Integrability and Cluster Algebras, Providence, August 2014

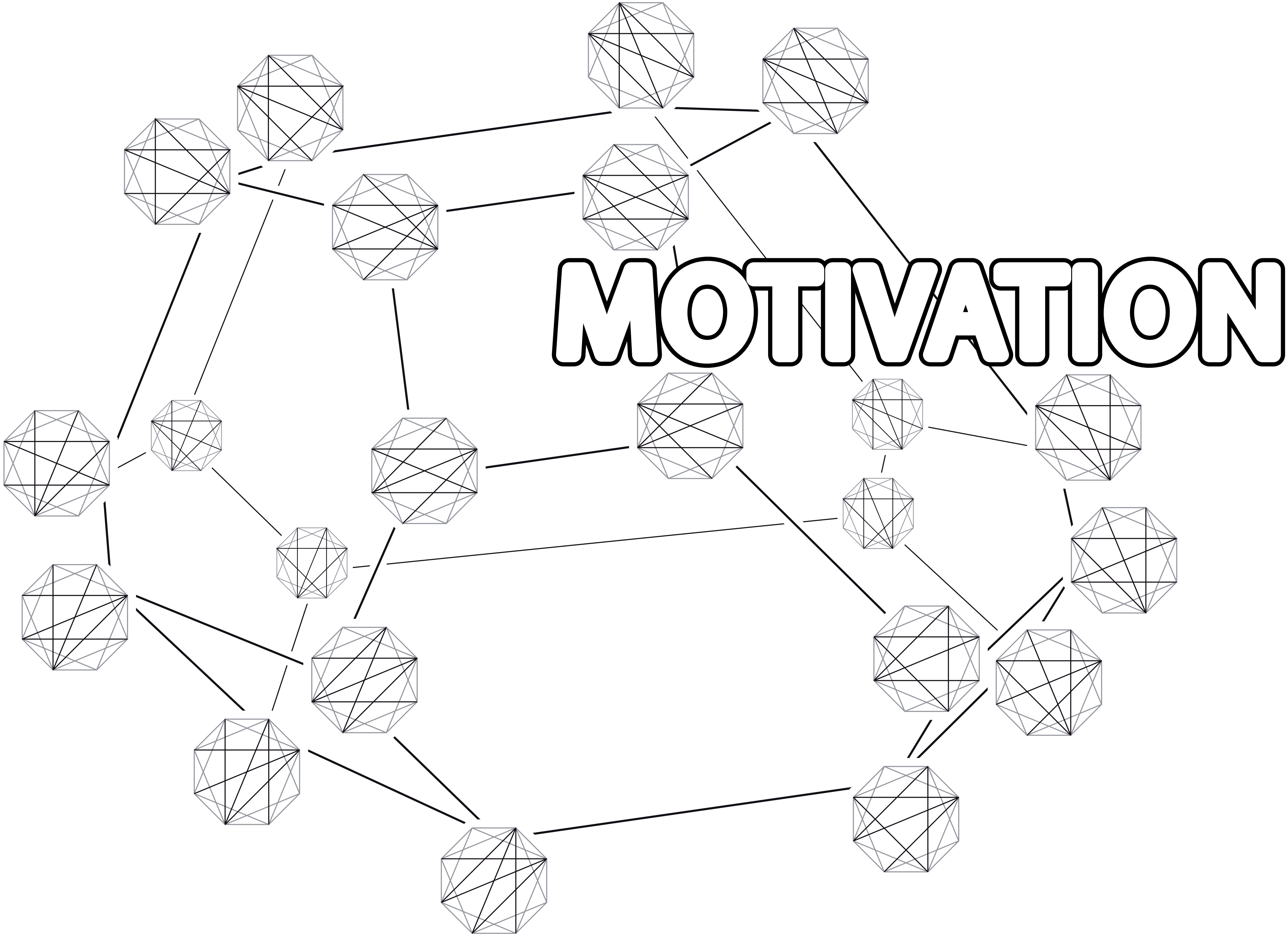
THE BRICK POLYTOPE

Vincent PILAUD
(CNRS & École Polytechnique)

Christian STUMP
(Freie Universität Berlin)

Michel POCCHIOLA
(Université Paris 6)

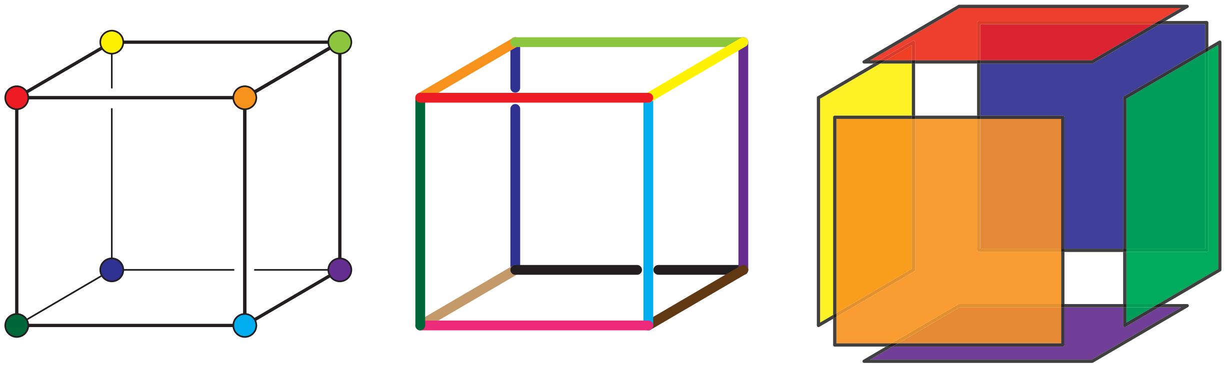
Francisco SANTOS
(Universidad de Cantabria)

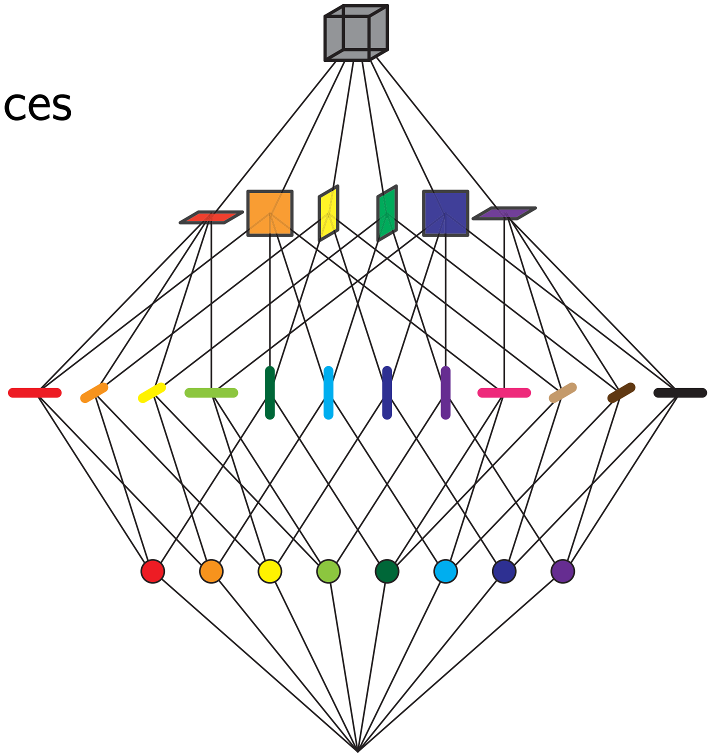

MOTIVATION

POLYTOPES WITH PRESCRIBED COMBINATORICS

polytope = convex hull of a finite set of $\mathbb{R}^d$
= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane
face lattice = all the faces with their inclusion relations





Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?

POLYTOPES OF DIMENSION ≥ 4

Polytopes of dimension 3 $\longleftrightarrow$ planar 3-connected graphs

Various open conjectures in dimension 4:

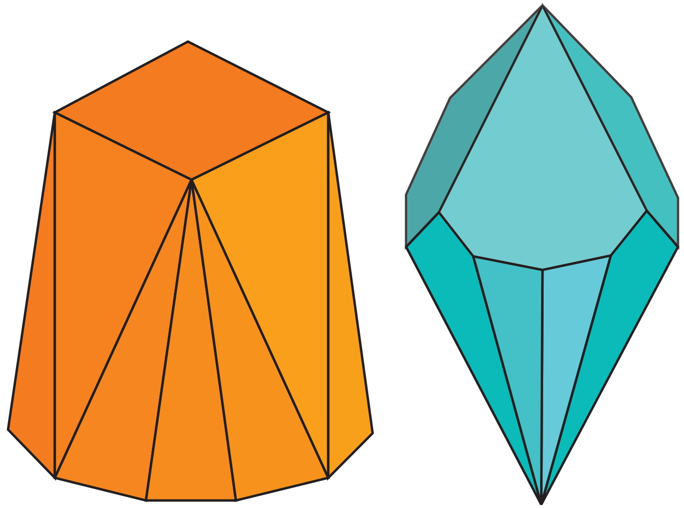
Hirsch conjecture

diameter $\leq$ #facets – dimension (Santos)

complexity of the simplex algorithm

3^d Conjecture (Kalai)

f-vecteur shape (Barany, Ziegler)



Prismatoïdes

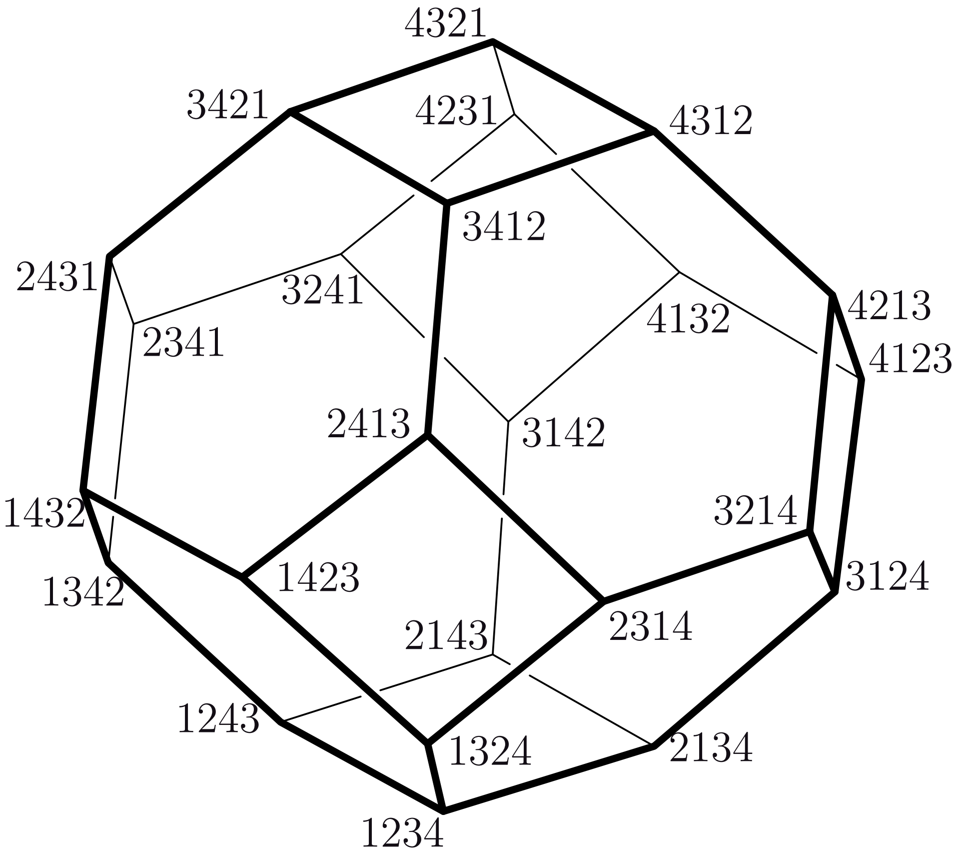
"Our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the good questions and in the lack of examples, methods of constructing them, and means of classifying them."

Kalai. Handbook of Discrete and Computational Geometry, 2004.

PERMUTAHEDRON

$\Pi_n = \text{conv}\left\{(\sigma(1), \ldots, \sigma(n))^T \;\middle|\; \sigma \in \mathfrak{S}_n\right\}$

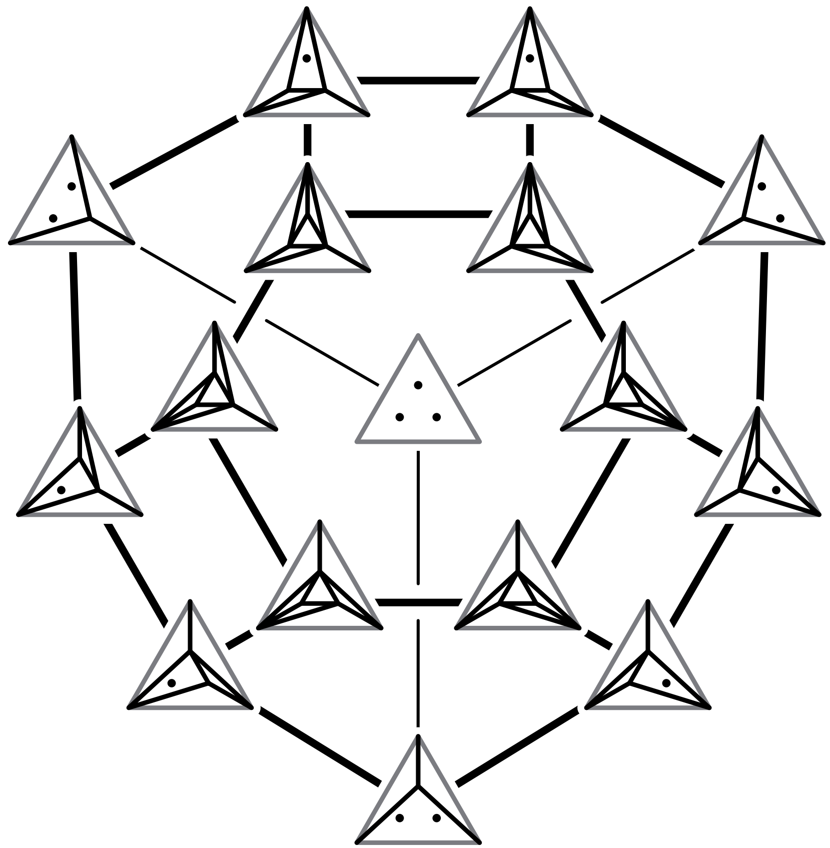
$\partial\Pi_n =$ refinement poset on ordered partitions of $[n]$

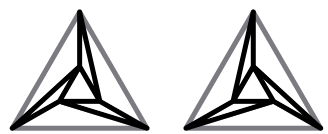


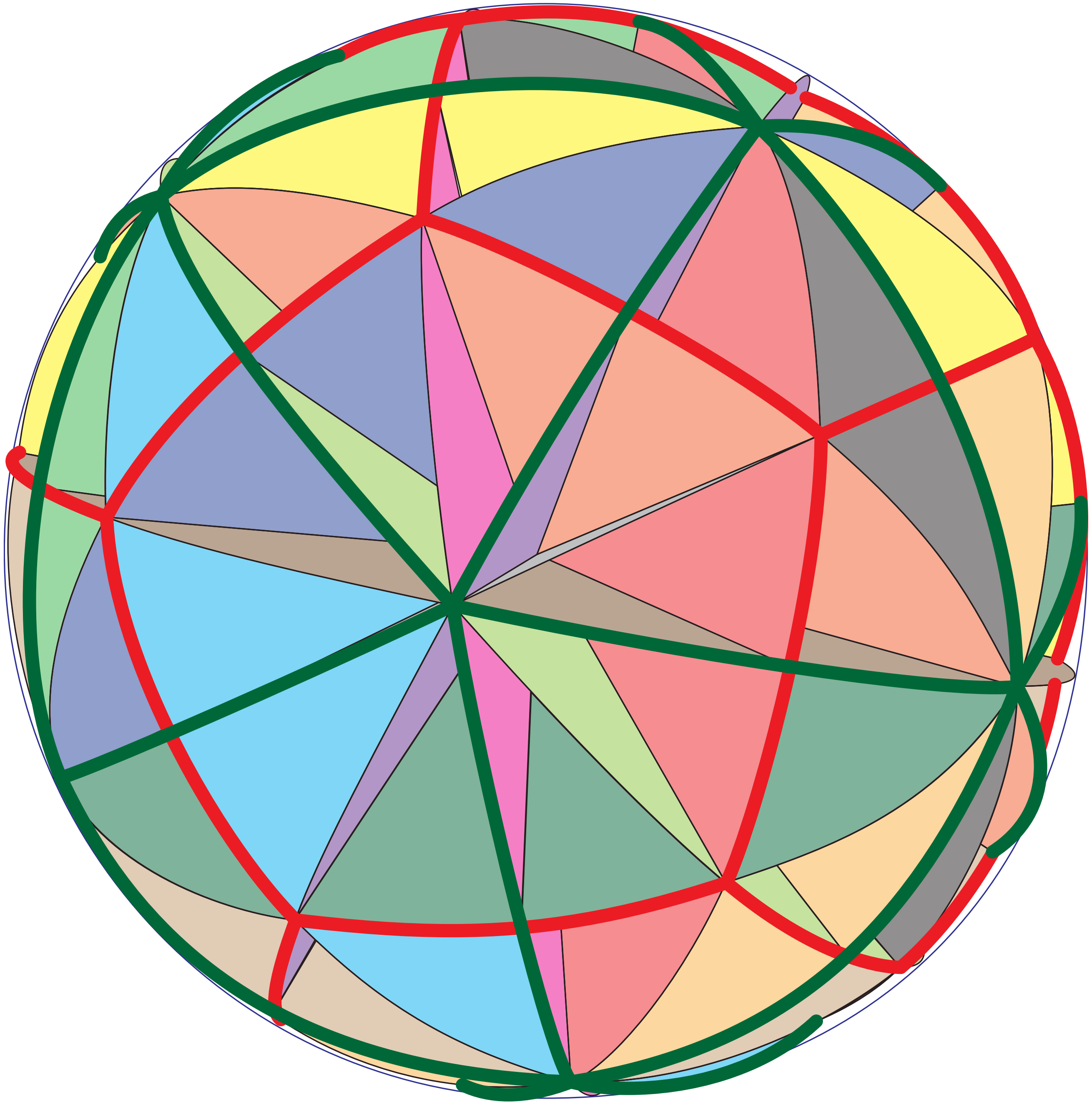
SECONDARY POLYTOPE

$\Sigma(P) = \text{conv}\left\{\sum_{p \in P} \text{vol}(T, p) e_p \;\middle|\; T \text{ triang. } P\right\}$

$\partial\Sigma(P) =$ refinement poset on regular polyhedral subdivisions of P



Triangulations  are non-regular

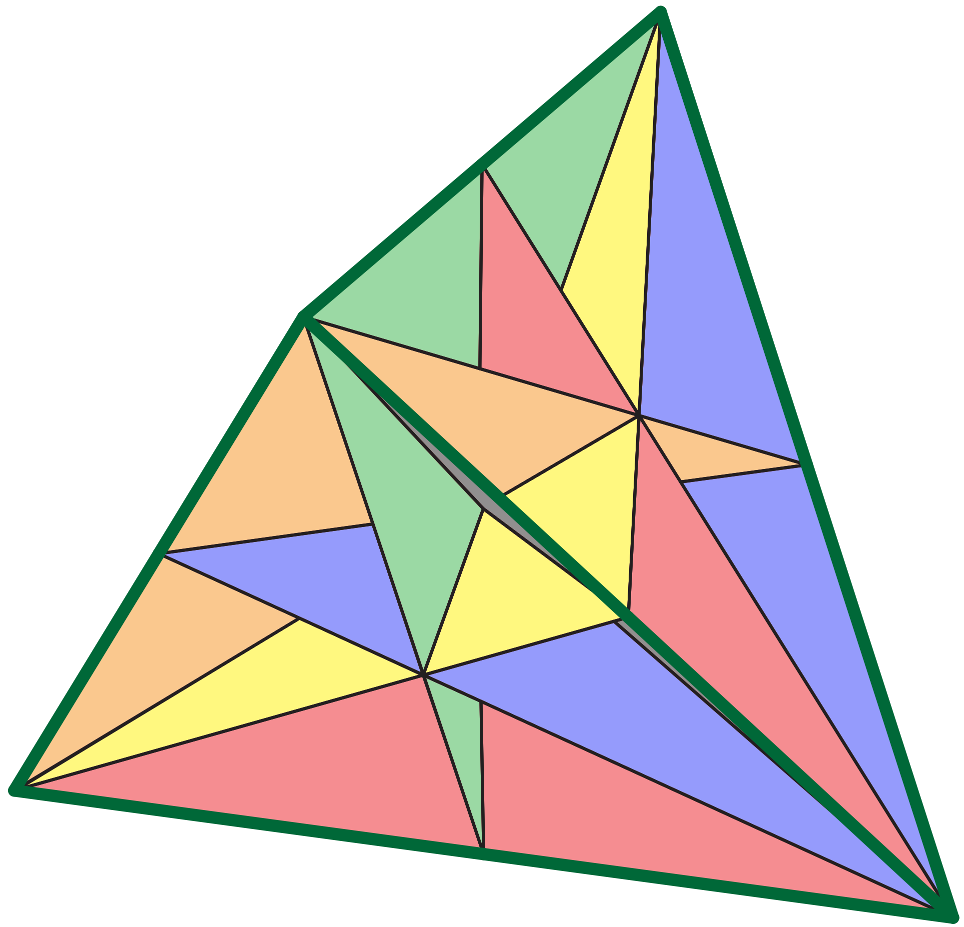


FINITE COXETER GROUPS

J. Humphreys, Reflection groups and Coxeter groups, 1990.

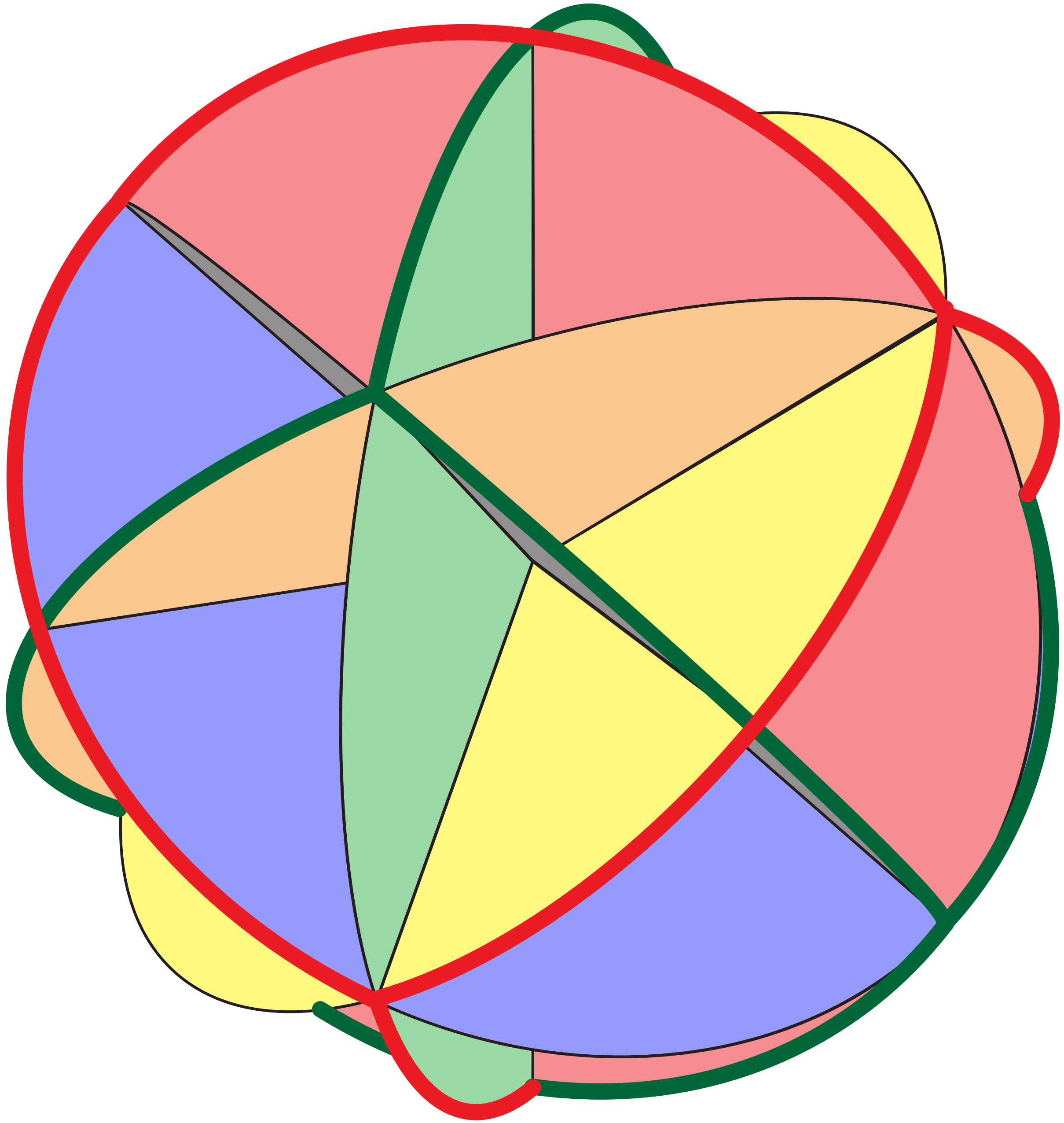
FINITE COXETER GROUPS

W = finite Coxeter group



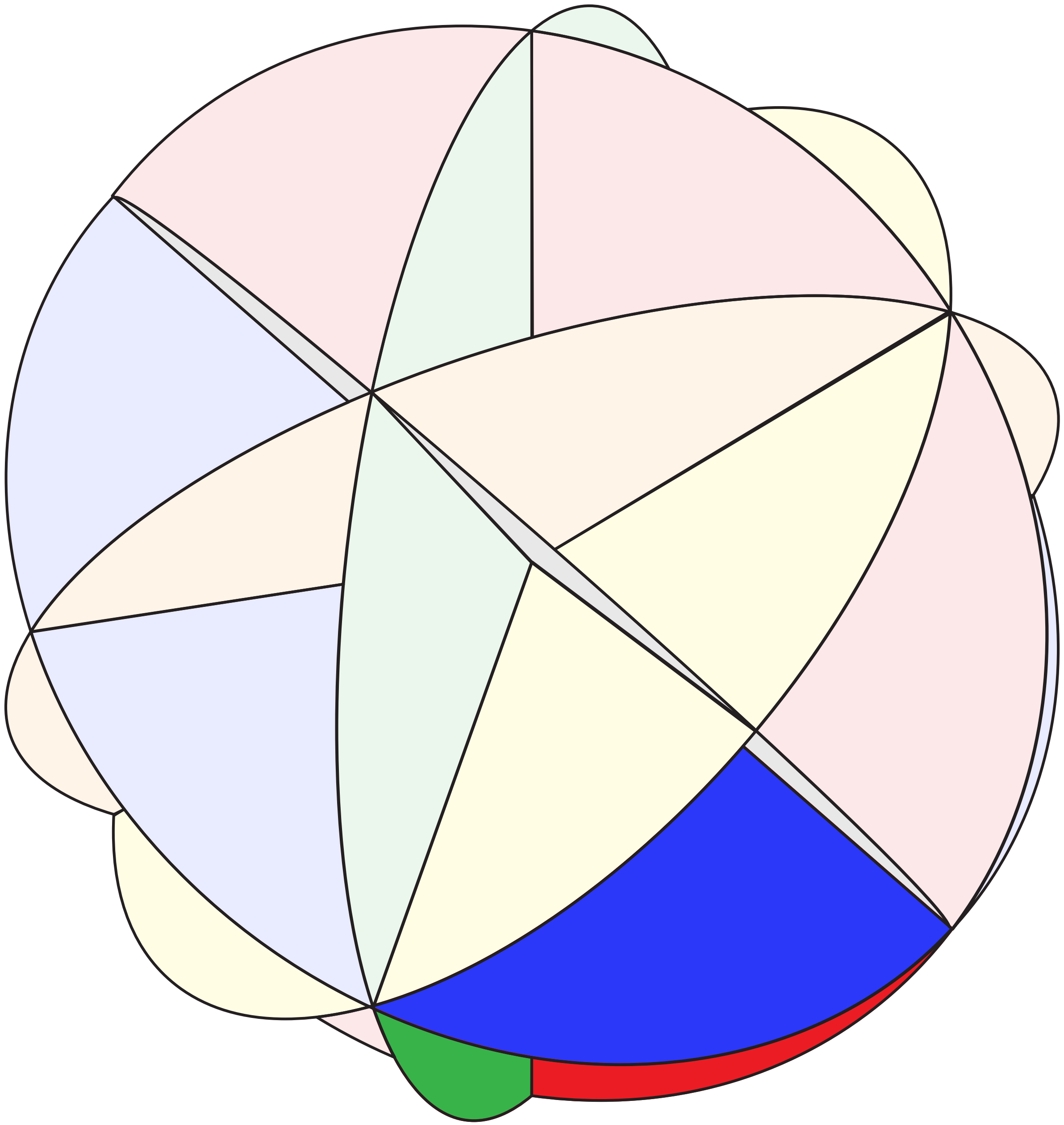
W = finite Coxeter group

Coxeter fan



FINITE COXETER GROUPS

W = finite Coxeter group
Coxeter fan
fundamental chamber



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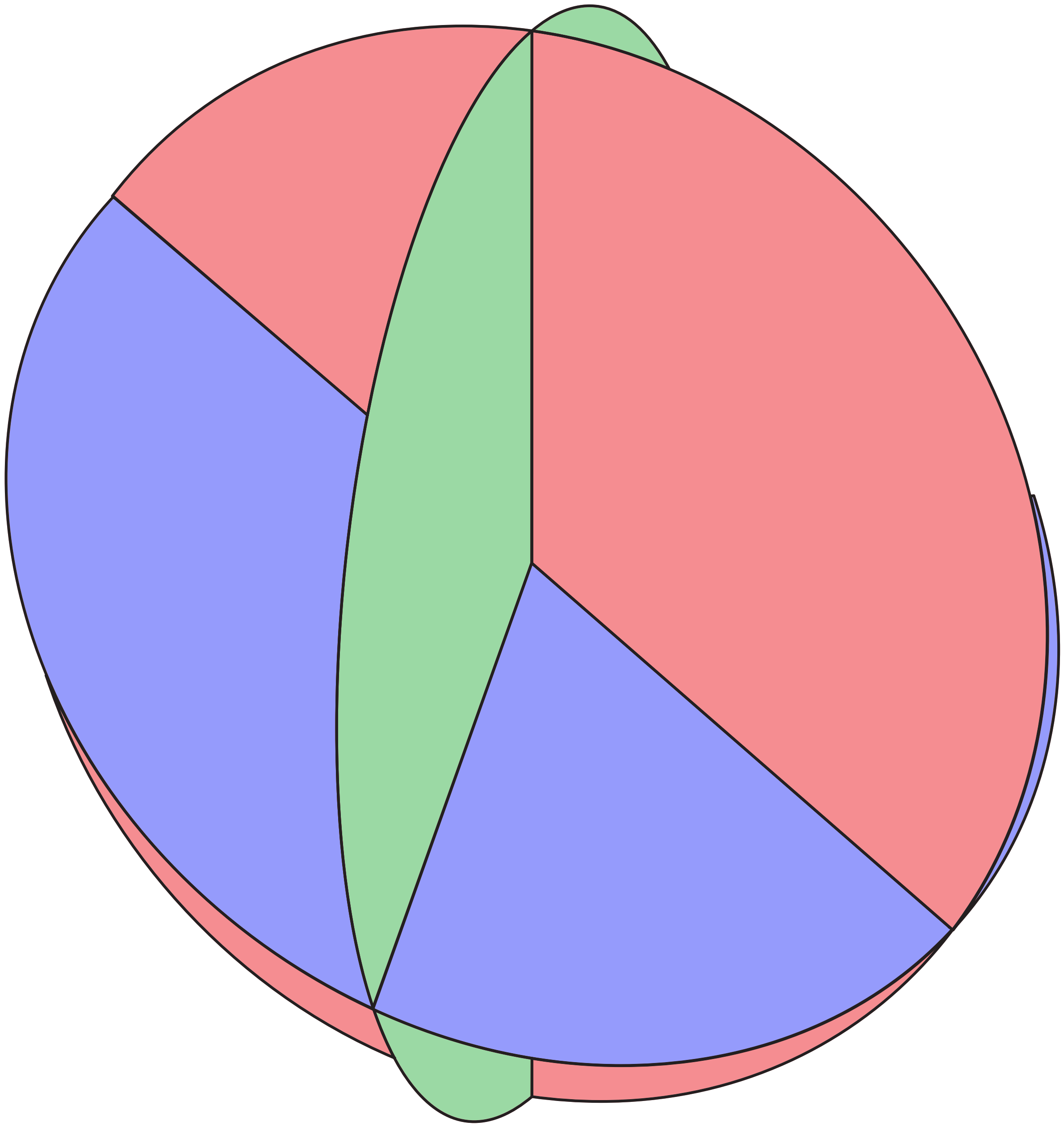

W = finite Coxeter group

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S = simple reflections

FINITE COXETER GROUPS

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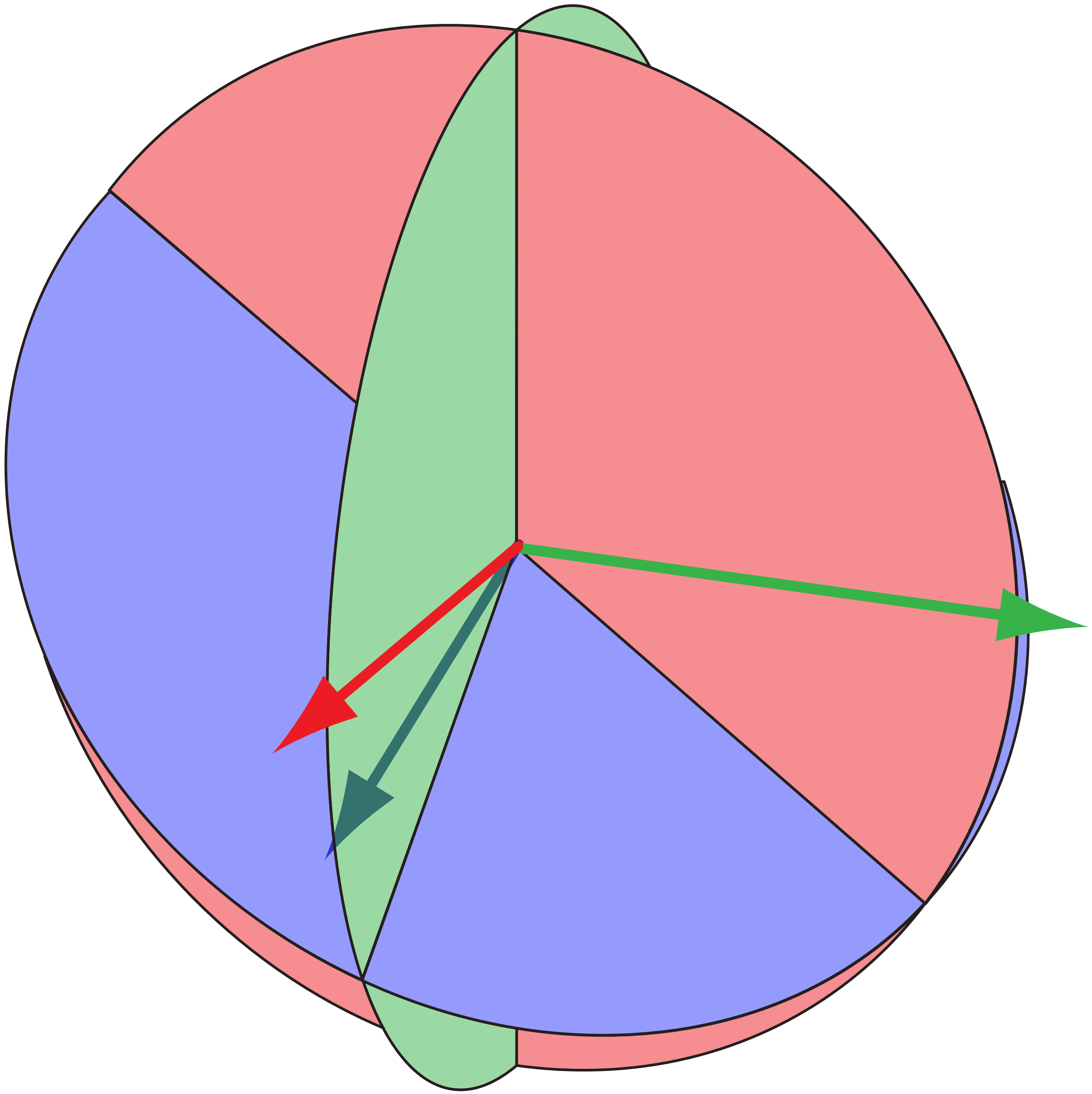
S = simple reflections
$\Delta = \{\alpha_s \mid s \in S\}$ = simple roots



FINITE COXETER GROUPS

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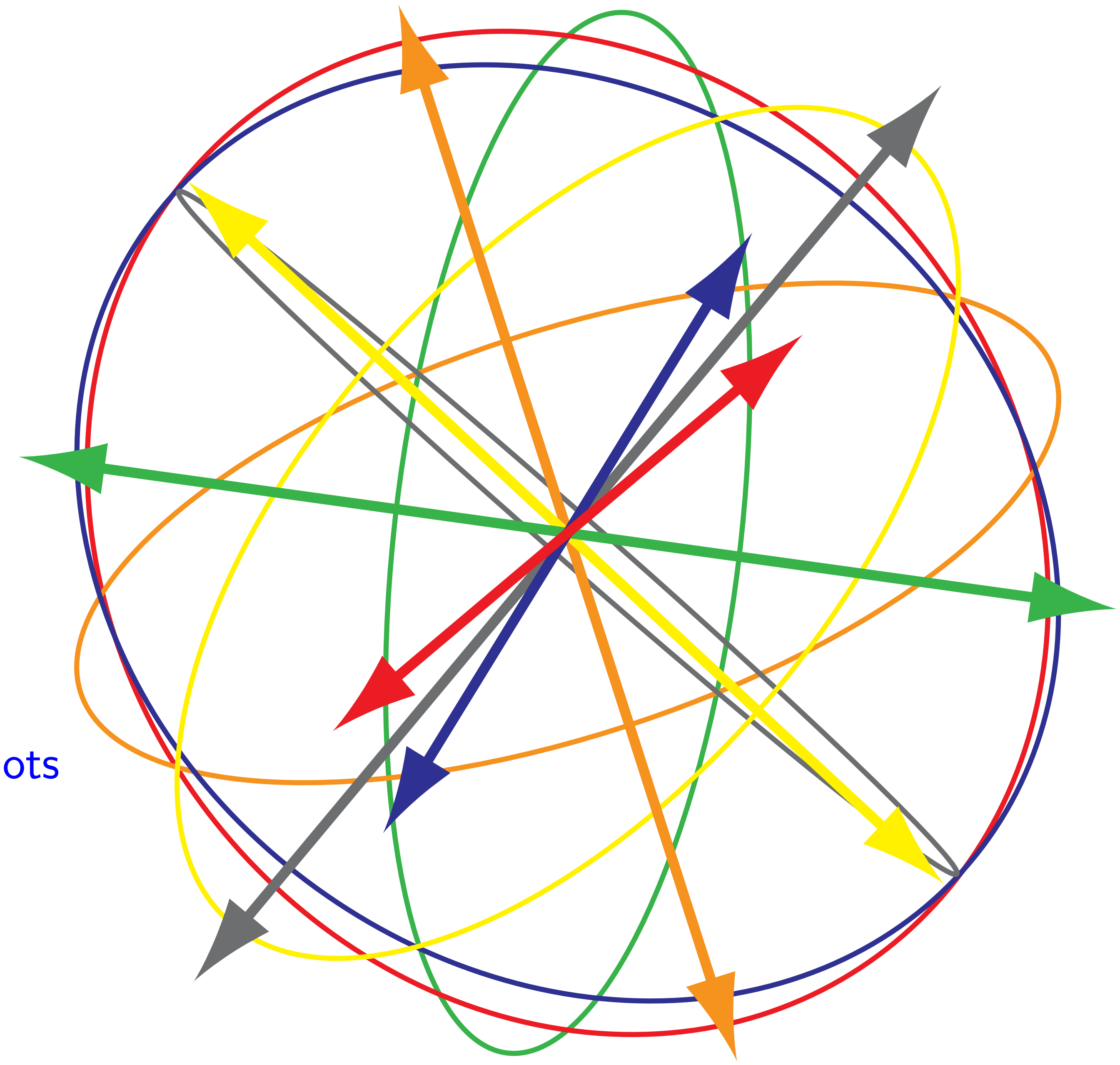
$\Phi = W(\Delta)$ = root system



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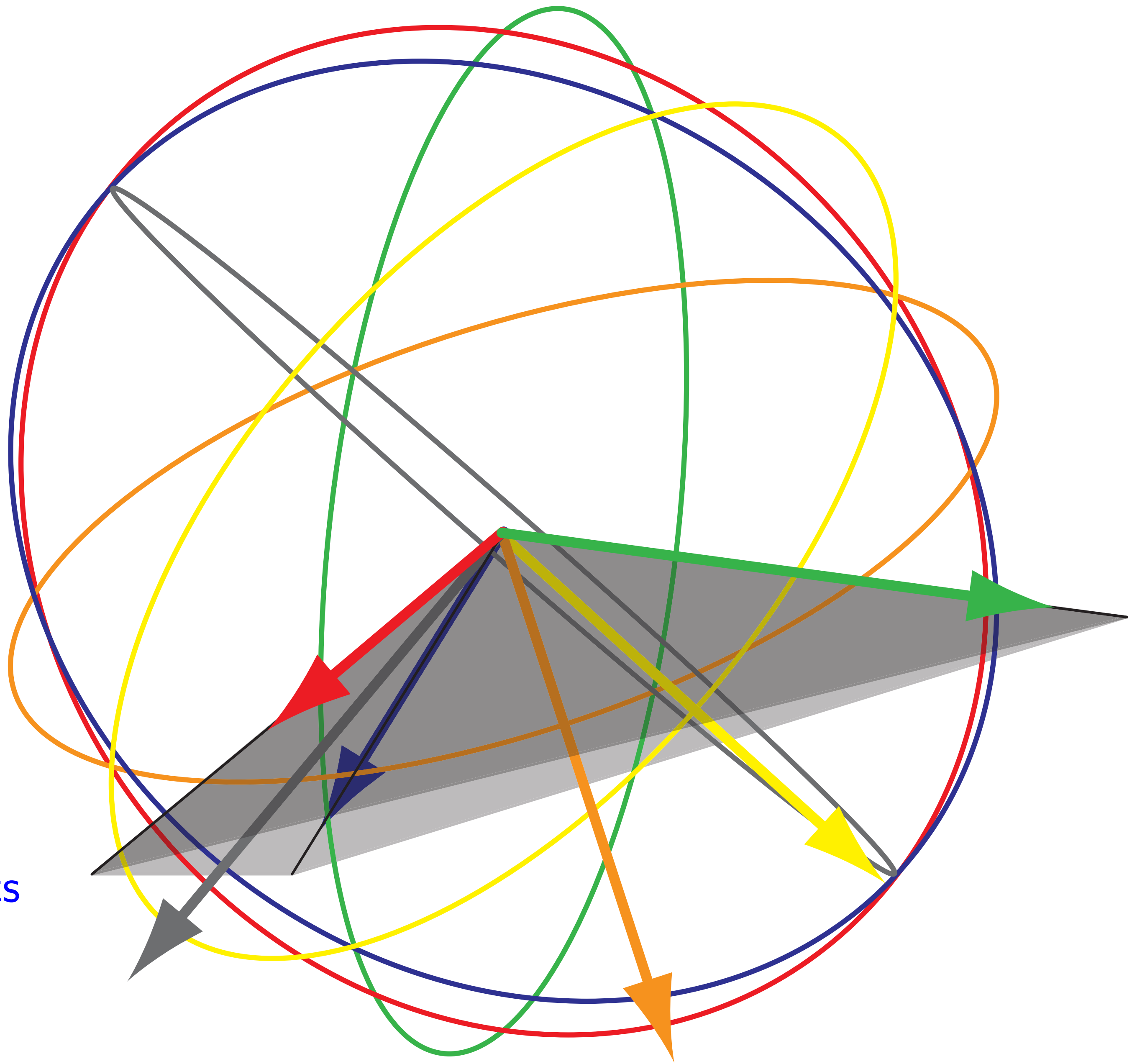
$\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta]$ = positive roots



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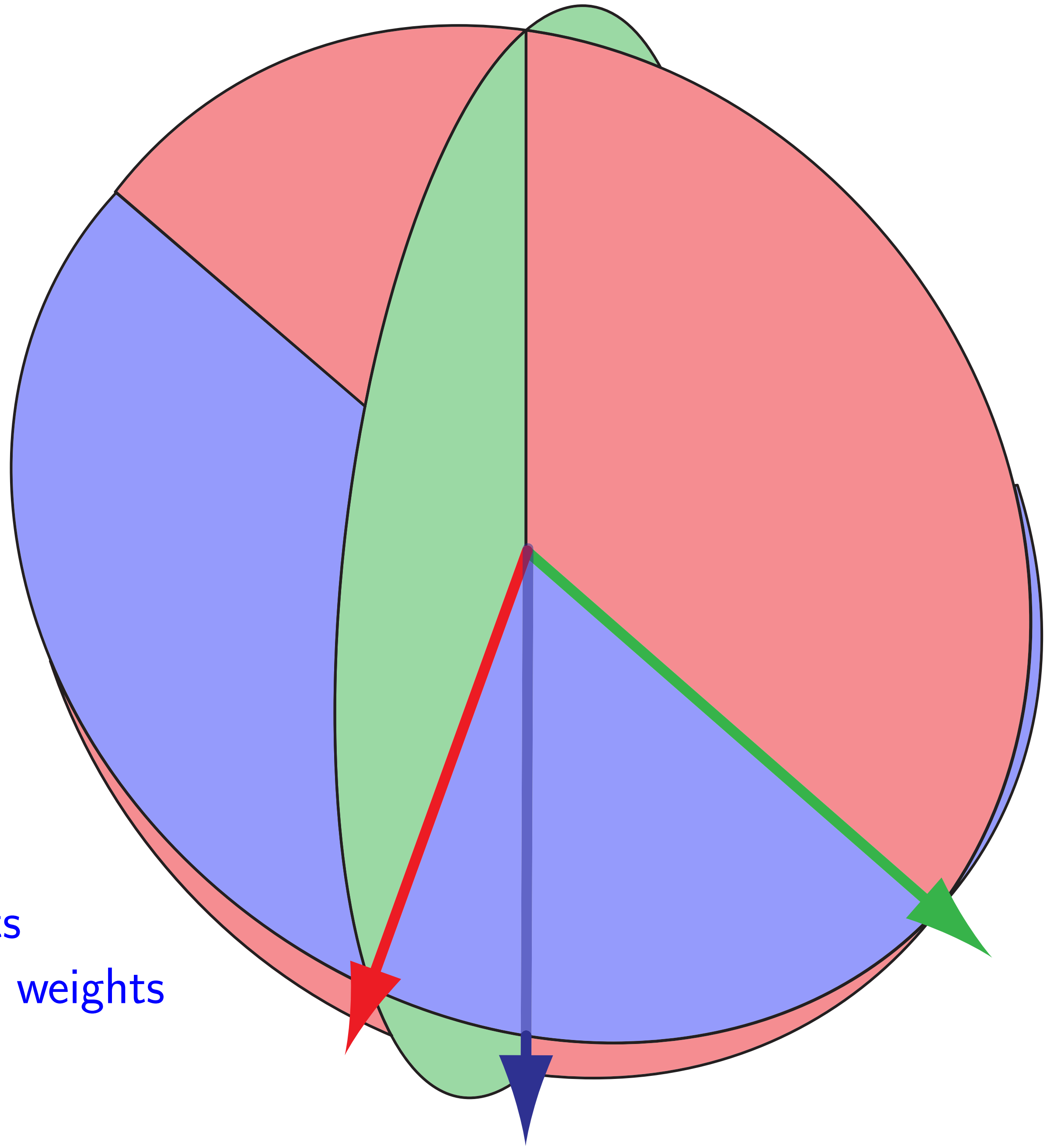

