BRICK POLYTOPES, LATTICE QUOTIENTS & HOPF ALGEBRAS

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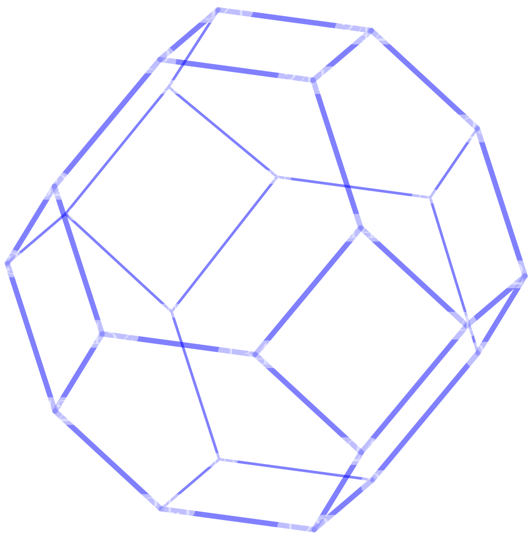
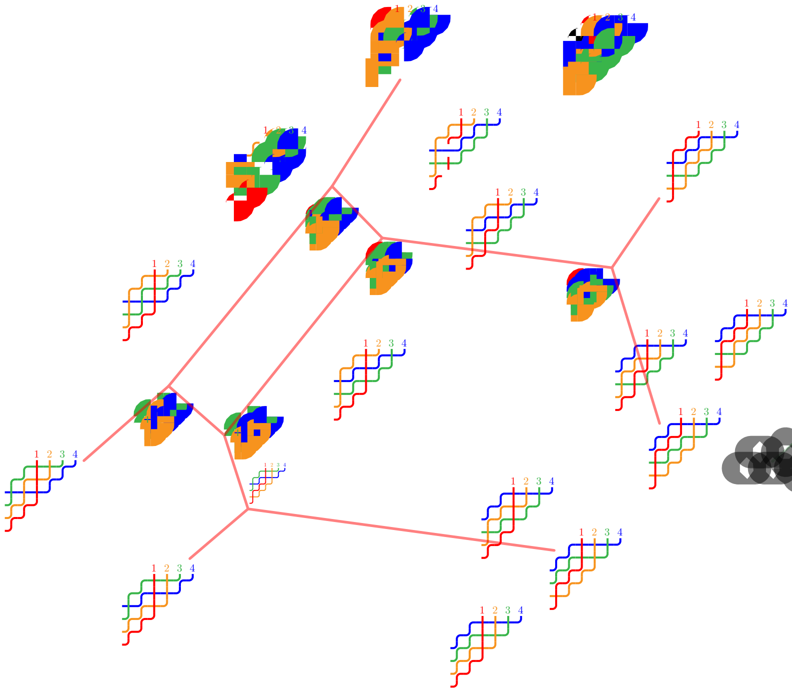
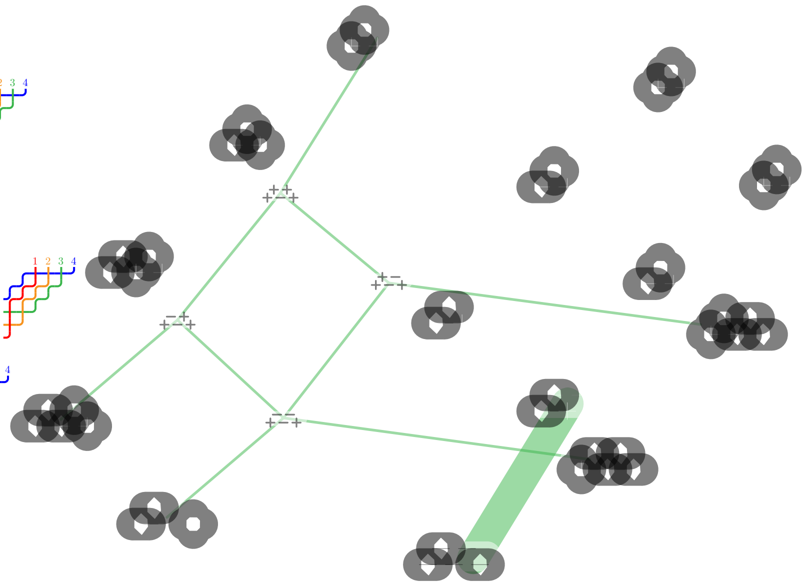
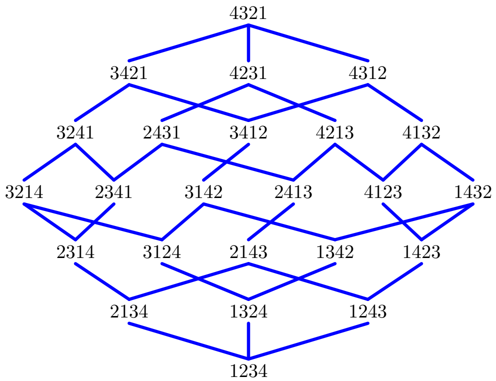
