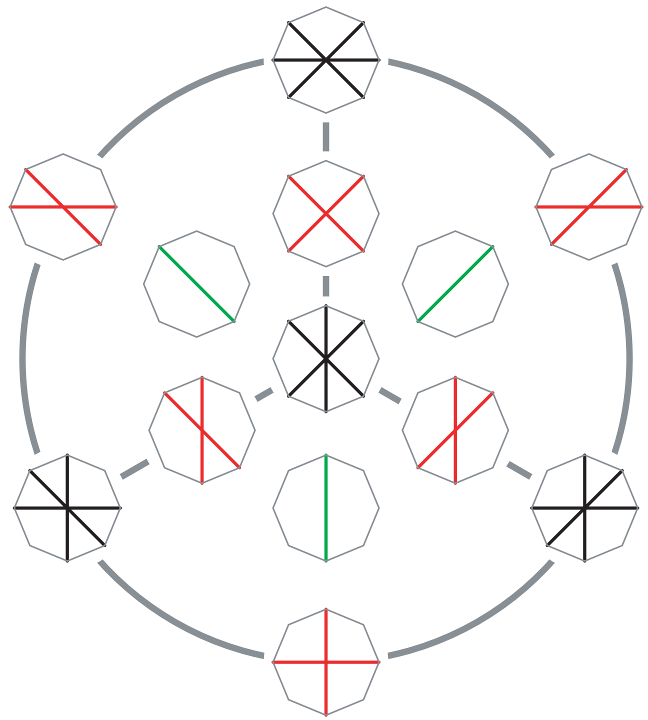

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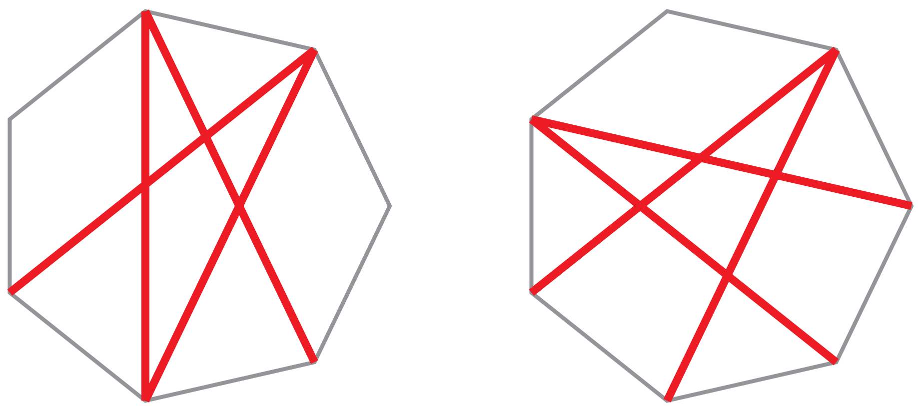
$\boxed{n = 2k + 2}$ $_{2k+1;k}$ = boundary complex of the k-simplex

$\boxed{n = 2k + 3}$ $_{2k+3;k}$ = boundary complex of the cyclic polytope of dimension $2k$ with $2k + 3$ vertices



POLYTOPE ?

$\boxed{k = 1}$ Maximal elements of $_{n;1}$ = triangulations of the n-gon

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$\boxed{\text{General } k}$ Maximal elements of $_{n;k}$ = k-triangulations of the n-gon