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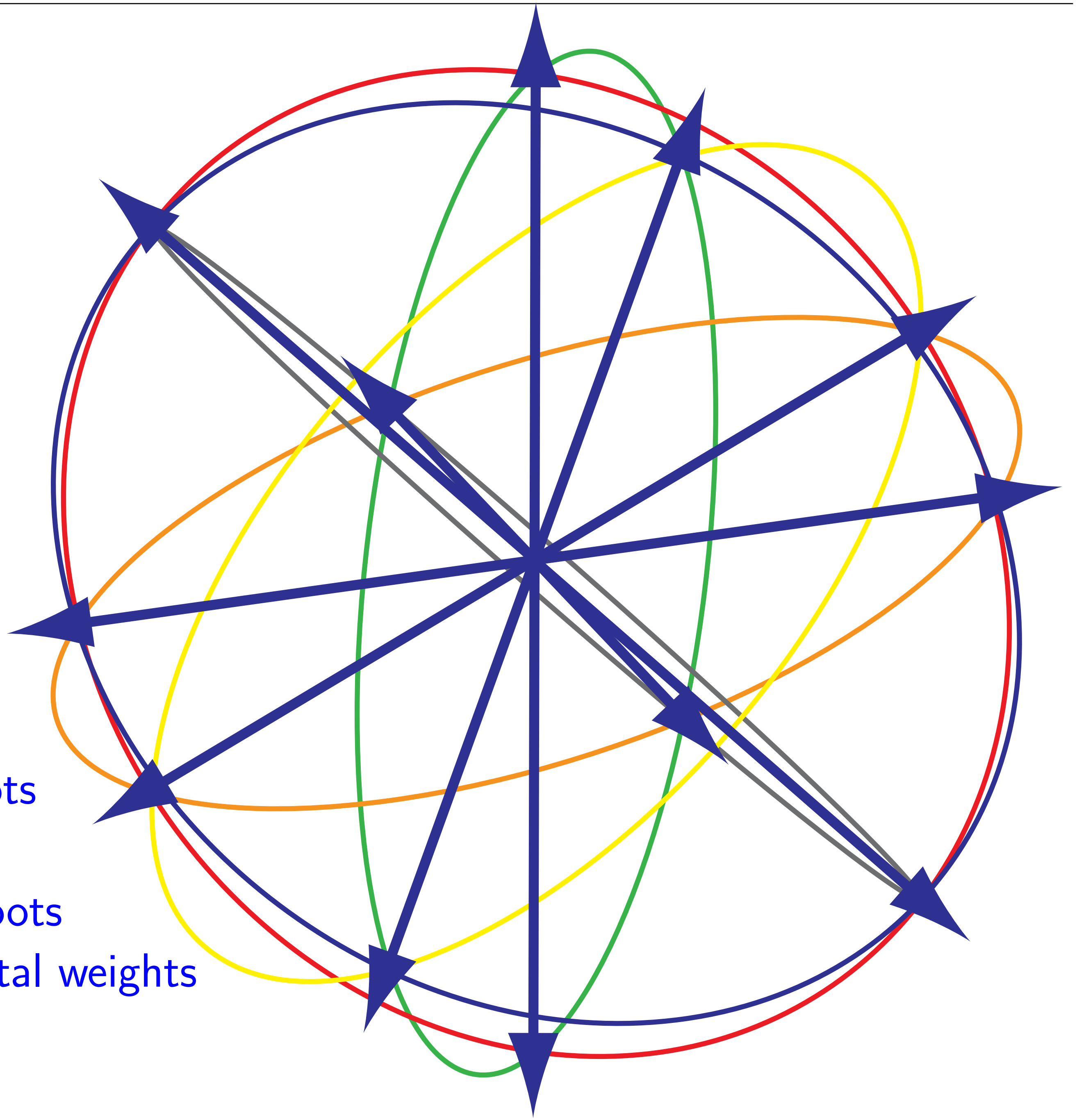

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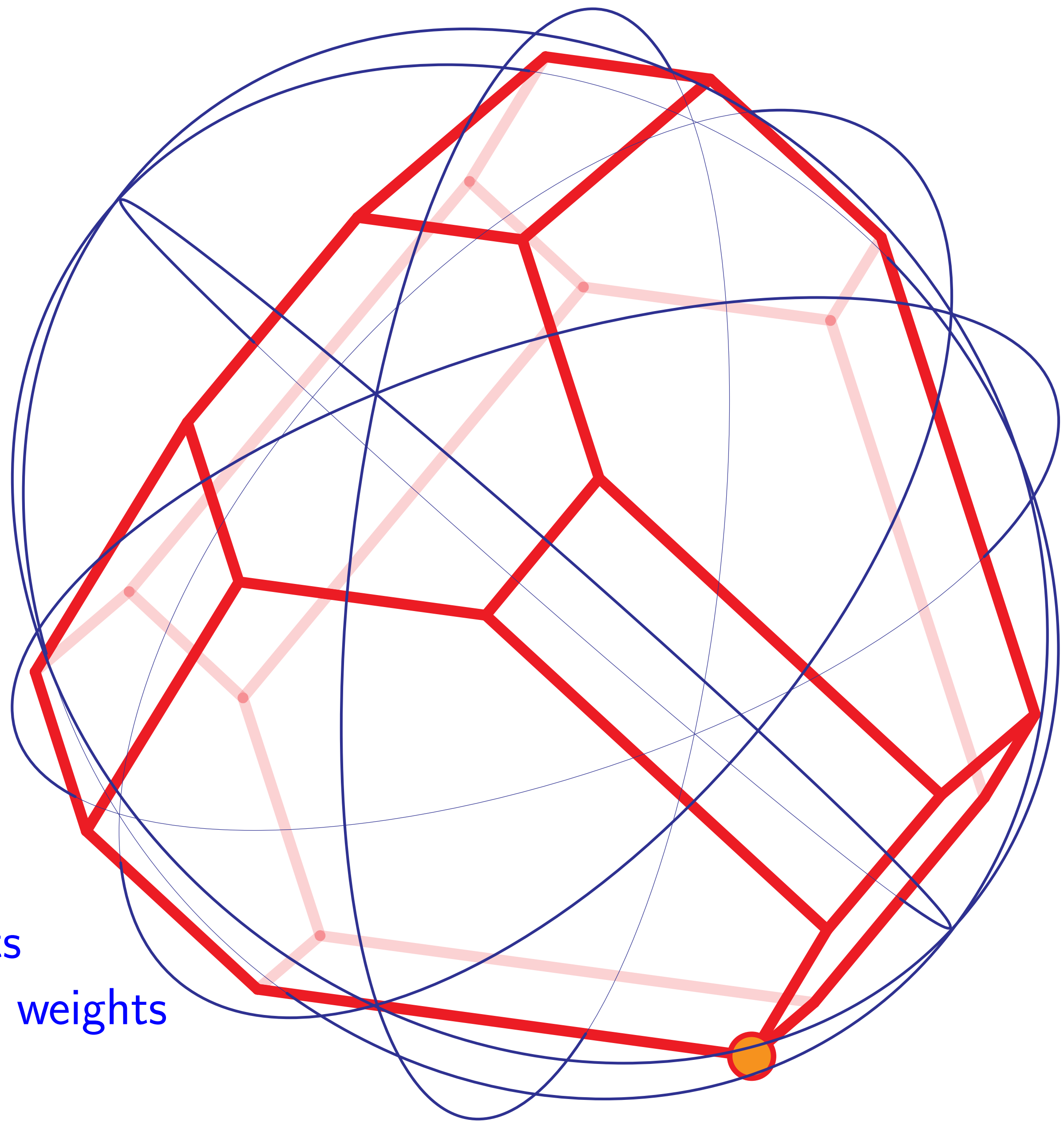
permutahedron



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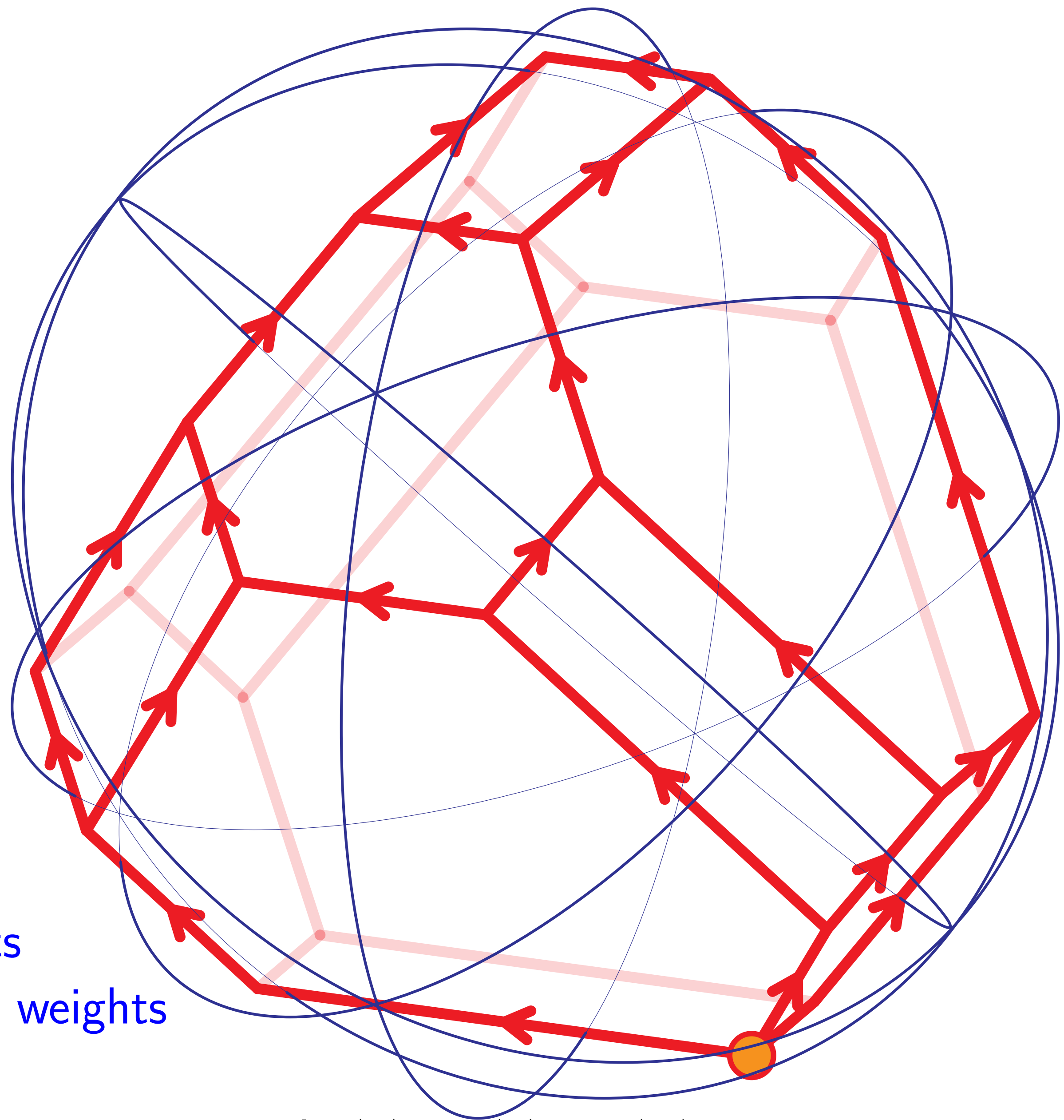

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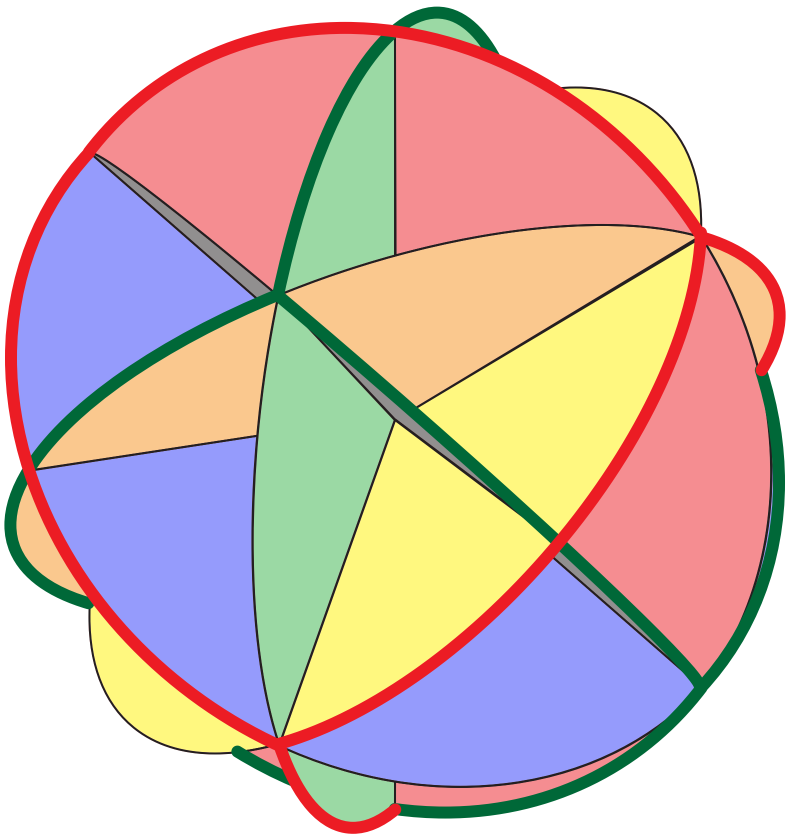
permutahedron

weak order = $u \leq w \iff \exists v \in W,\ uv = w$ and $\ell(u) + \ell(v) = \ell(w)$

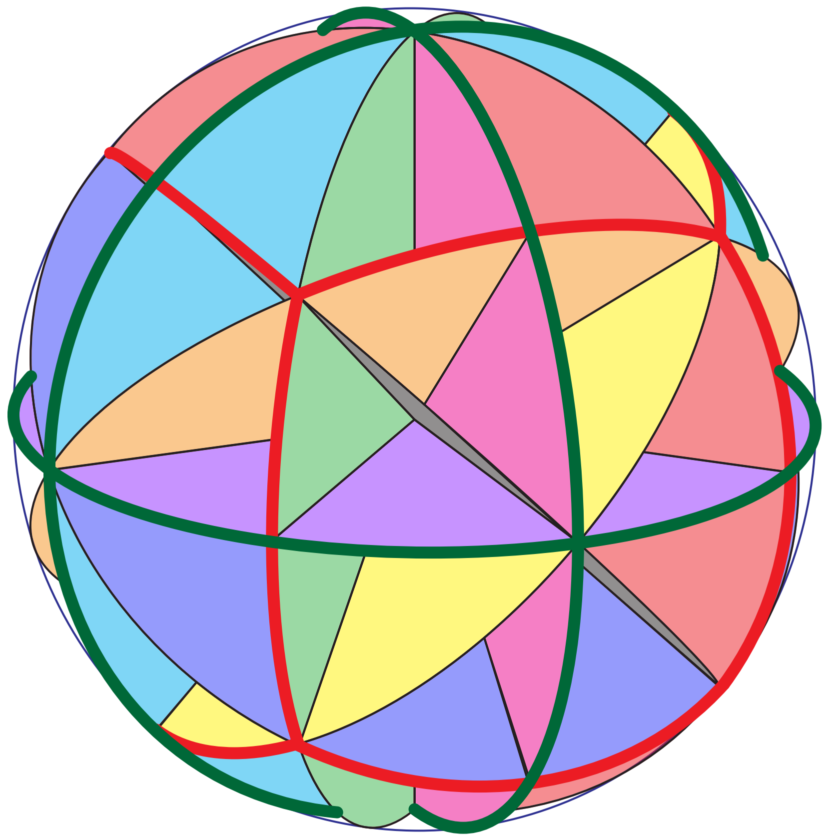
EXAMPLES: TYPE A AND B

TYPE A_n = symmetric group $\mathfrak{S}_{n+1}$



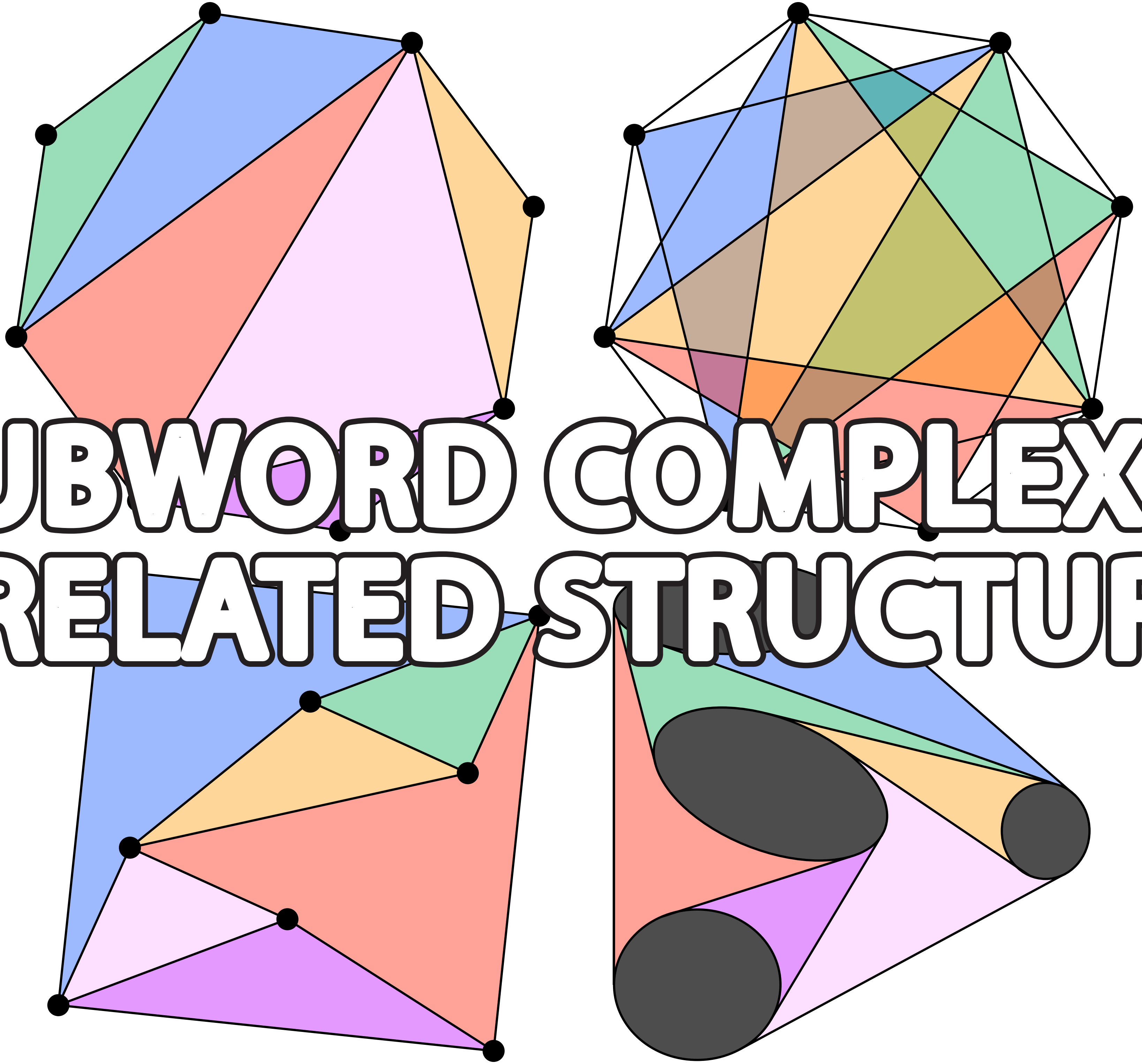
$$S = \{(i, i+1) \mid i \in [n]\}$$
$$\Delta = \{e_{i+1} - e_i \mid i \in [n]\}$$
$$\text{roots} = \{e_i - e_j \mid i, j \in [n+1]\}$$
$$\nabla = \left\{\sum_{j>i} e_j \;\middle|\; i \in [n]\right\}$$

TYPE B_n = semidirect product $\mathfrak{S}_n \rtimes (\mathbb{Z}_2)^n$



$$S = \{(i, i+1) \mid i \in [n-1]\} \cup \{\chi\}$$
$$\Delta = \{e_{i+1} - e_i \mid i \in [n-1]\} \cup \{e_1\}$$
$$\text{roots} = \{\pm e_i \pm e_j \mid i, j \in [n]\} \cup \{\pm e_i \mid i \in [n]\}$$
$$\nabla = \left\{\sum_{j \geq i} e_j \;\middle|\; i \in [n]\right\}$$

SUBWORD COMPLEXES & RELATED STRUCTURES



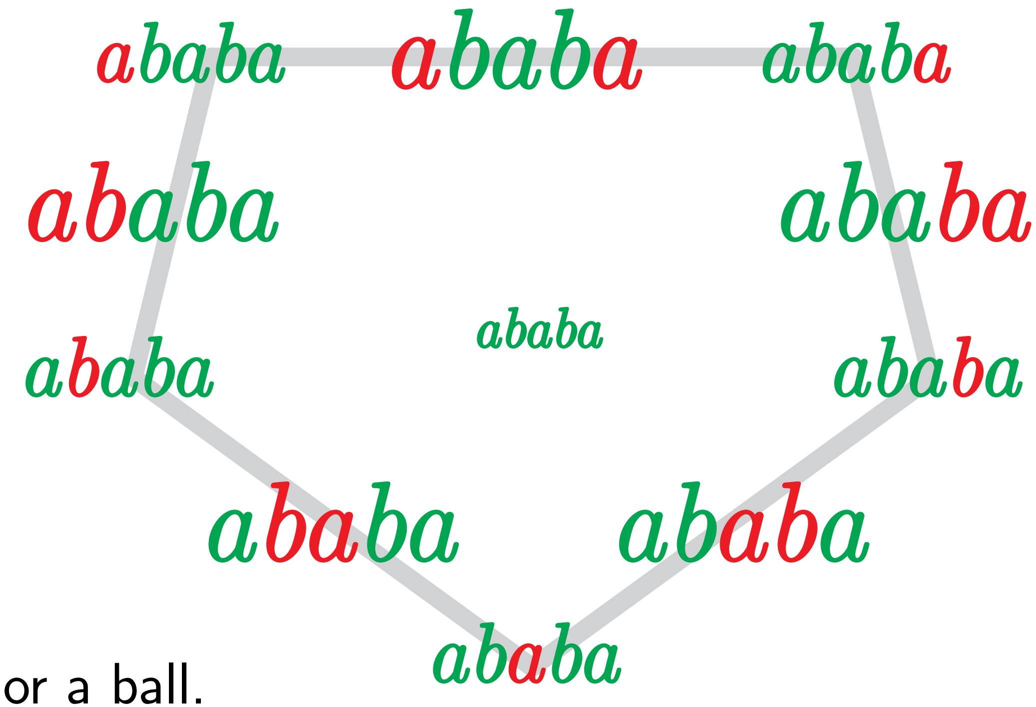
SUBWORD COMPLEX

(W, S) a finite Coxeter system, $\mathrm{Q} = q_1 q_2 \cdots q_m$ a word on S, ρ an element of W.

Subword complex $\mathcal{S}(\mathrm{Q}, \rho)$ = simplicial complex of subsets of positions of Q whose complement contains a reduced expression of ρ.

A. Knutson & E. Miller, Subword complexes in Coxeter groups, 2004.

$W = \mathfrak{S}_3$
$S = \{(1\ 2), (2\ 3)\} = \{a, b\}$
$\mathrm{Q} = ababa$
$\rho = aba = bab$



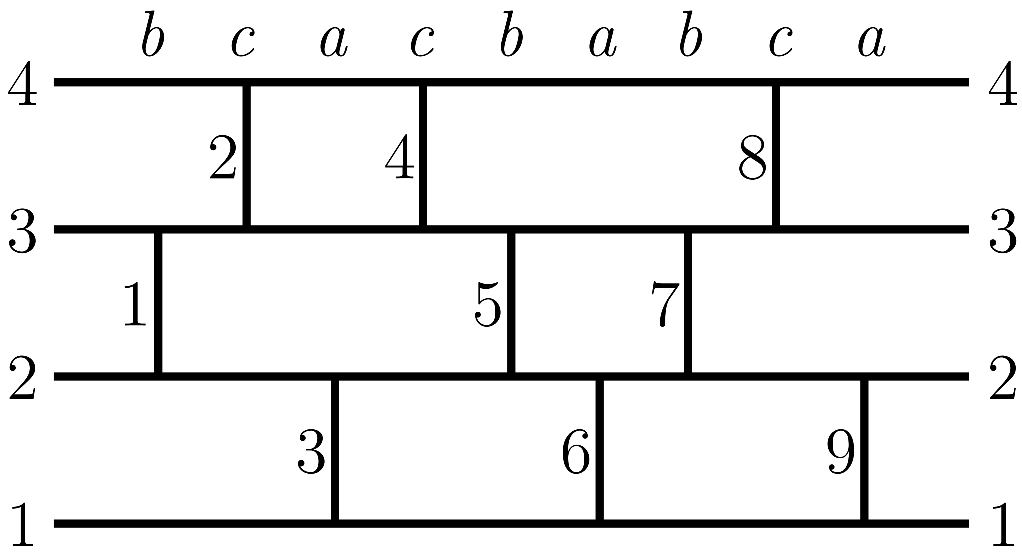
The subword complex is either a sphere (when the Demazure product of Q is ρ) or a ball.

QUESTION. Are all spherical subword complexes polytopal?

TYPE A: PRIMITIVE SORTING NETWORKS

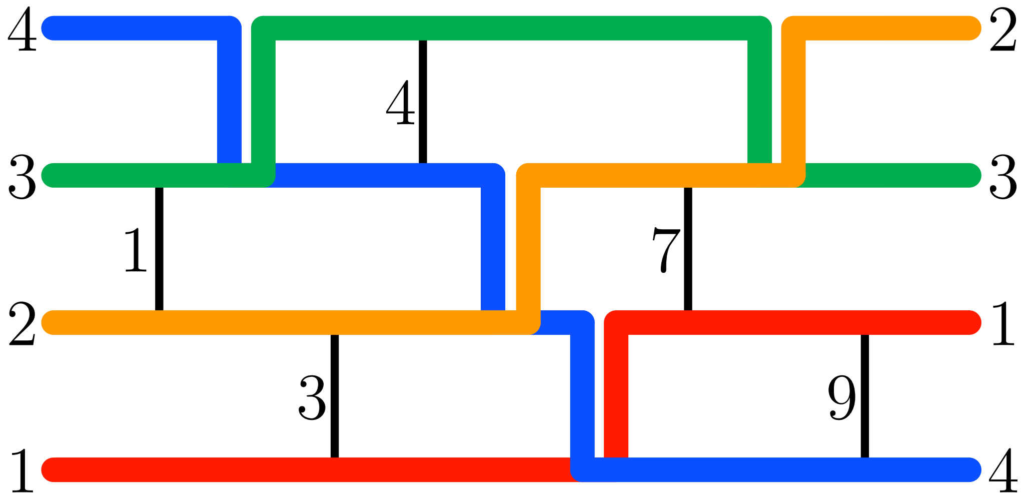
Classical situation of type A:

- Coxeter group $W = \mathfrak{S}_{n+1}$
- simple system $S = \{\tau_i \mid i \in [n]\}$, where $\tau_i = (i\ \ i+1)$
- word $\mathrm{Q} = q_1 q_2 \cdots q_m$ on S
- w element of W



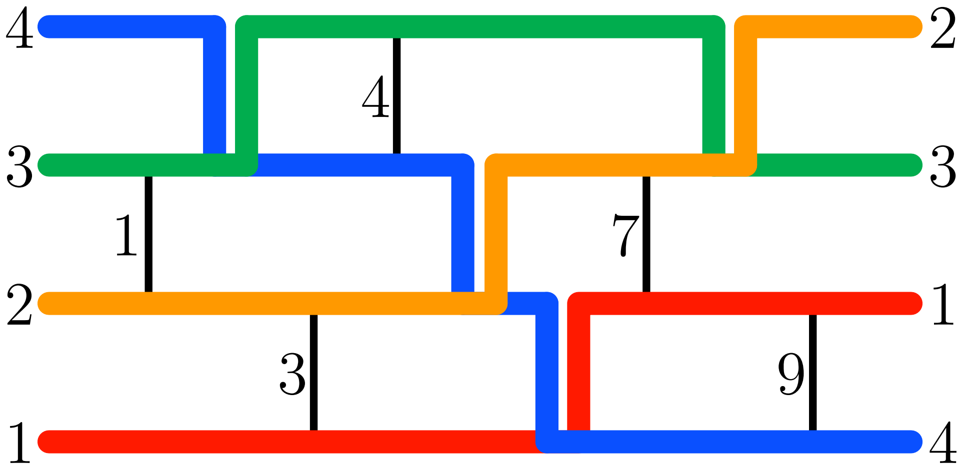
The subword complex can be interpreted with a primitive sorting network:

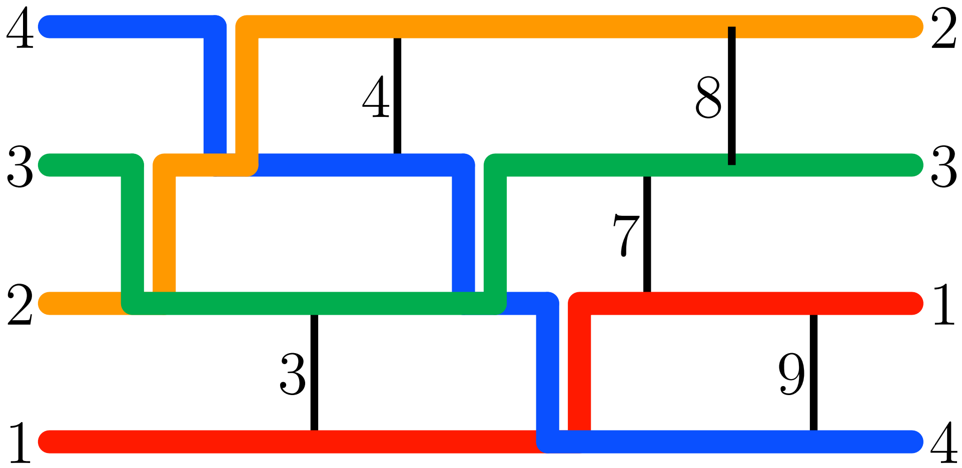
- $\mathcal{N}_{\mathrm{Q}}$ formed by $n+1$ levels and m commutators
- facets of $\mathcal{S}(\mathrm{Q}, w)$ $\longleftrightarrow$ pseudoline arrangements on $\mathcal{N}_{\mathrm{Q}}$



FLIPS

flip $=$ exchange a contact with the corresponding crossing





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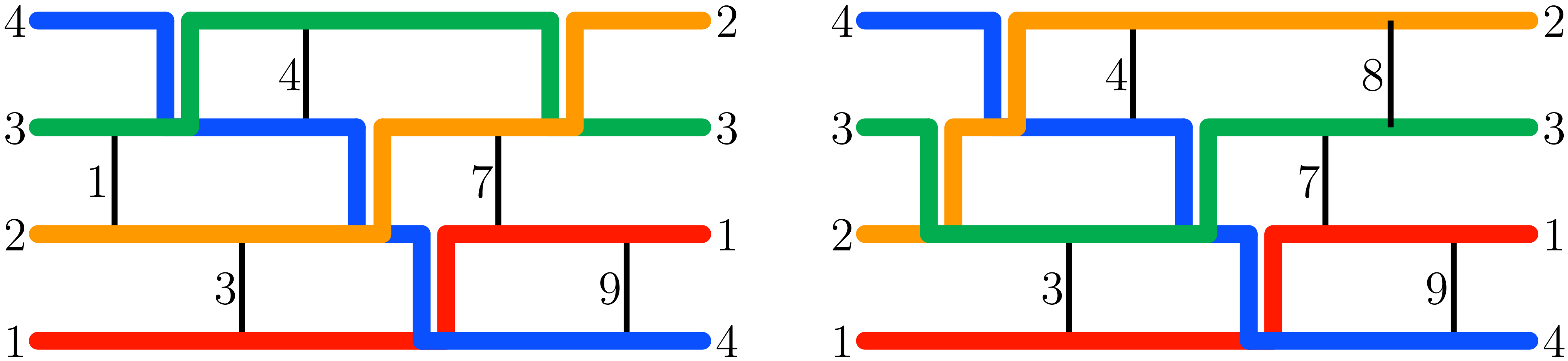
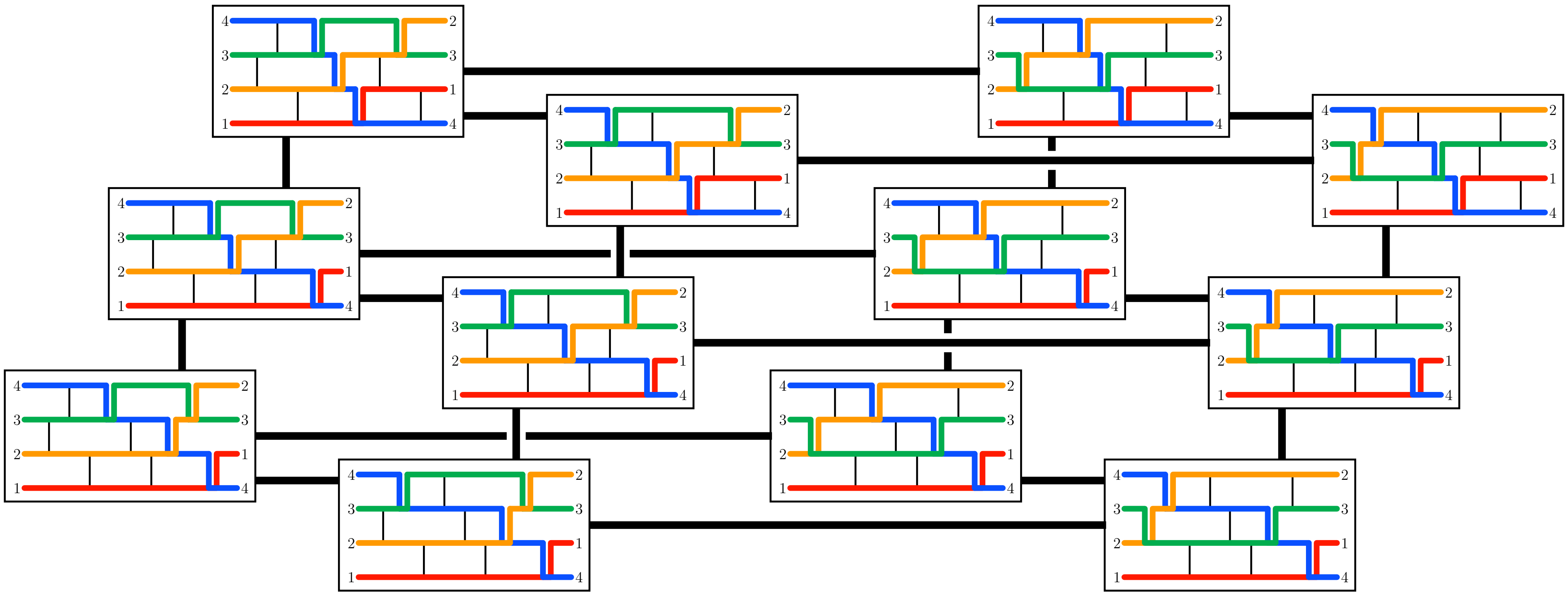
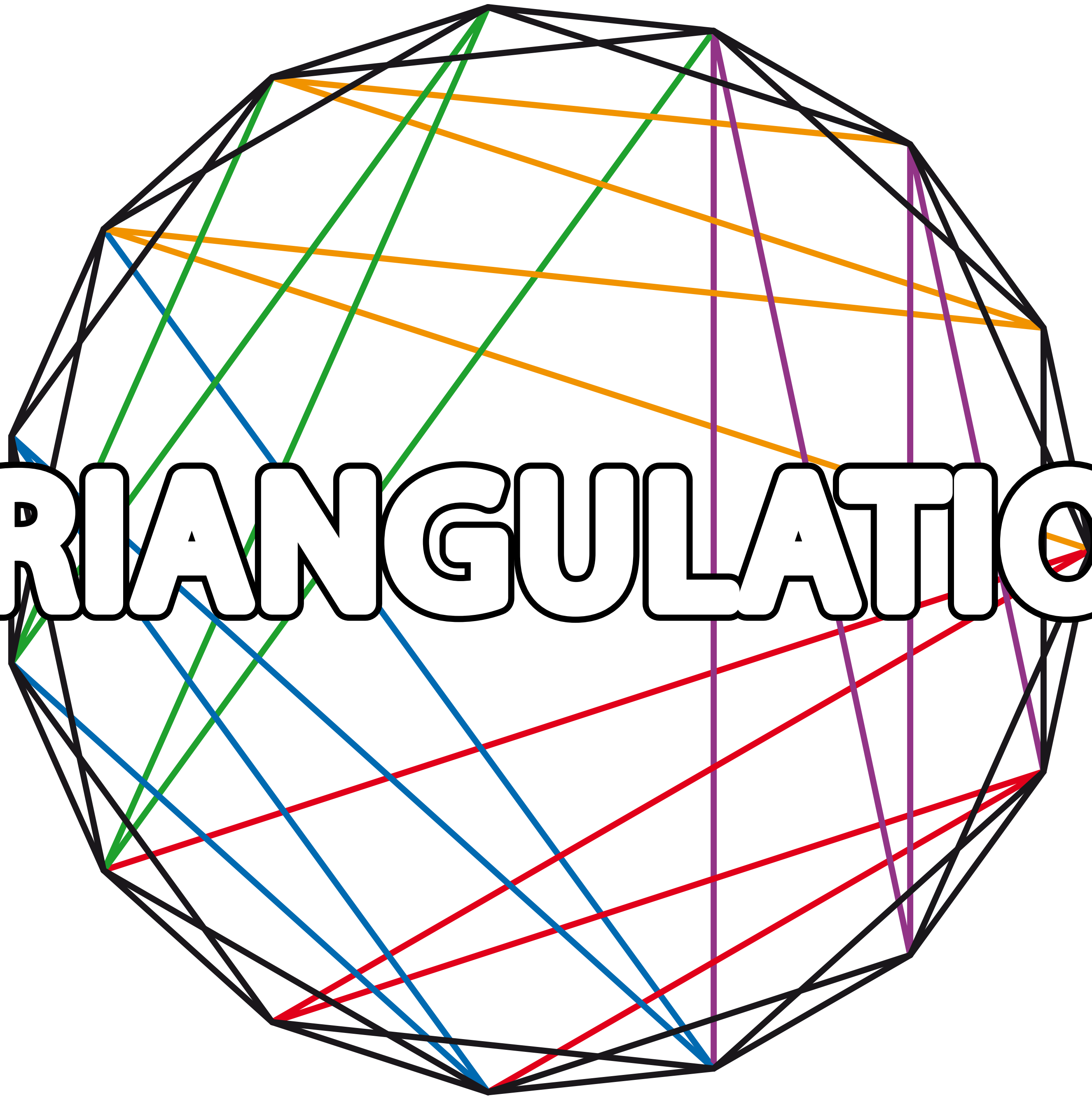


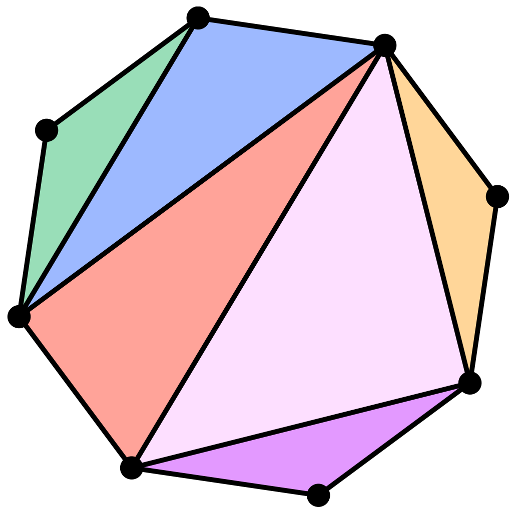


...TRIANGULATIONS

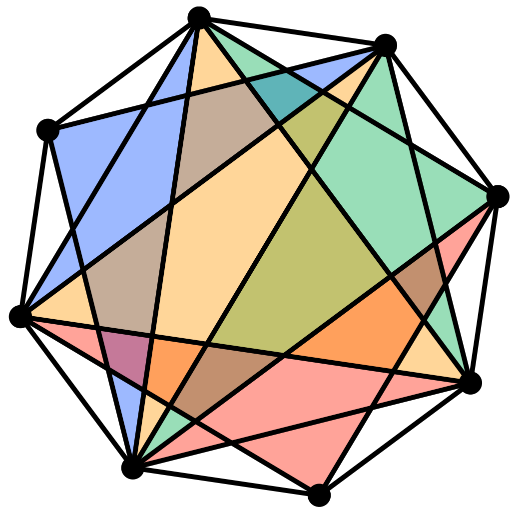


COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS

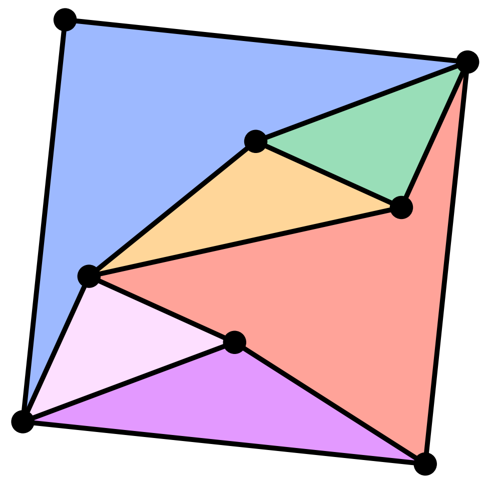
Type A subword complexes give combinatorial models for:



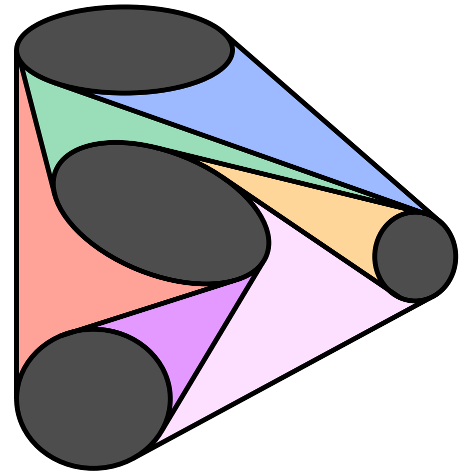
triangulations
of convex polygons



multitriangulations
of convex polygons



pseudotriangulations
of point sets in
general position



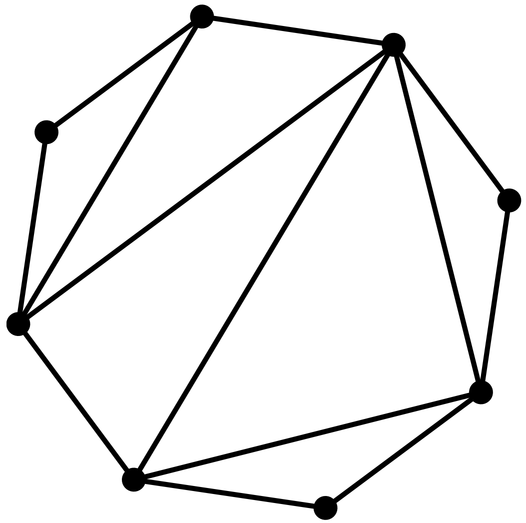
pseudotriangulations
of sets of disjoint
convex bodies

VP & M. Pocchiola, Pseudotriangulations, multitriangulations, and primitive sorting networks, 2012.

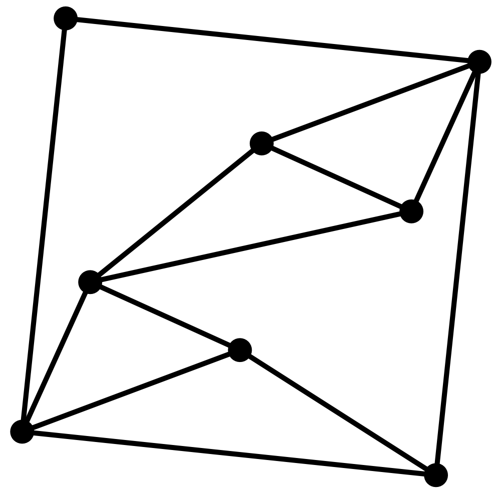
C. Stump, A new perspective on multitriangulations, 2011.