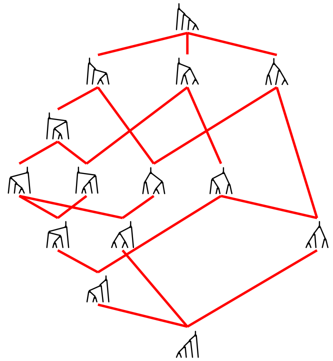
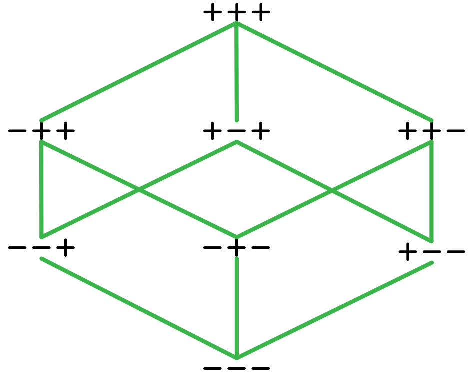
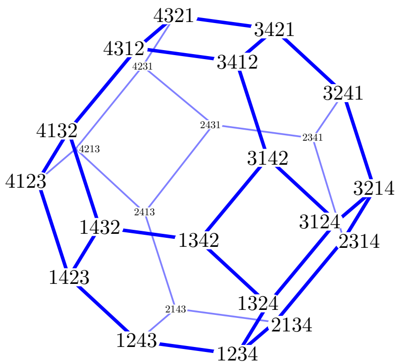
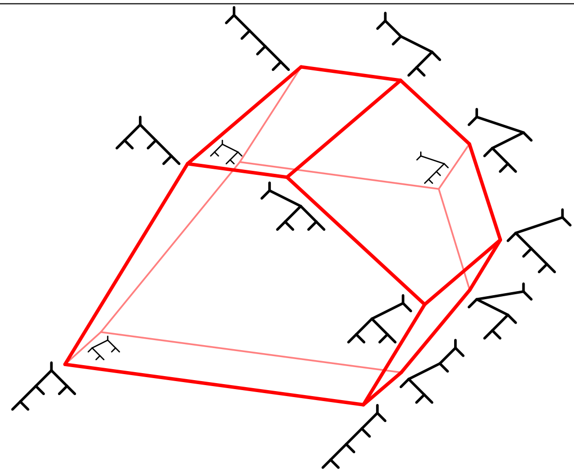
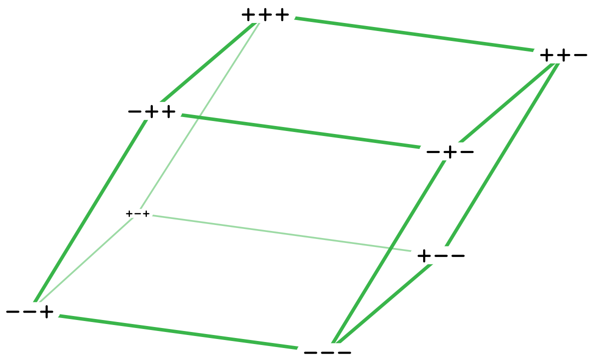
CNRS & École Polytechnique