$_{n;k}$ is pure of dimension $k(n - 2k - 1) - 1$

$_{n;k}$ is a topological sphere

V. Capoyleas & J. Pach, A Turan-type theorem on chords of a convex polygon, 1992

T. Nakamigawa, A generalization of diagonal ips in a convex polygon, 2000

J. Jonsson, Generalized triangulations of the n-gon, 2003

V. Pilaud & F. Santos, Multi-triangulations as complexes of star polygons, 2007

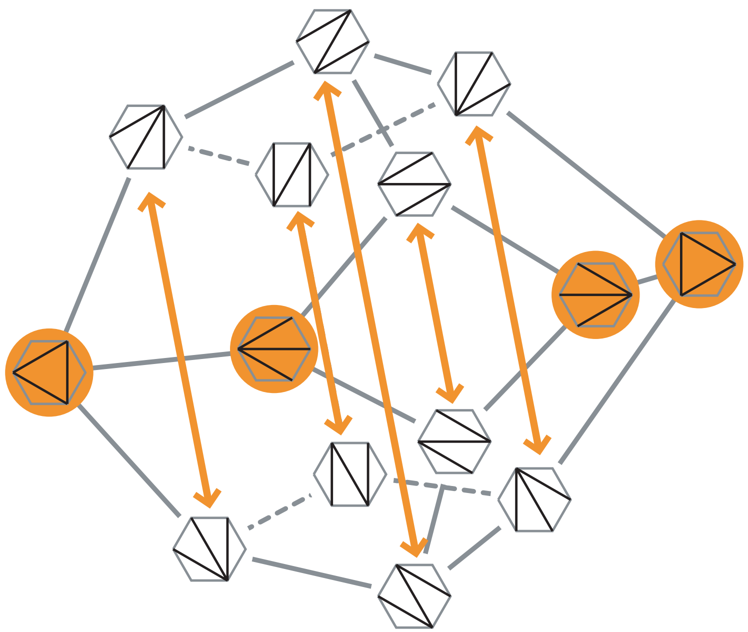
Q1. Is $_{n;k}$ the boundary complex of a $k(n - 2k - 1)$-dimensional simplicial polytope ?

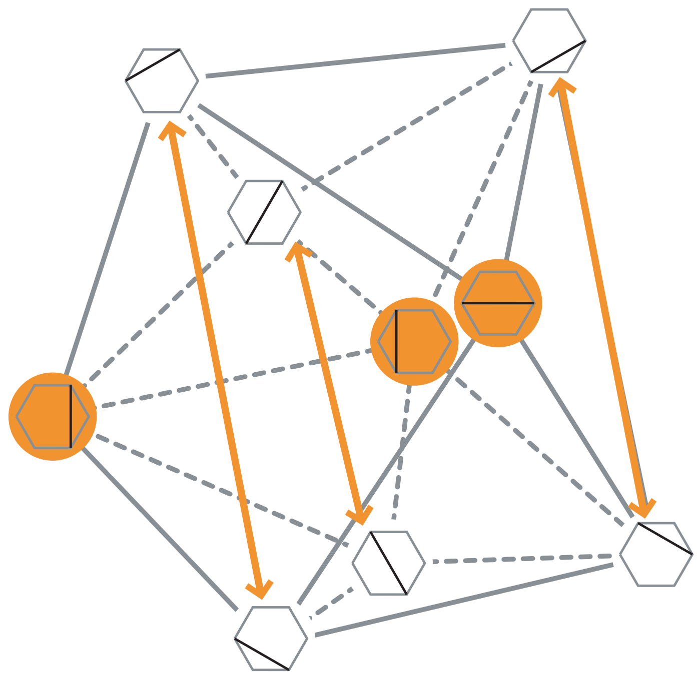
> The rst open case is $k = 2$ and $n = 8$
> f-vector should be $(12; 66; 192; 306; 252; 84)$! ! ?

USE SYMMETRY

$\mathbb{D}_n$ = dihedral group = isometries of the regular n-gon

Natural action of $\mathbb{D}_n$ on $_{n;k}$: $\begin{array}{ccc} \mathbb{D}_n \times \; _{n;k} & \longrightarrow & _{n;k} \\ (\; ; E) & \longmapsto & E = \{ \; e \,|\, e \in E \} \end{array}$