THREE GEOMETRIC STRUCTURES

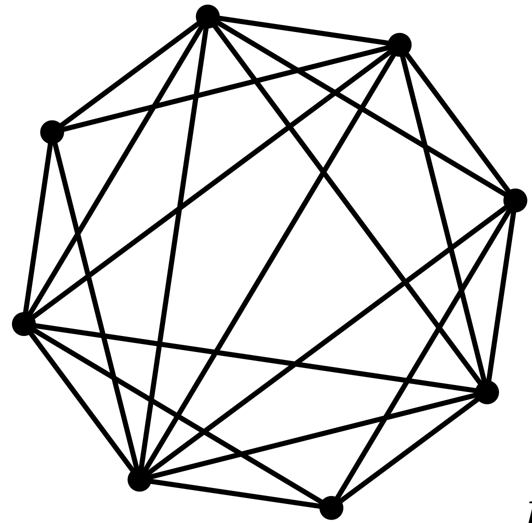
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

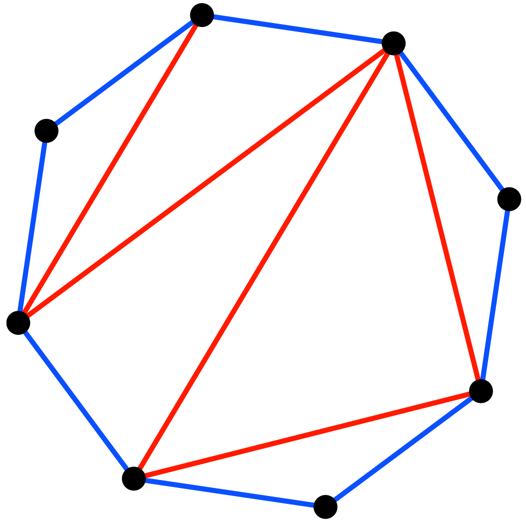
triangulation $=$ maximal crossing-free set of edges

pseudotriangulation $=$ maximal crossing-free pointed set of edges

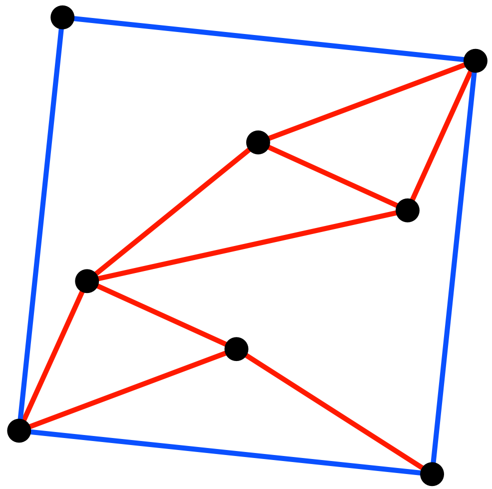
k-triangulation $=$ maximal $(k+1)$-crossing-free set of edges

THREE GEOMETRIC STRUCTURES

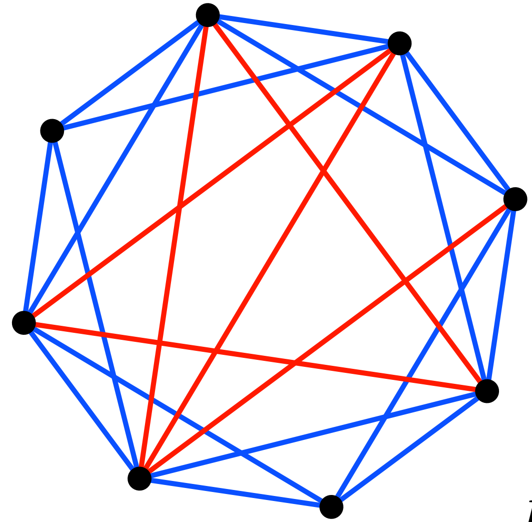
Triangulations



Pseudotriangulations



Multitriangulations



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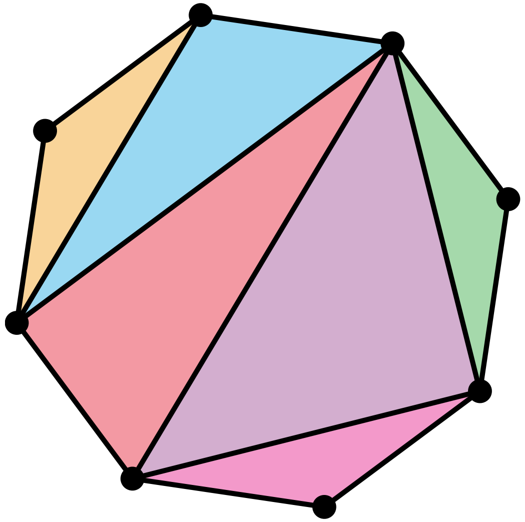
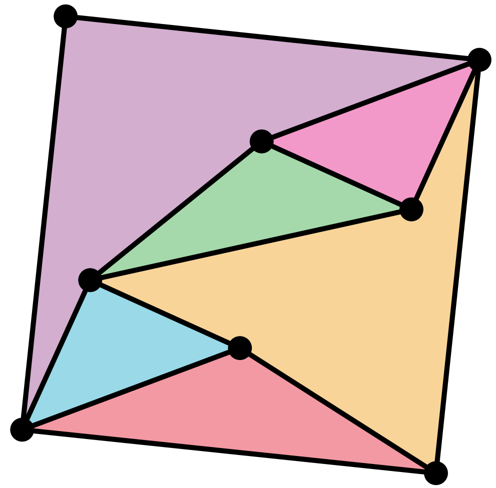
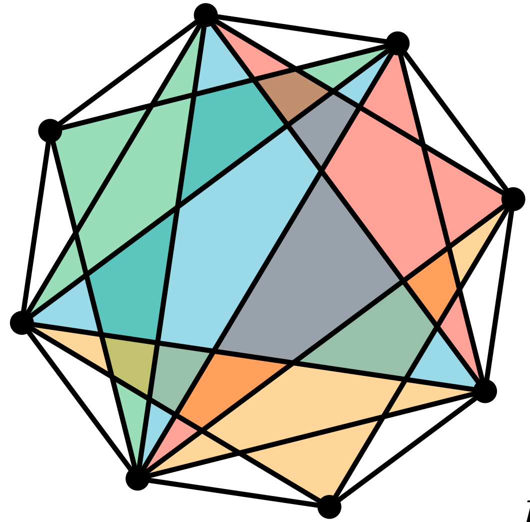


Pseudotriangulations



Multitriangulations



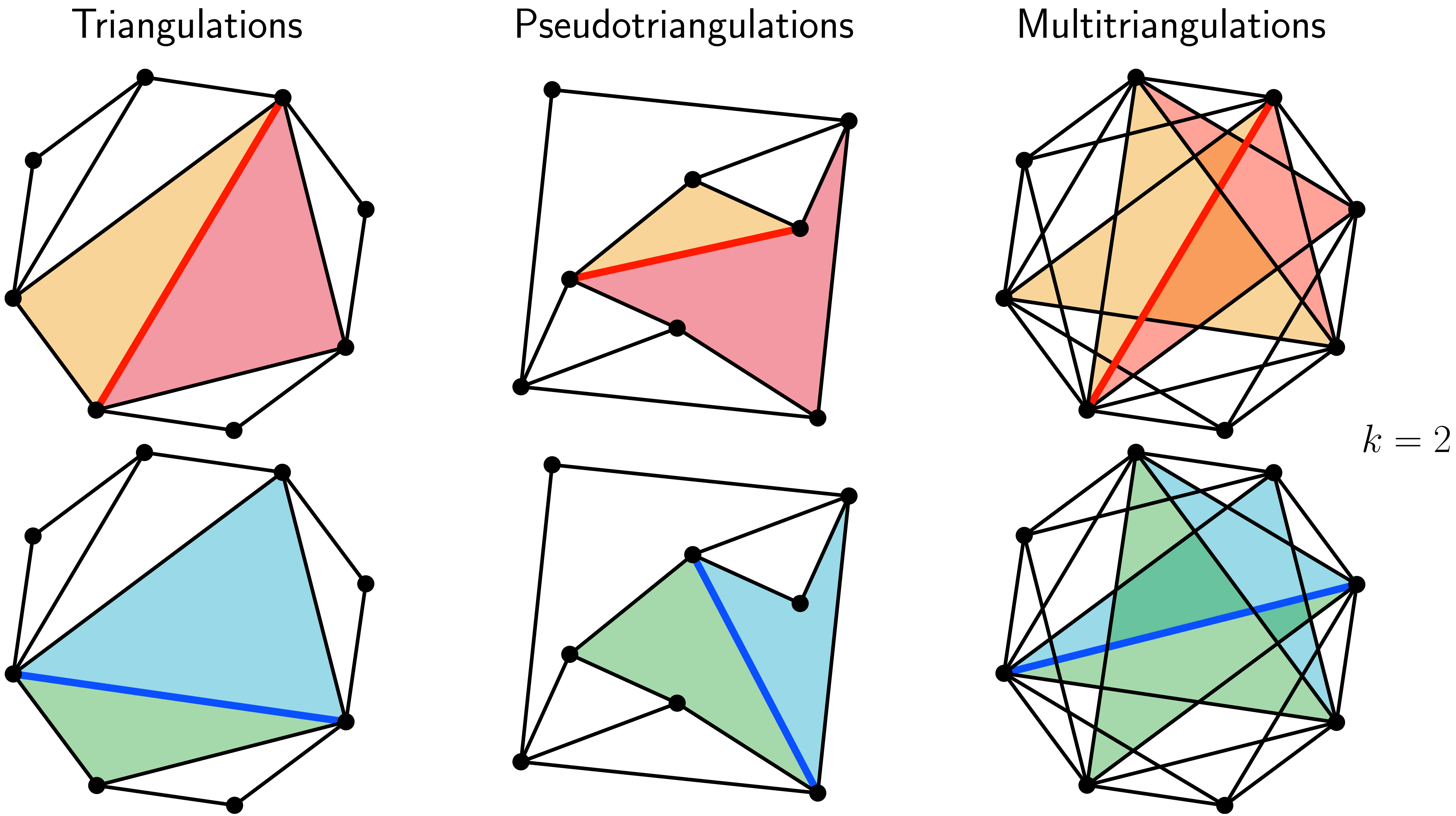
triangulation = maximal crossing-free set of edges
= decomposition into triangles

pseudotriangulation = maximal crossing-free pointed set of edges
= decomposition into pseudotriangles

k-triangulation = maximal $(k+1)$-crossing-free set of edges
= decomposition into k-stars

VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

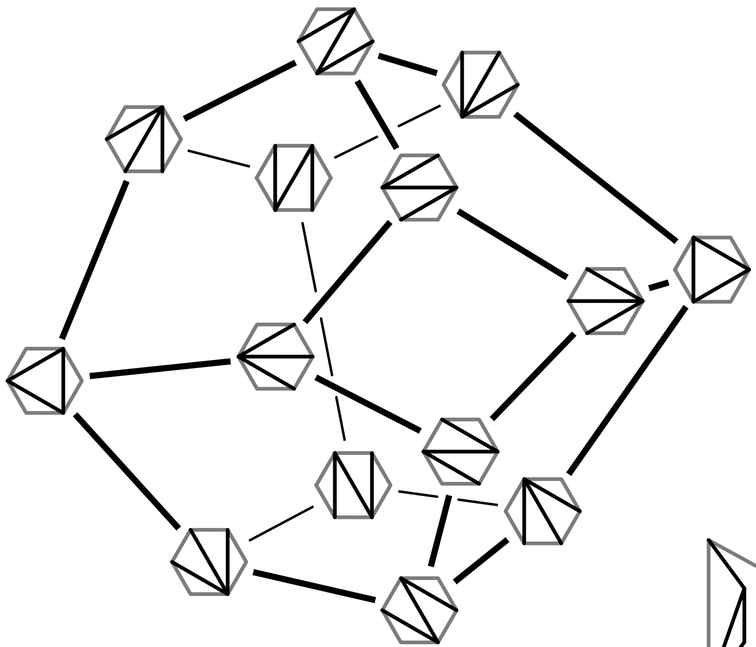
THREE GEOMETRIC STRUCTURES

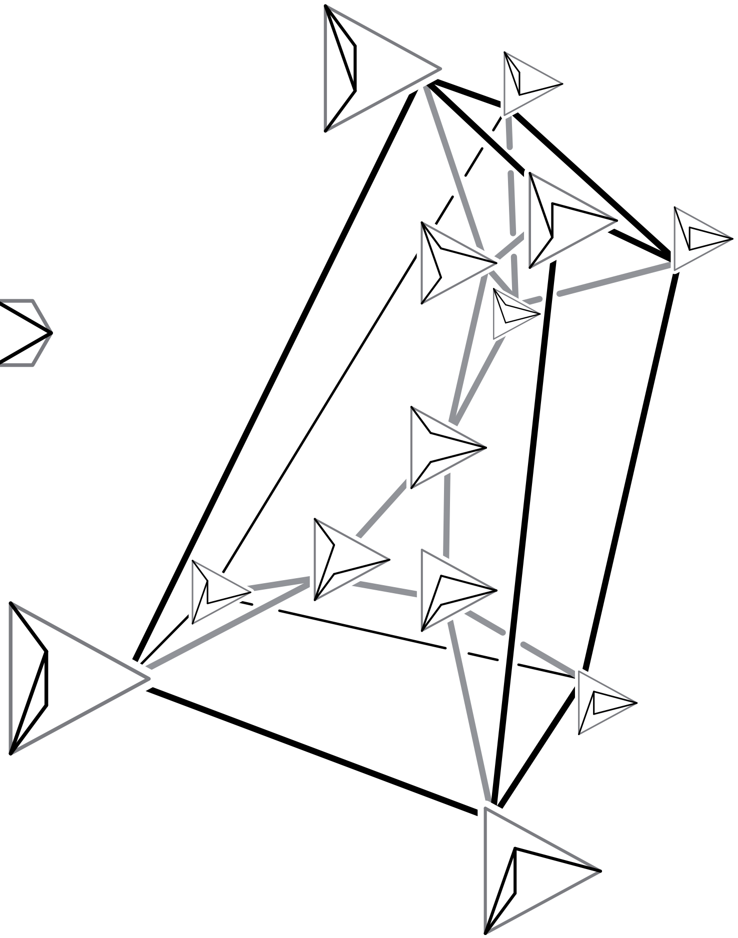


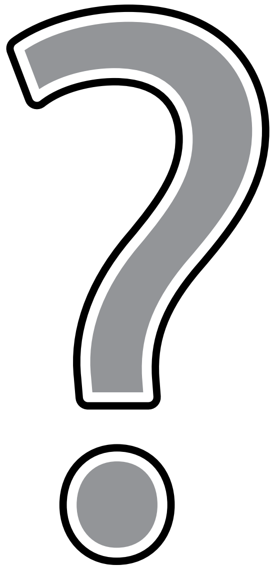
flip $=$ exchange an internal edge with the common bisector of the two adjacent cells

THREE GEOMETRIC STRUCTURES

Triangulations | Pseudotriangulations | Multitriangulations







associahedron $\longleftrightarrow$ crossing-free sets of internal edges

pseudotriangulations polytope $\longleftrightarrow$ pointed crossing-free sets of internal edges

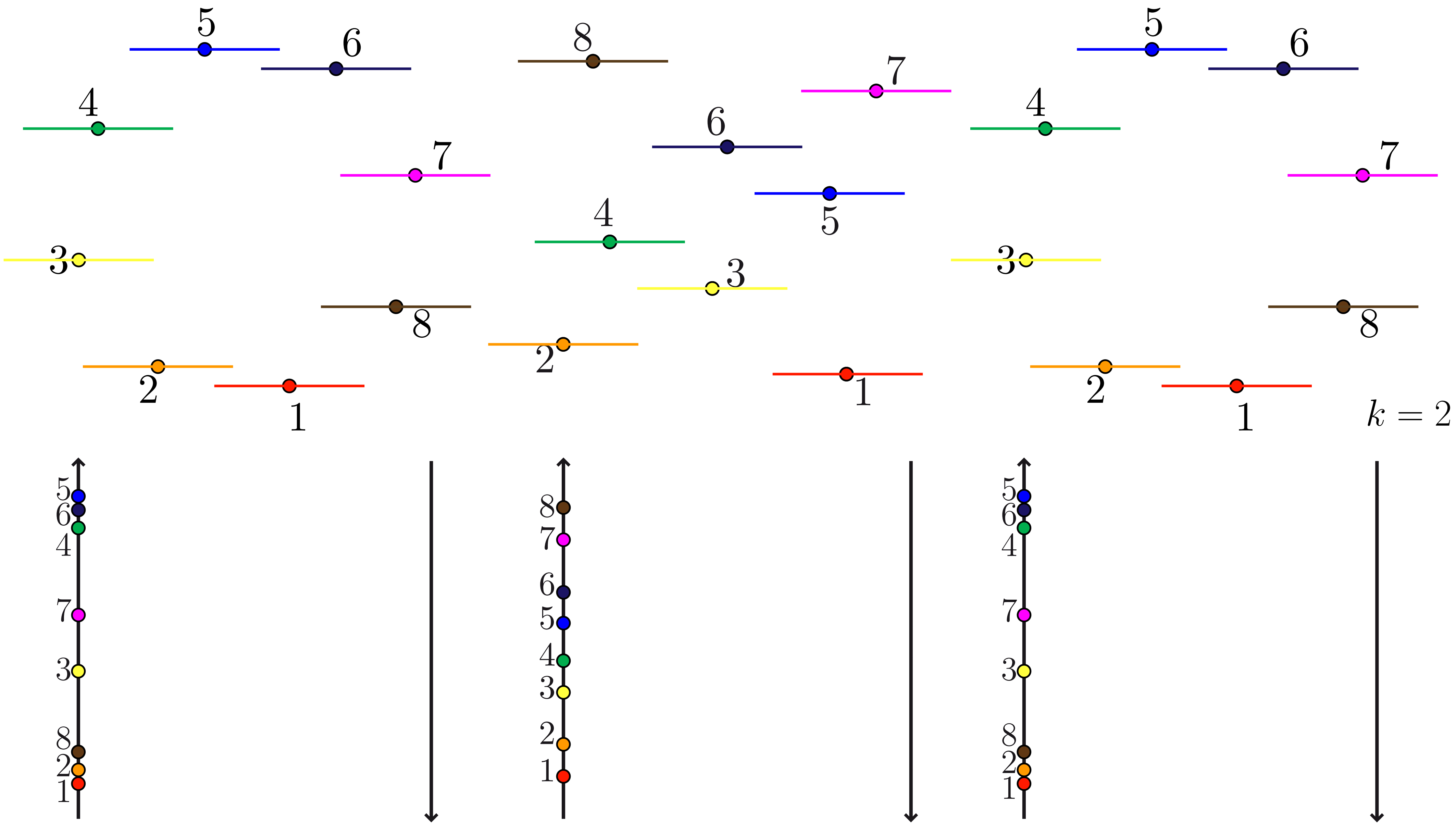
multiassociahedron $\longleftrightarrow$ $(k+1)$-crossing-free sets of k-internal edges

DUALITY

VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.

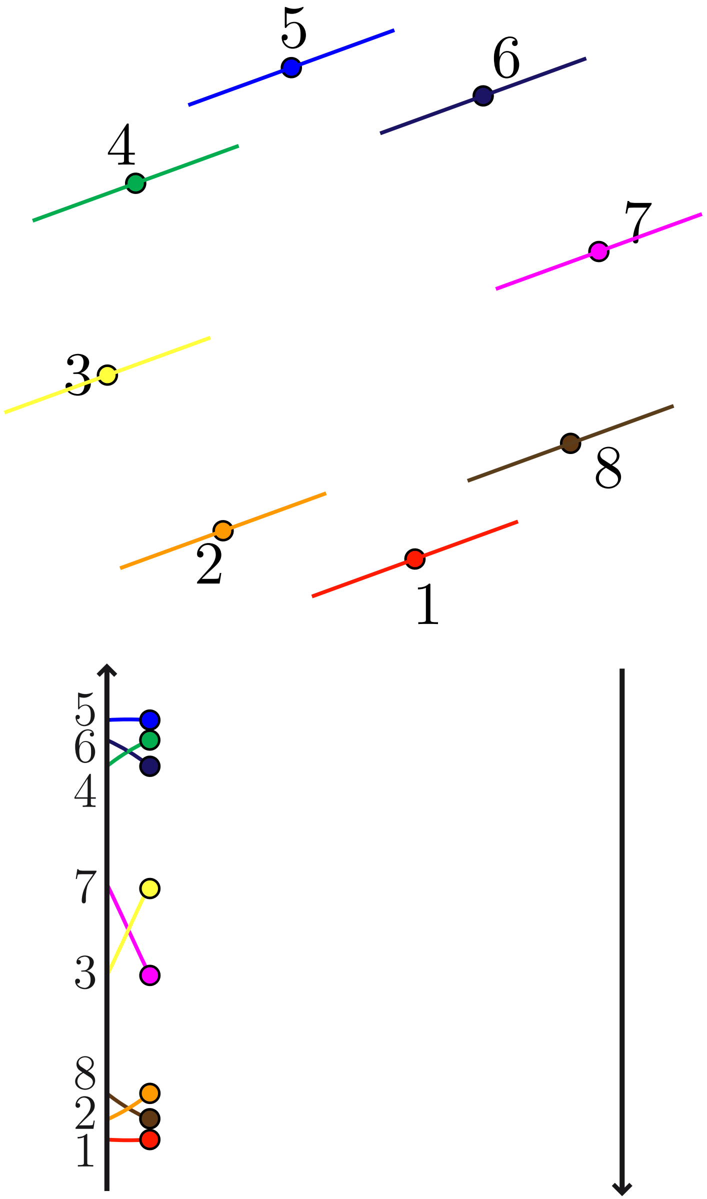
DUALITY

Triangulations | Pseudotriangulations | Multitriangulations

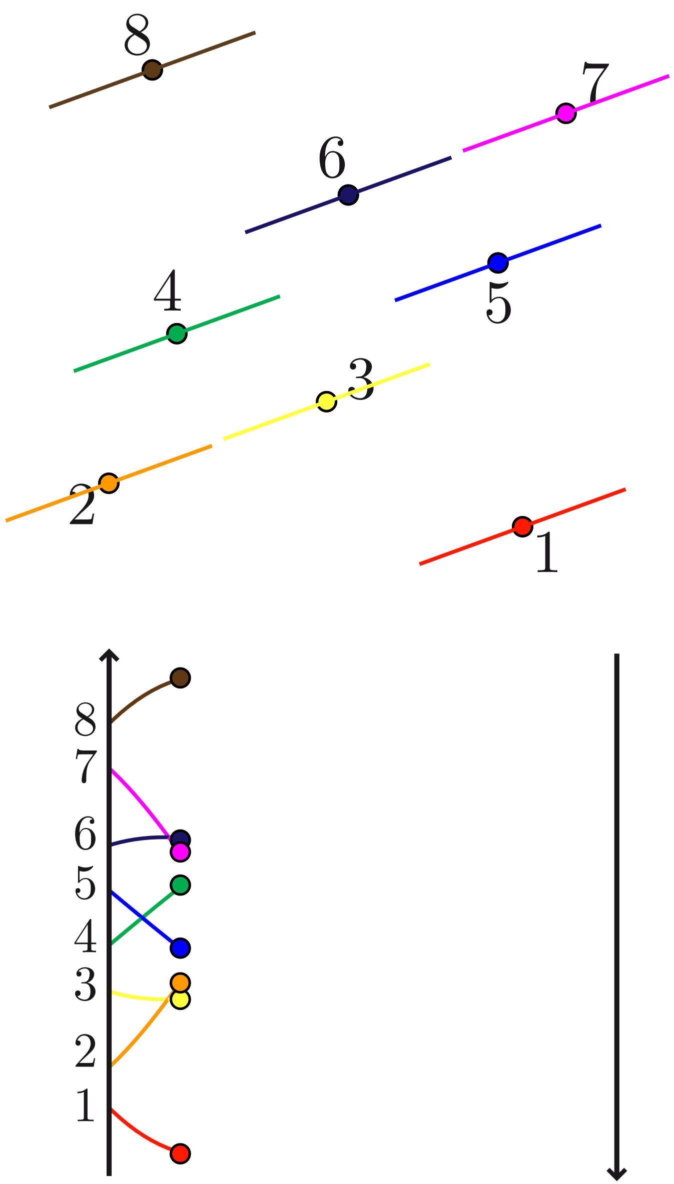


DUALITY

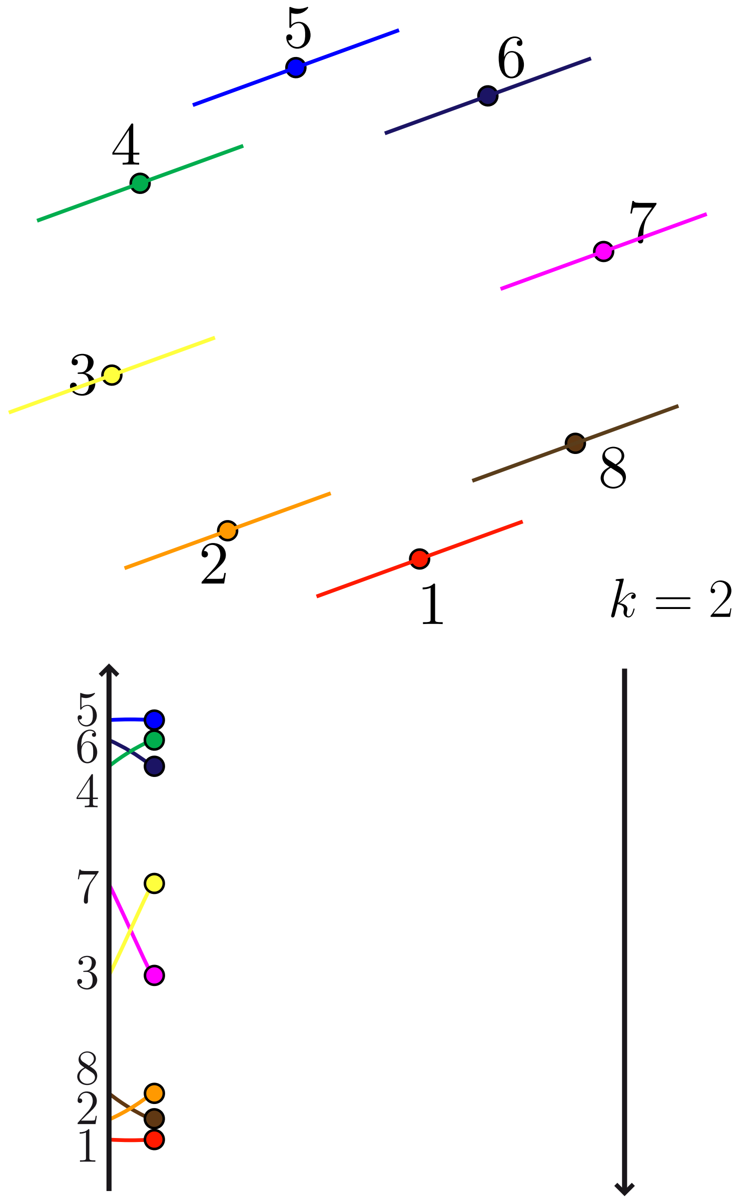
Triangulations



Pseudotriangulations

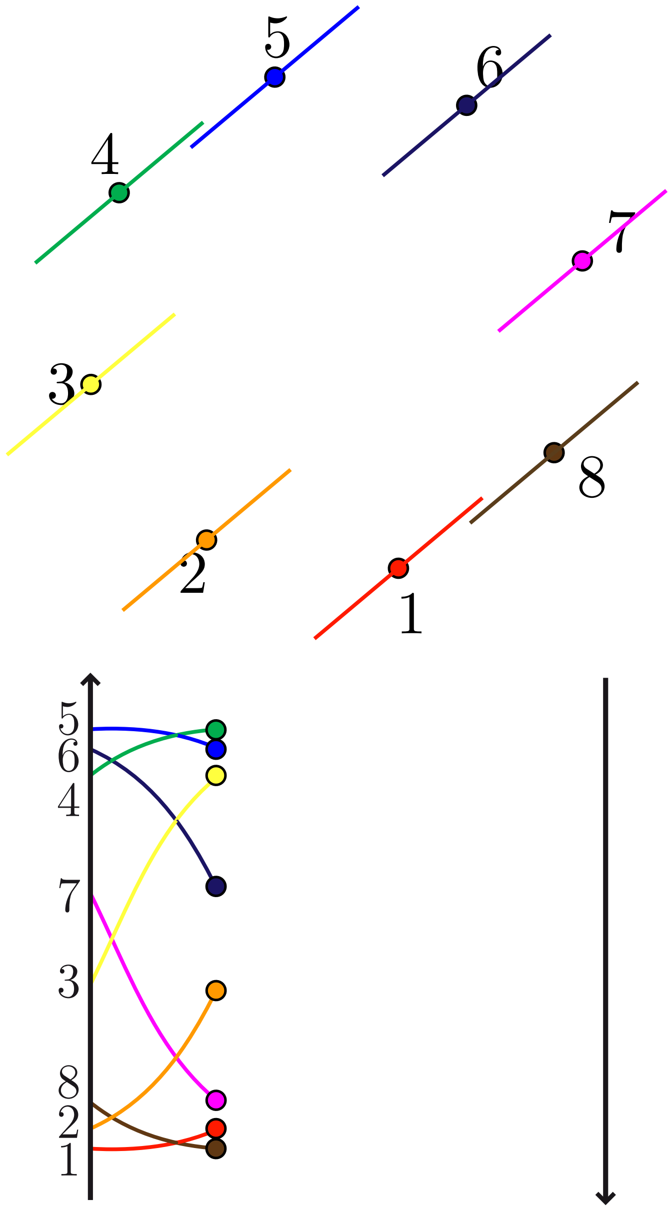


Multitriangulations

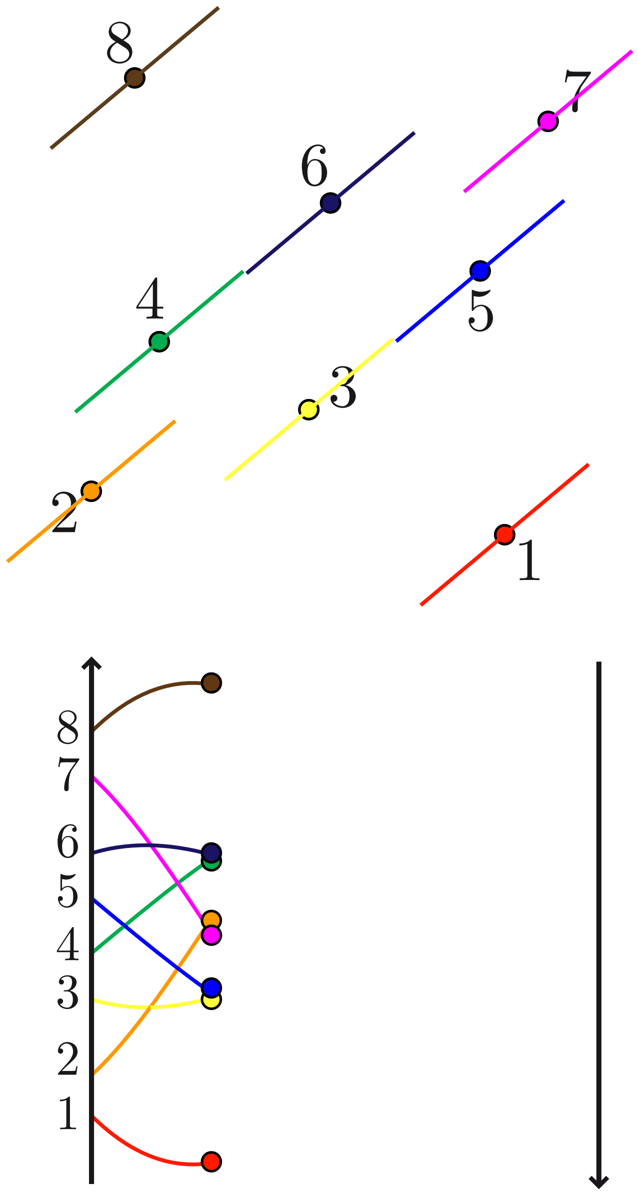


DUALITY

Triangulations



Pseudotriangulations



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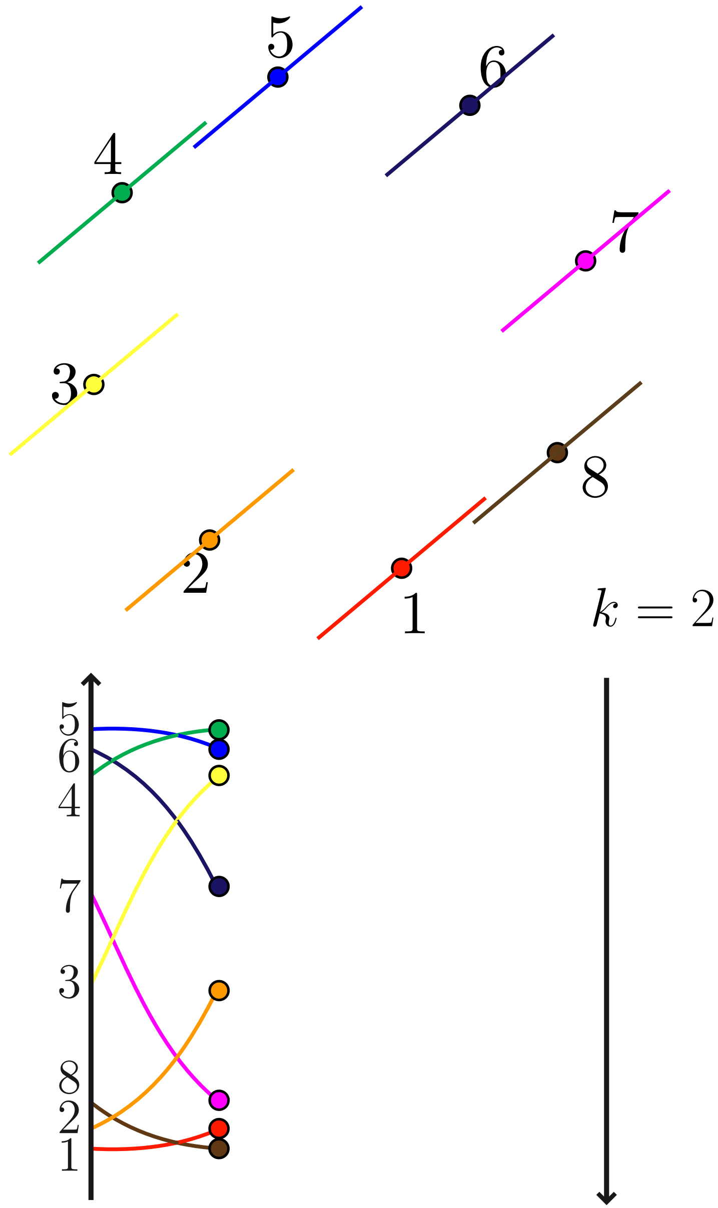

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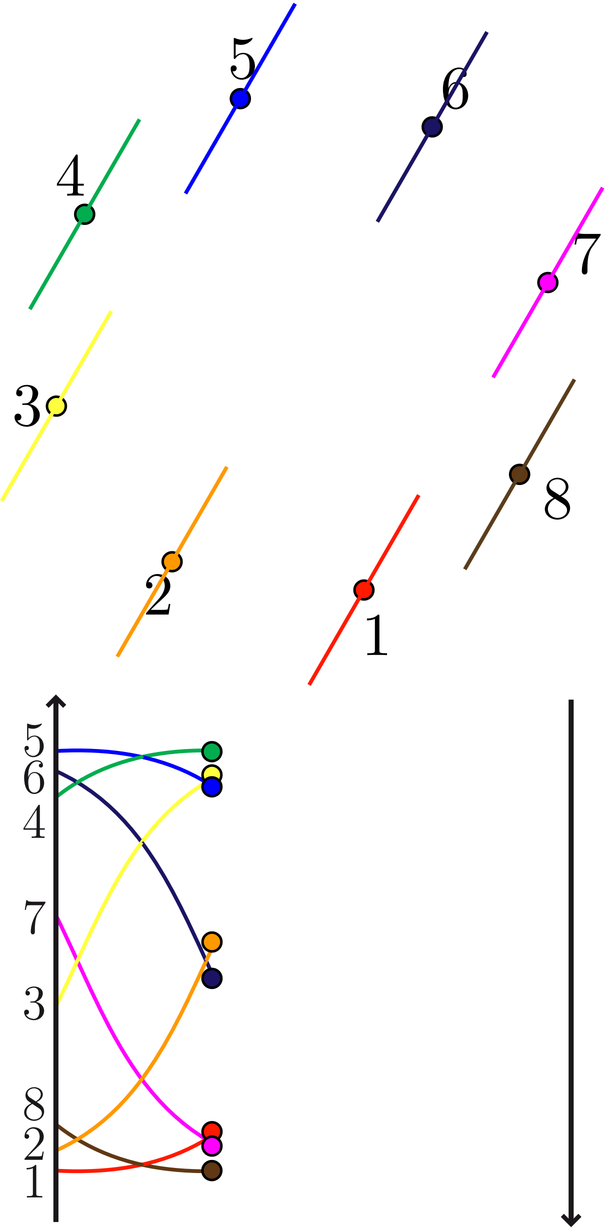

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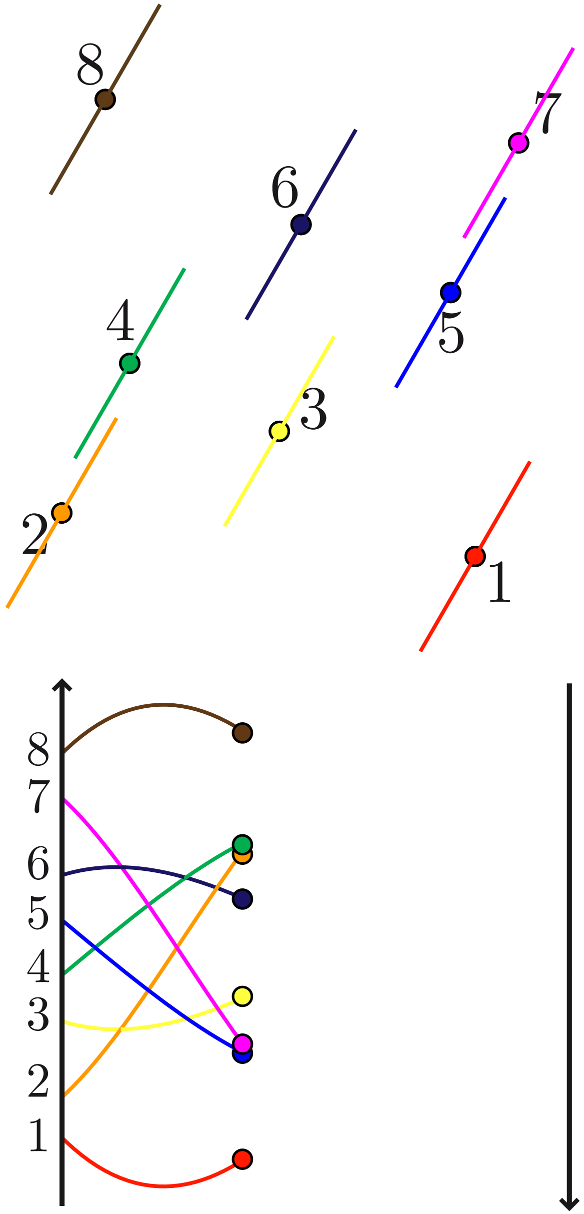

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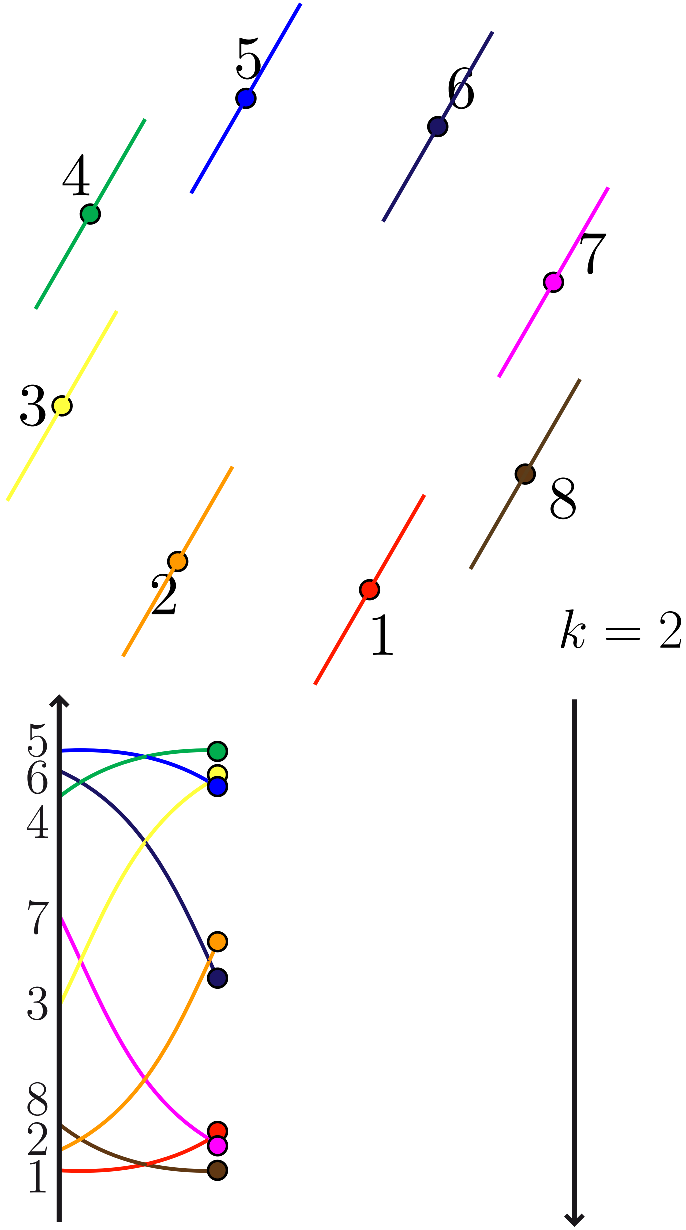

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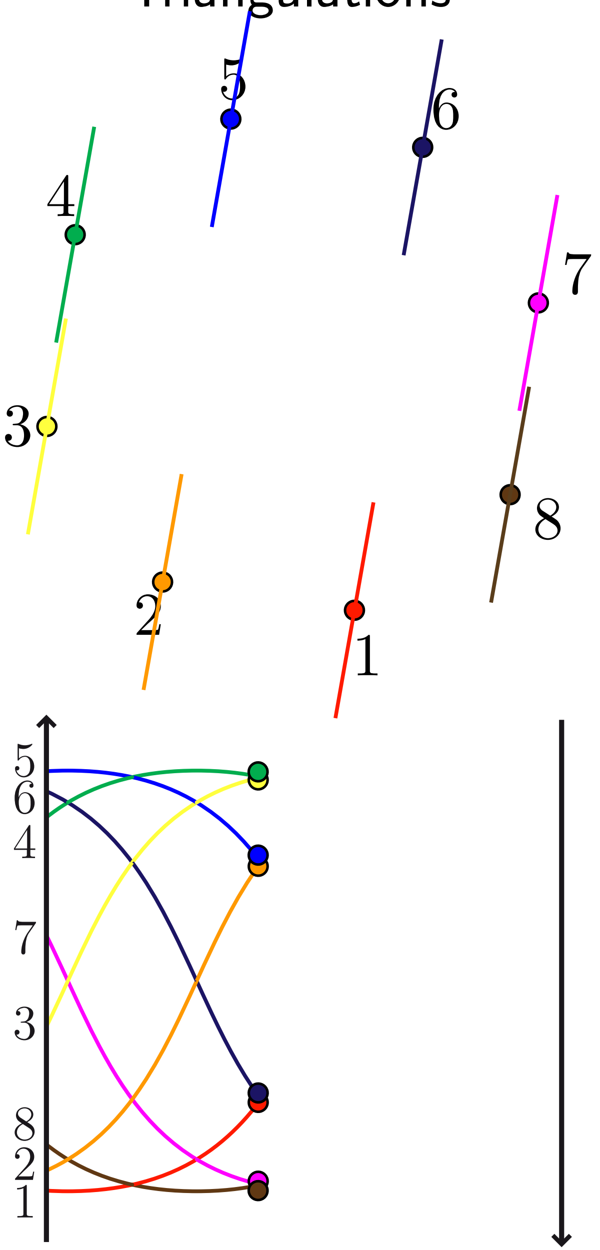

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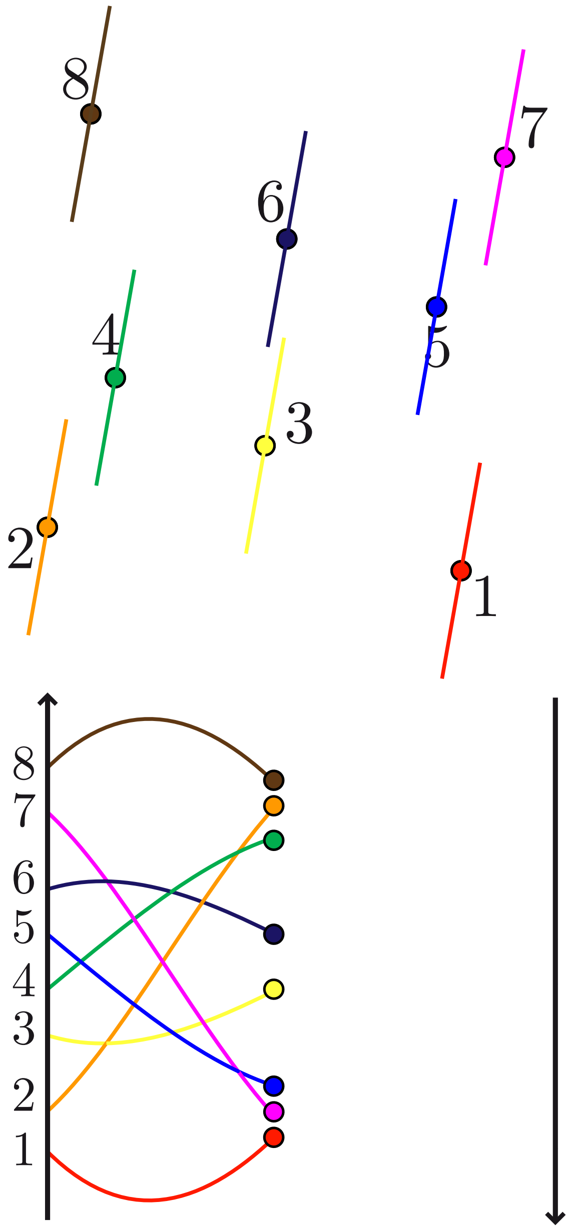

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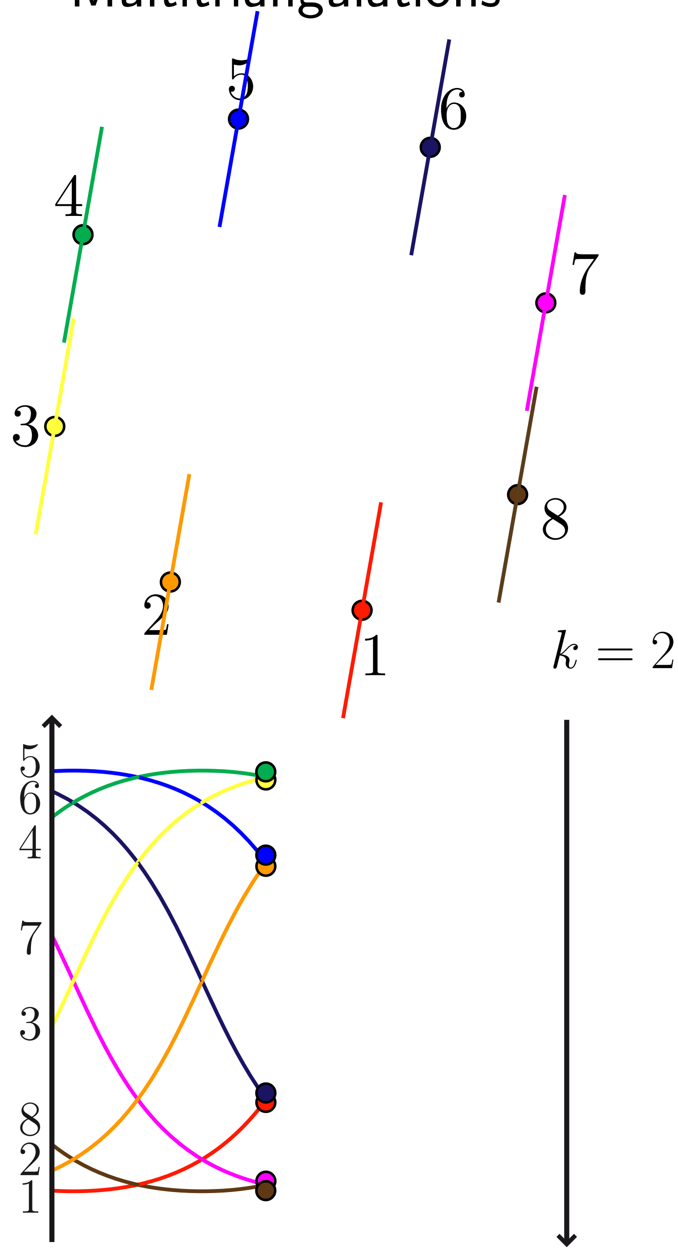