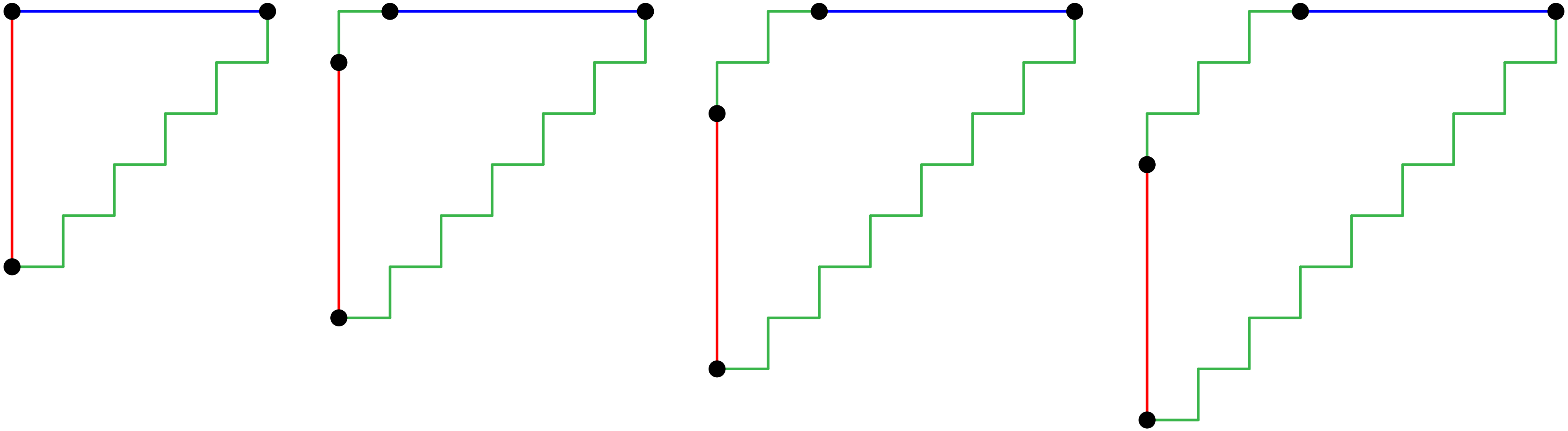


MOTIVATION

	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra FQSym $= \text{vect}\langle \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \rangle$ $\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma$ $\triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$	Loday-Ronco algebra PBT $= \text{vect}\langle \mathbb{P}_{\mathrm{T}} \mid \mathrm{T} \in \mathcal{BT} \rangle$ $\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{{}_{T}\nearrow^{T'} \le T'' \le {}^{T}\nwarrow_{T'}} \mathbb{P}_{T''}$ $\triangle \mathbb{F}_\gamma = \sum_{\gamma \text{ cut}} B(T,\gamma) \otimes A(T,\gamma)$	Solomon algebra Rec $= \text{vect}\langle \mathbb{X}_\eta \mid \eta \in \pm^* \rangle$ $\mathbb{X}_\eta \cdot \mathbb{X}_{\eta'} = \mathbb{X}_{\eta+\eta'} + \mathbb{X}_{\eta-\eta'}$ $\triangle \mathbb{X}_\eta = \sum_{\gamma \text{ cut}} B(\eta,\gamma) \otimes A(\eta,\gamma)$
Geometry			