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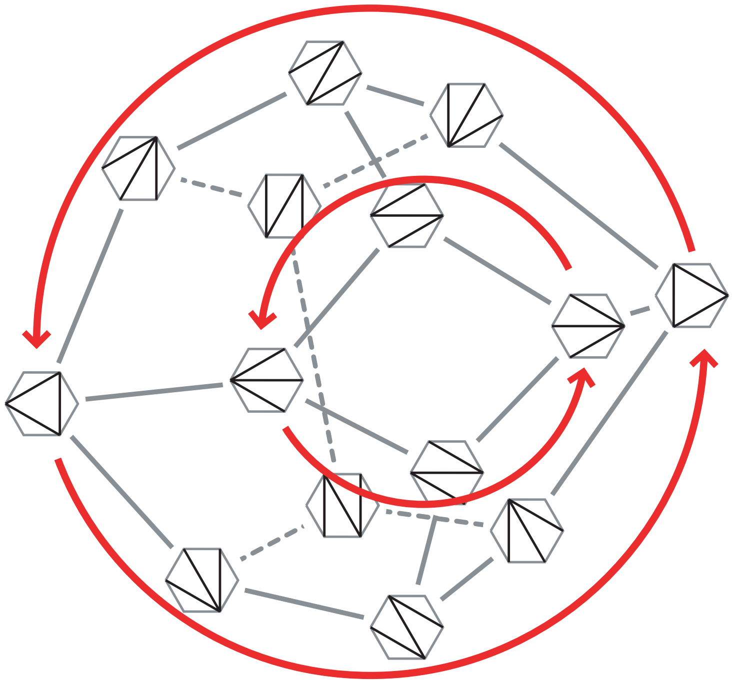

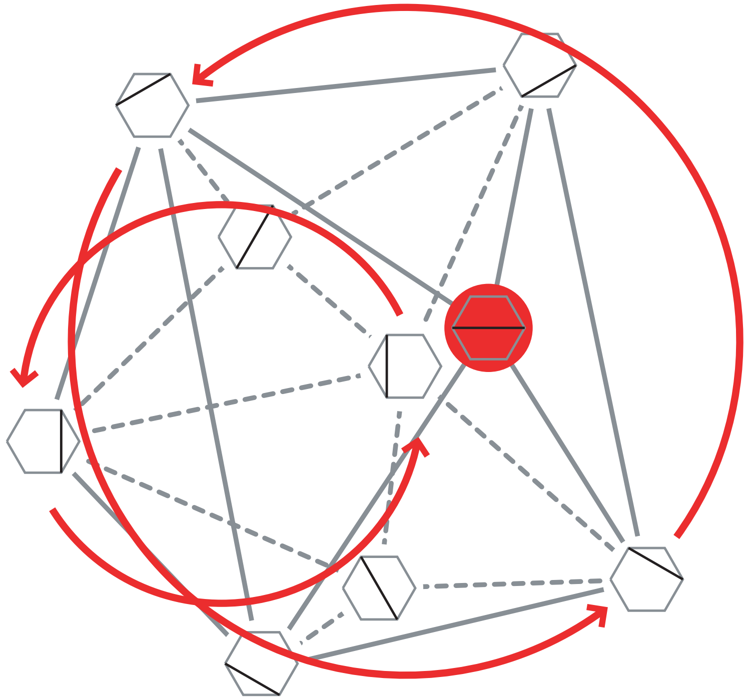

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DECOMPOSE INTO TWO STEPS

1. From face lattice to oriented matroids

   Find all possible symmetric oriented matroids realizing $\ {}_{n;k}$

2. From oriented matroids to polytopes

   Deduce the space of symmetric realizations of $\ {}_{n;k}$

a simplicial complex with an action of a group G

$P \subset \mathbb{R}^d$ a realization of symmetric under G, and V its vertex set

$$: \left\{ \begin{array}{lcl} V^{d+1} & \longrightarrow & \{-1; 0; +1\} \\ (v_0; v_1; \ldots; v_d) & \longmapsto & \begin{array}{c}\text{orientation of the simplex} \\ \text{spanned by } v_0; v_1; \ldots; v_d\end{array} = \operatorname{sign}\det \begin{array}{cccc} v_0 & v_1 & \ldots & v_d \\ 1 & 1 & 1 & 1 \end{array} \end{array} \right.$$