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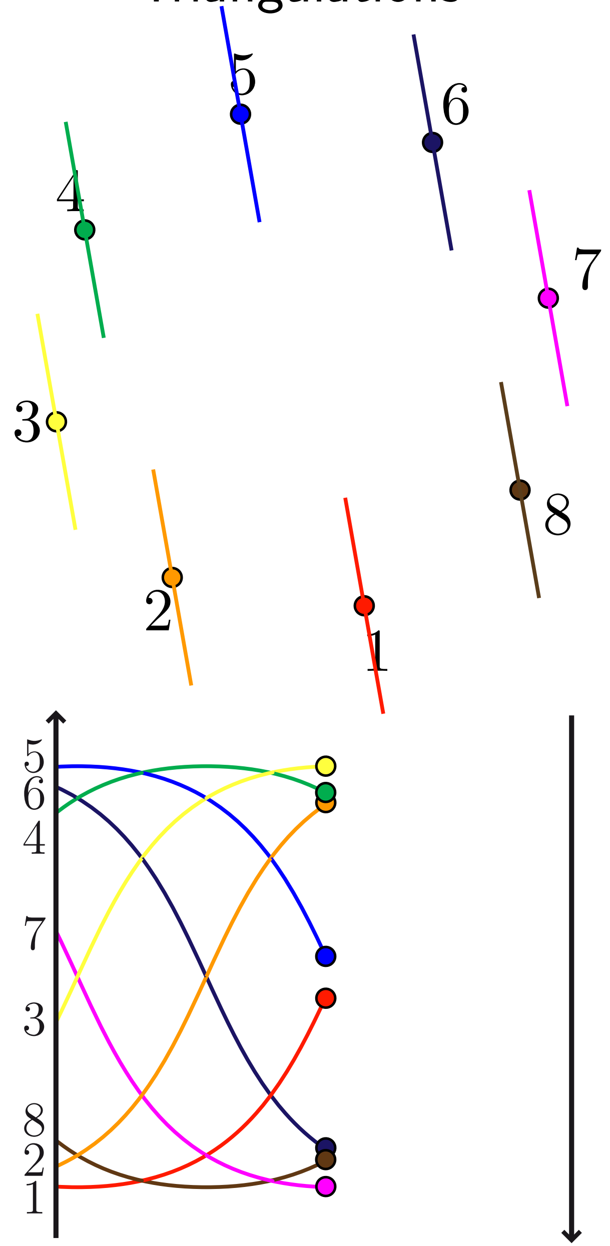

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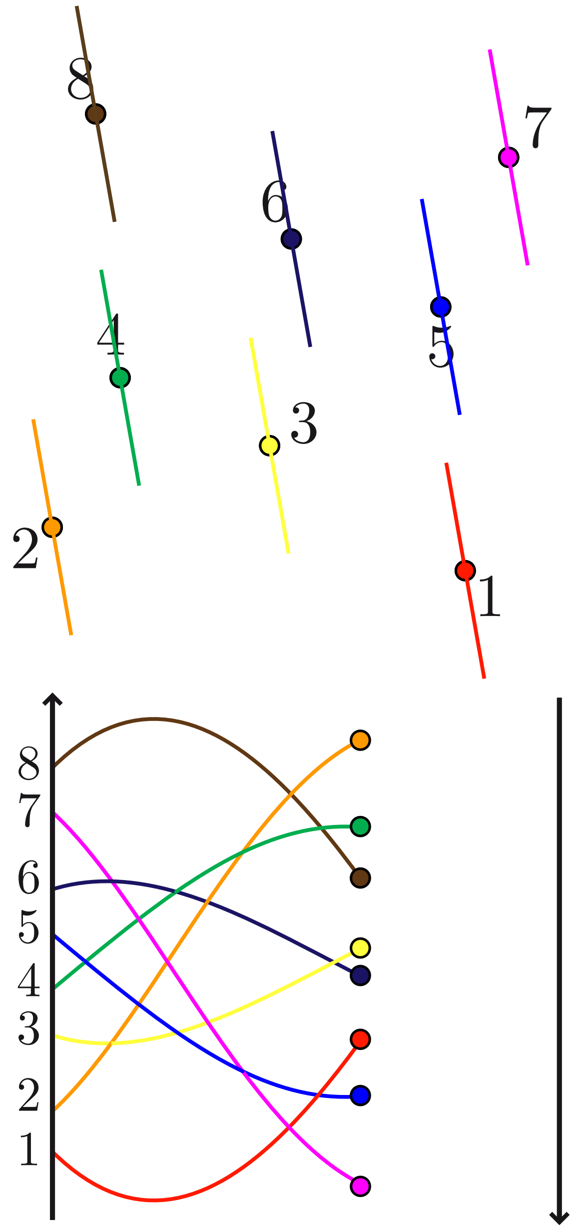

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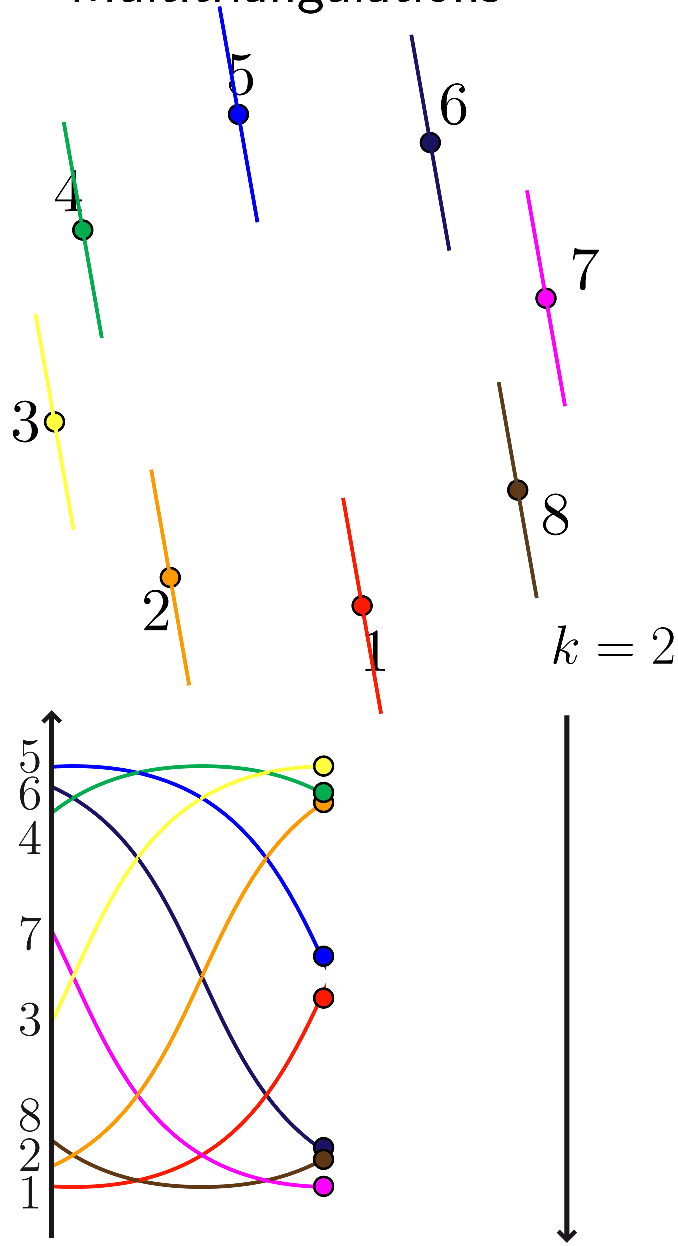

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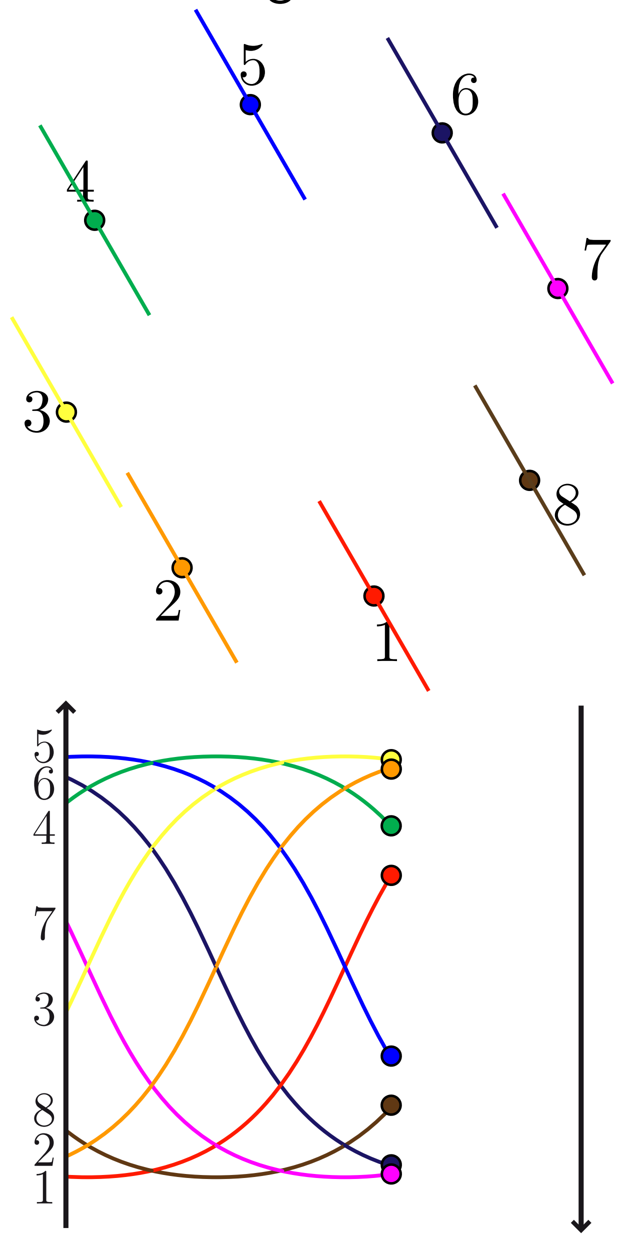

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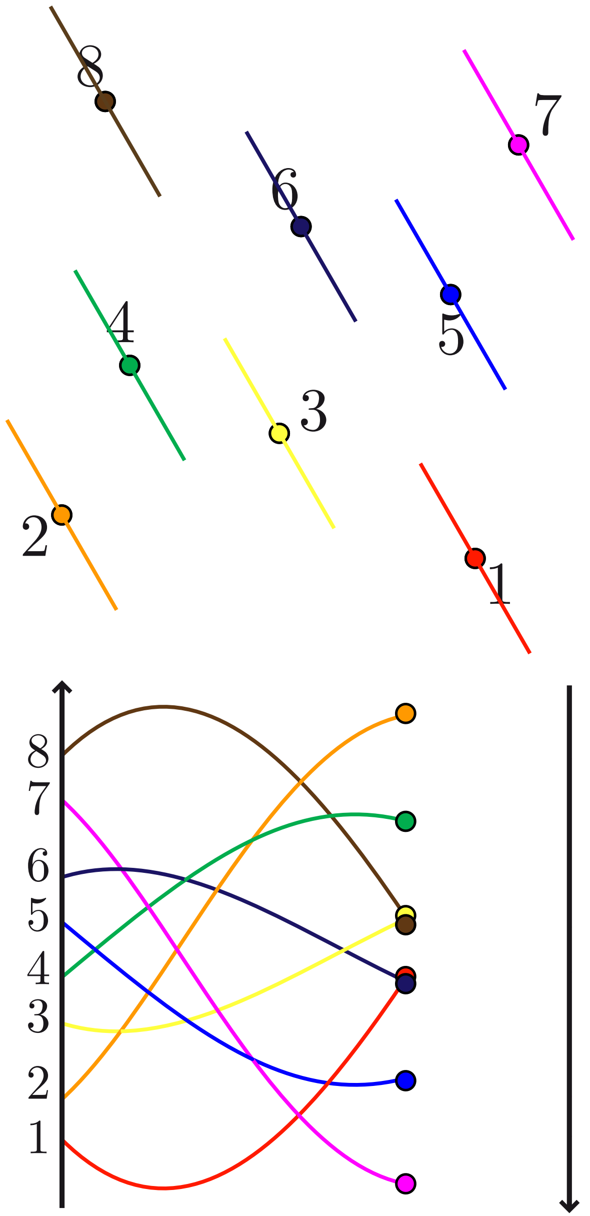

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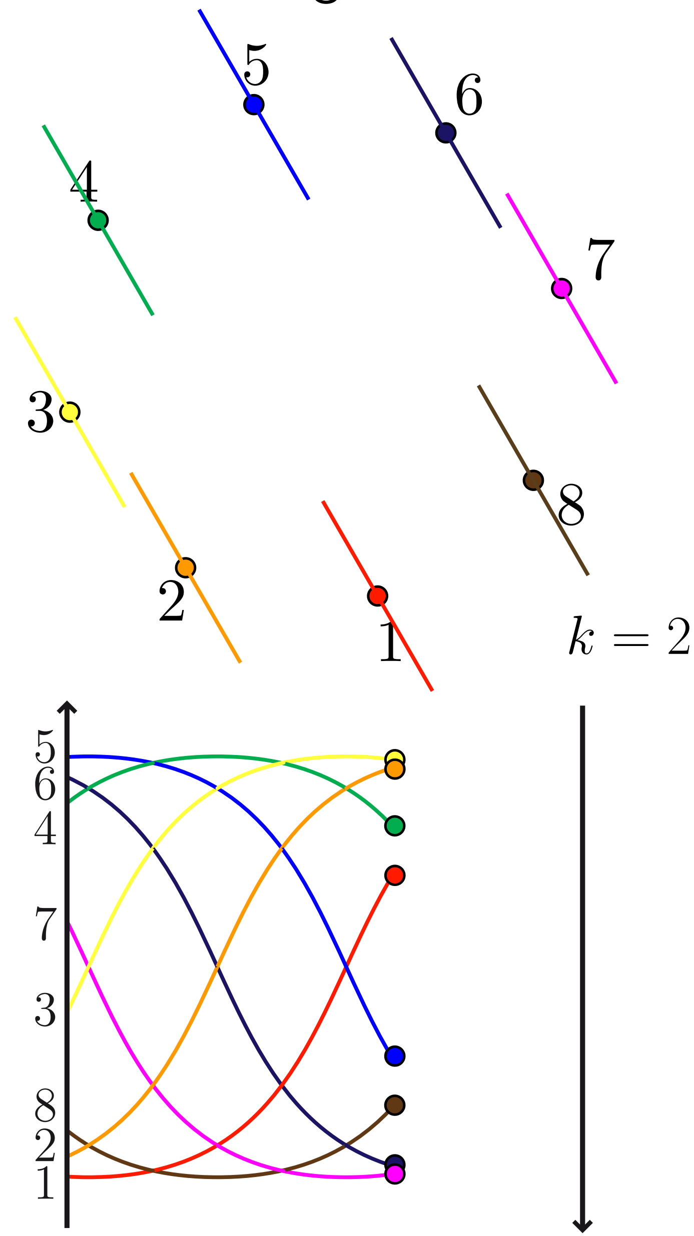
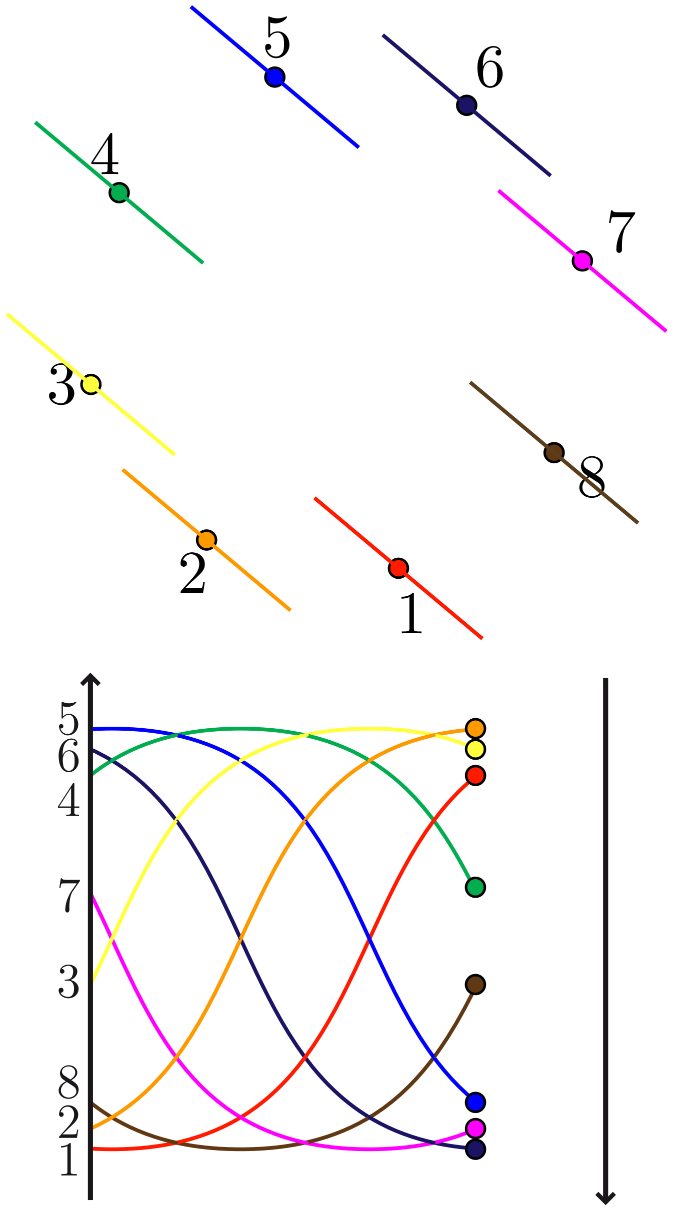
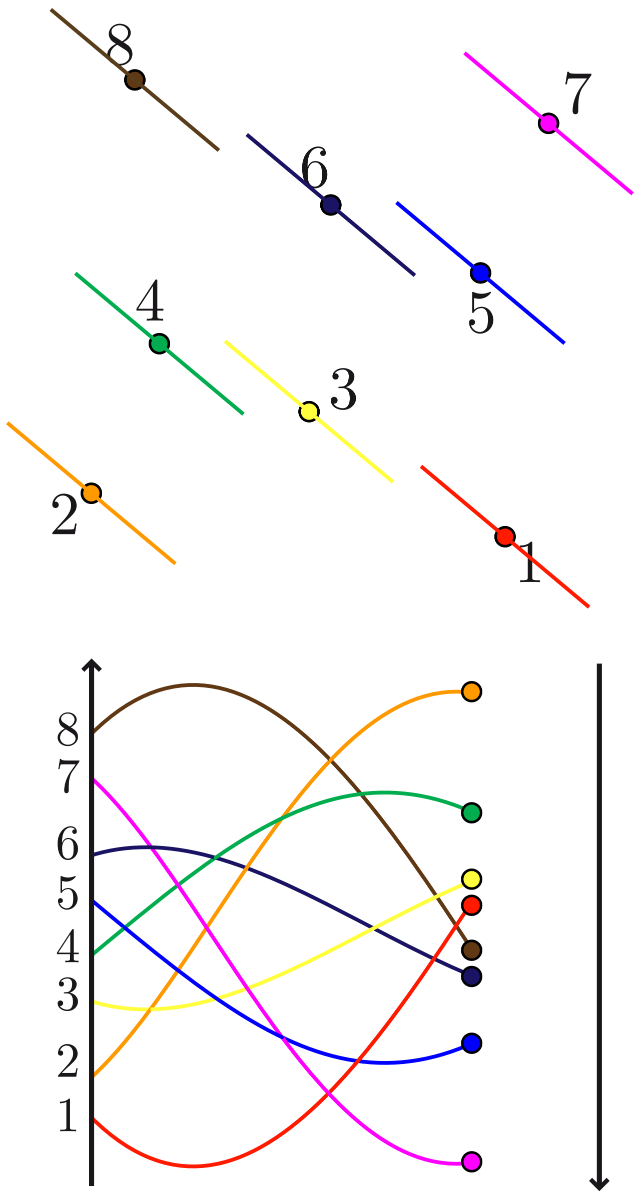
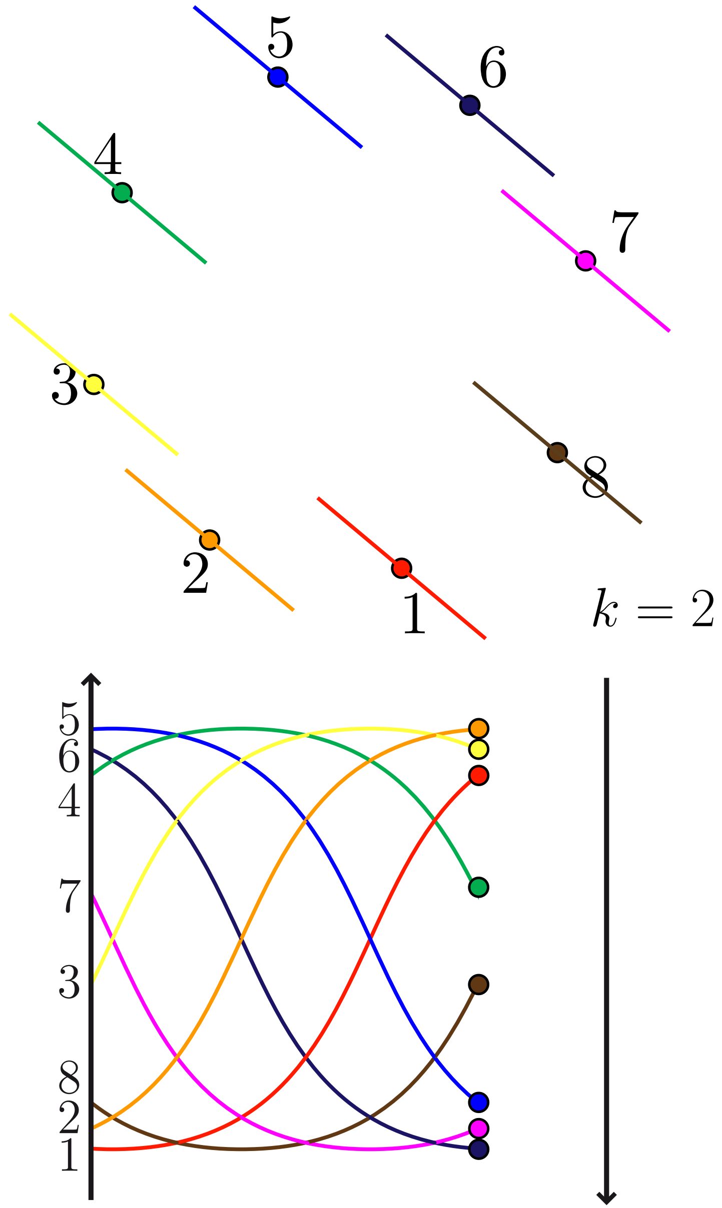


DUALITY

Triangulations	Pseudotriangulations	Multitriangulations
		

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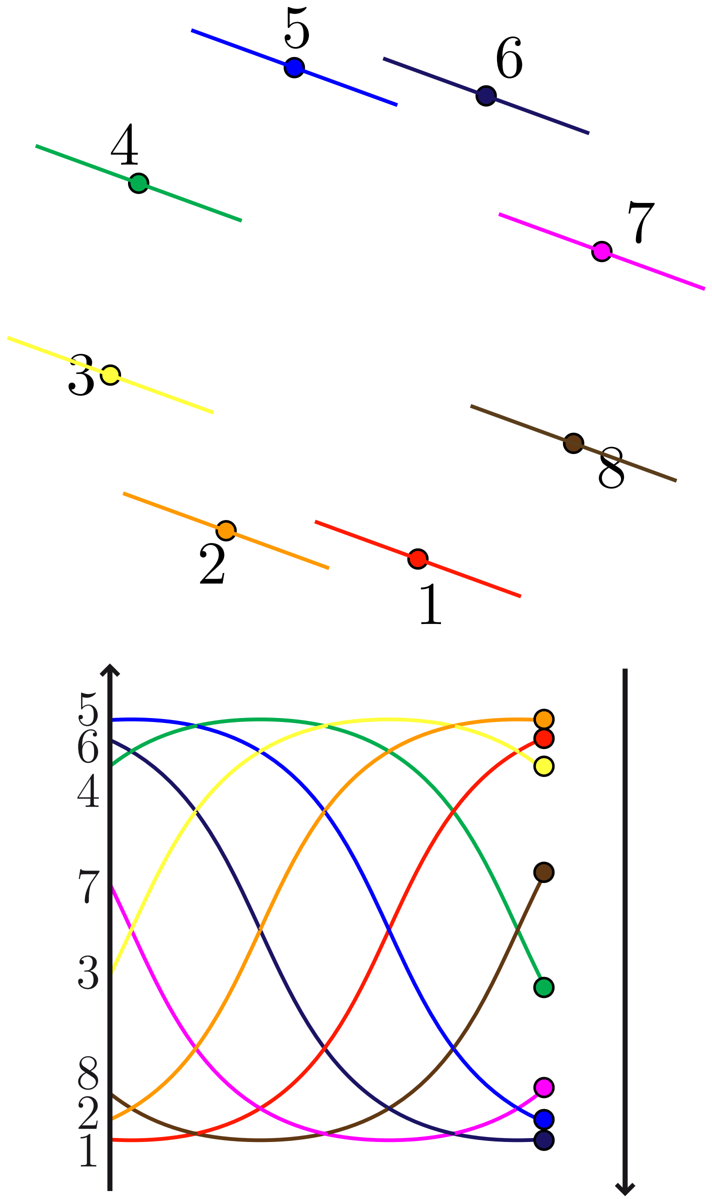

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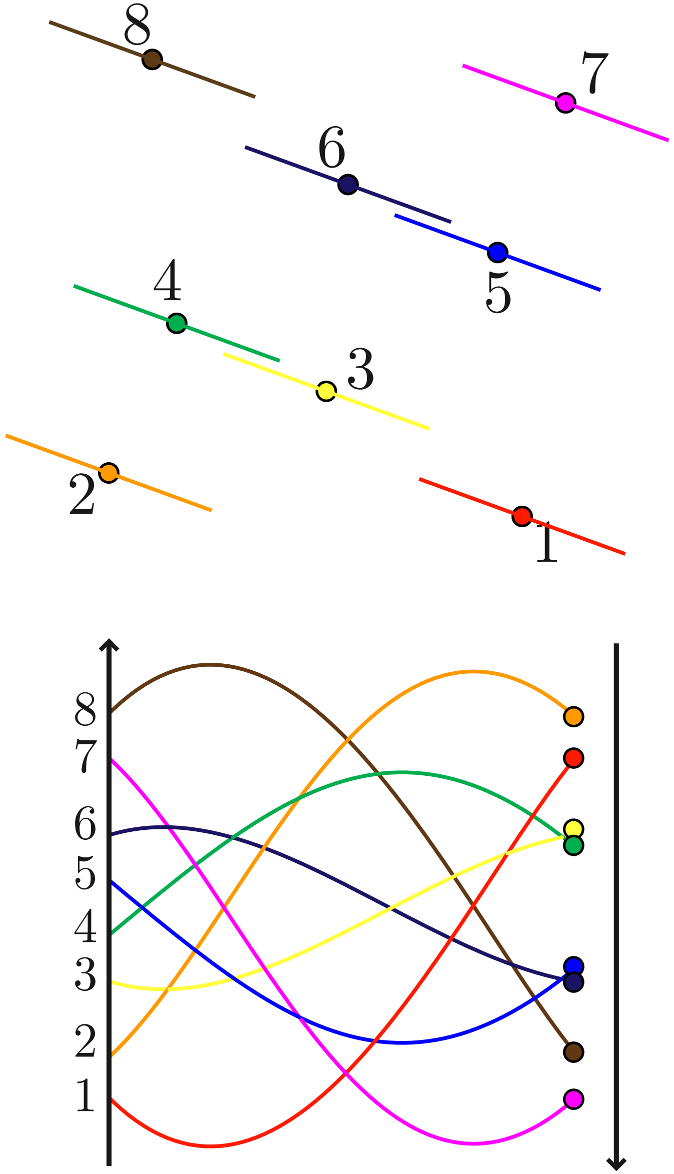

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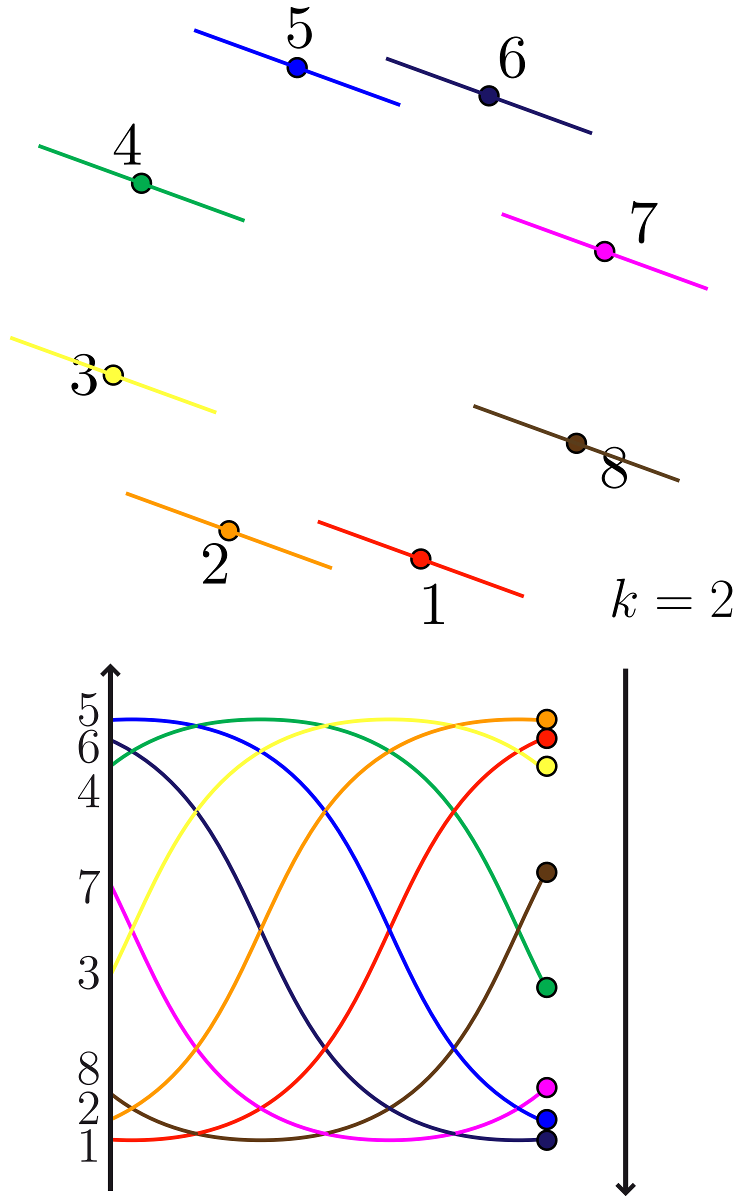

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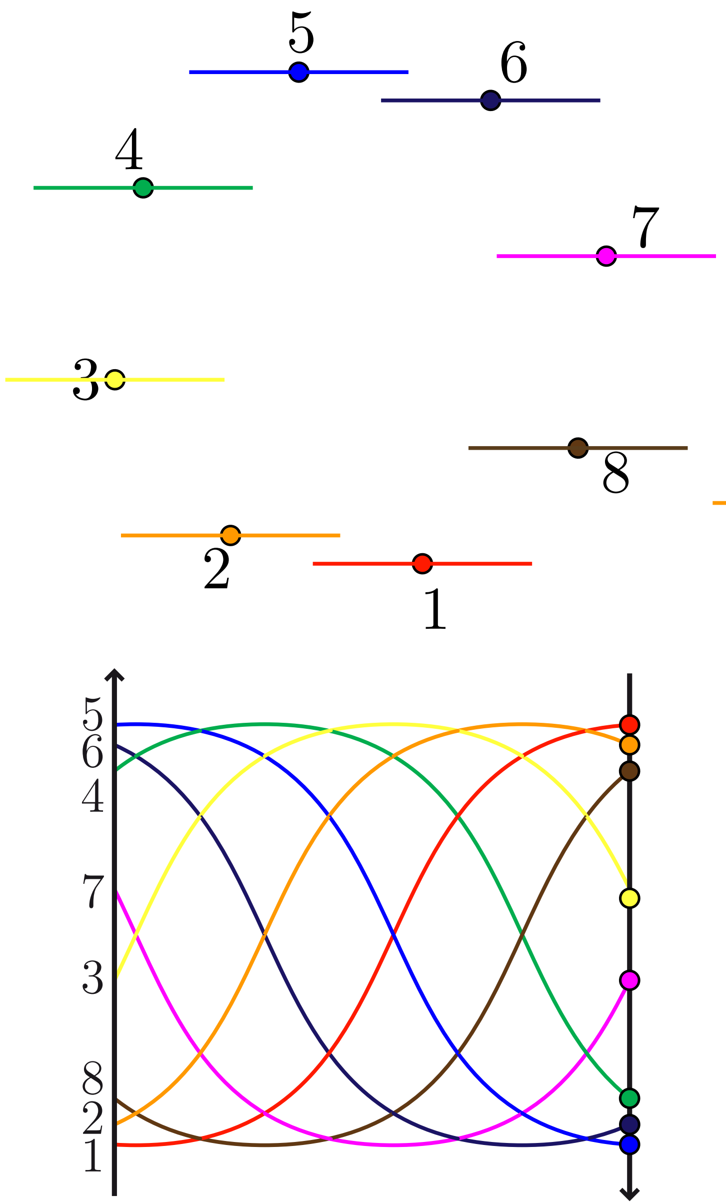

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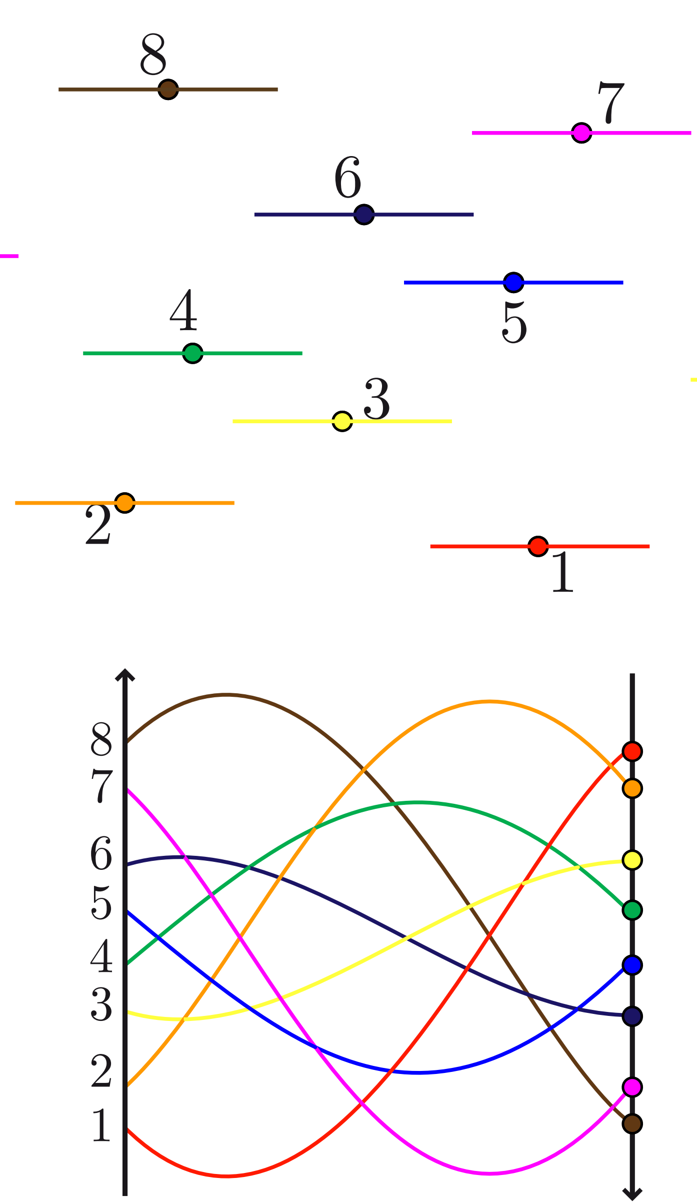

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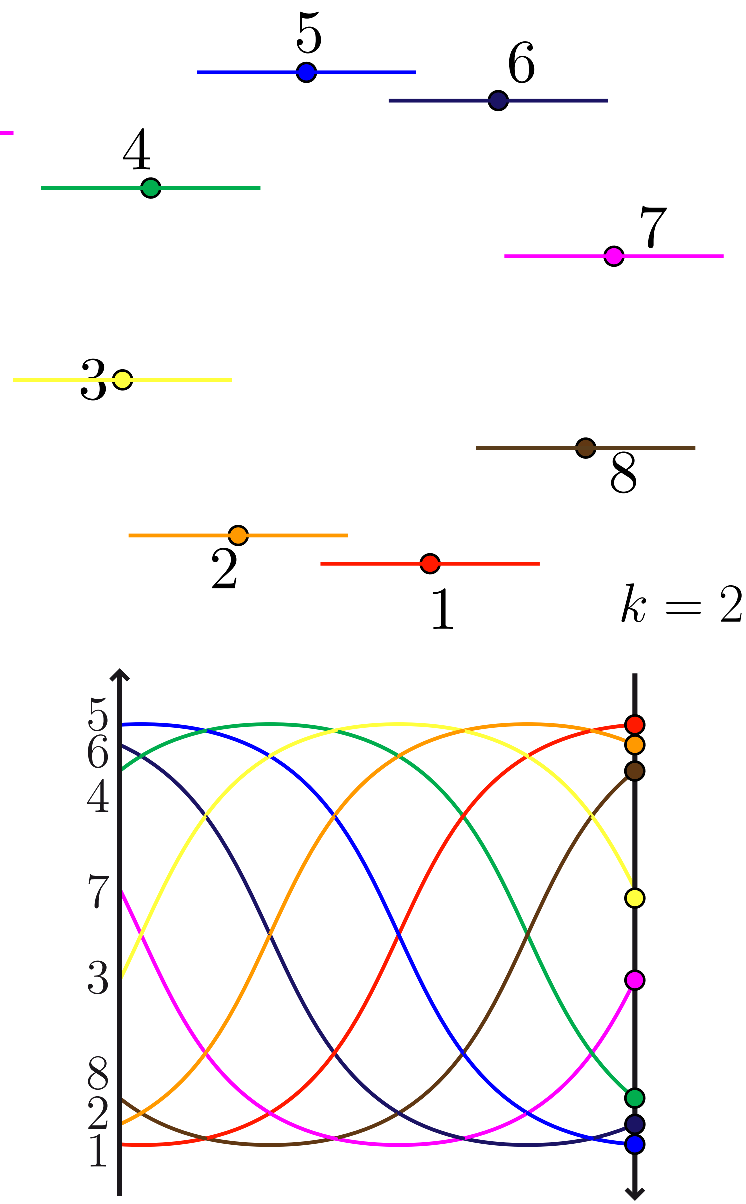

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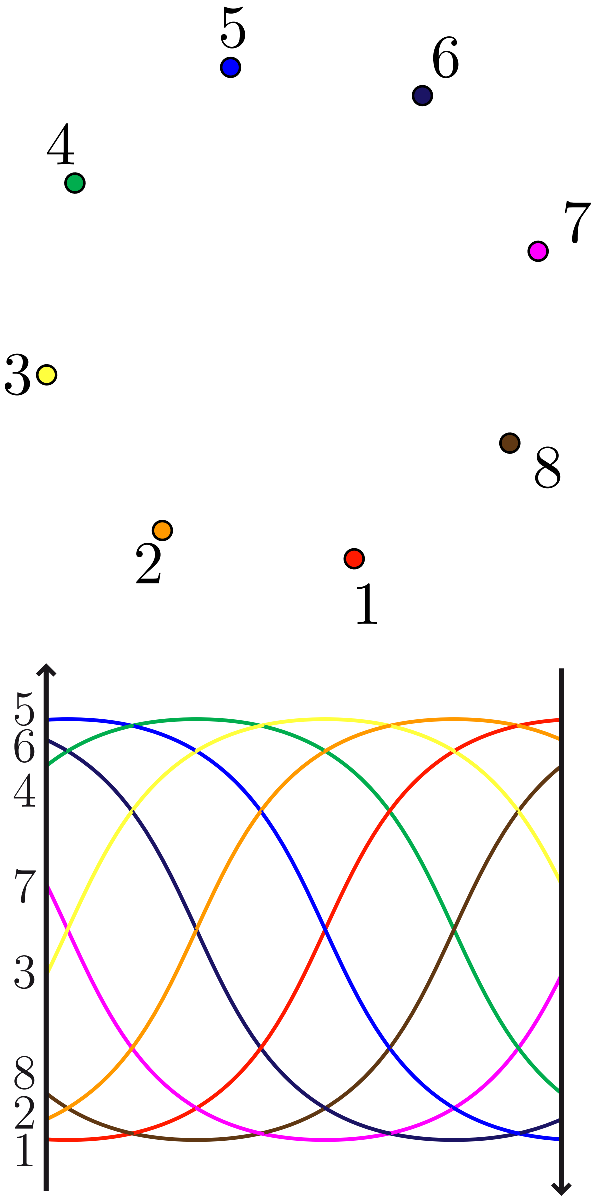

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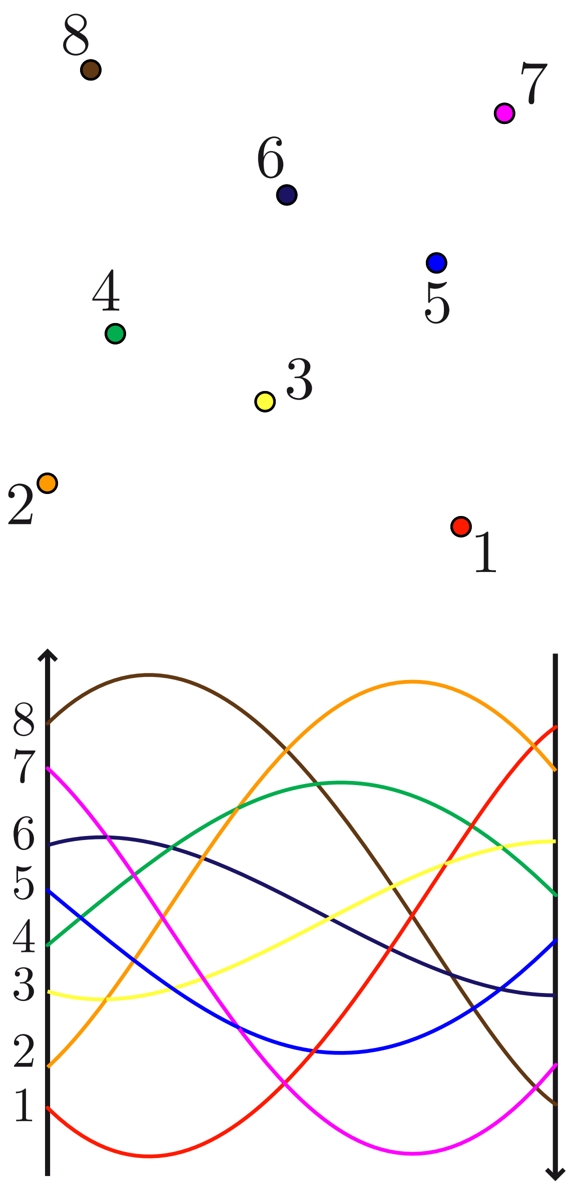

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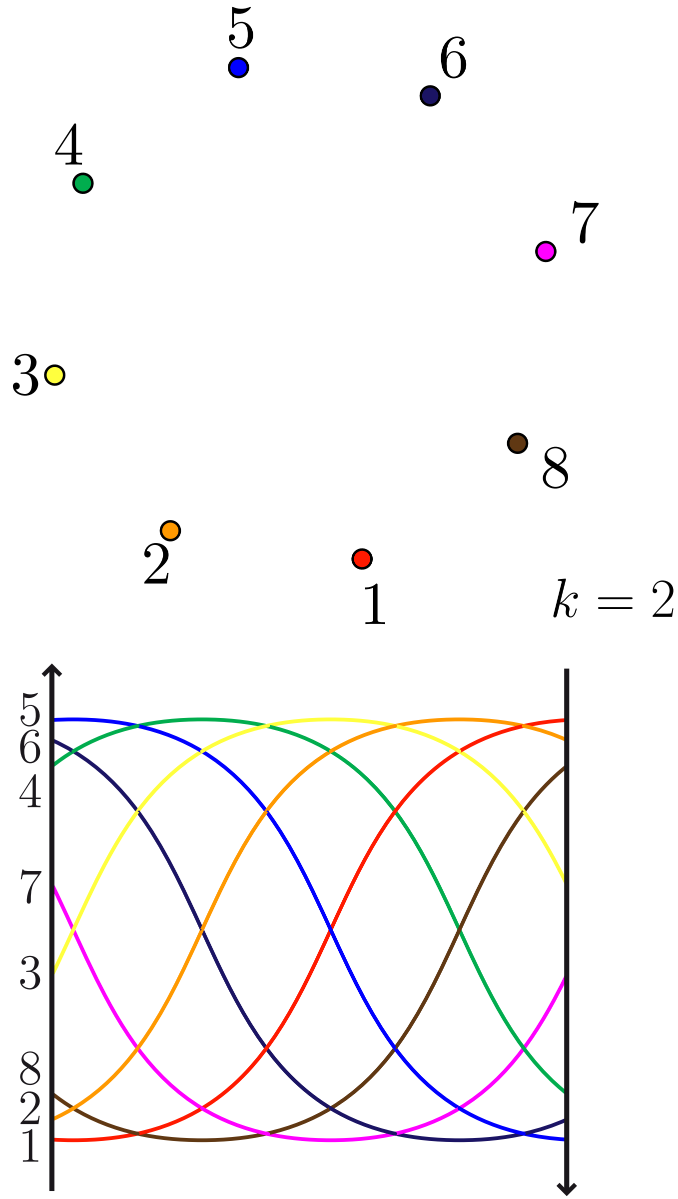

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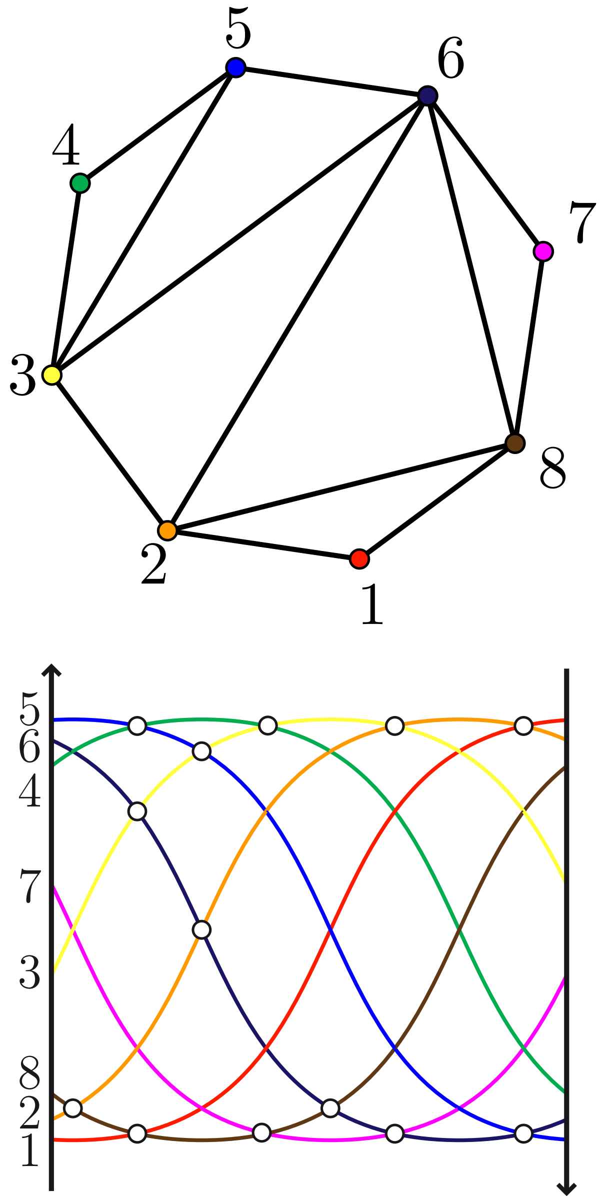