COMBINATORICS

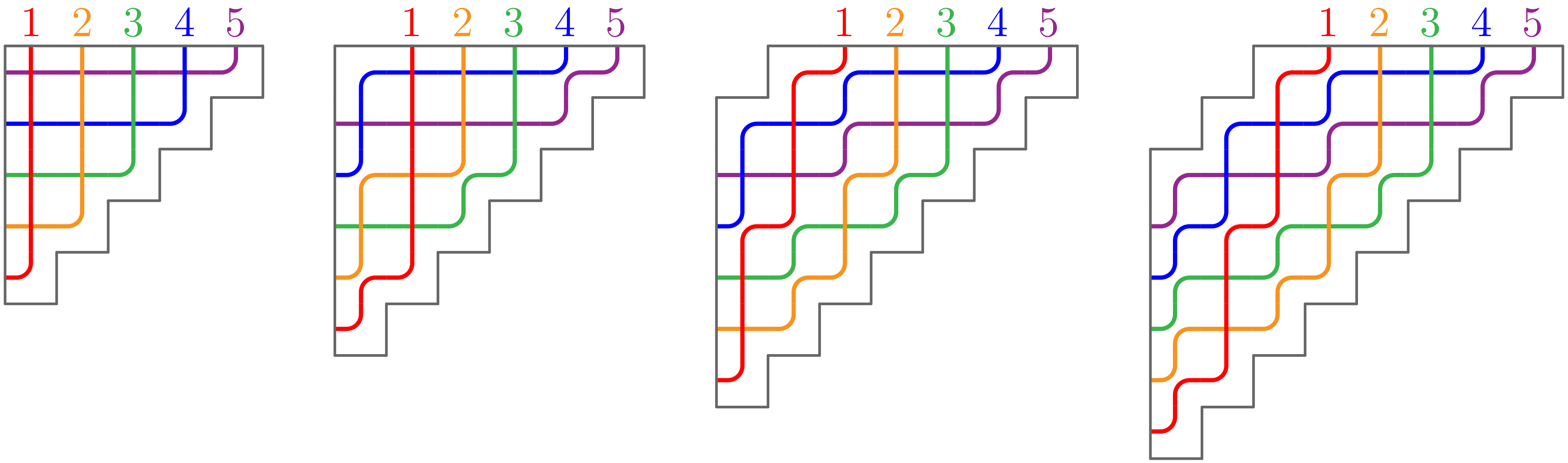
k-TWISTS



$k = 0$ $k = 1$ $k = 2$ $k = 3$

trapezoidal shape of height n and width k

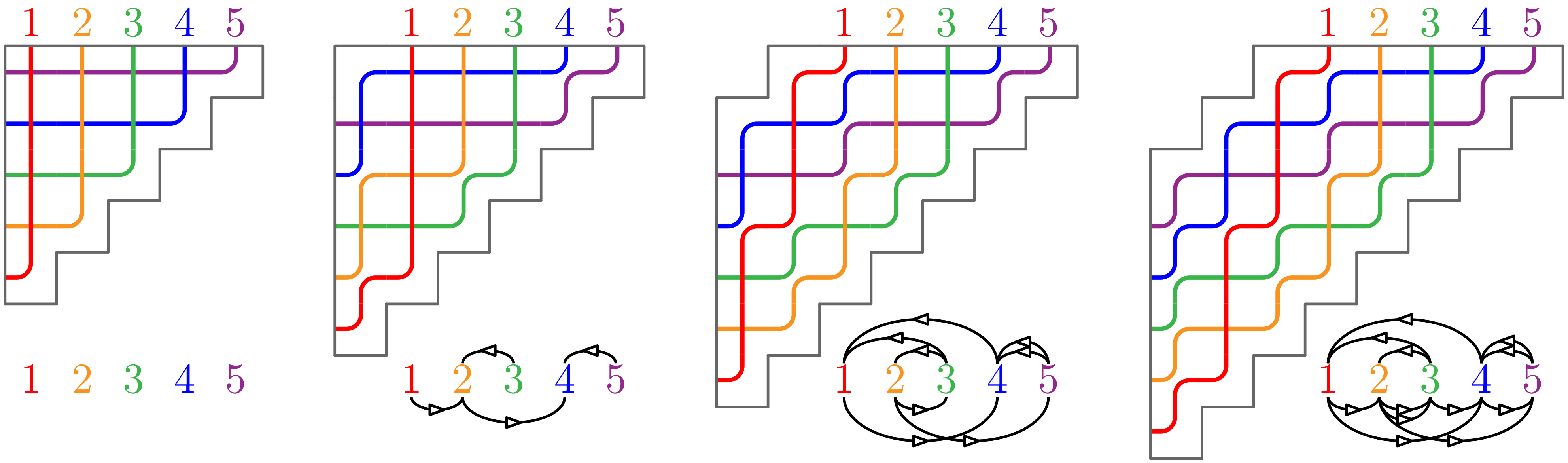
k-TWISTS



$k = 0$ $\quad$ $k = 1$ $\quad$ $k = 2$ $\quad$ $k = 3$

(k, n)-twist $=$ pipe dream in the trapezoidal shape of height n and width k

k-TWISTS



$k = 0$ $k = 1$ $k = 2$ $k = 3$

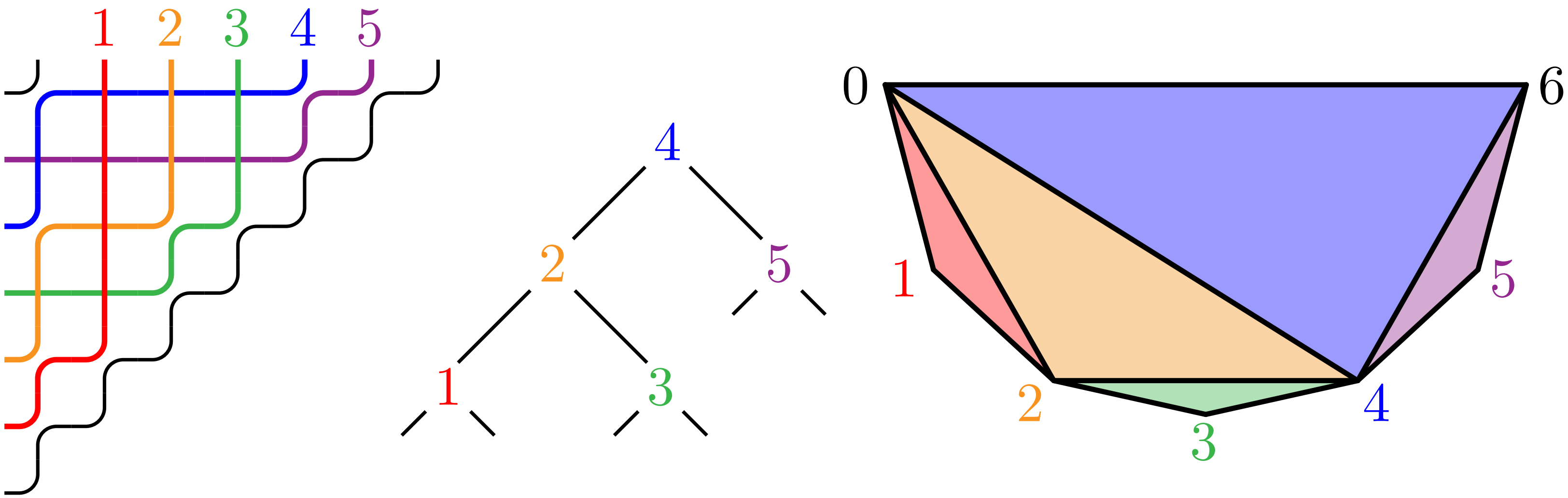
(k, n)-twist $=$ pipe dream in the trapezoidal shape of height n and width k

contact graph of a twist T $=$ vertices are pipes of T and arcs are elbows of T

1-TWISTS AND TRIANGULATIONS

Correspondence

elbow in row i and column j	$\longleftrightarrow$	diagonal $[i, j]$ of the $(n+2)$-gon
$(1, n)$-twist T	$\longleftrightarrow$	triangulation $\mathrm{T}^\star$ of the $(n+2)$-gon
pth relevant pipe of T	$\longleftrightarrow$	pth triangle of $\mathrm{T}^\star$
contact graph of T	$\longleftrightarrow$	dual binary tree of $\mathrm{T}^\star$
elbow flips in T	$\longleftrightarrow$	diagonal flips in $\mathrm{T}^\star$



Woo. Catalan numbers and Schubert Polynomials for $w = 1(n+1)\dots 2$. Unpub 2004
P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012