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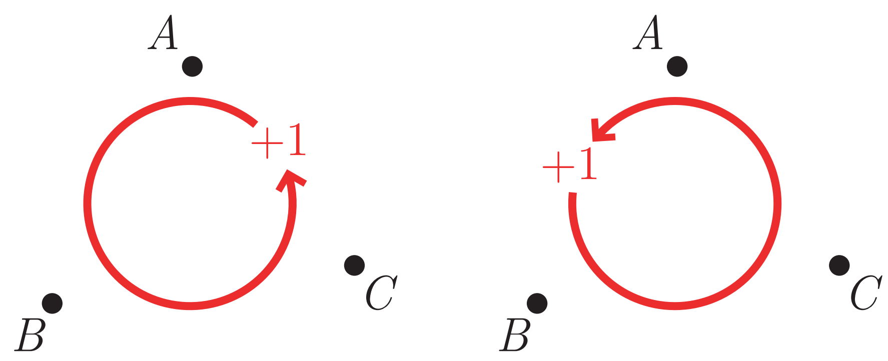
satis es the relations :

(i) Alternating relations



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$$
\begin{array}{cccc}
 & 2 & & \\
 & V^{d+1} & \longrightarrow & \{-1; 0; +1\} \\
: & 4 & & \\
 & (v_0; v_1; \dots; v_d) & \longmapsto & \text{orientation of the simplex spanned by } v_0; v_1; \dots; v_d = \operatorname{sign}\det \begin{matrix} v_0 & v_1 & \dots & v_d \\ 1 & 1 & 1 & 1 \end{matrix}
\end{array}
$$

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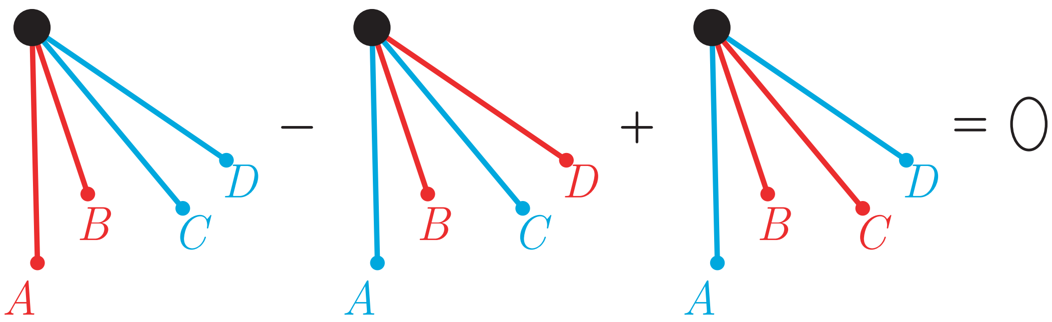
(ii) Grassmann-Plucker relations



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$$\begin{array}{llcll} & 2 & & & \\ & V^{d+1} & \longrightarrow & \{-1;0;+1\} & \\ : 4 & (v_0;v_1;:::;v_d) & \longmapsto & \begin{array}{c}\text{orientation of the simplex}\\ \text{spanned by } v_0;v_1;:::;v_d\end{array} & = \text{sign det} \begin{matrix} v_0 & v_1 & ::: & v_d \\ 1 & 1 & 1 & 1 \end{matrix} \end{array}$$