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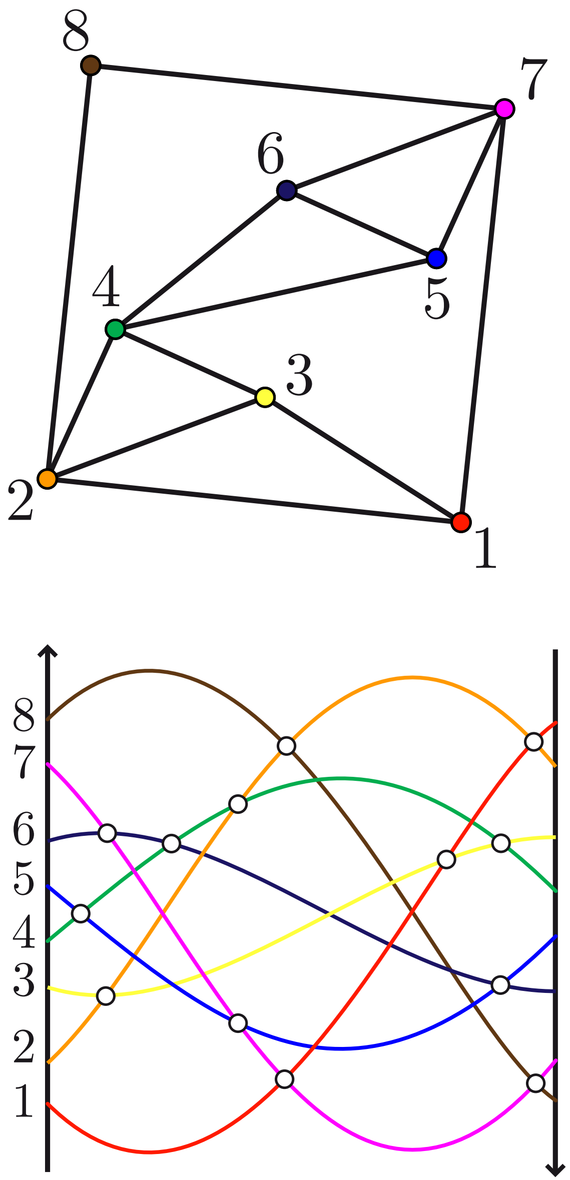

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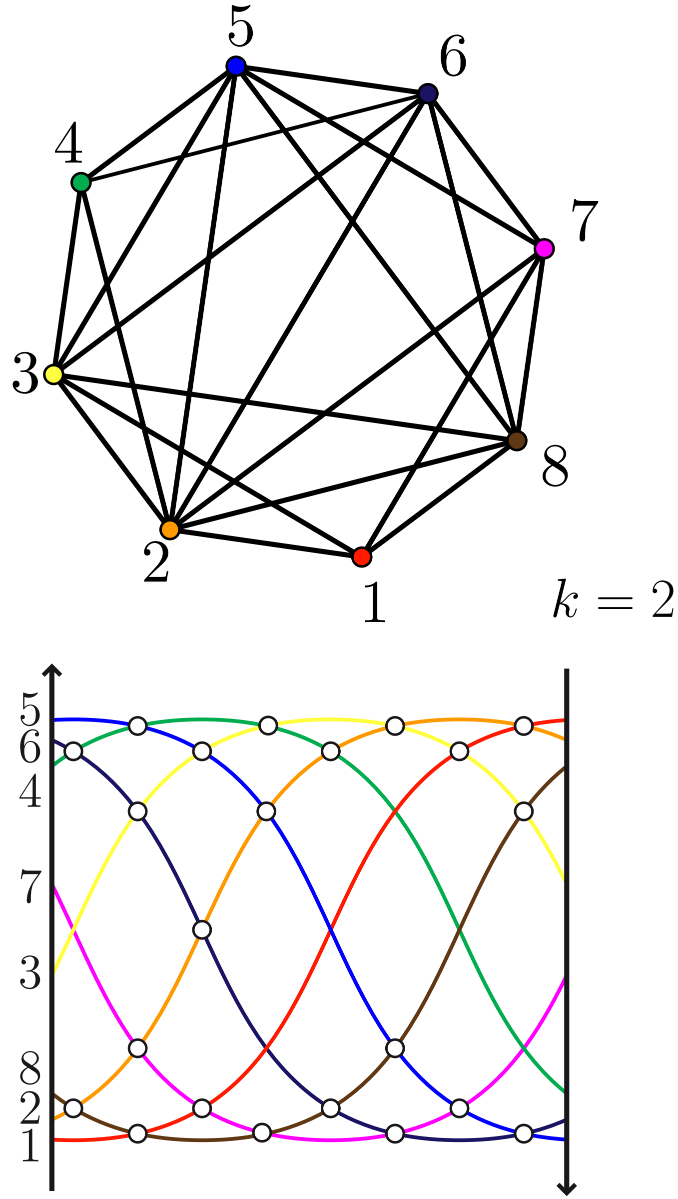
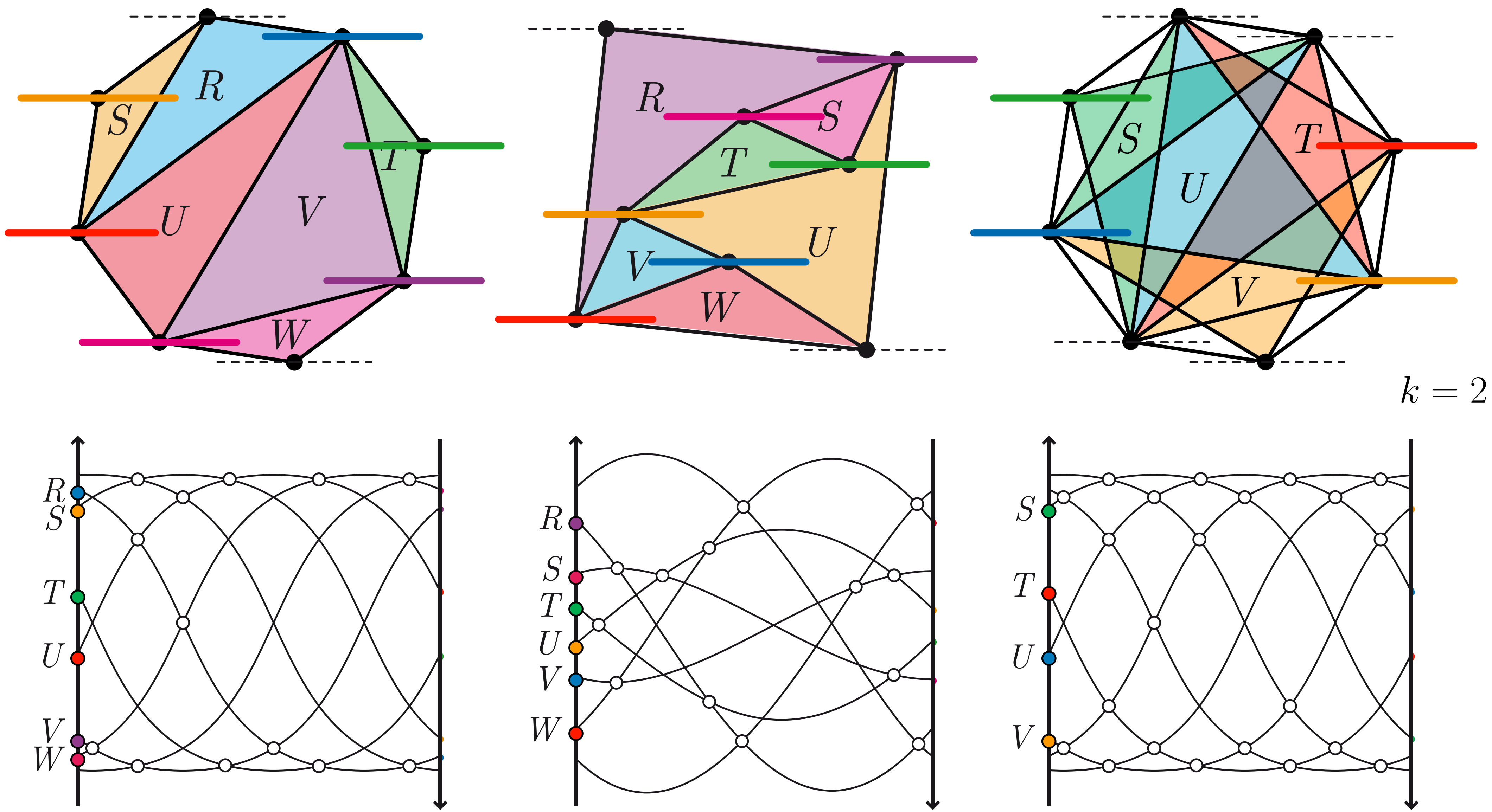


DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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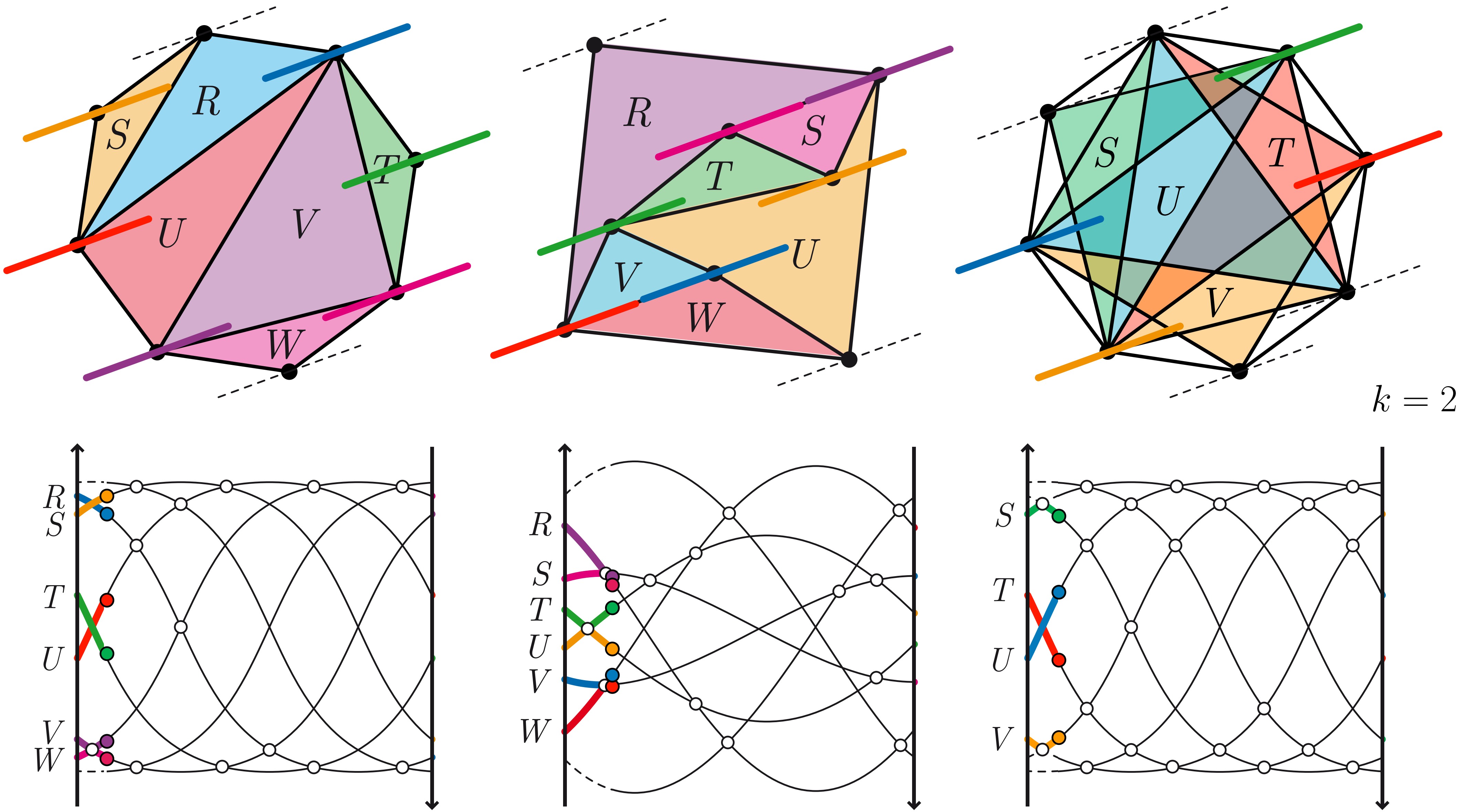
Triangulations | Pseudotriangulations | Multitriangulations



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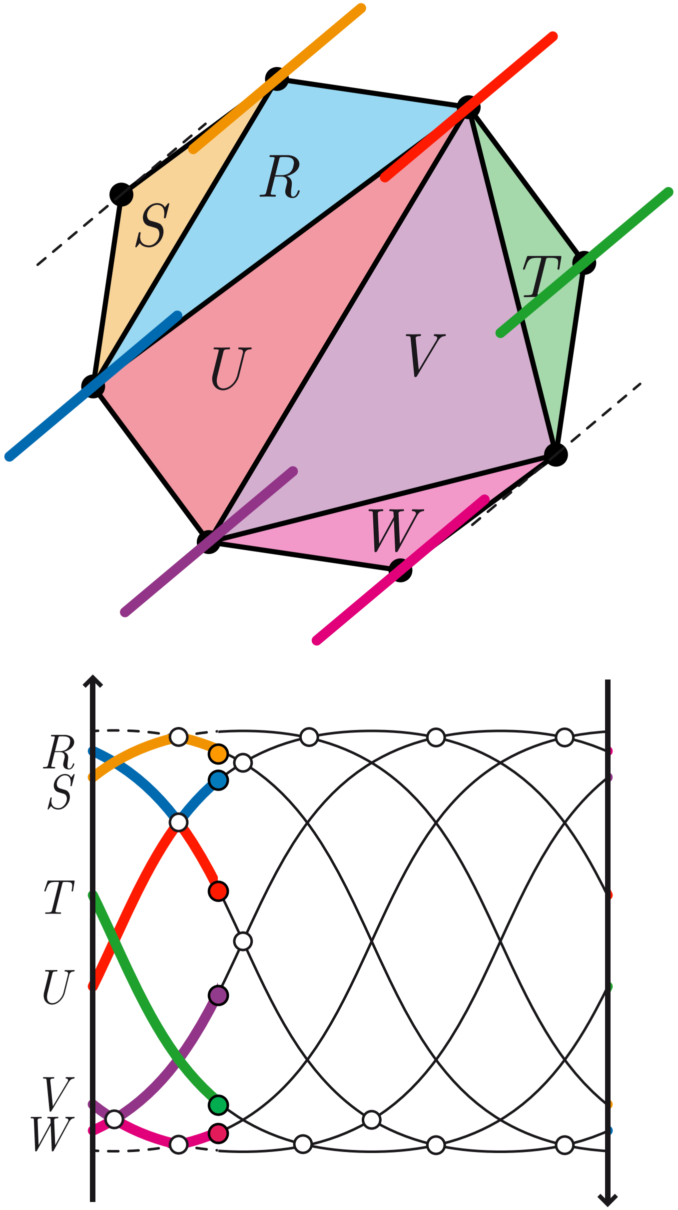

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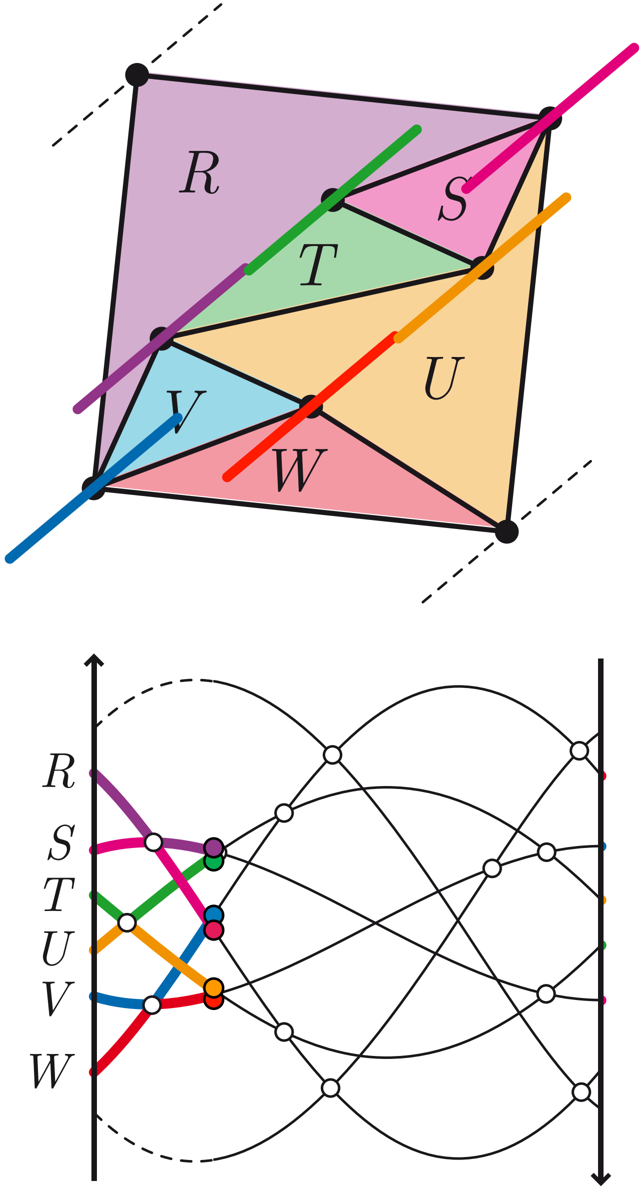

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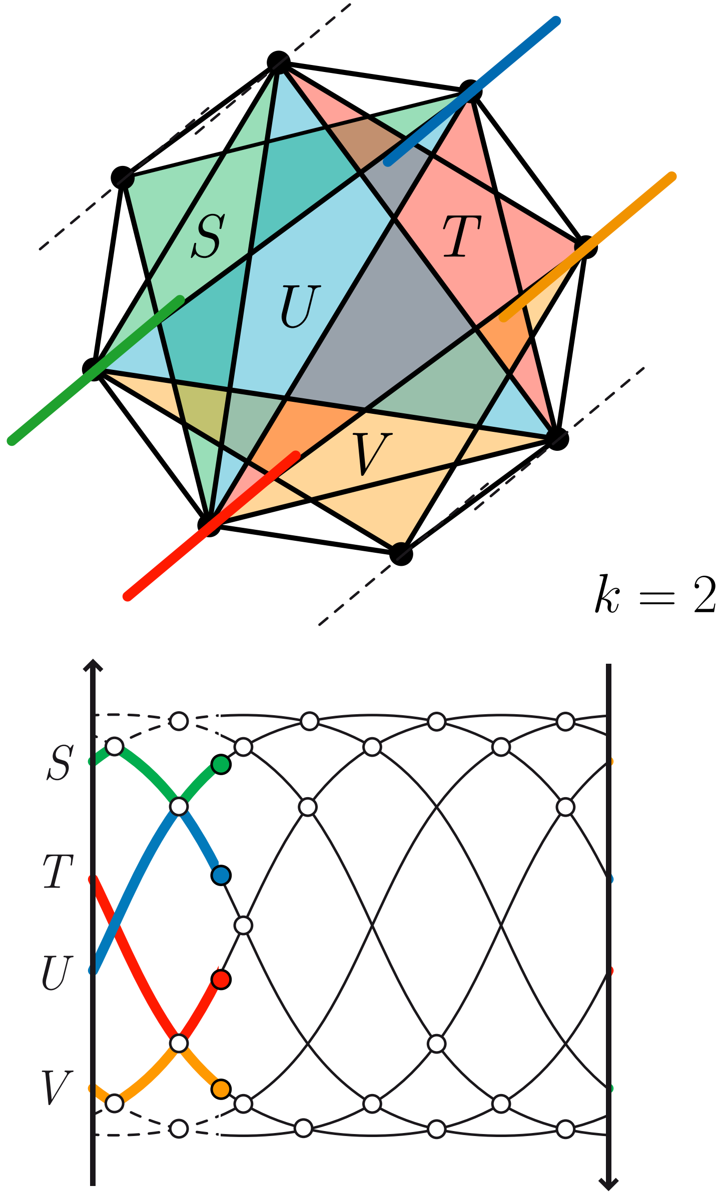

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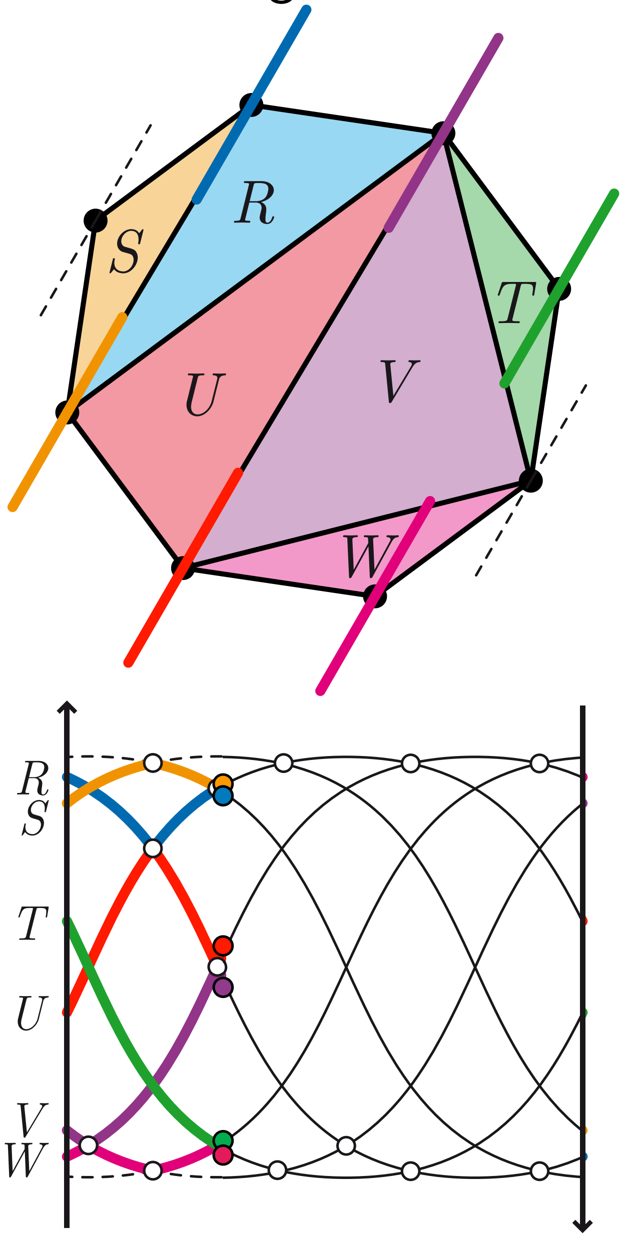

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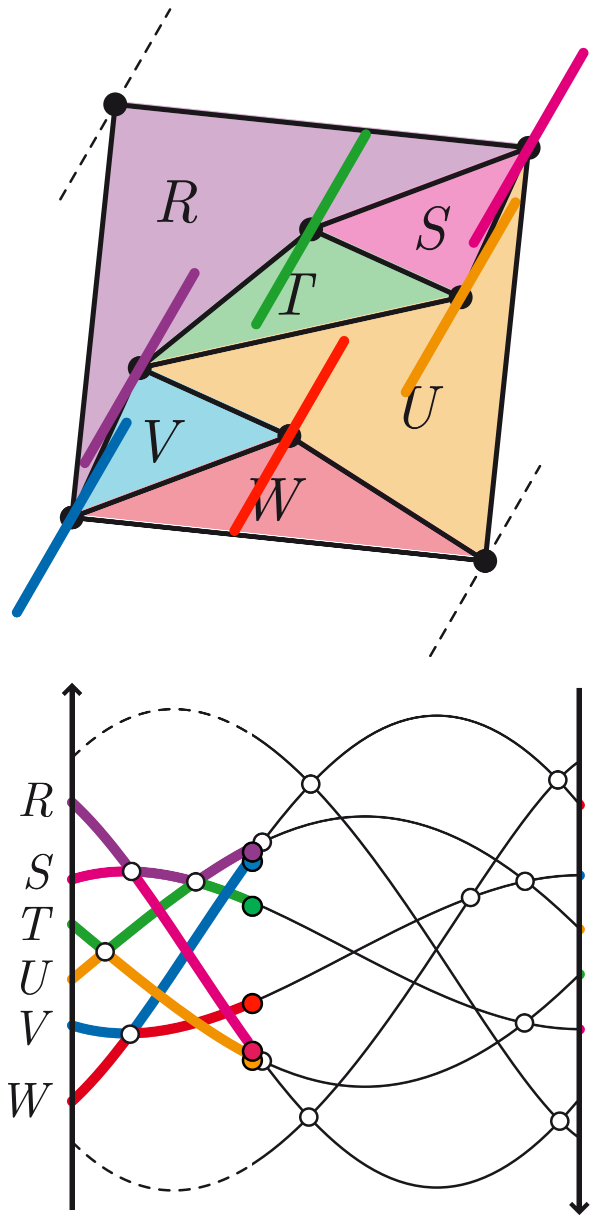

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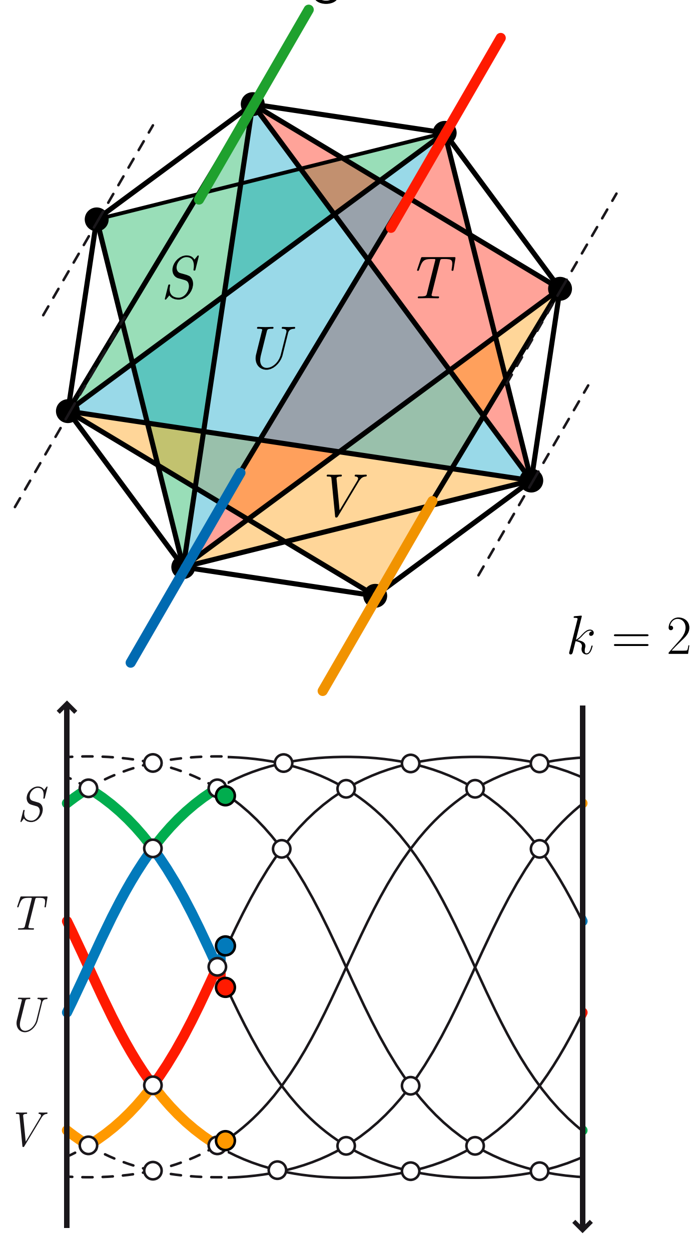

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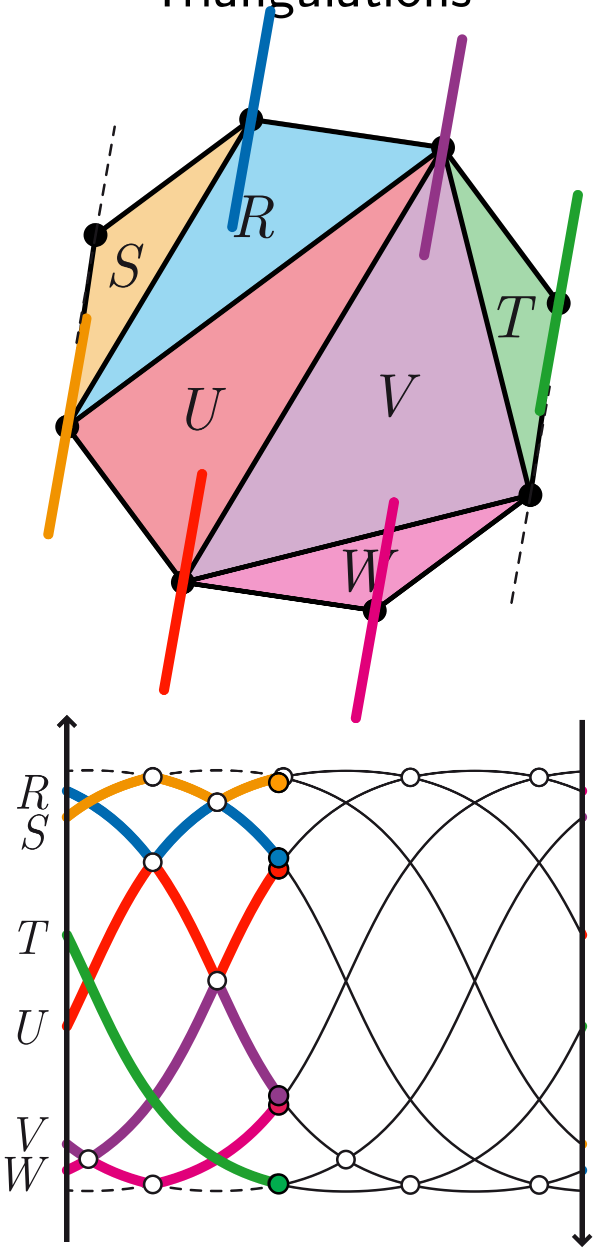

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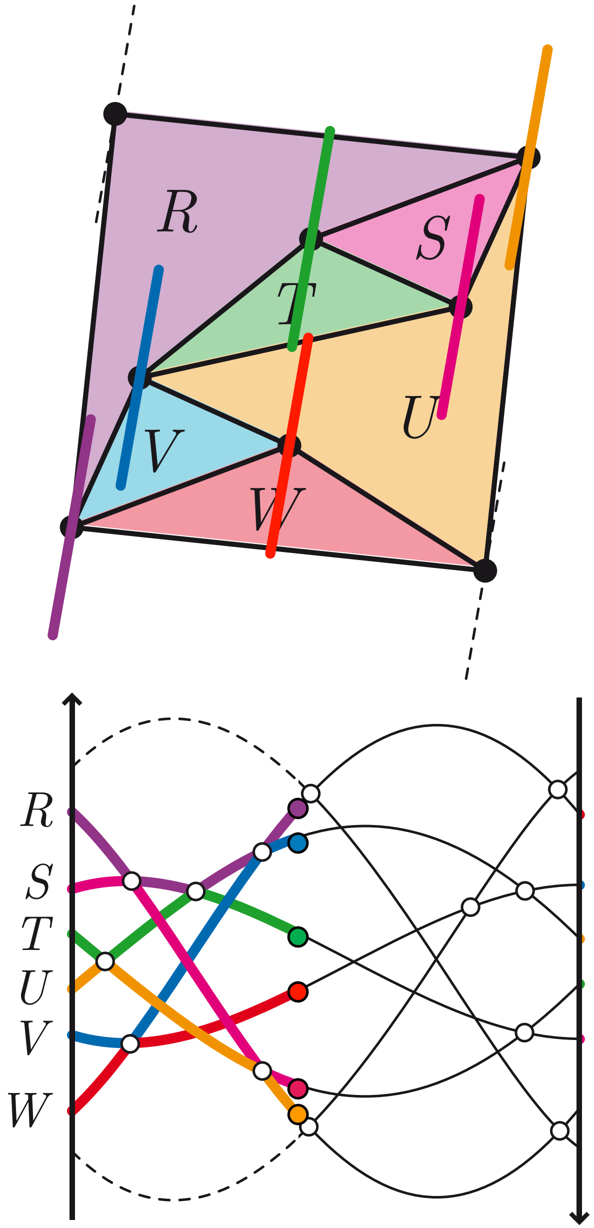

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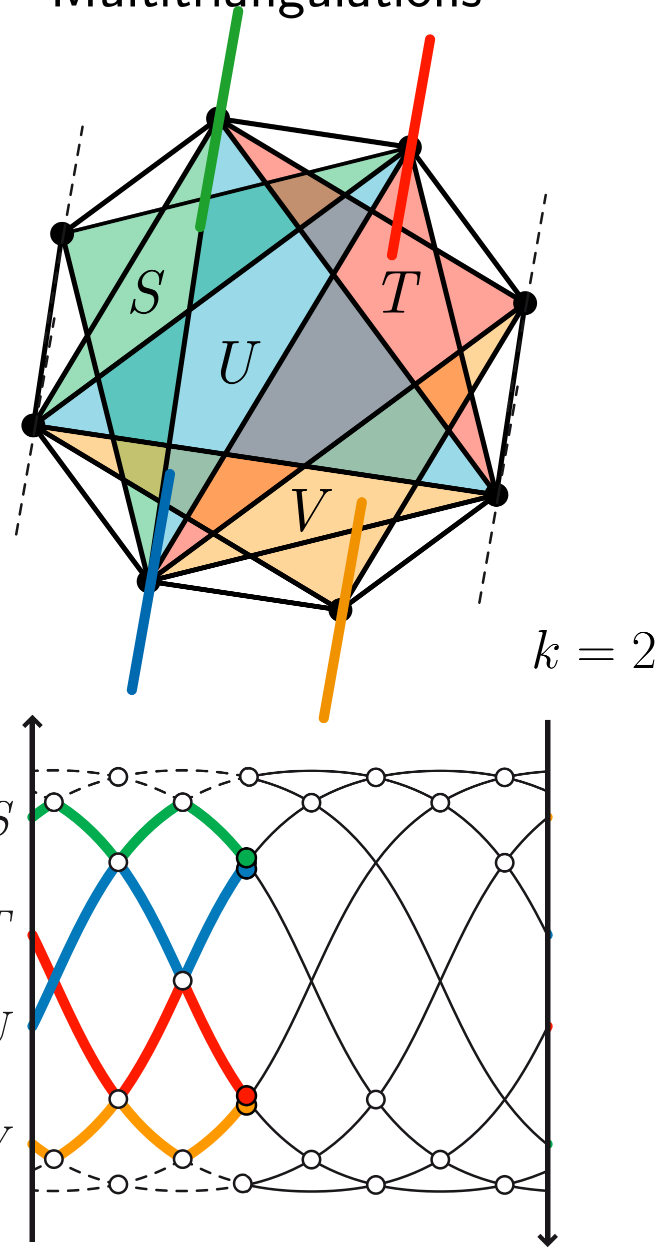

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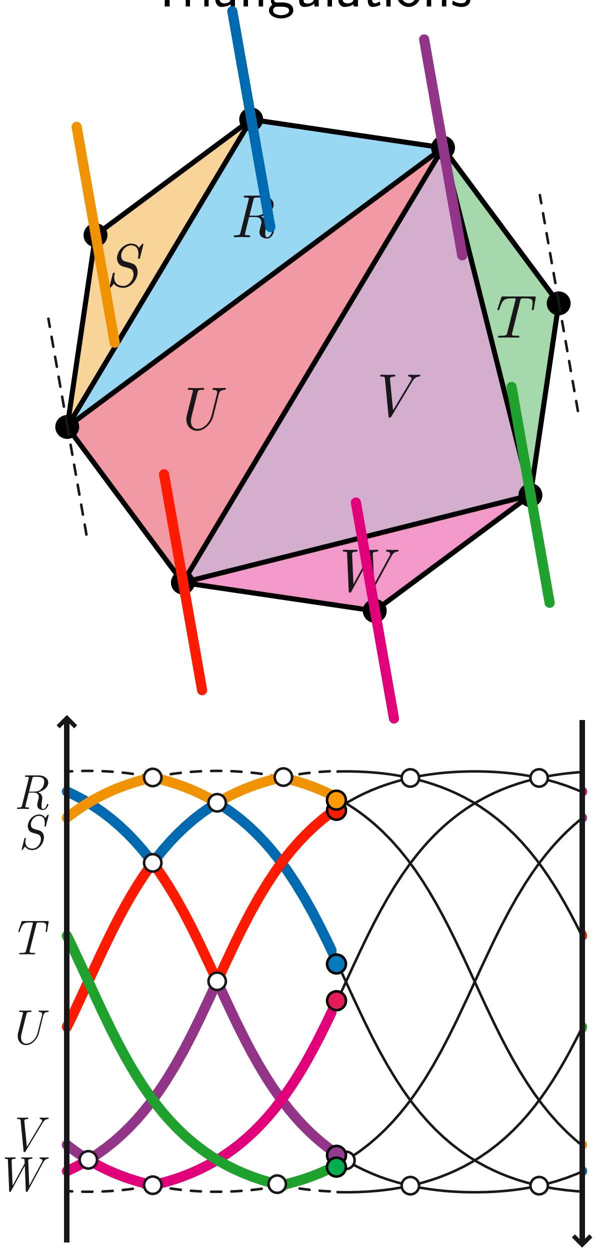

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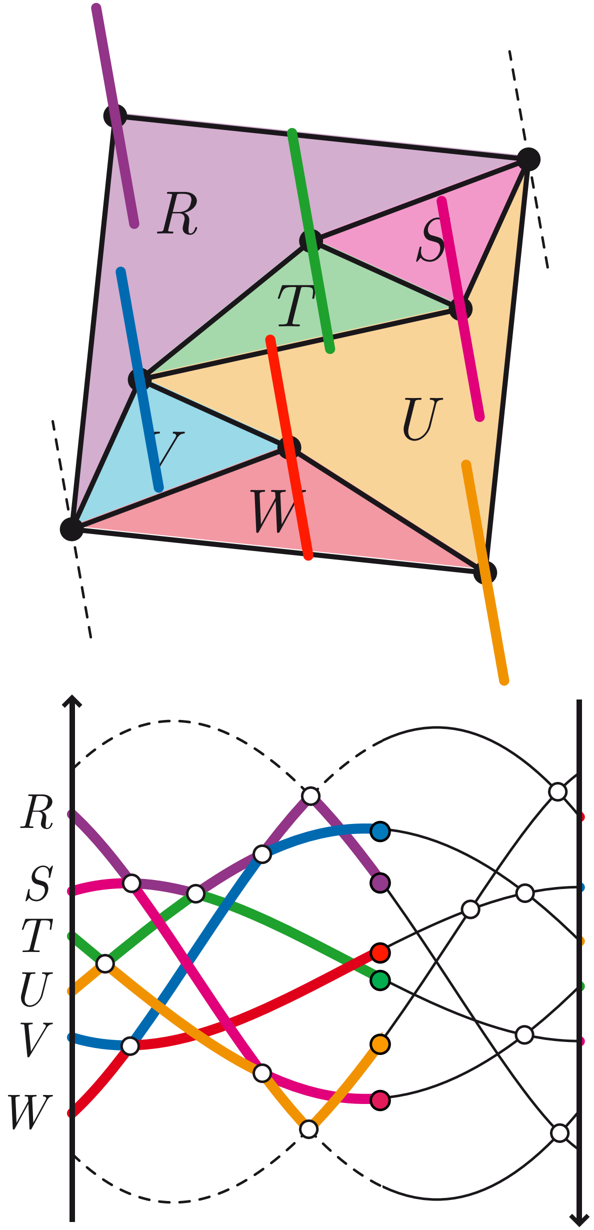

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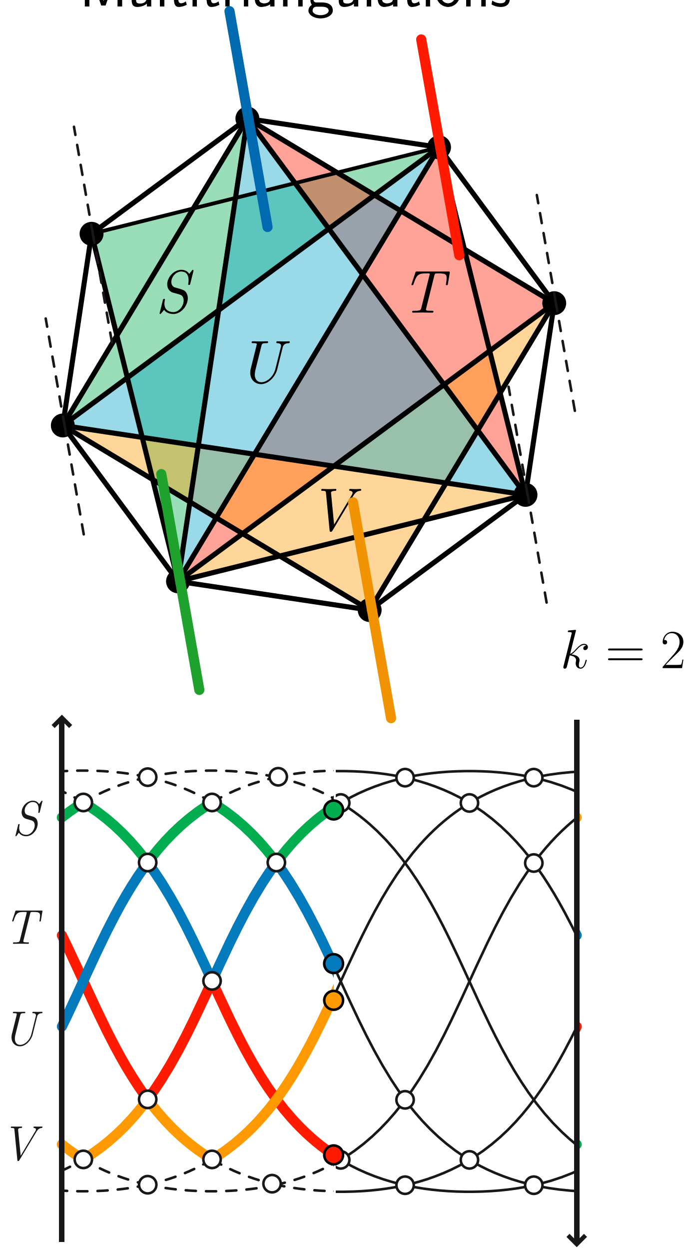

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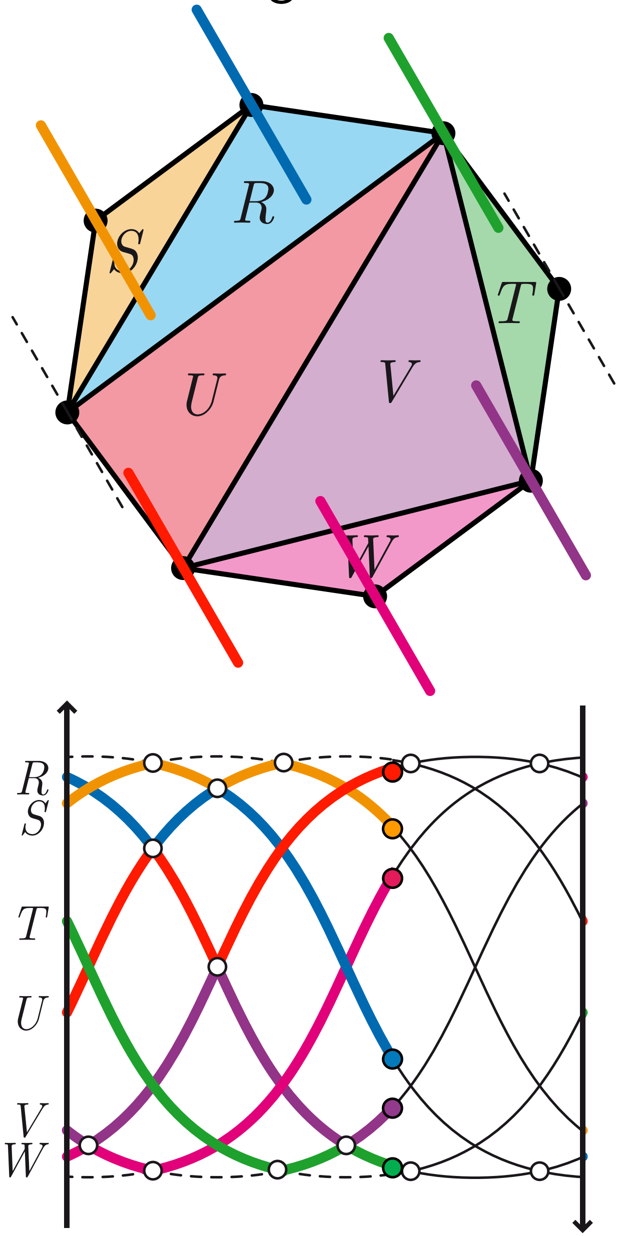

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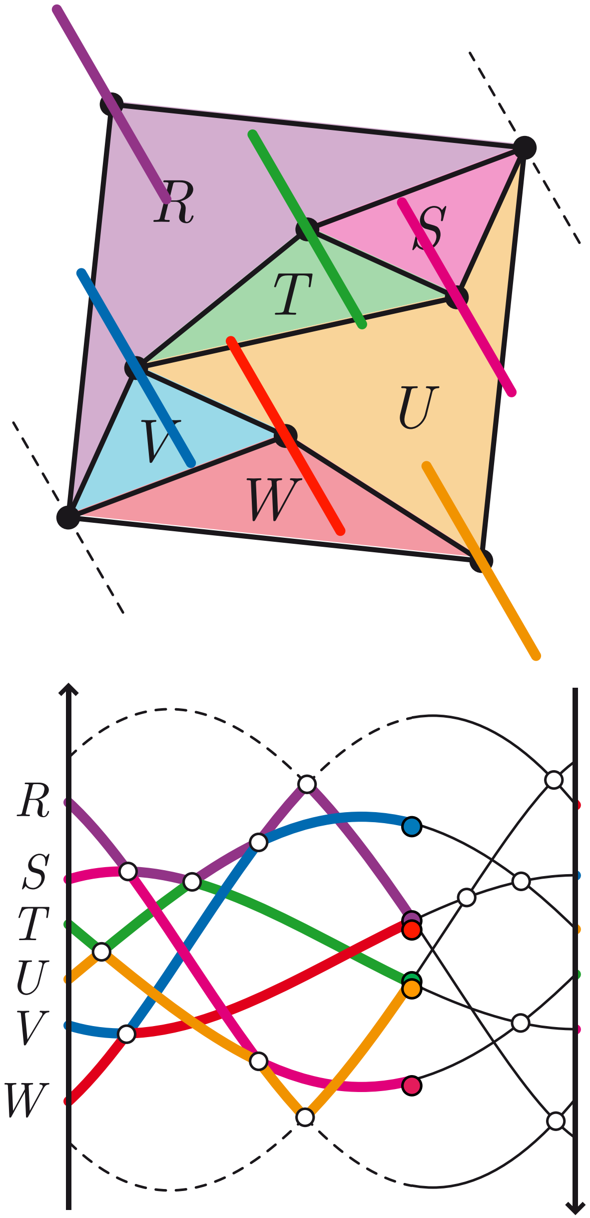