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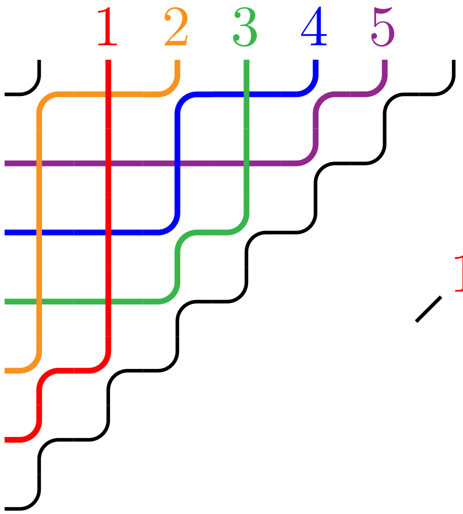
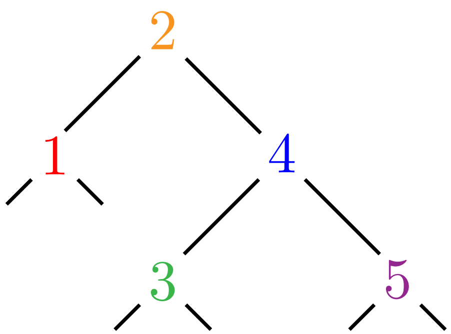
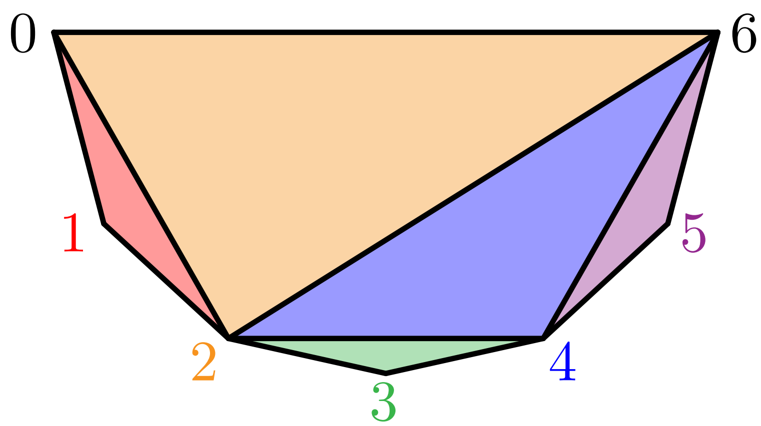


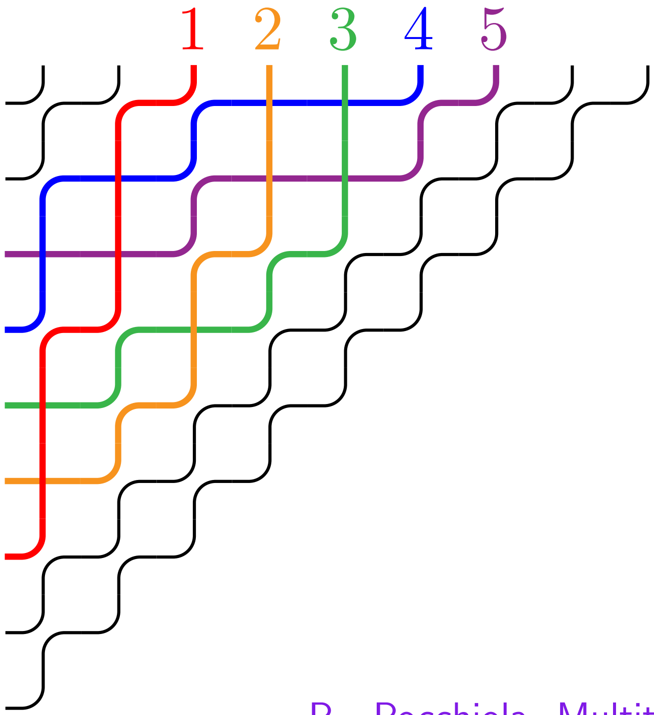


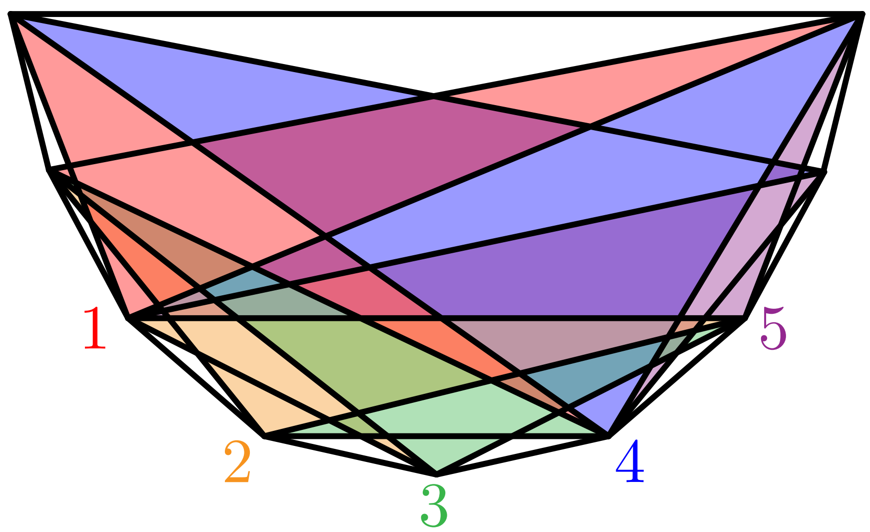


Woo. Catalan numbers and Schubert Polynomials for $w = 1(n+1)\ldots 2$. Unpub 2004
P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012