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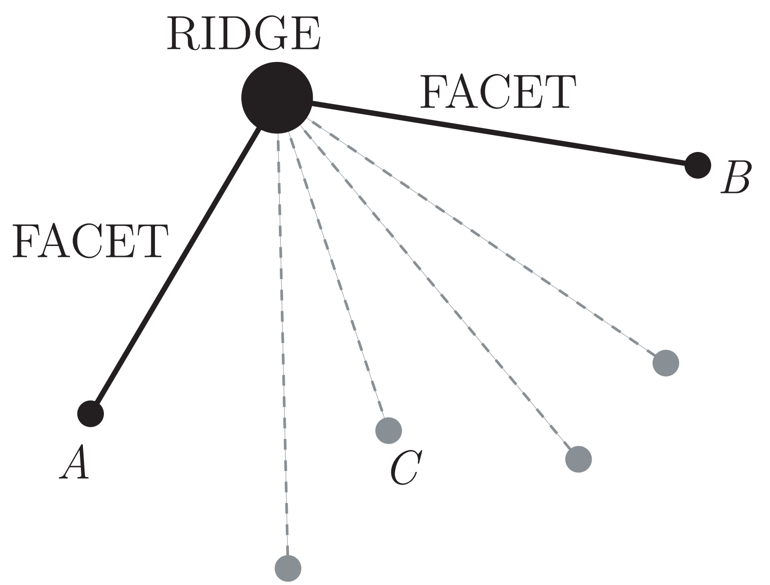
(iii) Necessary simplex orientations



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(iv) Symmetry

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satis es the relations :

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Example. Only one solution for $_{6,1}$

Proposition. 15 solutions for $_{8,2}$

FROM ORIENTED MATROIDS TO POLYTOPES

Problem. For a given oriented matroid, nd a matrix representing it or a proof that such a matrix is impossible to nd.

\On the one hand, there is a general algorithm to solve this problem. On the other hand, it is known that this algorithm from real algebraic geometry is far from applicable for our cases in the theory of oriented matroids."

J. Bokowski, Computational Oriented Matroids, 2006

$\Longrightarrow$ USE HEURISTICAL METHODS

Our heuristic is symmetry

Consider the matrix of homogeneous coordinates of the vertices of P

$$M = \begin{pmatrix} v_1 & v_2 & \dots & v_p \\ 1 & 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 & v_{d+1} \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} I_{d+1} & W \end{pmatrix}$$

We use symmetry on the determinants of the submatrices of M to compute coefficients (the volumes of the simplices spanned by vertices of P are preserved by isometries...)