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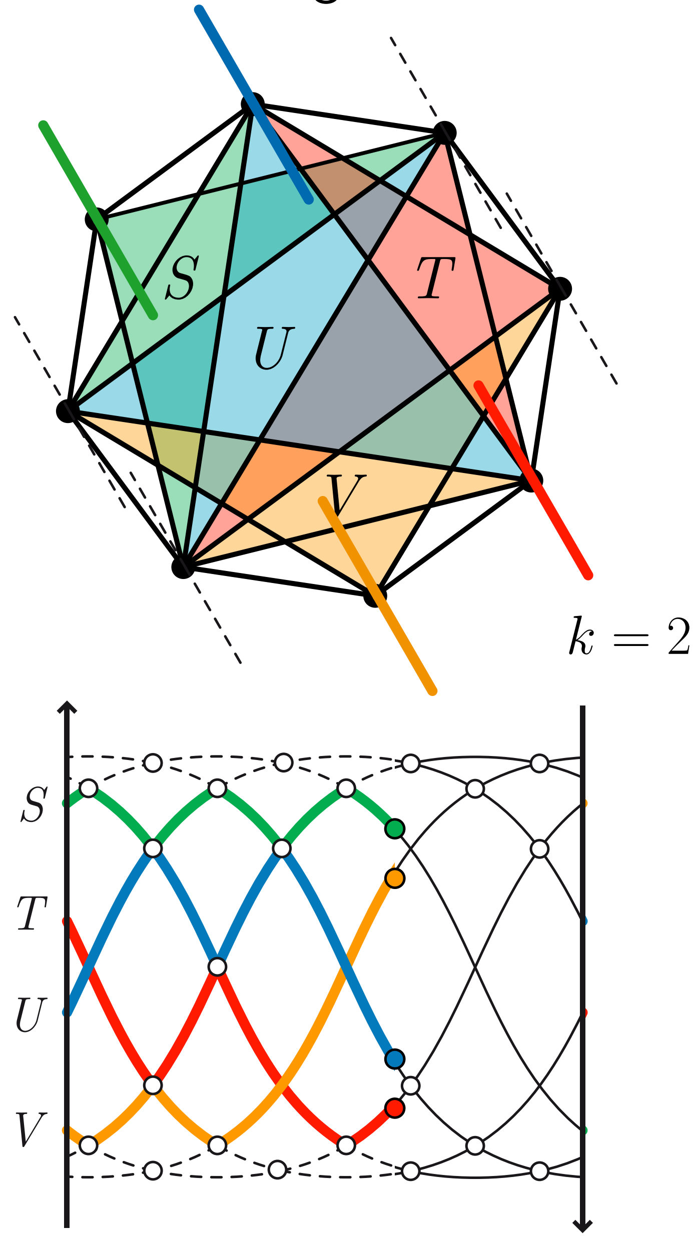

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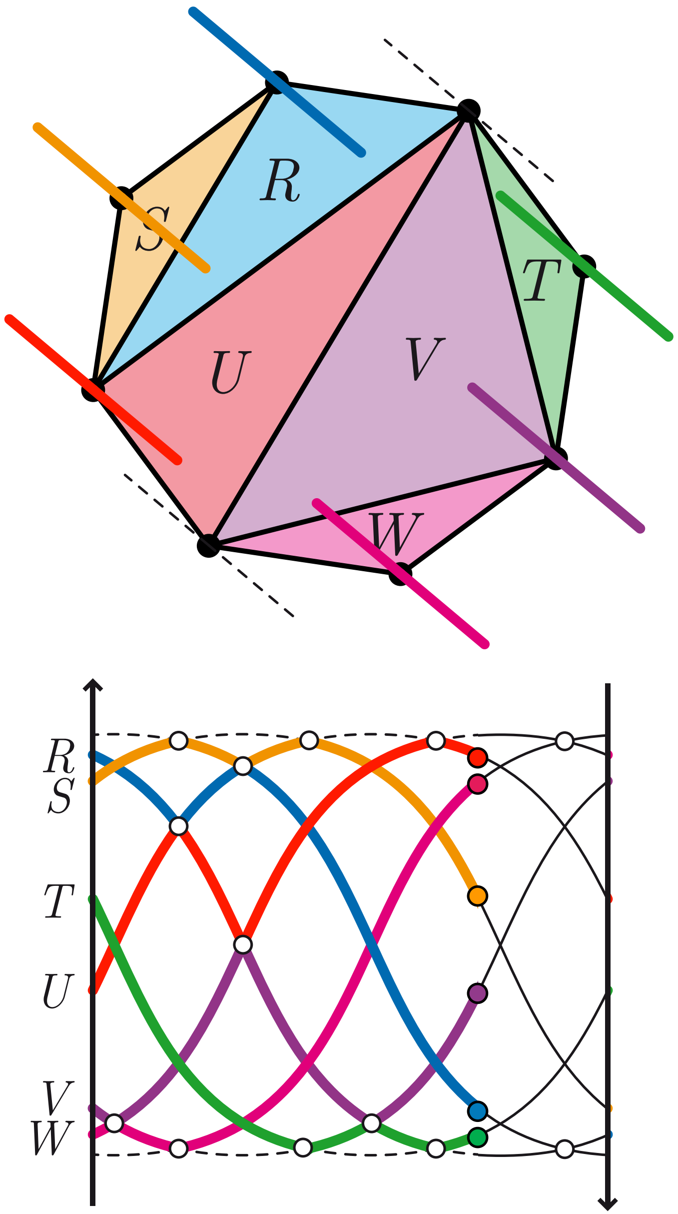

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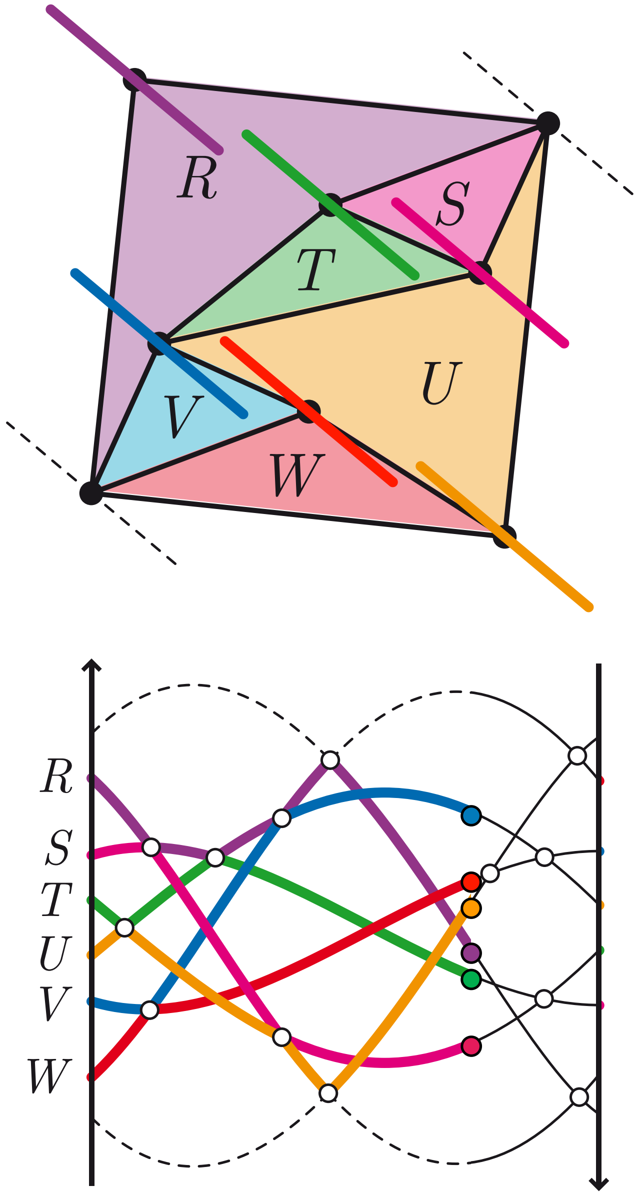

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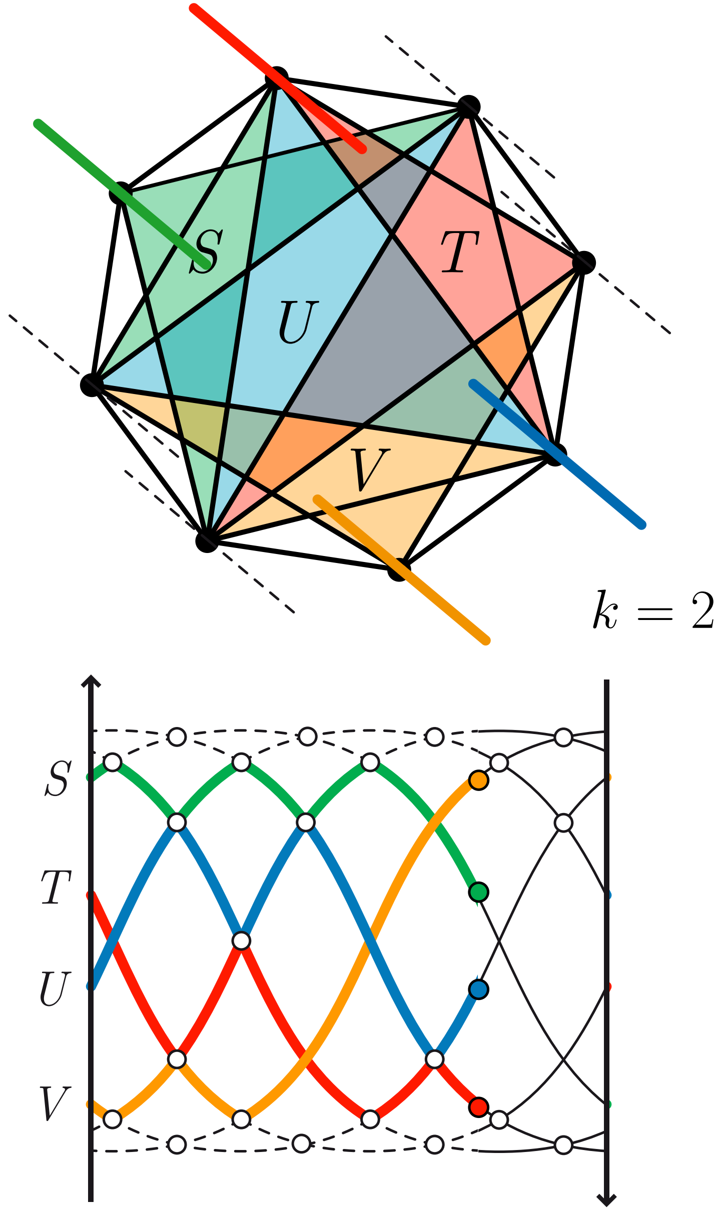

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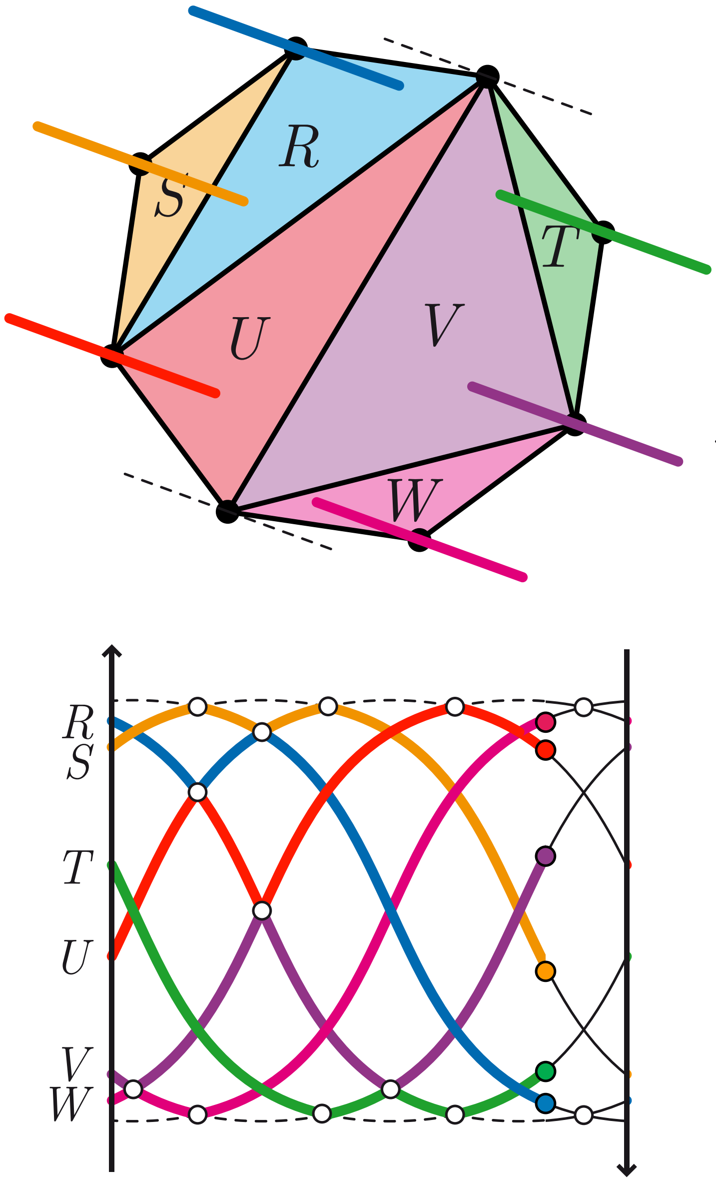

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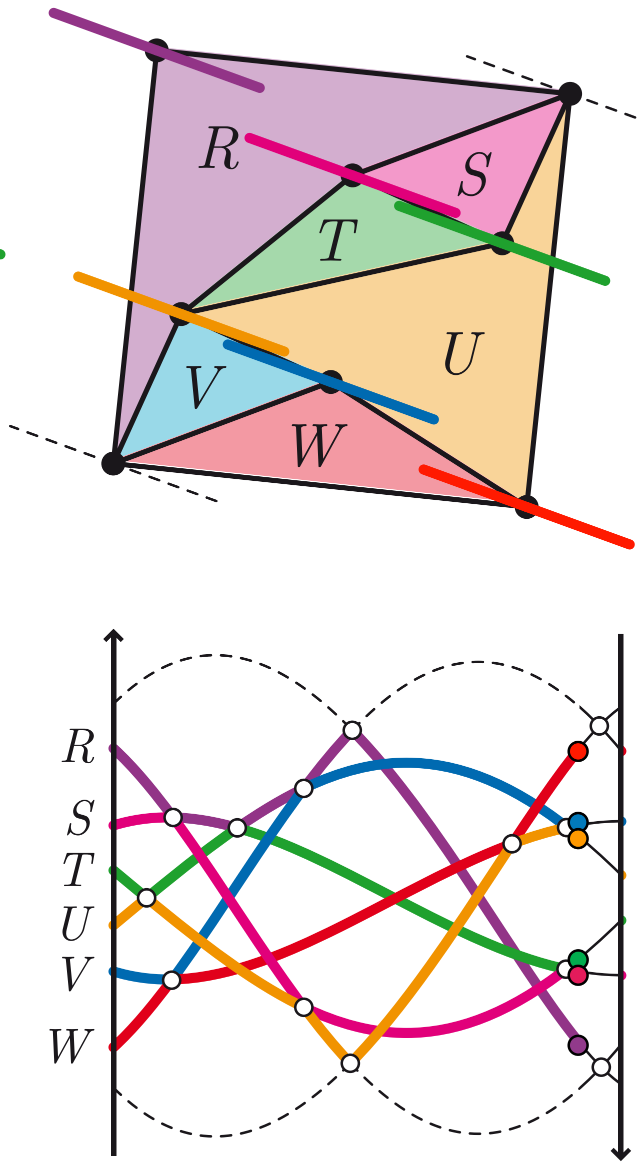

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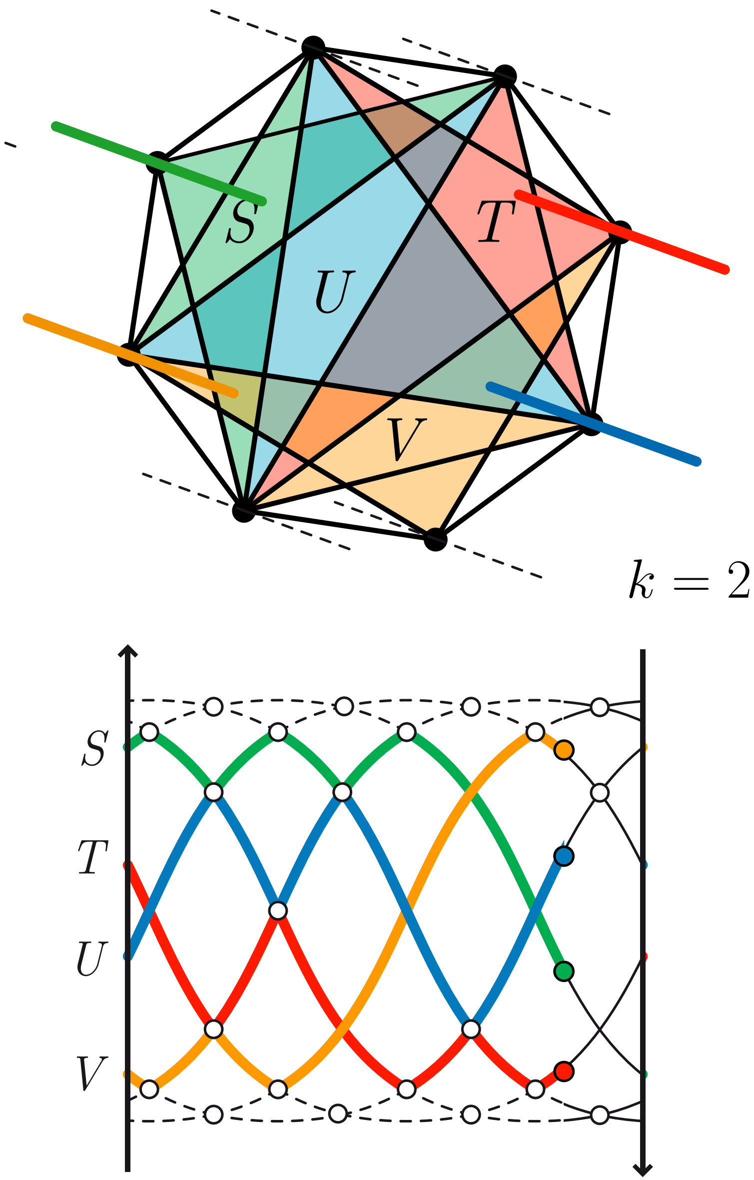
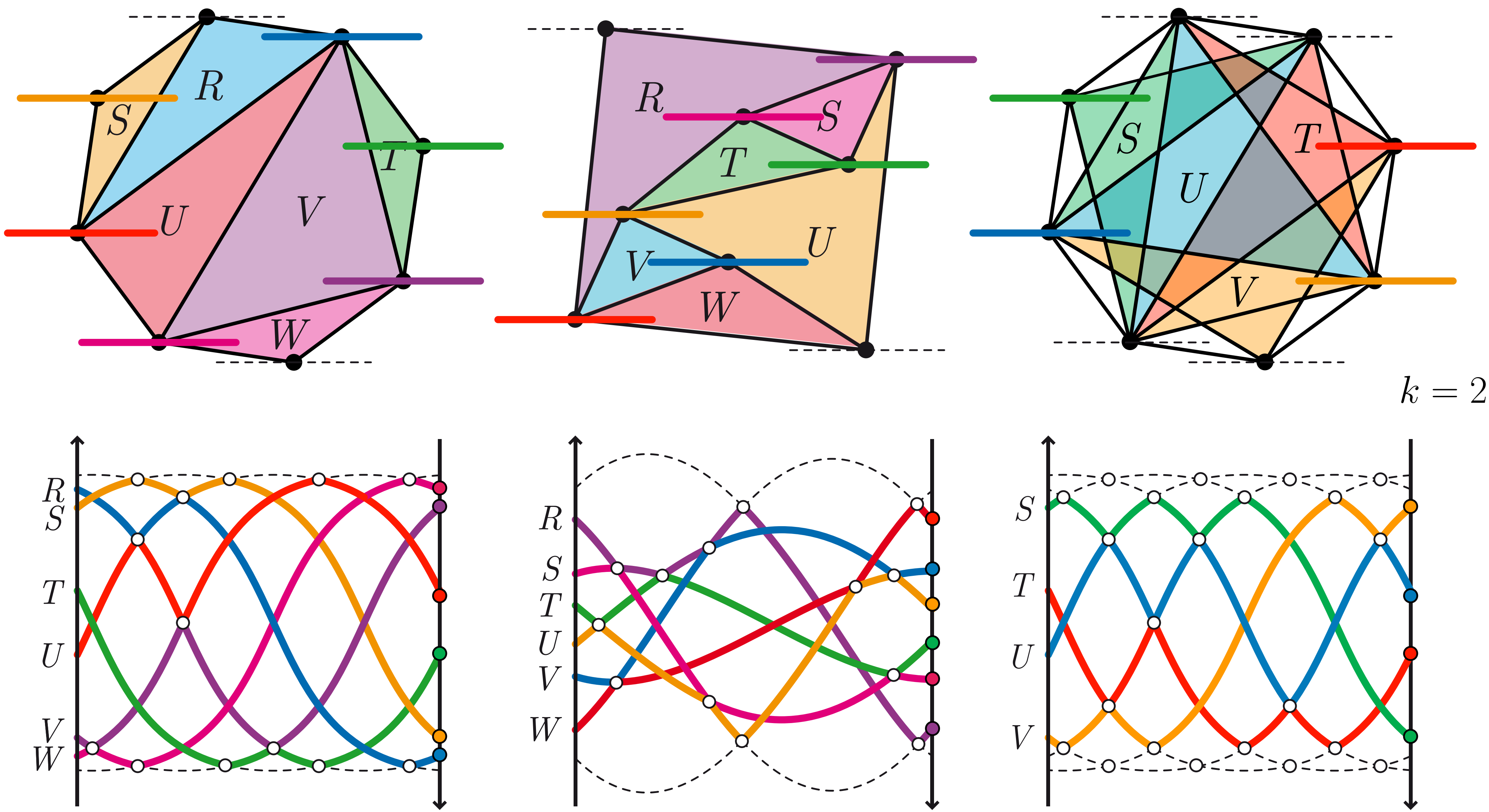


DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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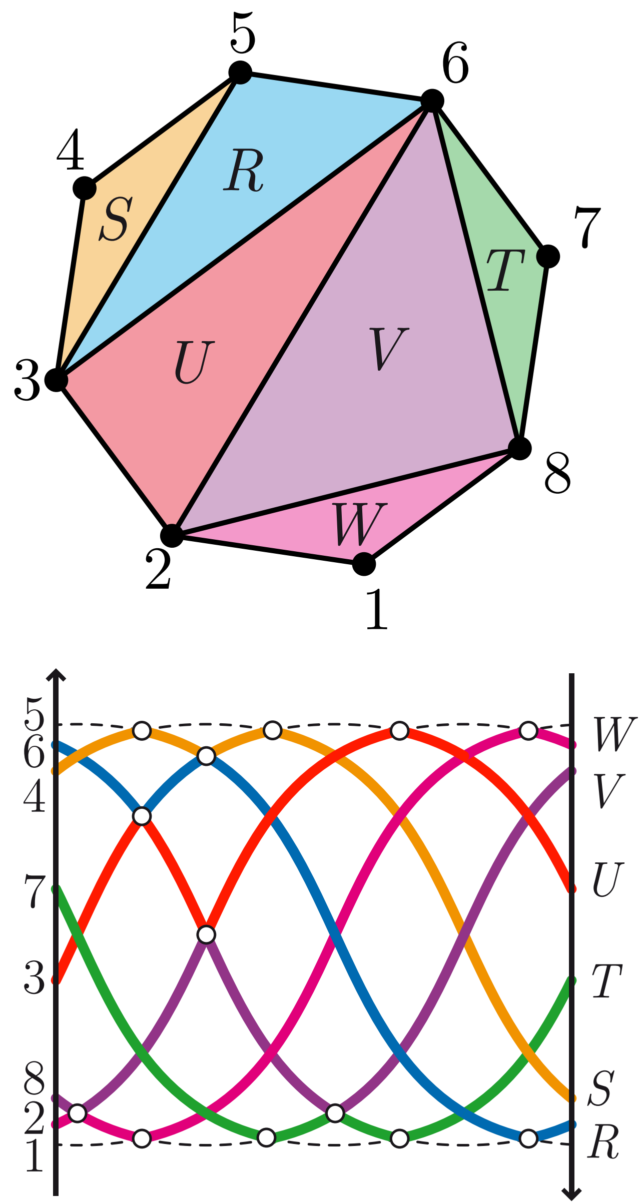

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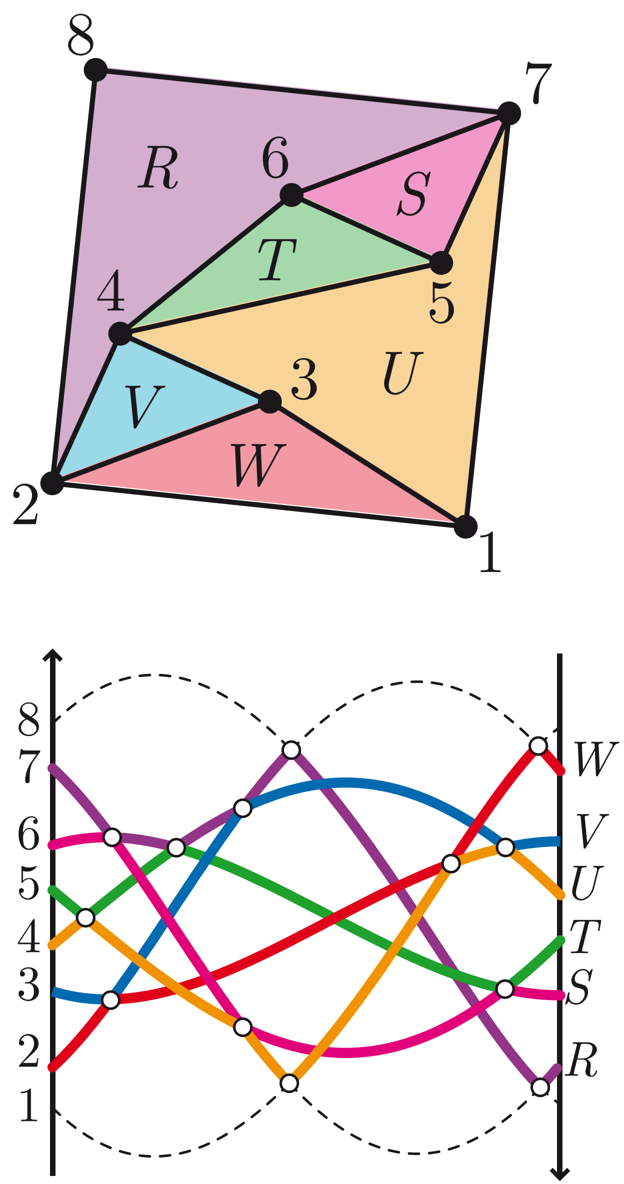

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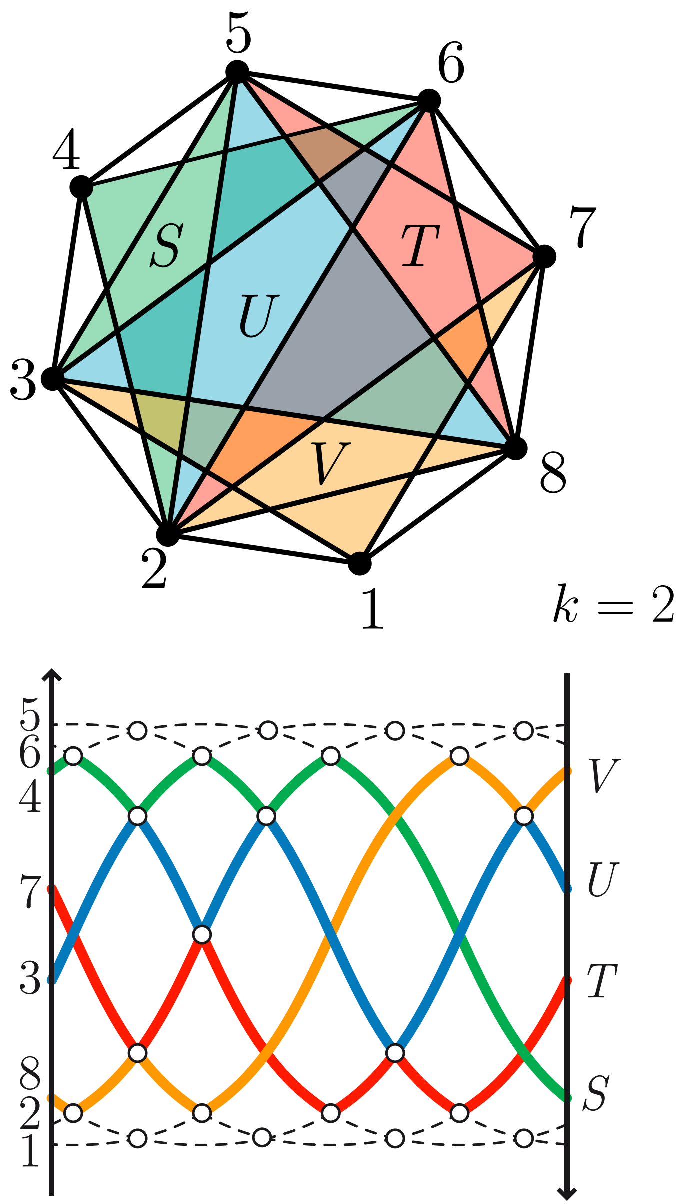
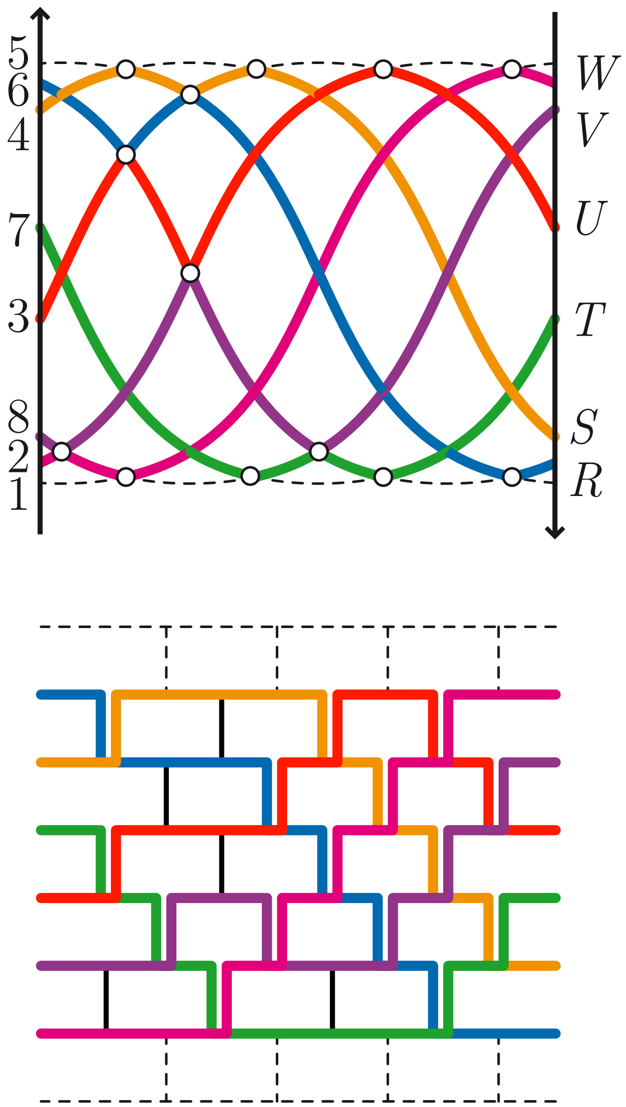


VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.

DUALITY

Triangulations



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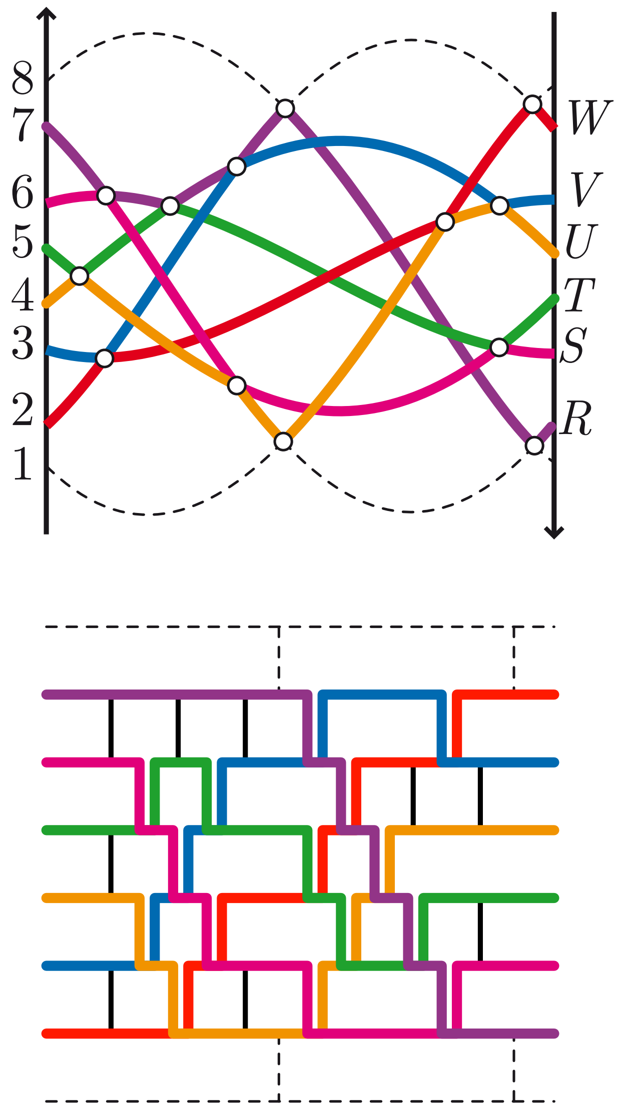

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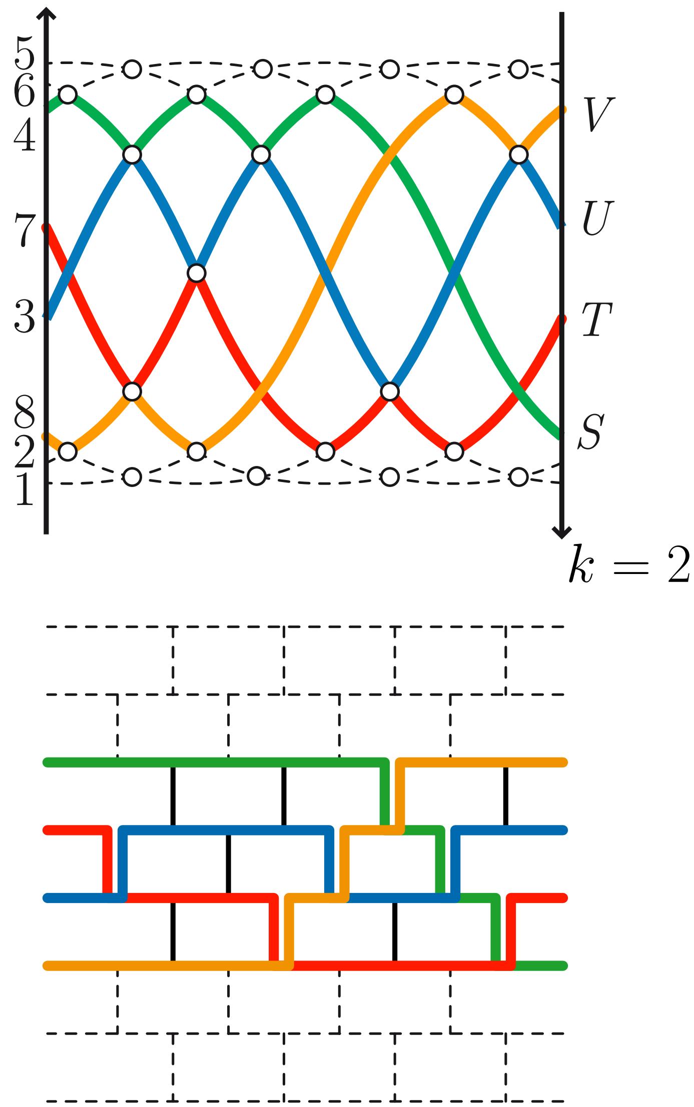
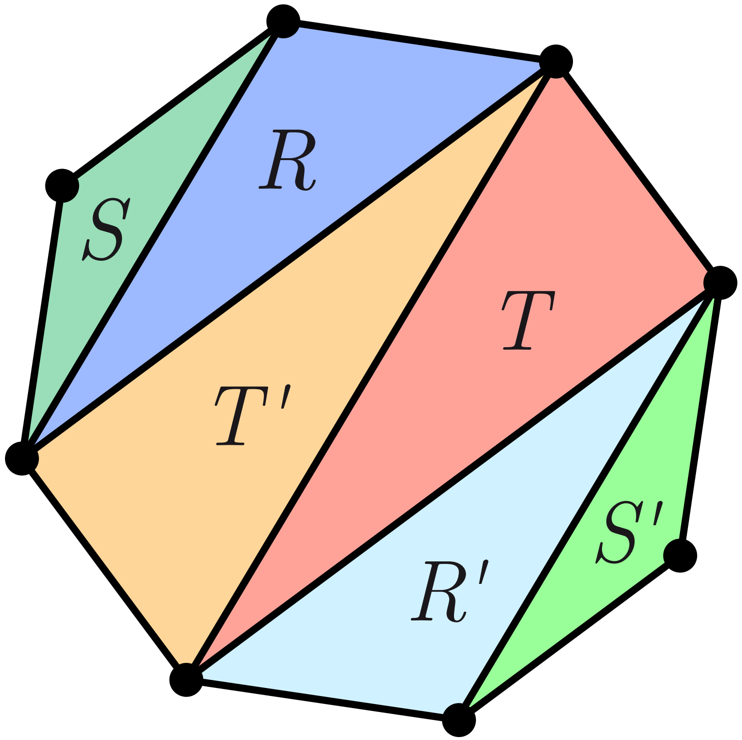
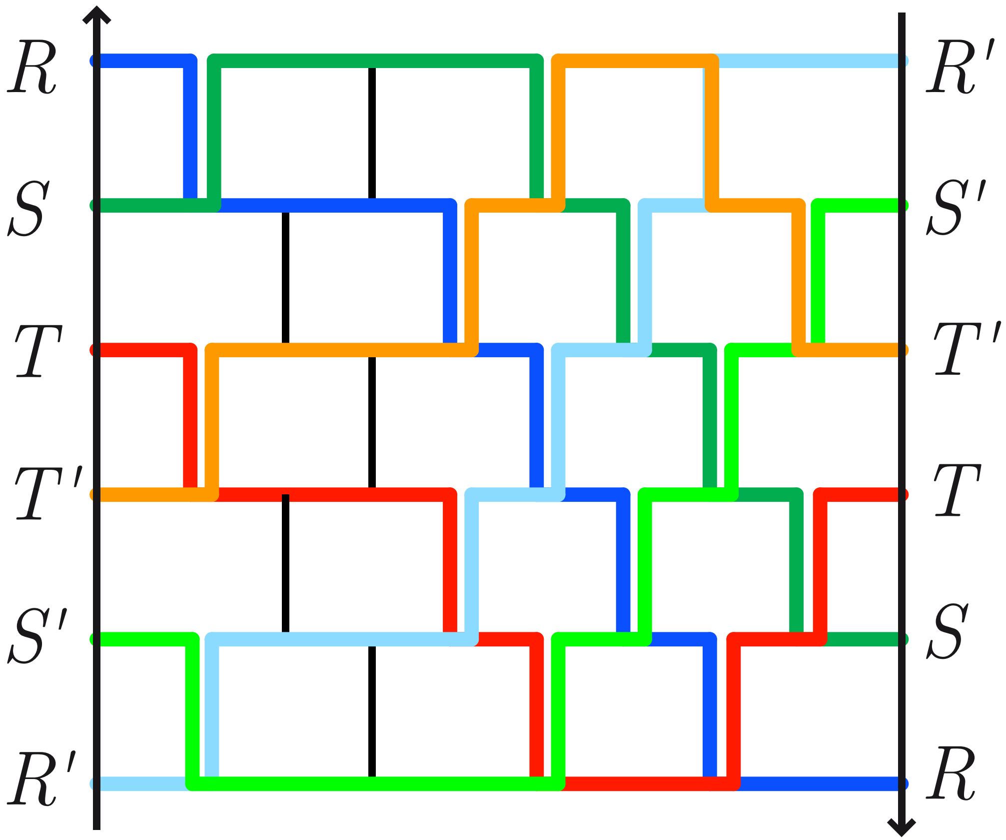


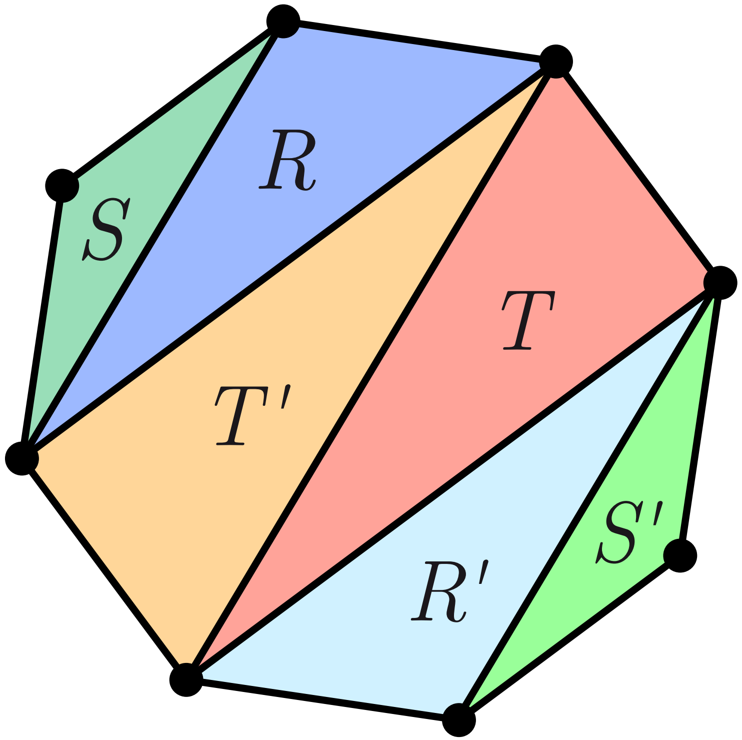
Type B subword complexes give models for centrally symmetric triangulations:

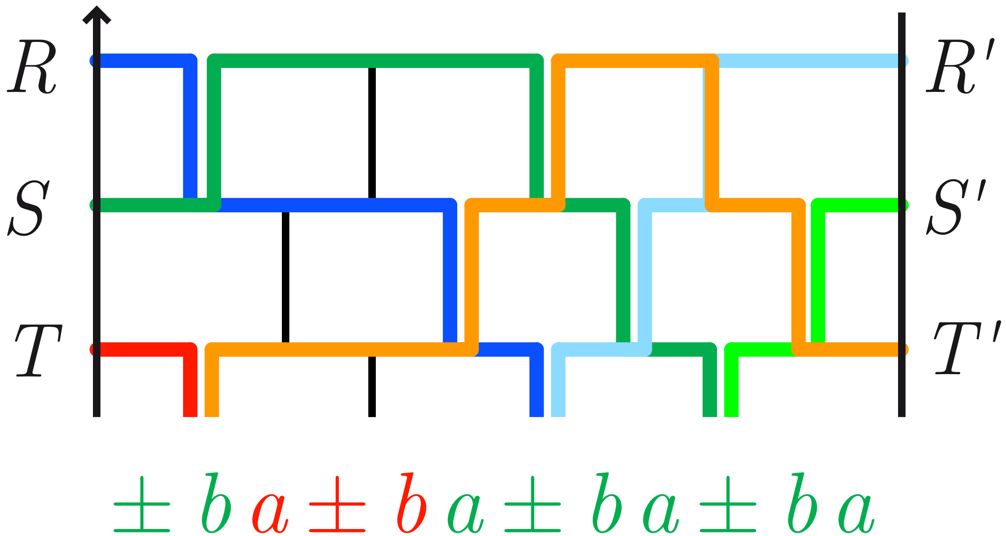


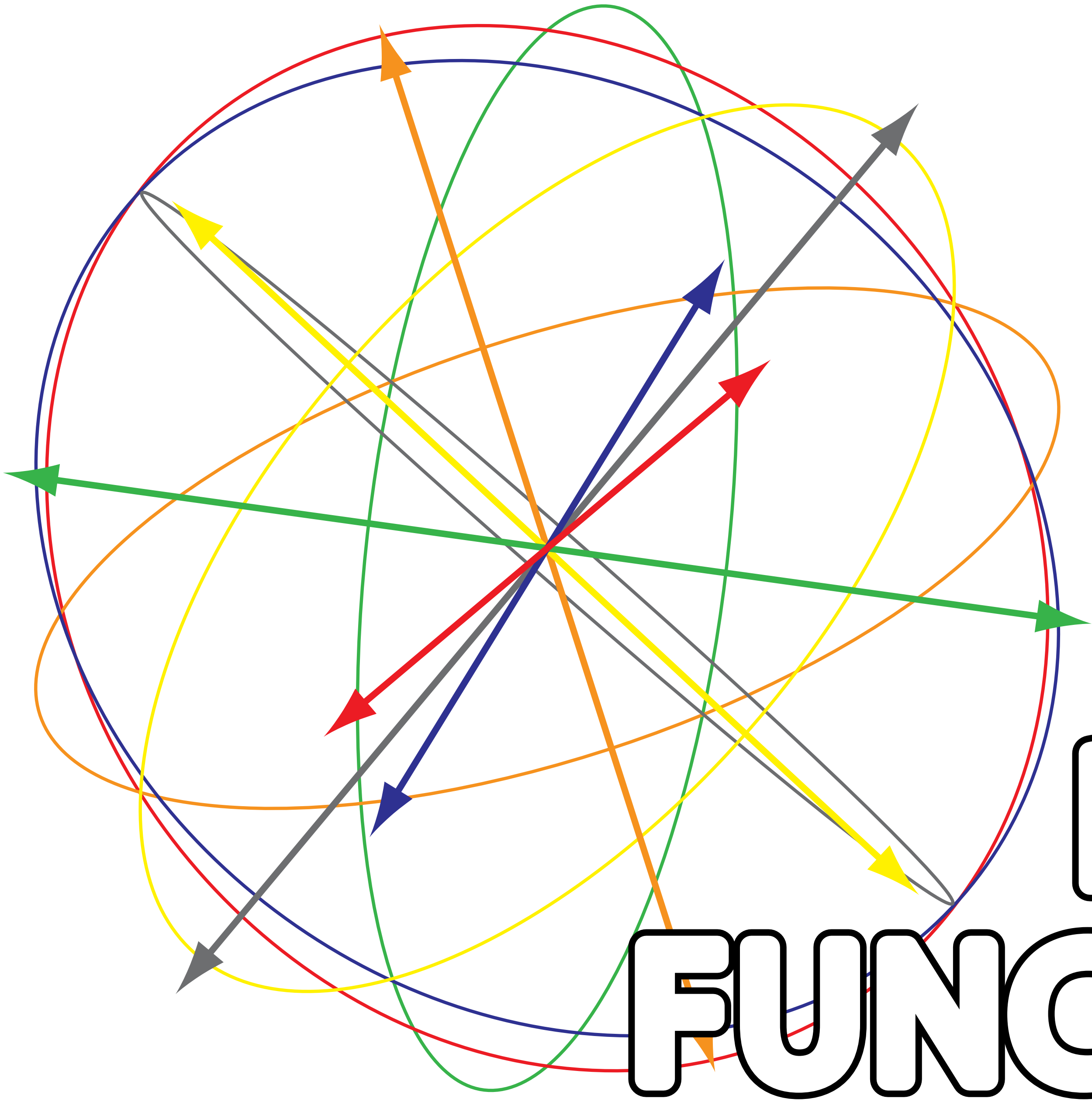


CENTRALLY SYMMETRIC GEOMETRIC GRAPHS

Type B subword complexes give models for centrally symmetric triangulations:





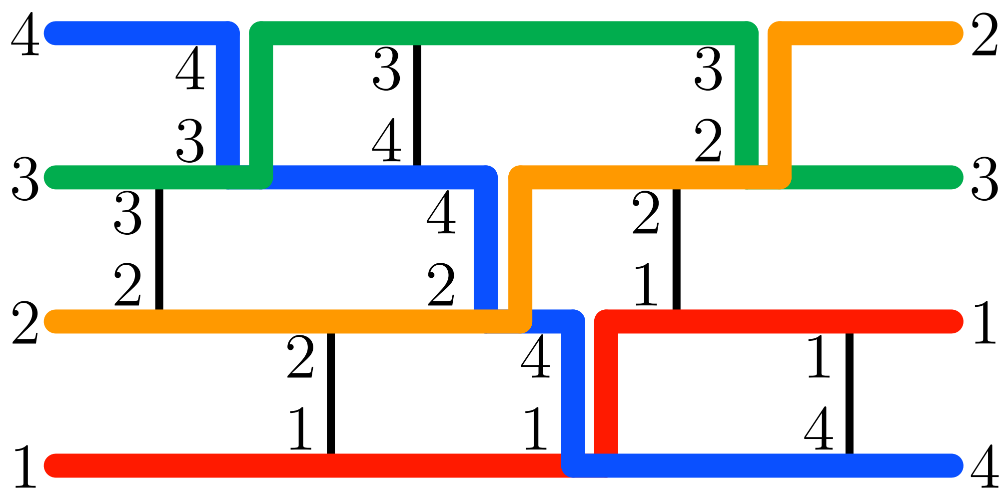


ROOT FUNCTION

C. Ceballos, JP. Labbé & C. Stump, Subword complexes, cluster complexes, & gener. multiassoc., 2011.

VP & C. Stump, Brick polytopes of spherical subword complexes, 2012^{+}.