Correspondence

elbow in row i and column j	$\longleftrightarrow$	diagonal $[i, j]$ of the $(n + 2k)$-gon
(k, n)-twist T	$\longleftrightarrow$	k-triangulation $\mathrm{T}^\star$ of the $(n + 2k)$-gon
pth relevant pipe of T	$\longleftrightarrow$	pth k-star of $\mathrm{T}^\star$
contact graph of T	$\longleftrightarrow$	dual graph of $\mathrm{T}^\star$
elbow flips in T	$\longleftrightarrow$	diagonal flips in $\mathrm{T}^\star$





P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012

NUMEROLOGY

THM. The (k,n)-twists are counted by $\det(C_{n+2k-i-j})_{i,j\in[k]}$, where $C_m = \frac{1}{m+1}\binom{2m}{m}$.

Jonsson. Generalized triangulations and diagonal-free subsets of stack polyominoes. 2005
P. – Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. 2012
Stump. A new perspective on k-triangulations. 2011

QU. What is the number of acyclic (k,n)-twists?

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	.	2	5	14	42	132	429	1430	4862	16796
2	.	.	6	22	92	420	2042	10404	54954	298648
3	.	.	.	24	114	612	3600	22680	150732	1045440
4	.	.	.	.	120	696	4512	31920	242160	1942800
5	.	.	.	.	.	720	4920	37200	305280	2680800
6	.	.	.	.	.	.	5040	39600	341280	3175200
7	.	.	.	.	.	.	.	40320	357840	3457440
8	.	.	.	.	.	.	.	.	362880	3588480
9	.	.	.	.	.	.	.	.	.	3628800