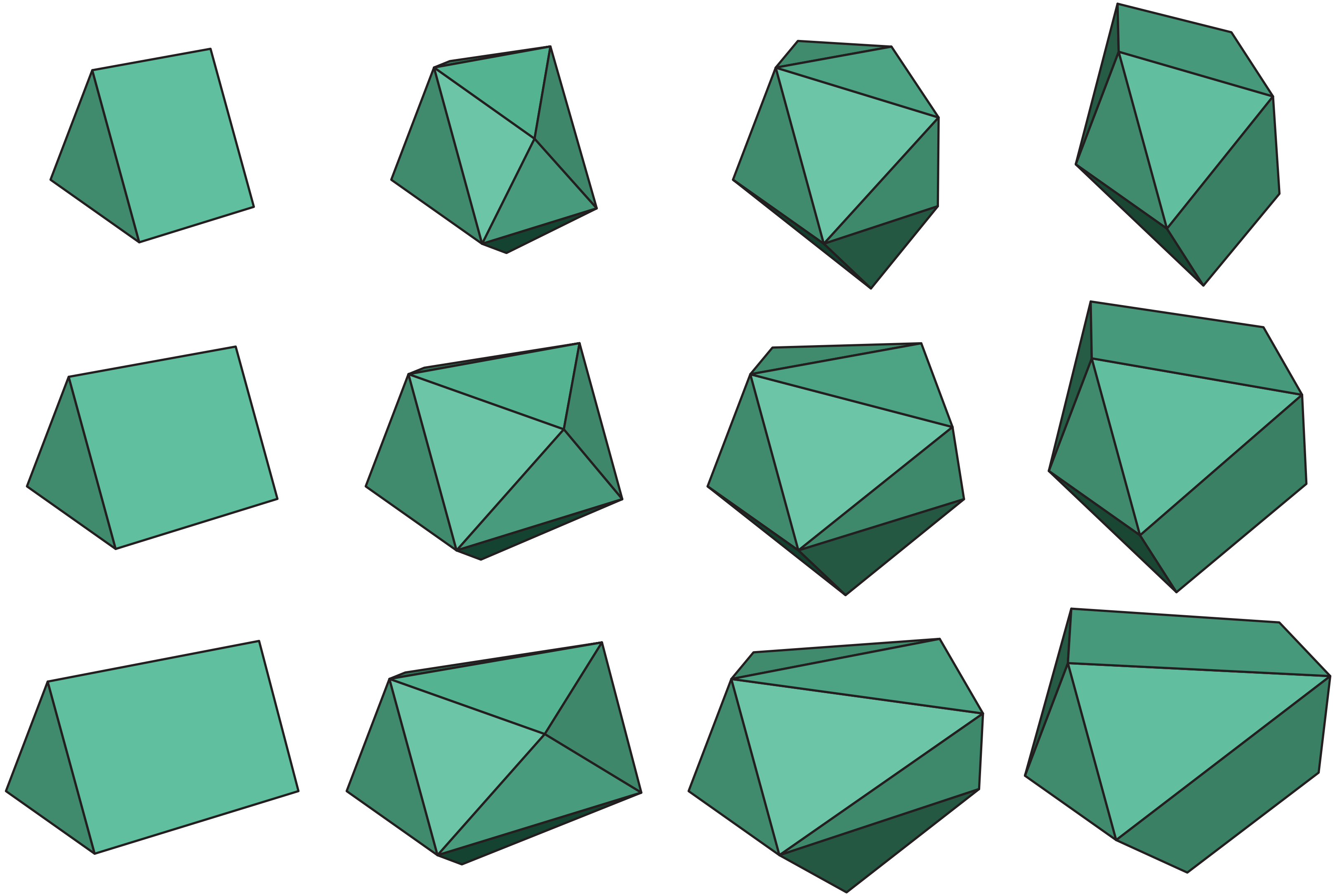
Example. For ${}_{6;1}$, we obtain

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & 1 & 0 & 0 \\ a+3x-2 & 0 & 1 & 0 \\ a & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -2x+2 & 3x-2 & -2x+2 & 0 & x & 1 & 0 & 0 \\ 0 & x & -3x+2 & 4x-2 & -3x+2 & x & 0 & 1 & 0 \\ 0 & x & 0 & -2x+2 & 3x-2 & -2x+2 & 0 & 0 & 1 \end{pmatrix}$$

with $0 < x < \frac{1}{2}$ and $a + x \neq 1$

Reciprocally...

THE SPACE OF SYMMETRIC REALIZATIONS OF $_{6;1}$



Proposition. The space of symmetric realizations of $_{8;2}$ has dimension 4

Example. With some arbitrary values of the 4 parameters, we obtain a particular symmetric realization of $_{8;2}$:

$$M \simeq \begin{pmatrix} 0{:}21 & 0 & 0 & -0{:}21 & 0{:}52 & -0{:}74 & 0{:}74 & -0{:}52 & 0 & 0 & 0 & 0 \\ -0{:}95 & 0{:}66 & 0 & -0{:}63 & 0{:}8 & -0{:}4 & -0{:}3 & 0{:}68 & 0 & 0 & 0 & 0 \\ -0{:}17 & 0{:}75 & -1 & 0{:}77 & -0{:}21 & -0{:}4 & 0{:}60 & -0{:}34 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0{:}55 & 0{:}55 & -0{:}55 & -0{:}55 & 0{:}5 & 0{:}7 & -0{:}7 & -0{:}4 & 1 & 0 & -1 & 0 \\ -0{:}55 & 0{:}55 & 0{:}55 & -0{:}4 & -0{:}7 & 0{:}7 & 0{:}5 & -0{:}55 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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> polymake multiassociahedron82 F_VECTOR
F_VECTOR
12 66 192 306 252 84
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THANK YOU.