ROOT FUNCTION

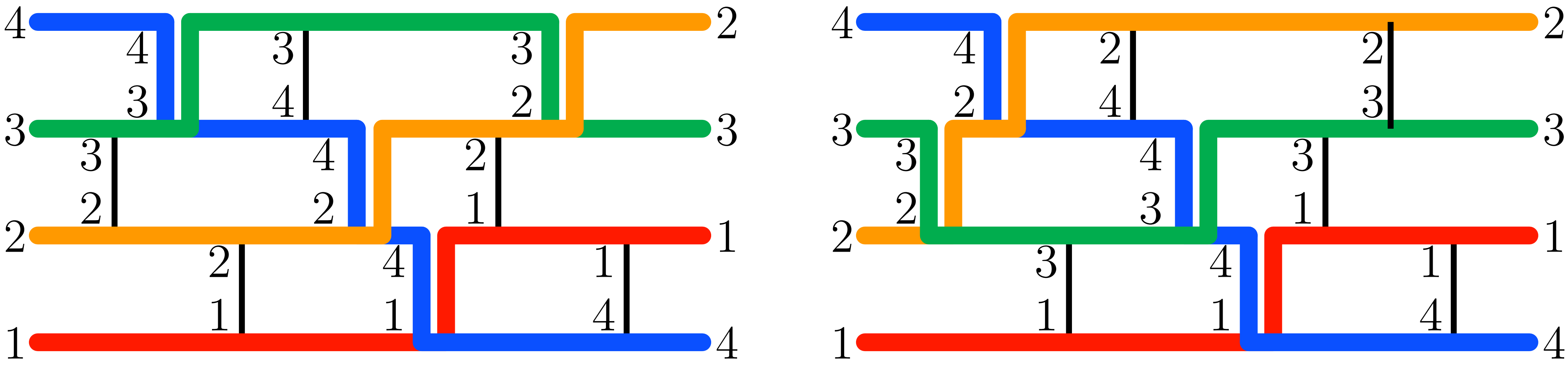


For a facet I of $\mathcal{S}(\mathrm{Q}, \rho)$ and a position $k \in [m]$, define the root $\mathsf{r}(I, k) = \mathrm{Q}_{[k-1] \setminus I}(\alpha_{q_k})$, where $\mathrm{Q}_{[k-1] \setminus I}$ is the product of all reflections q_j for j from 1 to $k-1$ but not in I.

The root function of the facet I is $\mathsf{r}(I, \cdot) : [m] \longrightarrow \Phi$

The root configuration of I is $\mathsf{R}(I) = \{\mathsf{r}(I, i) \mid i \in I\}$

ROOT FUNCTION & FLIPS

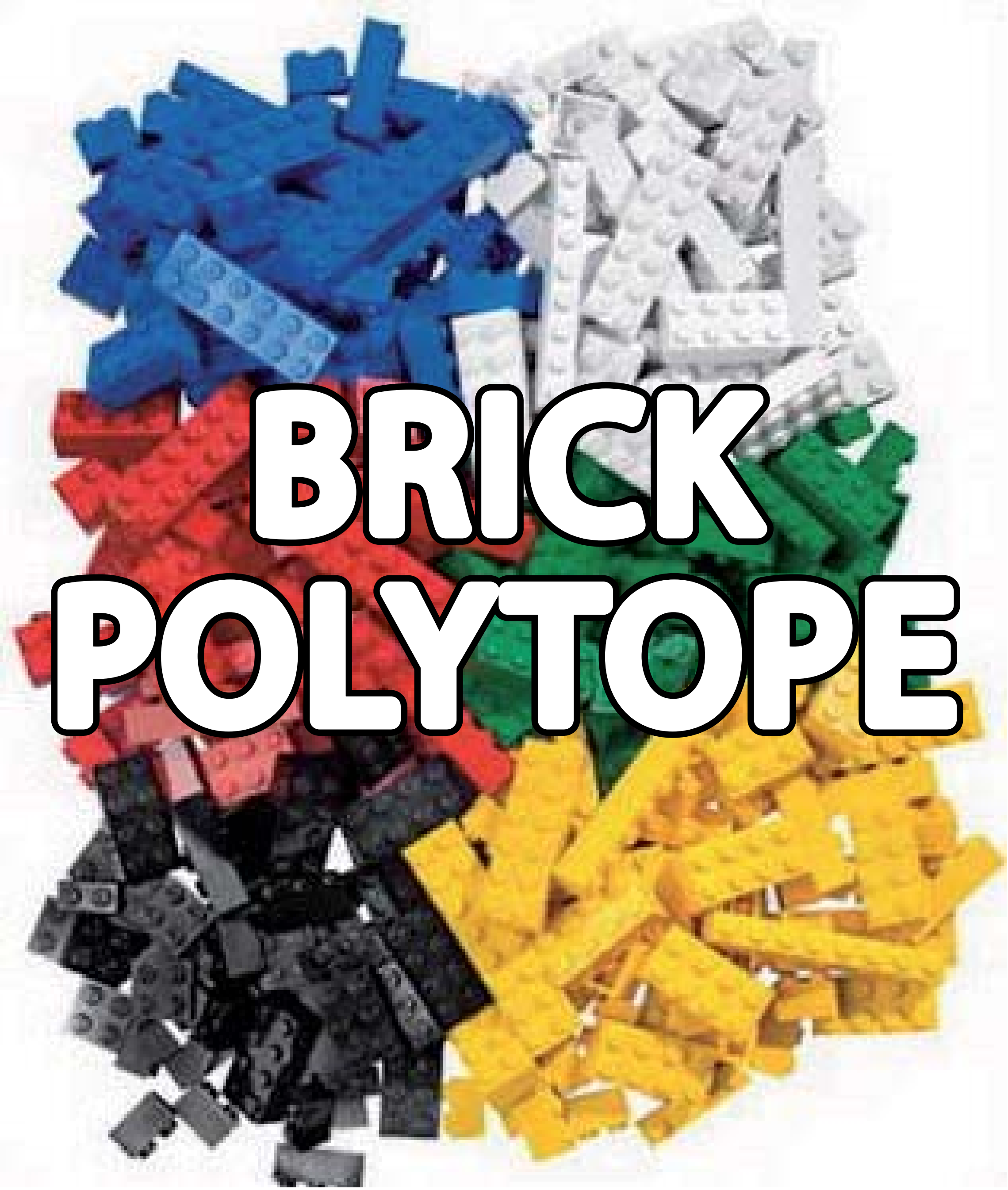


PROPOSITION. The root function encodes flips in subword complexes:

1. The map $\mathsf{r}(I, \cdot)$ is a bijection from the complement of I to $\mathrm{inv}(\rho)$.
2. If I and J are two adjacent facets of $\mathcal{S}(\mathrm{Q})$ with $I \smallsetminus i = J \smallsetminus j$, then j is the unique position in the complement of I such that $\mathsf{r}(I, i) = \pm\mathsf{r}(I, j)$.
3. In the situation of 2, the root function of J is obtained from that of I by

$$\mathsf{r}(J, k) = \begin{cases} s_{\mathsf{r}(I,i)}(\mathsf{r}(I, k)) & \text{if } \min(i, j) < k \le \max(i, j), \\ \mathsf{r}(I, k) & \text{otherwise.} \end{cases}$$

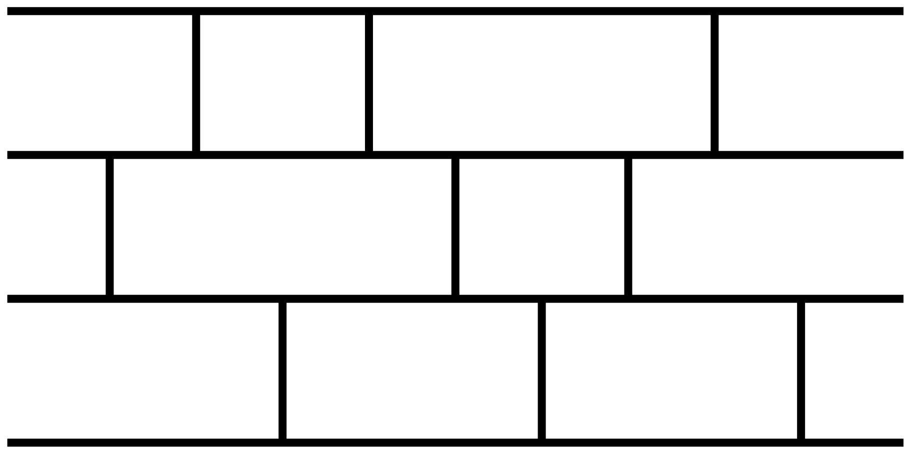
C. Ceballos, JP. Labbé & C. Stump, Subword complexes, cluster complexes, & gener. multiassoc., 2011.



VP & F. Santos, The brick polytope of a sorting network, 2012.

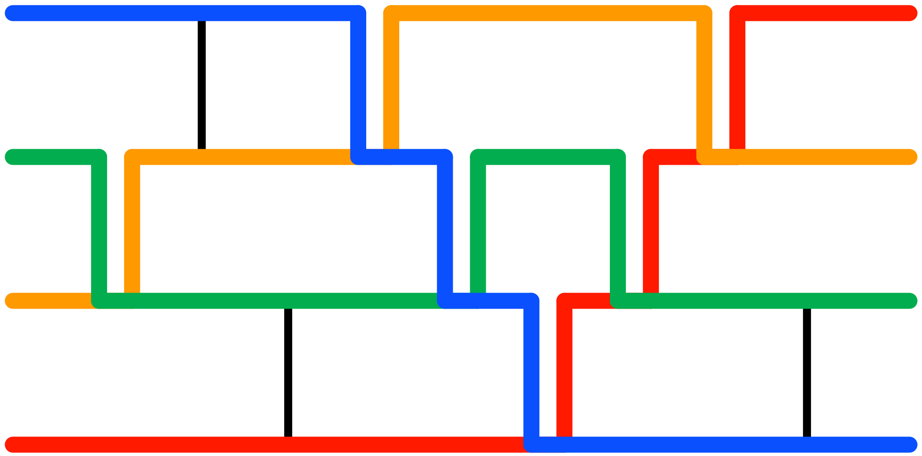
VP & C. Stump, Brick polytopes of spherical subword complexes, 2012^{+}.

BRICK POLYTOPE



$\mathcal{N}$ a sorting network with $n+1$ levels

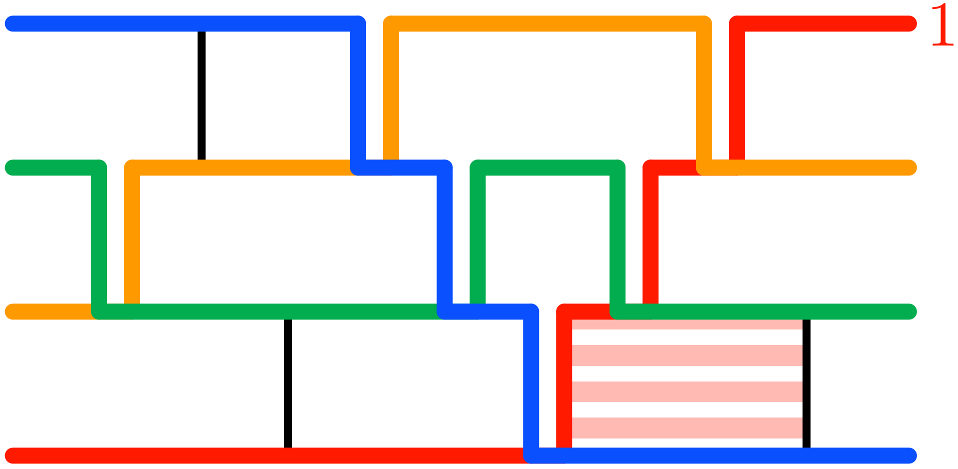
BRICK POLYTOPE



$\mathcal{N}$ a sorting network with $n+1$ levels

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\mathrm{B}(\Lambda) \in \mathbb{R}^{n+1}$

BRICK POLYTOPE

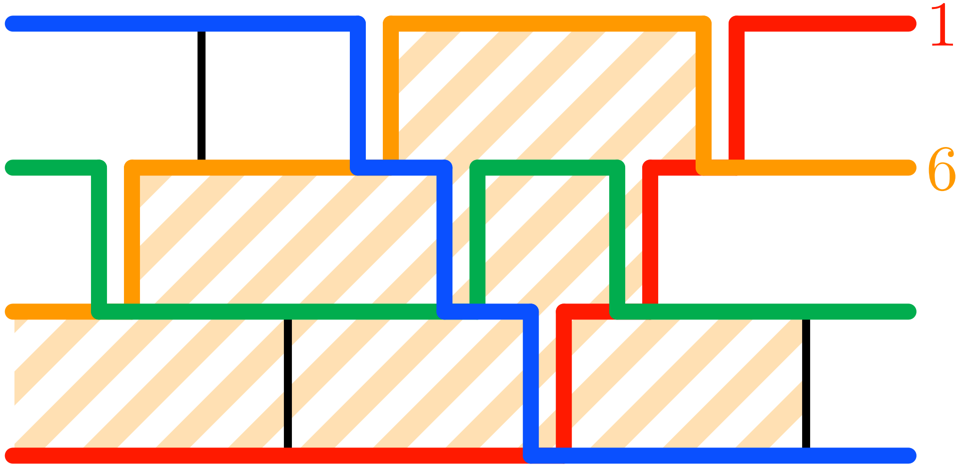


$\mathcal{N}$ a sorting network with $n+1$ levels

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\mathsf{B}(\Lambda) \in \mathbb{R}^{n+1}$

$\mathsf{B}(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

BRICK POLYTOPE



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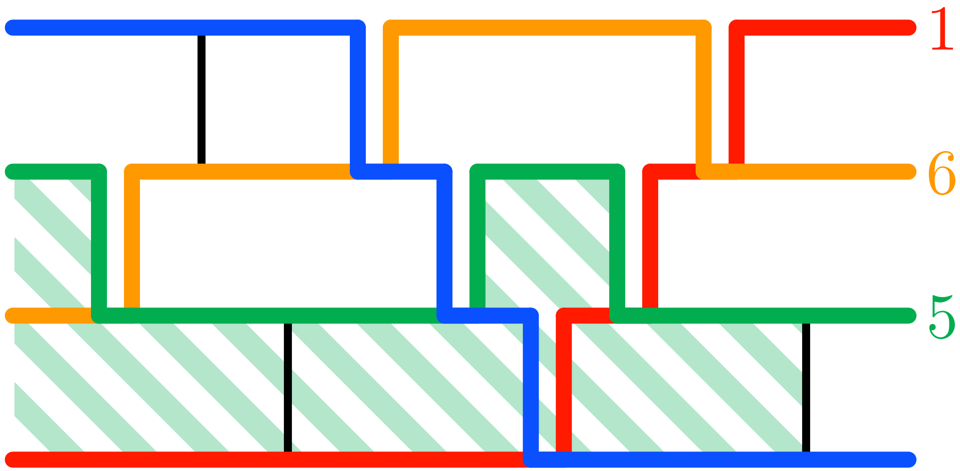

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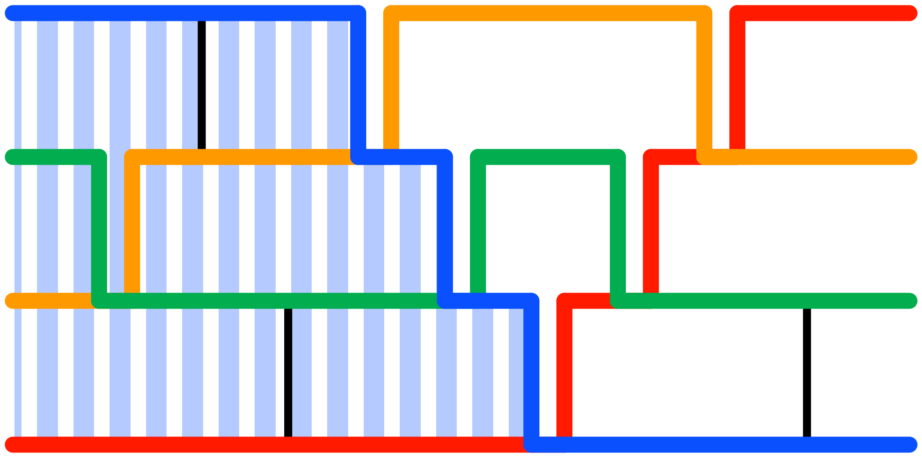

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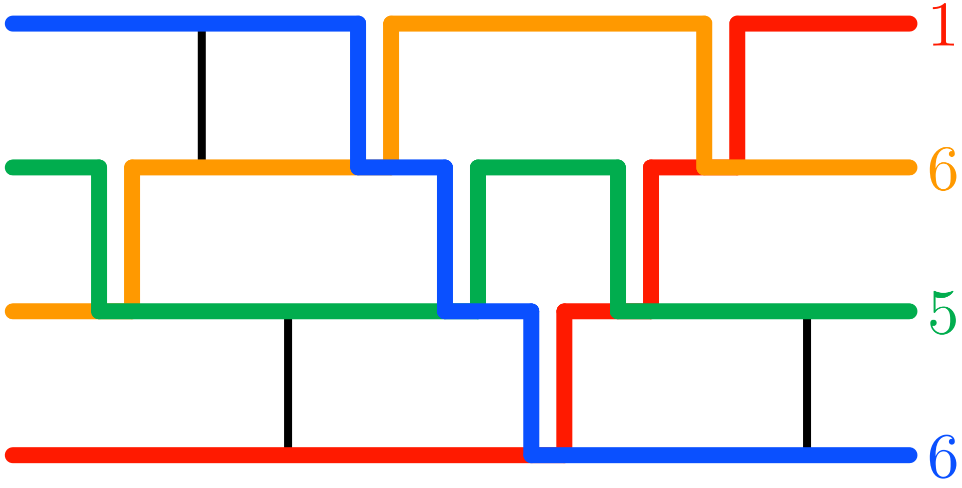


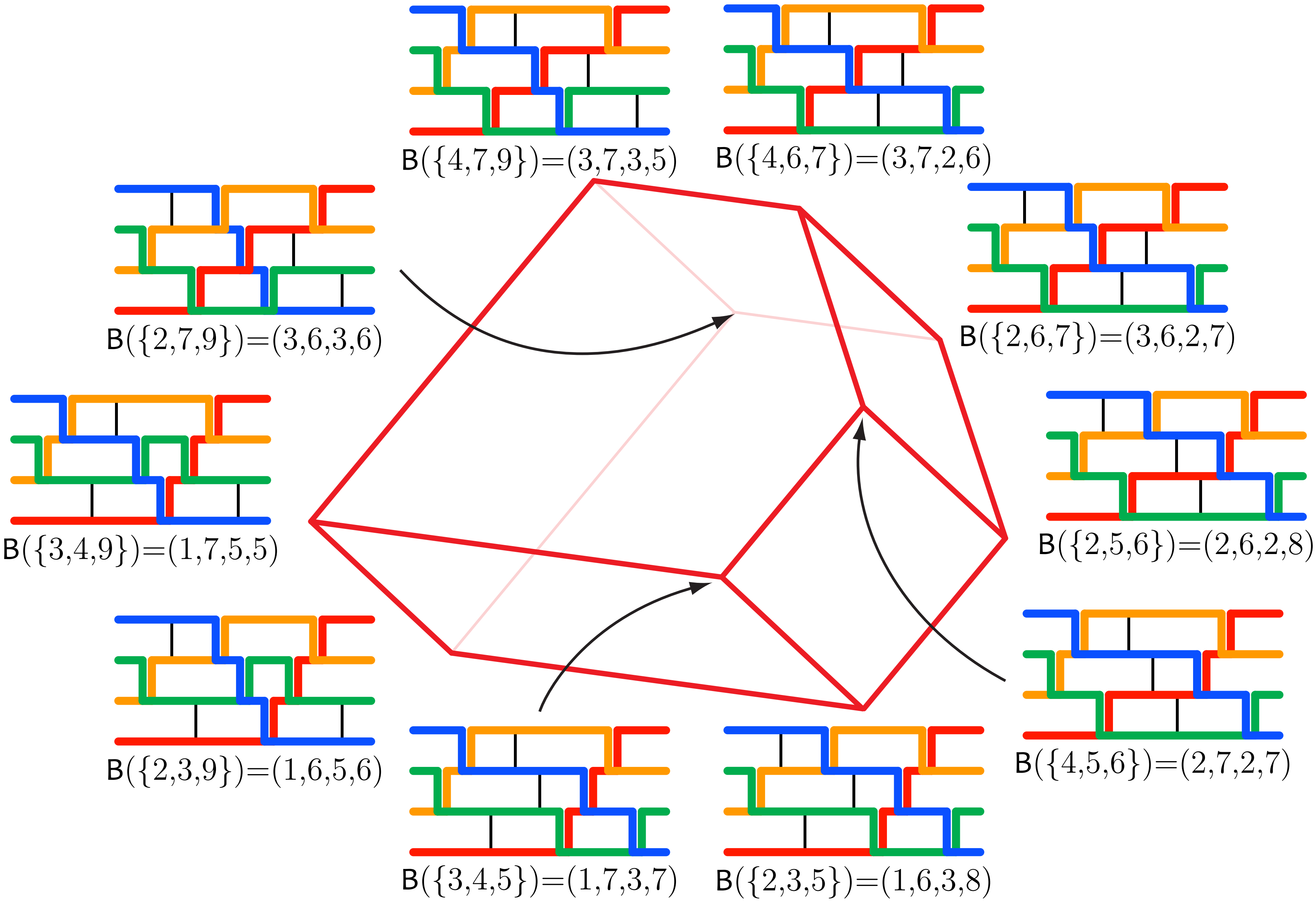
$\mathcal{N}$ a sorting network with $n+1$ levels

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\mathrm{B}(\Lambda) \in \mathbb{R}^{n+1}$

$\mathrm{B}(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

Brick polytope $\mathcal{B}(\mathcal{N}) = \mathrm{conv}\,\{\mathrm{B}(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

BRICK POLYTOPE



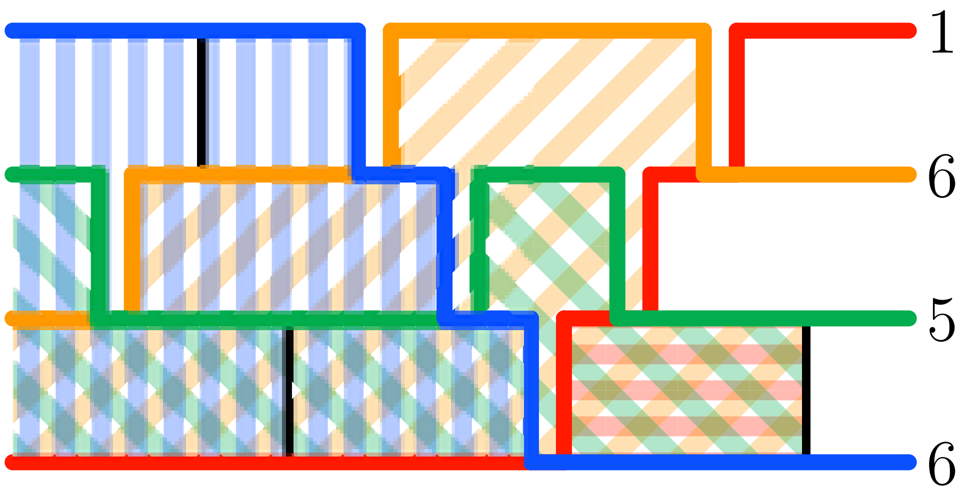
WEIGHT FUNCTION, BRICK VECTOR & BRICK POLYTOPE

(W, S) a finite Coxeter system, $\mathrm{Q} = q_1 q_2 \cdots q_m$ a word on S, $w_\circ$ longest element of W.
$\mathcal{S}(\mathrm{Q}) = \mathcal{S}(\mathrm{Q}, w_\circ)$ spherical subword complex.

To a facet I of $\mathcal{S}(\mathrm{Q})$ and a position $k \in [m]$, associate a weight $\mathsf{w}(I, k) = \mathrm{Q}_{[k-1] \smallsetminus I}(\omega_{q_k})$, where $\mathrm{Q}_{[k-1] \smallsetminus I}$ is the product of all reflections q_j for j from 1 to $k-1$ but not in I.

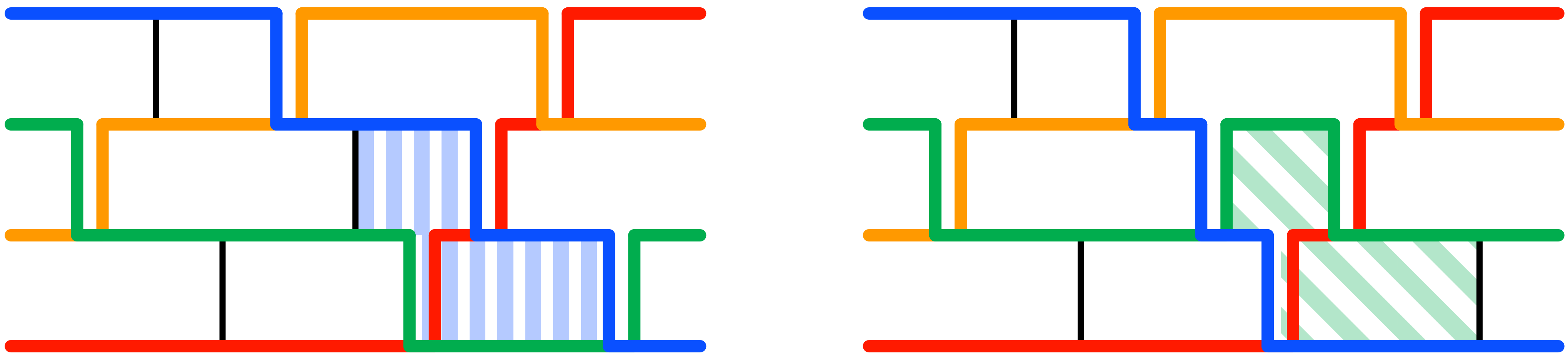
The brick vector of I is the vector $\mathsf{B}(I) = \sum_{k \in [m]} \mathsf{w}(I, k)$.

The brick polytope is the convex polytope $\mathcal{B}(\mathrm{Q}) = \mathrm{conv}\left\{\mathsf{B}(I) \mid I \text{ facet of } \mathcal{S}(\mathrm{Q})\right\}$.



In type A, $\mathsf{w}(I, k)$ = characteristic vector of the pseudolines passing above the kth brick.
$\mathsf{B}(I) = (\text{number of bricks below the } j\text{th pseudoline of } I)_{j \in [n+1]}$

BRICK VECTORS AND FLIPS



If Λ and Λ' are two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip between their ith and jth pseudolines, then $\mathsf{B}(\Lambda) - \mathsf{B}(\Lambda') \in \mathbb{N}_{>0}\,(e_j - e_i)$.

THEOREM. The cone of the brick polytope $\mathcal{B}(\mathrm{Q})$ at the brick vector $\mathsf{B}(I)$ is generated by $-\mathsf{R}(I)$, for any facet I of $\mathcal{S}(\mathrm{Q})$.

BRICK POLYTOPE

The brick polytope is the convex polytope $\mathcal{B}(\mathrm{Q}) = \mathrm{conv}\,\{\mathrm{B}(I) \mid I \text{ facet of } \mathcal{S}(\mathrm{Q})\}$.

THEOREM. The polar of the brick polytope $\mathcal{B}(\mathrm{Q})$ realizes the subword complex $\mathcal{S}(\mathrm{Q})$
$\iff$ Q is such that $\mathrm{R}(I)$ is linearly independent, for I facet of $\mathcal{S}(\mathrm{Q})$.

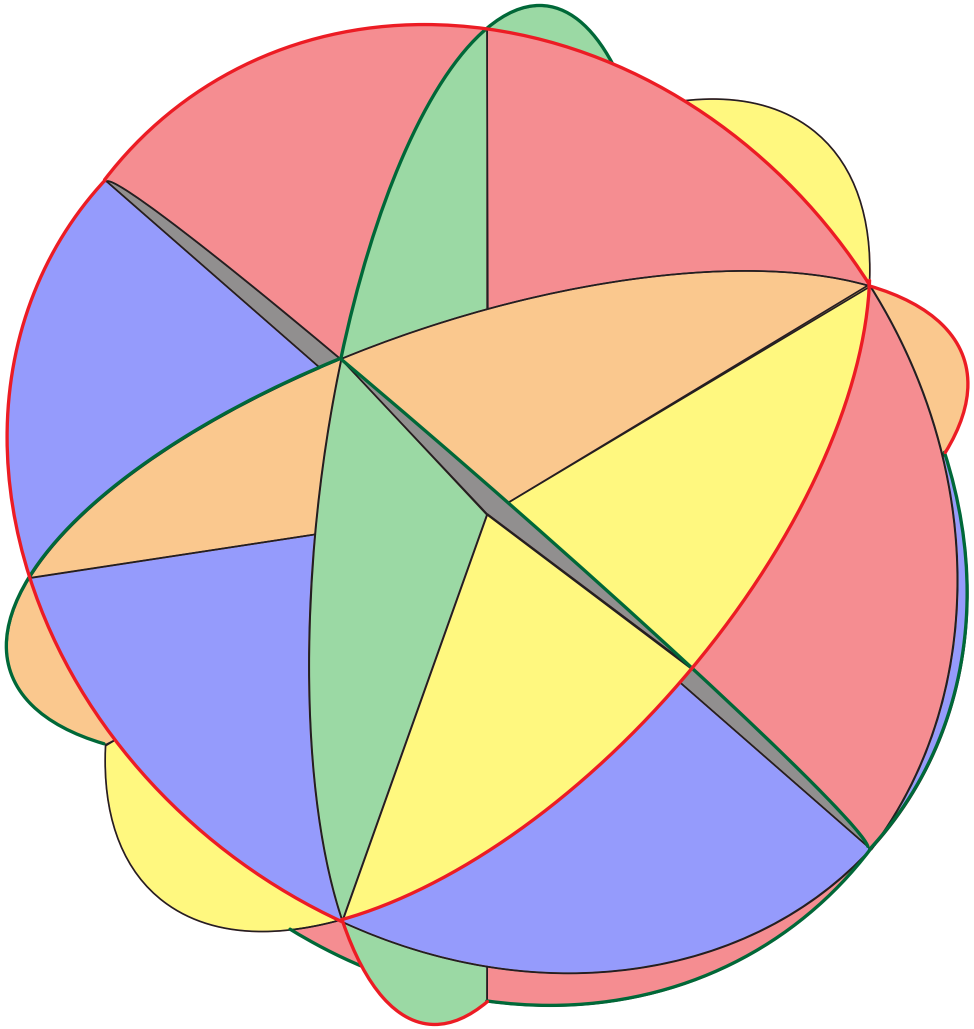
THEOREM. If Q is realizing, the cone of the brick polytope $\mathcal{B}(\mathrm{Q})$ at the brick vector $\mathrm{B}(I)$ is generated by $-\mathrm{R}(I)$, for any facet I of $\mathcal{S}(\mathrm{Q})$.

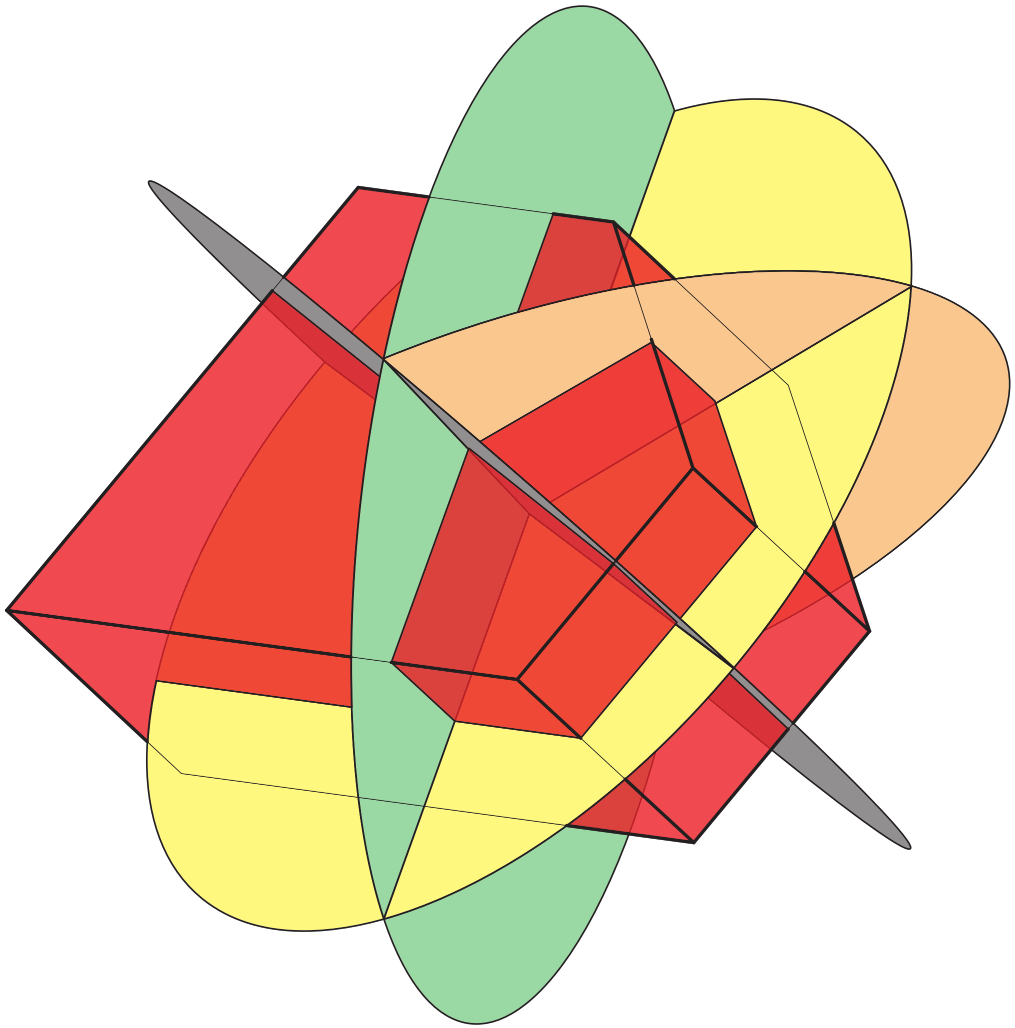
THEOREM. If Q is realizing, the Coxeter fan refines the normal fan of the brick polytope. More precisely,

$$\text{normal cone of } \mathrm{B}(I) \text{ in } \mathcal{B}(\mathrm{Q}) = \bigcup_{\substack{w \in W \\ \mathrm{R}(I) \subset w(\Phi^+)}} w(\text{ fundamental cone }).$$

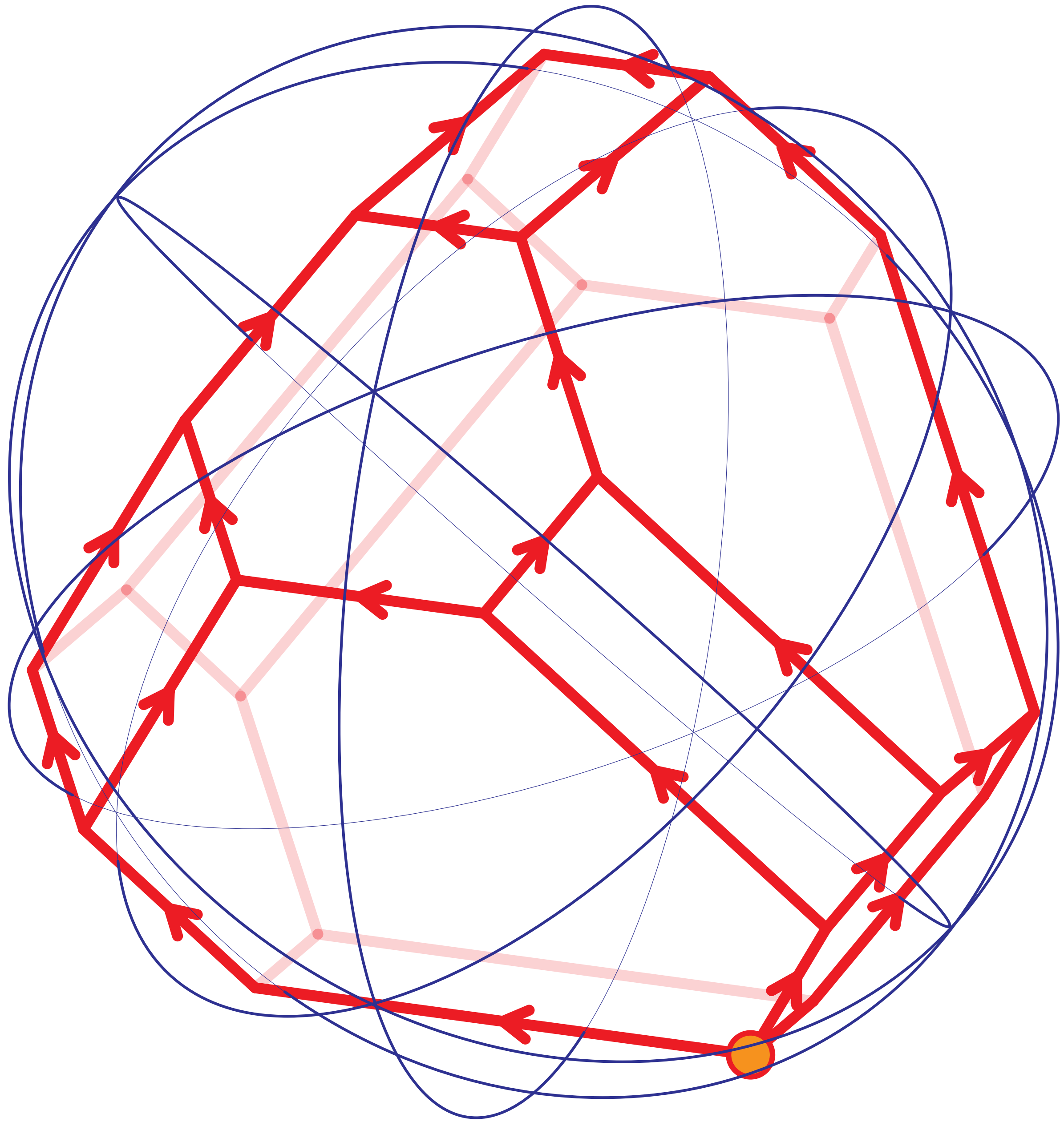
NORMAL FAN

THEOREM. If $\mathbb{Q}$ is realizing, the Coxeter fan refines the normal fan of the brick polytope.



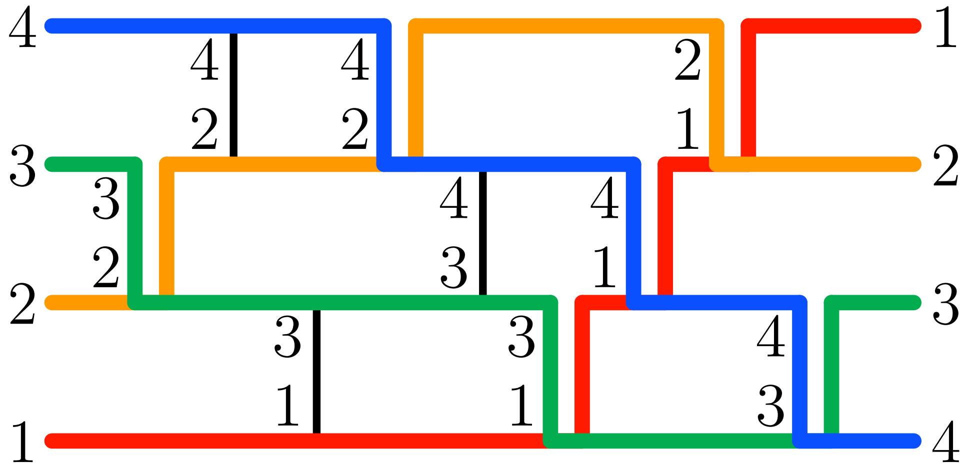


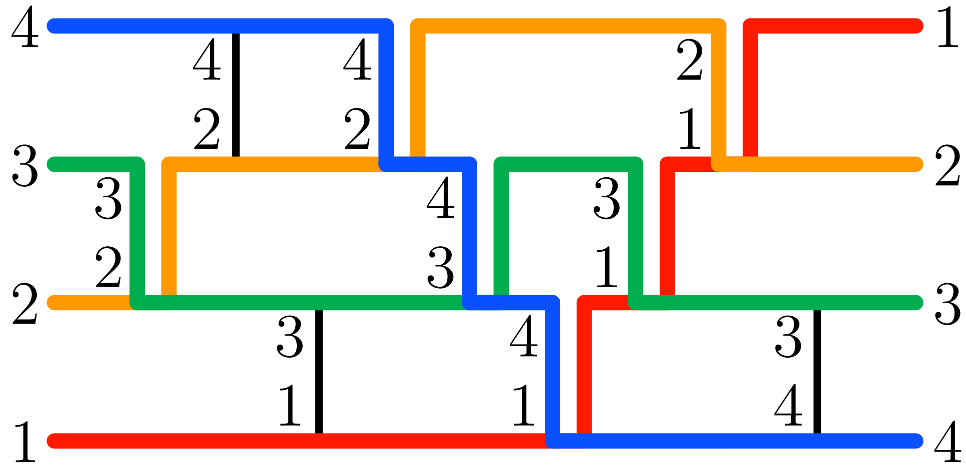
REMEMBER THE RIGHT WEAK ORDER



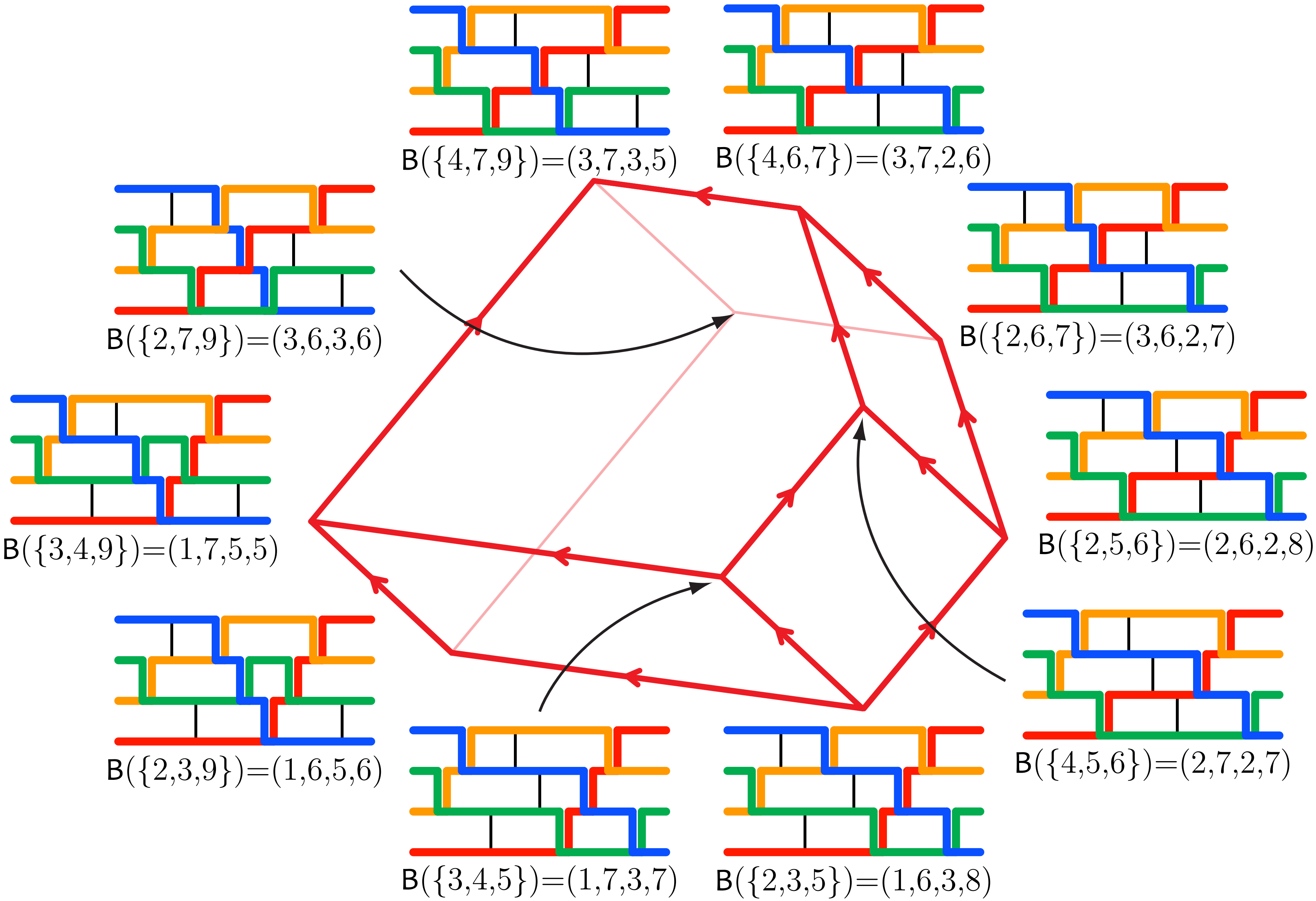
INCREASING FLIP GRAPH

I, J two adjacent facets of $\mathcal{S}(\mathrm{Q})$, with $I \smallsetminus i = J \smallsetminus j$.
The flip from I to J is increasing if $i < j$.





INCREASING FLIP GRAPH



INCREASING FLIP GRAPH

I, J two adjacent facets of $\mathcal{S}(\mathrm{Q})$, with $I \smallsetminus i = J \smallsetminus j$.
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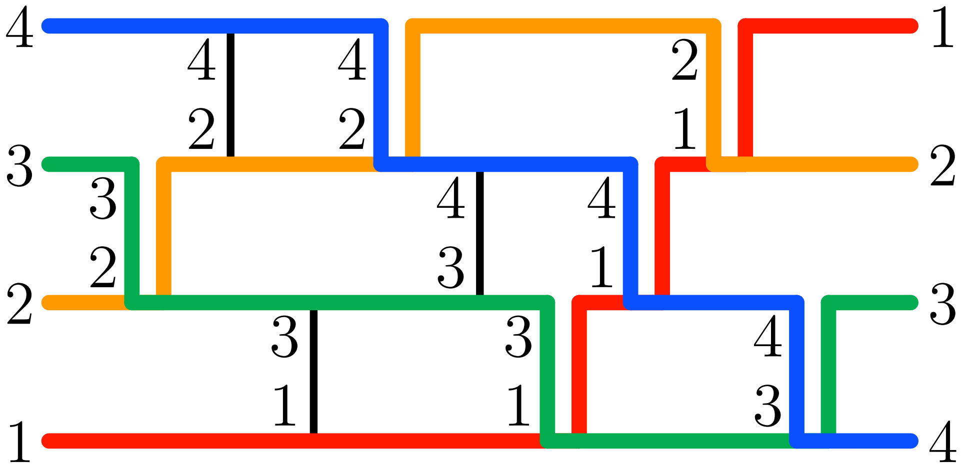
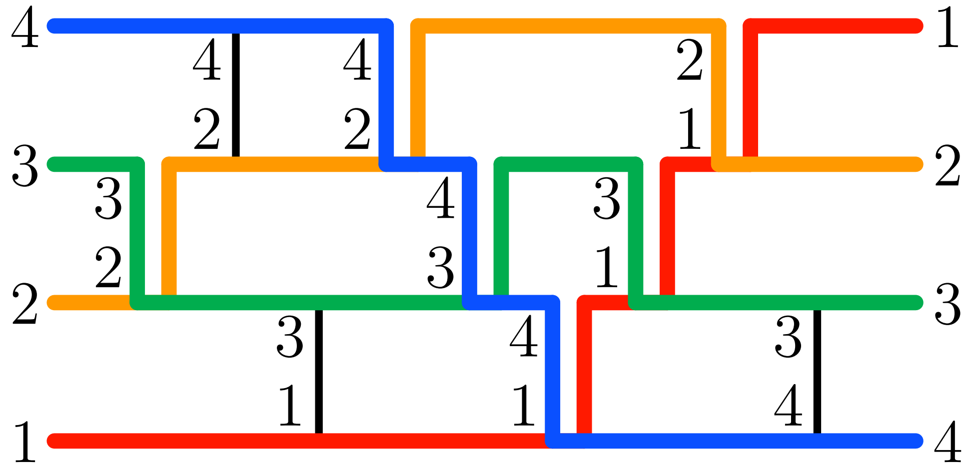
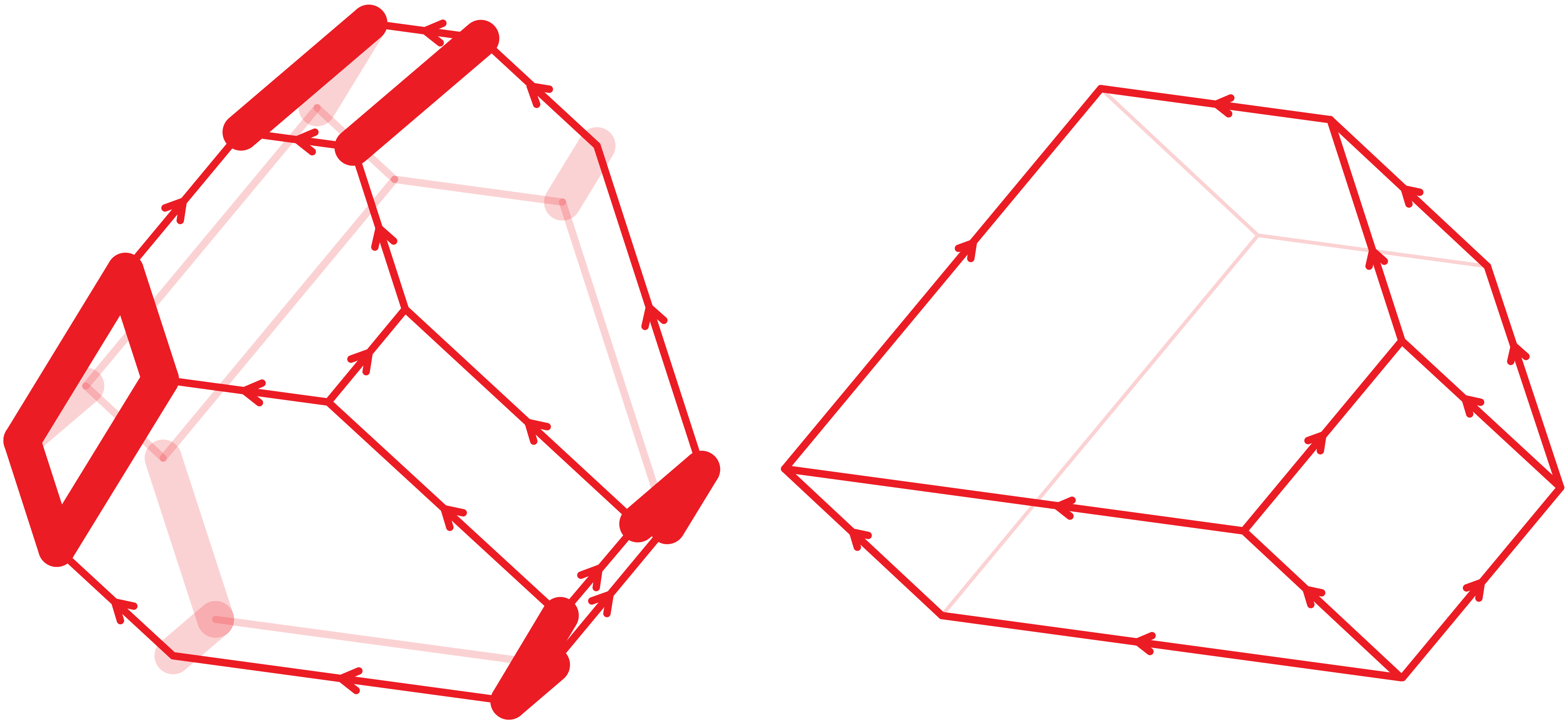




THEOREM. Assume that Q is realizing. Then I is covered by J in increasing flip order iff there exists $w_I, w_J \in W$ with $\mathrm{R}(I) \subset w_I(\Phi^+)$, $\mathrm{R}(J) \subset w_J(\Phi^+)$ and w_I is covered by w_J in weak order.

In other words, the oriented graph of the brick polytope is a quotient of the oriented graph of the permutohedron.

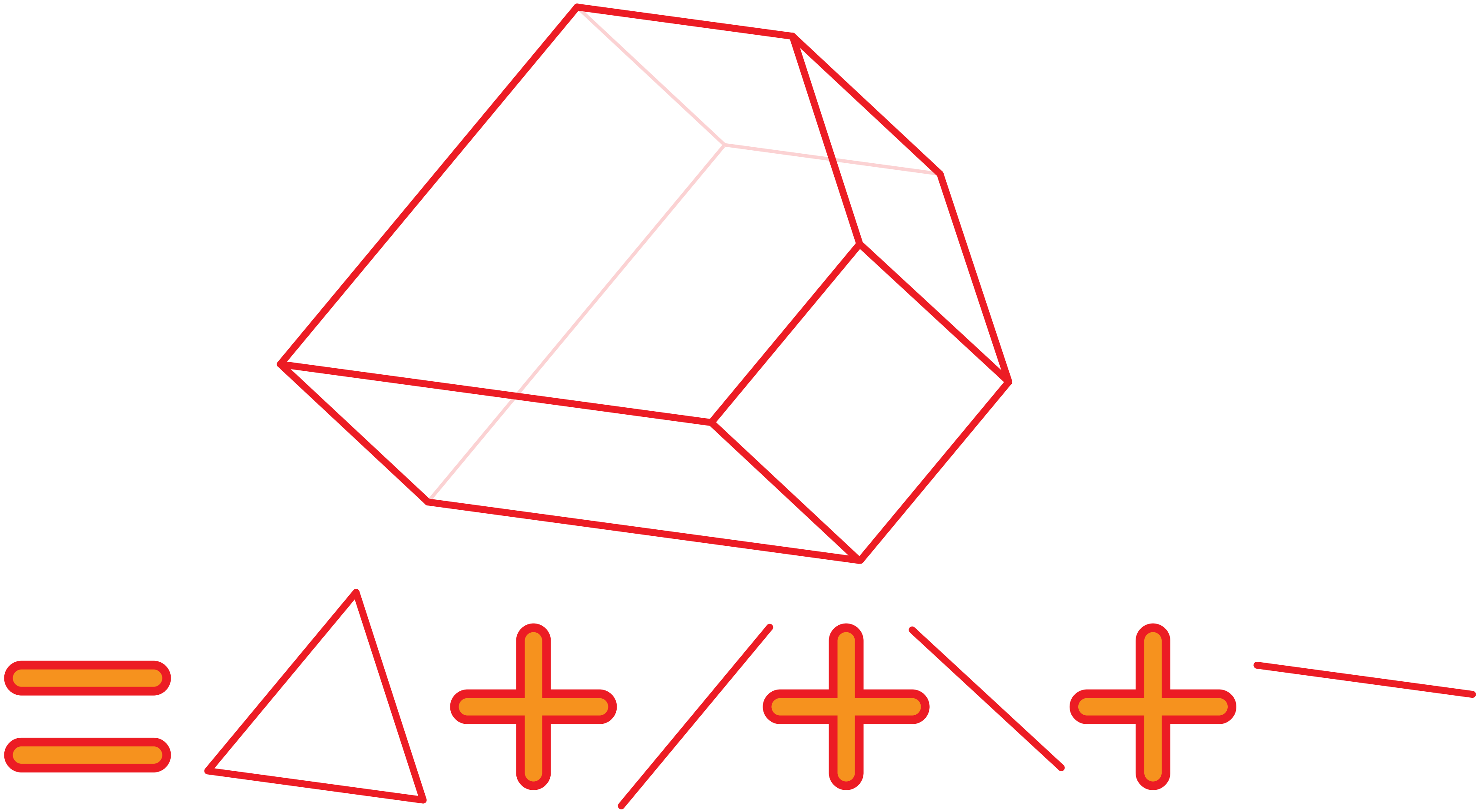
THEOREM. If $\mathbb{Q}$ is realizing, the Hasse diagram of the increasing flip order is a quotient of the Hasse diagram of the weak order.



MINKOWSKI SUM

THEOREM. If Q is realizing, the brick polytope $\mathcal{B}(\mathrm{Q})$ is the Minkowski sum of the polytopes $\mathcal{B}(\mathrm{Q}, k) = \mathrm{conv}\,\{\mathsf{w}(I,k) \mid I \text{ facet of } \mathcal{S}(\mathrm{Q})\}$. In other words,

$$\mathcal{B}(\mathrm{Q}) = \mathrm{conv}_I \sum_k \mathsf{w}(I,k) = \sum_k \mathrm{conv}_I \mathsf{w}(I,k) = \sum_k \mathcal{B}(\mathrm{Q}, k).$$



GENERALIZED ASSOCIAHEDRA

F. Chapoton, S. Fomin & A. Zelevinsky, Polytopal realizations of generalized associahedra, 2002.
C. Hohlweg, C. Lange & H. Thomas, Permutahedra and generalized associahedra, 2011.
S. Stella, Polyhedral models for generalized associahedra via Coxeter elements, 2011.
VP & C. Stump, Brick polytopes of spherical subword complexes, 2012^{+}.
C. Hohlweg, Permutahedra and associahedra, 2013.

CLUSTER ALGEBRAS

cluster algebra $=$ commutative ring generated by distinguished cluster variables grouped into overlapping clusters

clusters computed by a mutation process :

cluster seed $=$ algebraic data $\{x_1, \dots, x_n\}$, combinatorical data B (matrix or quiver)

cluster mutation $= (\{x_1, \dots, x_k, \dots, x_n\}, B) \overset{\mu_k}{\longleftrightarrow} (\{x_1, \dots, x'_k, \dots, x_n, \mu_k(B))$

$$x_k \cdot x'_k = \prod_{i,\, b_{ik} > 0} x_i^{b_{ik}} + \prod_{i,\, b_{ik} < 0} x_i^{-b_{ik}}$$

$$\big(\mu_k(B)\big)_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + |b_{ik}| \cdot b_{kj} & \text{if } k \notin \{i, j\} \text{ and } b_{ik} \cdot b_{kj} > 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

cluster complex $=$ simplicial complex w/ vertices $=$ cluster variables & facets $=$ clusters

S. Fomin & A. Zelevinsky, Cluster Algebras I, II, III, IV, 2002 – 2007.

CLUSTER ALGEBRAS

THEOREM. (Laurent phenomenon)
All cluster variables are Laurent polynomials in the variables of the initial cluster seed.

S. Fomin & A. Zelevinsky, Cluster algebras I: Fundations, 2002.

THEOREM. (Classification)
Finite type cluster algebras are classified by the Cartan-Killing classification for crystallographic root systems.

S. Fomin & A. Zelevinsky, Cluster algebras II: Finite type classification, 2003.

In fact, for a root system Φ, there is a bijection

cluster variables	$\longleftrightarrow$	$\Phi_{\geq -1} = \Phi^+ \cup -\Delta$
$y = \dfrac{F(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$	$\longleftrightarrow$	$\beta = d_1\alpha_1 + \cdots + d_n\alpha_n$
cluster	$\longleftrightarrow$	$\mathfrak{c}$-cluster
cluster complex	$\longleftrightarrow$	$\mathfrak{c}$-cluster complex

GENERALIZED ASSOCIAHEDRA

THEOREM. The cluster complex is polytopal.

F. Chapoton, S. Fomin & A. Zelevinsky, Polytopal realizations of generalized associahedra, 2002.

C. Hohlweg, C. Lange & H. Thomas, Permutahedra and generalized associahedra, 2011.

S. Stella, Polyhedral models for generalized associahedra via Coxeter elements, 2013.

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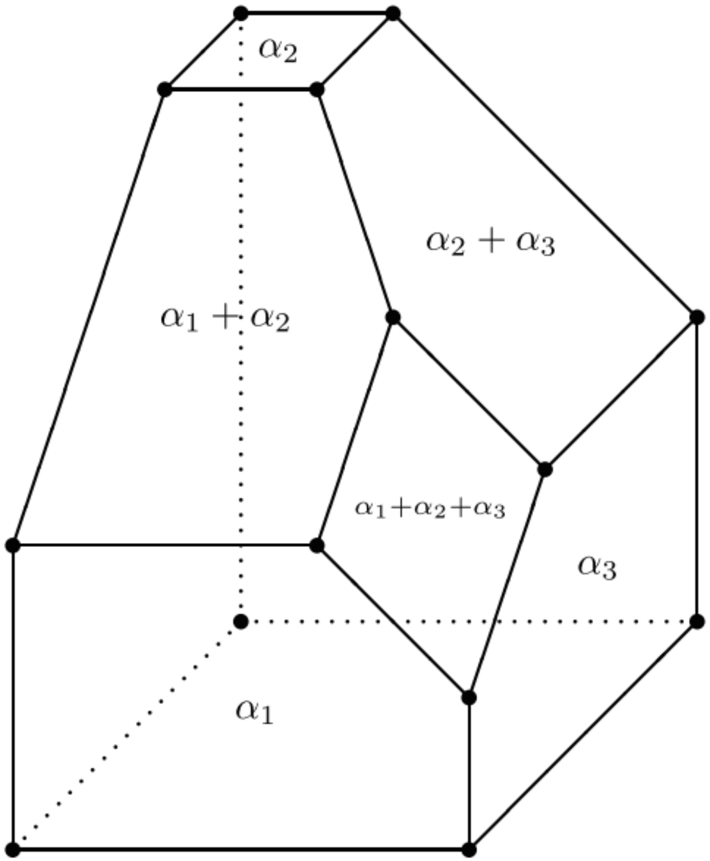
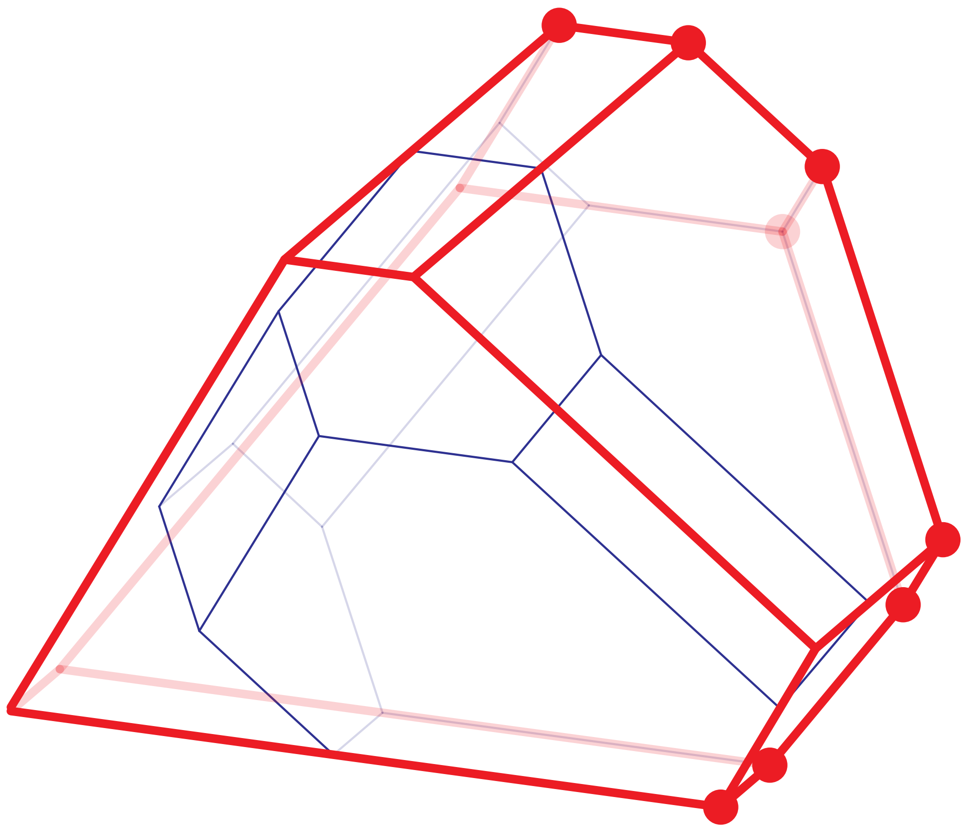
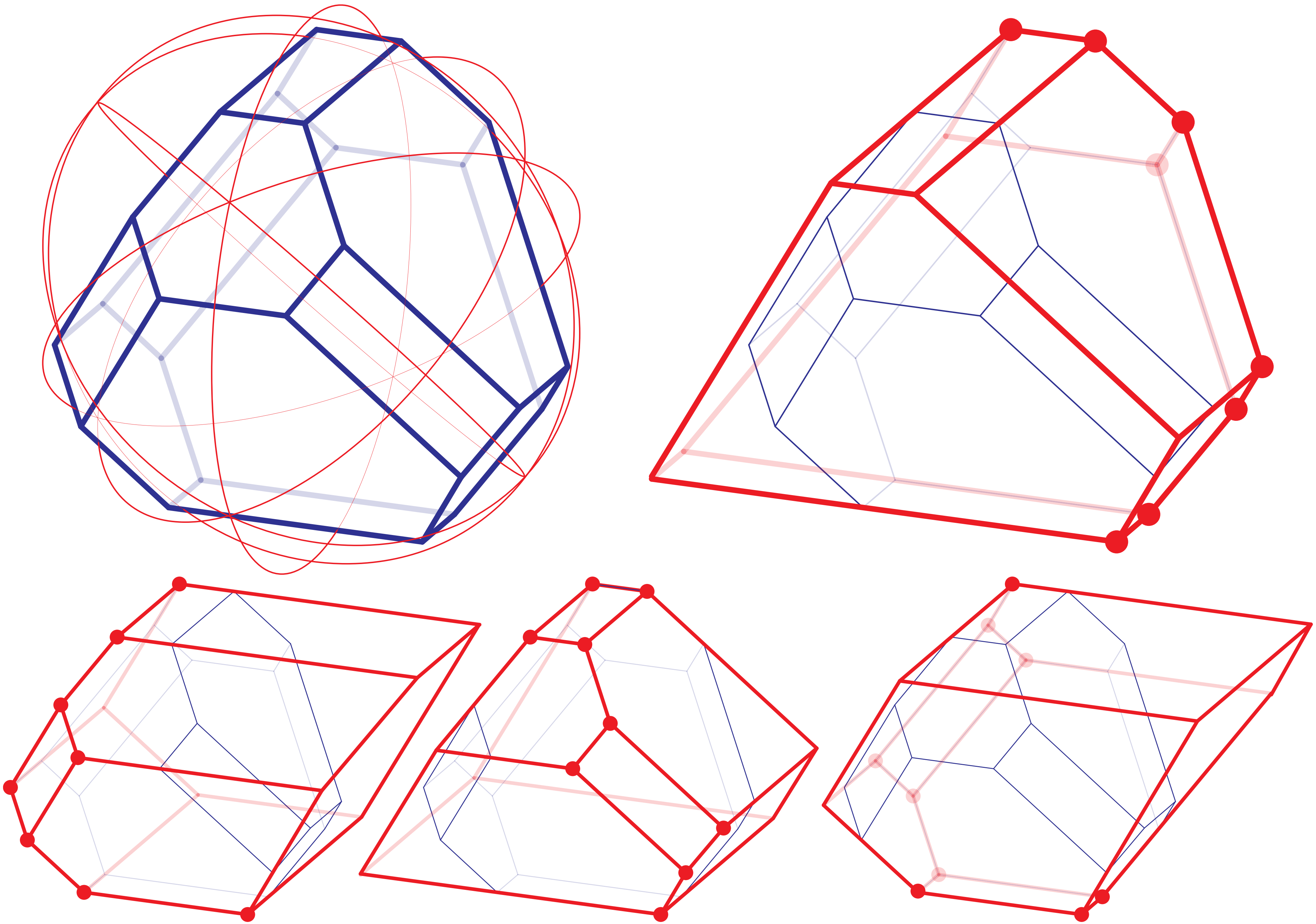
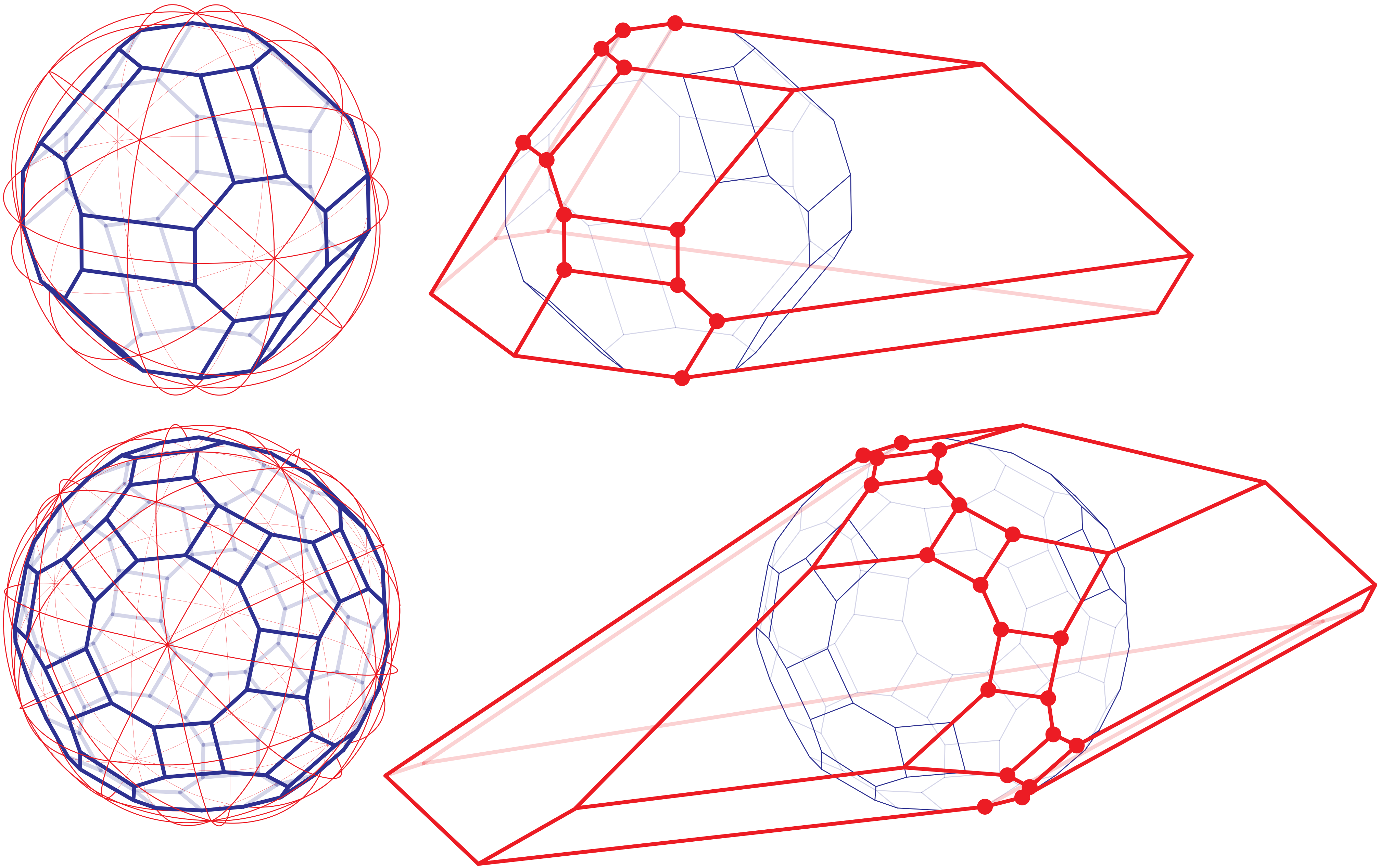




GENERALIZED ASSOCIAHEDRA



GENERALIZED ASSOCIAHEDRA



GENERALIZED ASSOCIAHEDRA ARE BRICK POLYTOPES

New approach to the combinatorics and geometry of the cluster complex:

THEOREM. The subword complex $\mathcal{S}(\mathrm{cw}_\circ(\mathrm{c}))$ is isomorphic to the cluster complex.

C. Ceballos, JP. Labbé & C. Stump, Subword complexes, cluster complexes, & gener. multiassoc., 2011.

cluster variables	$\longleftrightarrow$	$\Phi_{\geq -1} = \Phi^+ \cup -\Delta$	$\longleftrightarrow$	position in $\mathrm{cw}_\circ(\mathrm{c})$
$y = \dfrac{F(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$	$\longleftrightarrow$	$\beta = d_1\alpha_1 + \cdots + d_n\alpha_n$	$\longleftrightarrow$	$\begin{cases} i & \text{if } \beta = -\alpha_{c_i} \\ j & \text{if } \beta = \mathsf{r}([n], j) \end{cases}$
cluster	$\longleftrightarrow$	c-cluster	$\longleftrightarrow$	facet of $\mathcal{S}(\mathrm{cw}_\circ(\mathrm{c}))$
cluster complex	$\longleftrightarrow$	c-cluster complex	$\longleftrightarrow$	subword complex $\mathcal{S}(\mathrm{cw}_\circ(\mathrm{c}))$

THEOREM. The brick polytope $\mathcal{B}(\mathrm{cw}_\circ(\mathrm{c}))$ realizes the subword complex $\mathcal{S}(\mathrm{cw}_\circ(\mathrm{c}))$.

THEOREM. The brick polytope $\mathcal{B}(\mathrm{cw}_\circ(\mathrm{c}))$ is a translate of the known realizations of the generalized associahedron.

FURTHER PROPERTIES OF GENERALIZED ASSOCIAHEDRA

CAMBRIAN LATTICES & FANS. The graph of the associahedron $\mathrm{Asso}_c(W)$, oriented from e to $w_\circ$ is the Hasse diagram of the c-Cambrian lattice.

The normal fan of the associahedron $\mathrm{Asso}_c(W)$ is the c-Cambrian fan, obtained by coarsening the braid fan.

Reading, Sortable elements and Cambrian lattices, 2007.

Reading & Speyer, Cambrian fans, 2009.

DIAMETER. The diameter of the

type A_n	associahedron is	$2n-4$ for $n \geq 9$.
type D_n		$2n-2$ for all n.

All type $A_n, B/C_n, D_n, H_3, H_4, F_4, E_6$ associahedra fulfill the non-leaving face property: every geodesic connecting two vertices stays in the minimal face containing them.

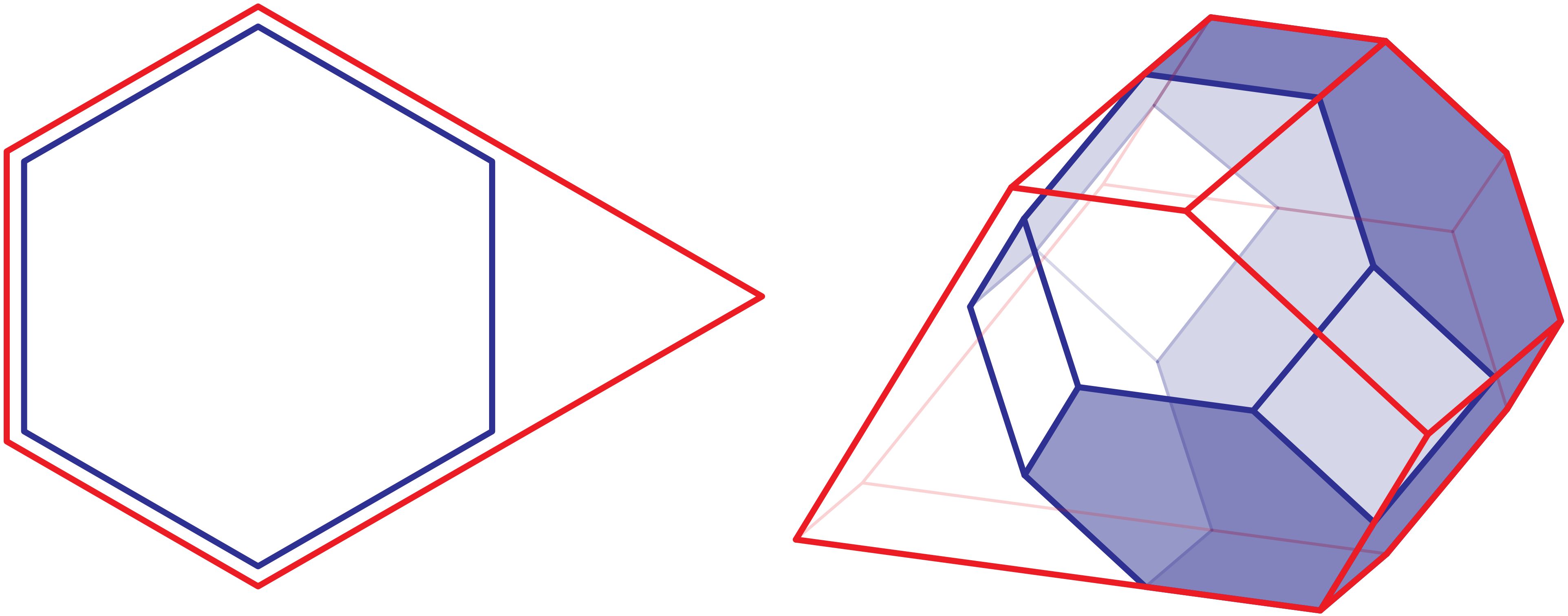
L. Pournin, The diameter of associahedra, 2014.

VP & C. Ceballos, The diameter of type D associahedra and the non-leaving face property, 2014^+.

BARYCENTER. The vertex barycenters of the permutahedron and associahedron coincide.

VP & C. Stump, Vertex barycenter of generalized associahedra, 2013.

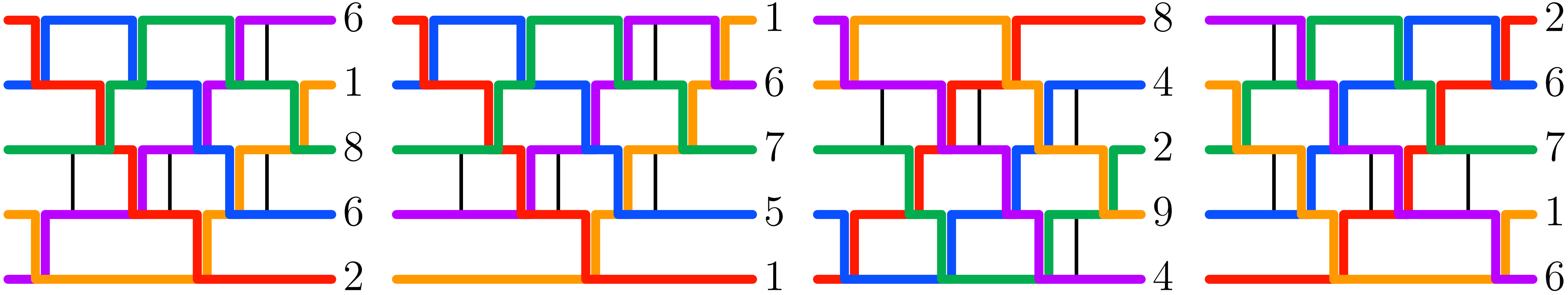
BARYCENTER



VP & C. Stump, Vertex barycenter of generalized associahedra, 2013.

THREE OPERATIONS

Evolution of the brick vector $\mathsf{B}_{\mathcal{N}}(\Lambda)$ under three operations:

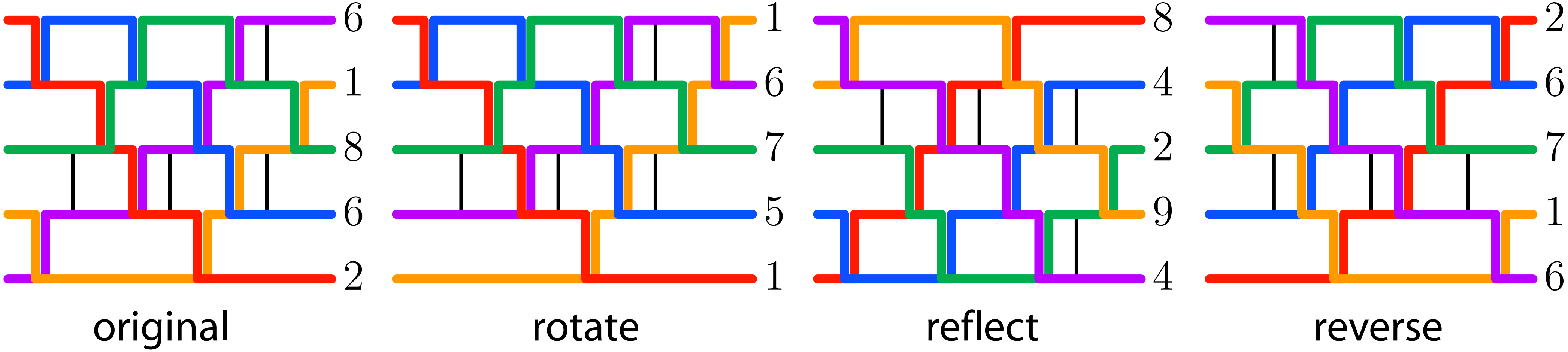


original rotate reflect reverse

1. Rotate: $\mathsf{B}_{\mathcal{N}^{\circlearrowleft}}(\Lambda^{\circlearrowleft}) - \mathsf{B}_{\mathcal{N}}(\Lambda) \in \omega_i + \mathbb{R}(e_{i+1} - e_i)$
2. Reflect: $\mathsf{B}_{\mathcal{N}^{\updownarrow}}(\Lambda^{\updownarrow}) = \#\{\text{bricks of } \mathcal{N}\}\,.\,\mathbb{1} - (\mathsf{B}_{\mathcal{N}}(\Lambda))^{\leftrightarrow}$
3. Reverse: $\mathsf{B}_{\mathcal{N}^{\leftrightarrow}}(\Lambda^{\leftrightarrow}) = (\mathsf{B}_{\mathcal{N}}(\Lambda))^{\leftrightarrow}$

THREE OPERATIONS

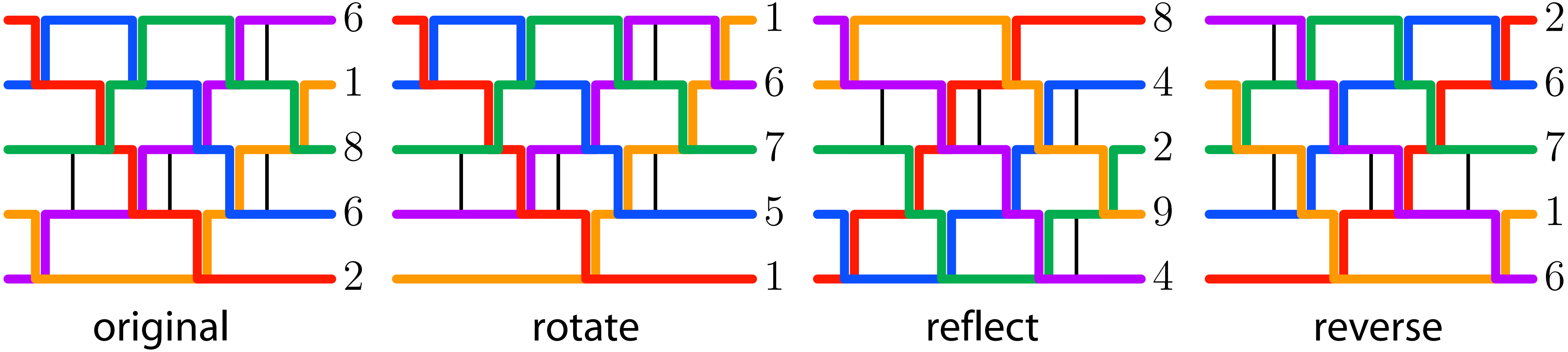
Evolution of the translated brick vector $\bar{\mathsf{B}}_c(\Lambda) = \mathsf{B}_c(\Lambda) - \Omega_c$ under three operations:



1. Rotate: $\bar{\mathsf{B}}_{c^{\circlearrowleft}}(\Lambda^{\circlearrowleft}) - \bar{\mathsf{B}}_c(\Lambda) \in \mathbb{R}(e_{i+1} - e_i)$
2. Reflect: $\bar{\mathsf{B}}_{c^{\updownarrow}}(\Lambda^{\updownarrow}) = -(\bar{\mathsf{B}}_c(\Lambda))^{\leftrightarrow}$
3. Reverse: $\bar{\mathsf{B}}_{c^{\leftrightarrow}}(\Lambda^{\leftrightarrow}) = (\bar{\mathsf{B}}_c(\Lambda))^{\leftrightarrow}$

THREE OPERATIONS

Evolution of the translated brick vector $\bar{\mathsf{B}}_c(\Lambda) = \mathsf{B}_c(\Lambda) - \Omega_c$ under three operations:

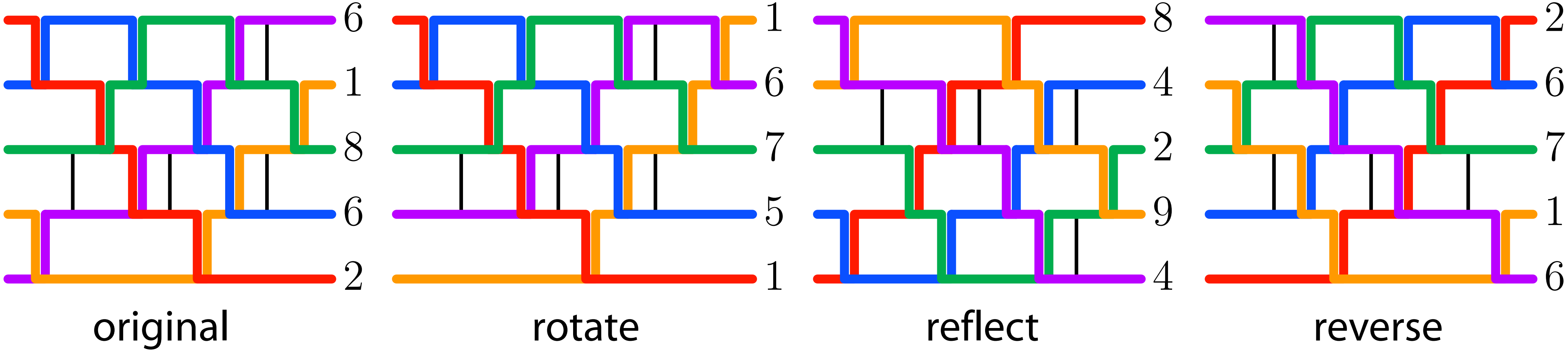


1. Rotate: $\bar{\mathsf{B}}_{c^{\circlearrowleft}}(\Lambda^{\circlearrowleft}) - \bar{\mathsf{B}}_c(\Lambda) \in \mathbb{R}(e_{i+1} - e_i)$

All associahedra Asso_c have the same barycenter

THREE OPERATIONS

Evolution of the translated brick vector $\bar{\mathsf{B}}_c(\Lambda) = \mathsf{B}_c(\Lambda) - \Omega_c$ under three operations:



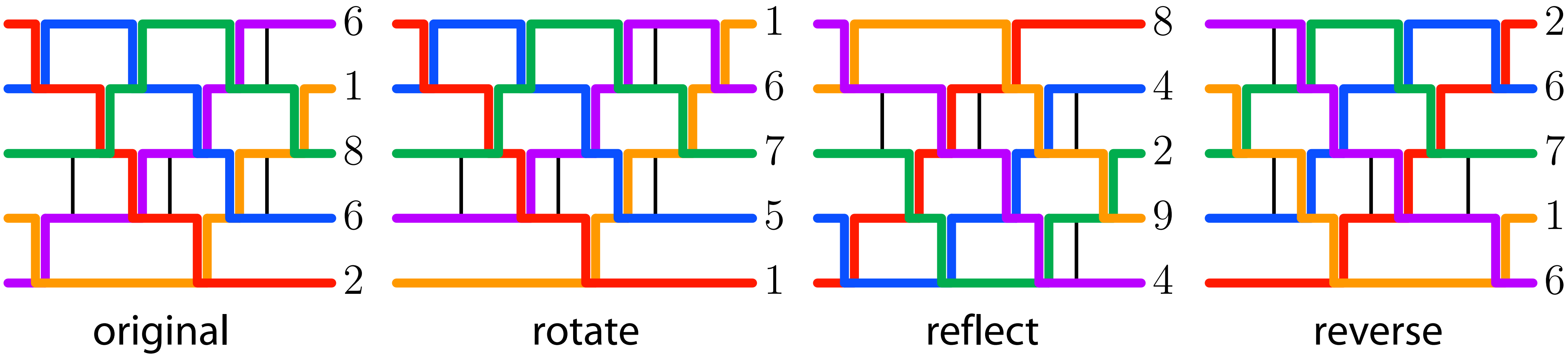
2. Reflect: $\bar{\mathsf{B}}_{c^{\updownarrow}}(\Lambda^{\updownarrow}) = -(\bar{\mathsf{B}}_c(\Lambda))^{\leftrightarrow}$

3. Reverse: $\bar{\mathsf{B}}_{c^{\leftrightarrow}}(\Lambda^{\leftrightarrow}) = (\bar{\mathsf{B}}_c(\Lambda))^{\leftrightarrow}$

The barycenter of the superposition of the vertices of $\mathrm{Asso}_{c^{\updownarrow}}$ and $\mathrm{Asso}_{c^{\leftrightarrow}}$ is the origin

THREE OPERATIONS

Evolution of the translated brick vector $\bar{\mathsf{B}}_c(\Lambda) = \mathsf{B}_c(\Lambda) - \Omega_c$ under three operations:



All associahedra Asso_c have the same barycenter

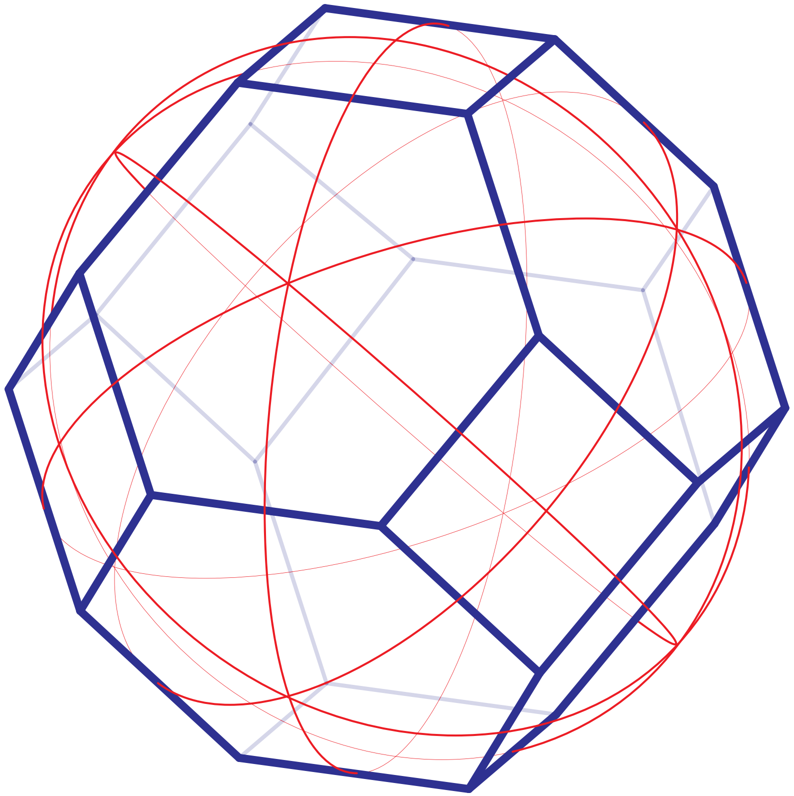
The barycenter of the superposition of the vertices of $\mathrm{Asso}_{c^{\updownarrow}}$ and $\mathrm{Asso}_{c^{\leftrightarrow}}$ is the origin

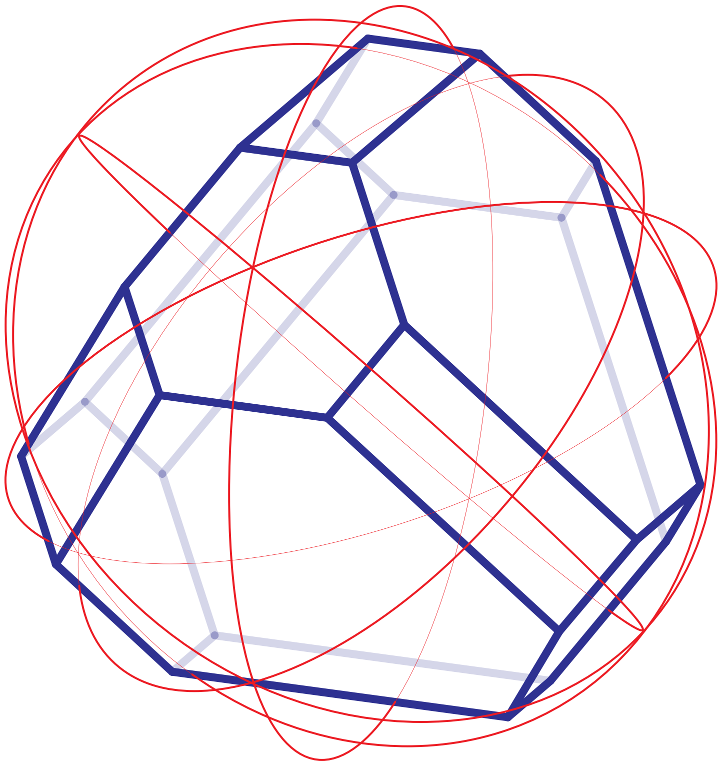
THEOREM. All associahedra Asso_c have vertex barycenter at the origin

. . . and the same method works for fairly balanced and generalized associahedra.

BARYCENTER

THEOREM. For | any finite Coxeter group W, any Coxeter element c, any fairly-balanced point u, the vertex barycenters of the generalized associahedron $\mathrm{Asso}_c^u(W)$ and of the permutahedron $\mathrm{Perm}^u(W)$ coincide.





The point u is fairly balanced if $w_\circ(u) = -u$, where $w_\circ$ is the longest element in W.

THANK YOU