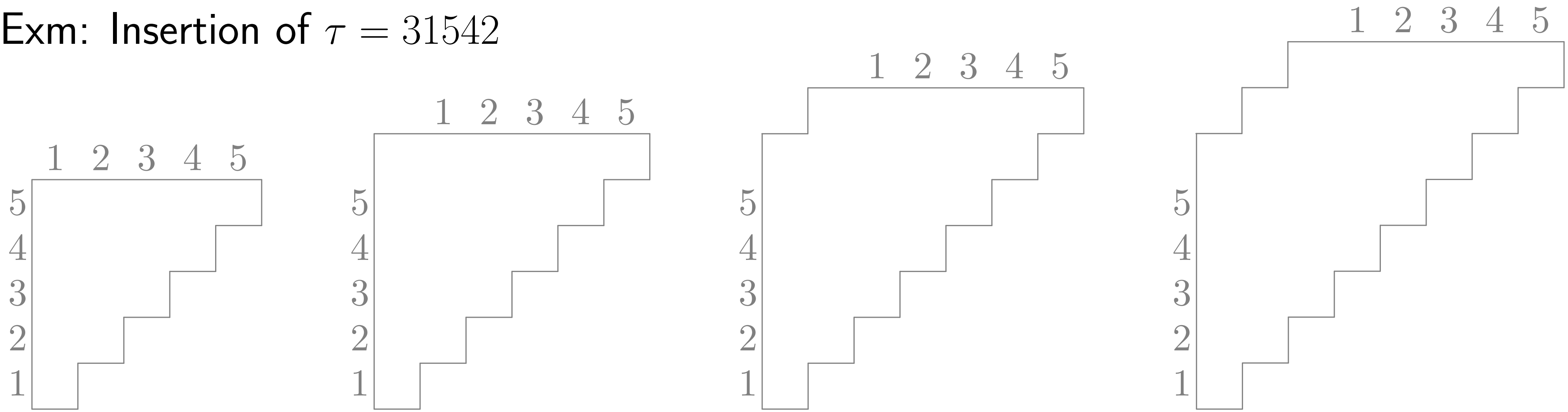
k-TWIST INSERTION

Input: a permutation $\tau = \tau_1 \cdots \tau_n$

Algo: Insert pipes one by one (from right to left) as northwest as possible

Output: an acyclic (k, n)-twist $\text{ins}^k(\tau)$

Exm: Insertion of $\tau = 31542$



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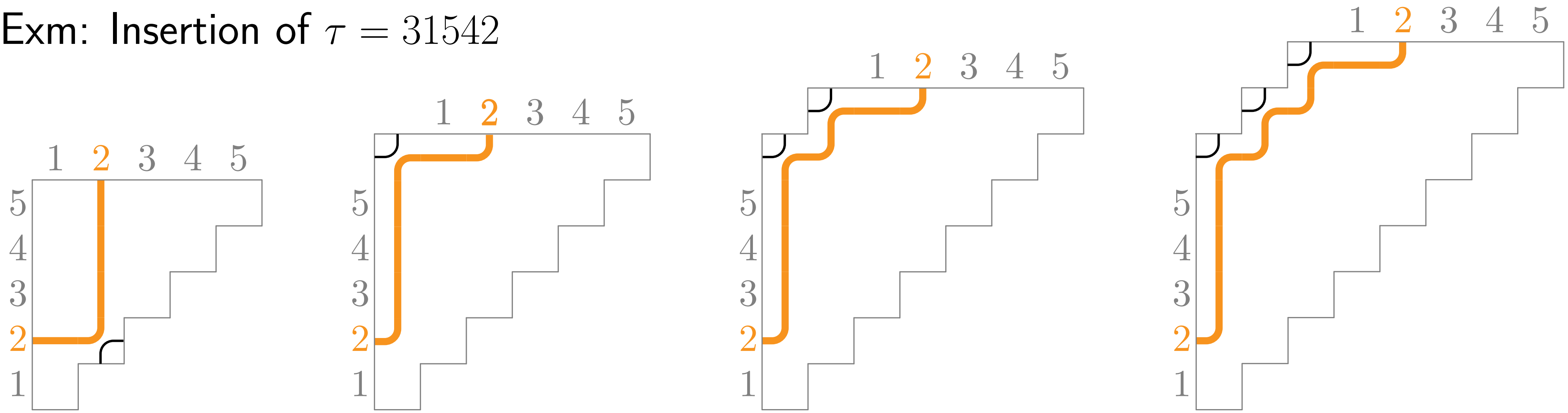

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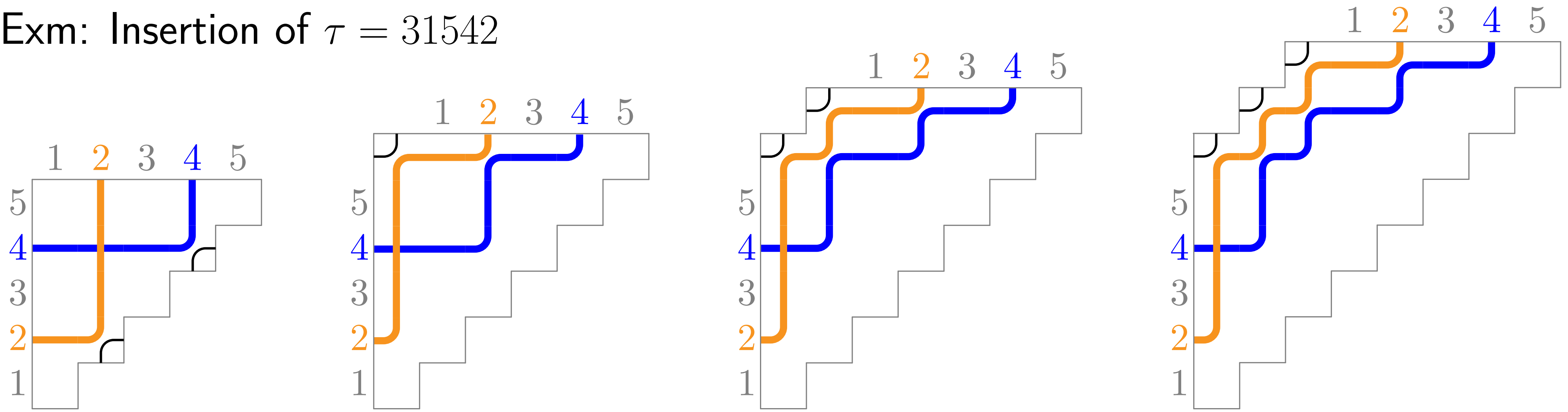

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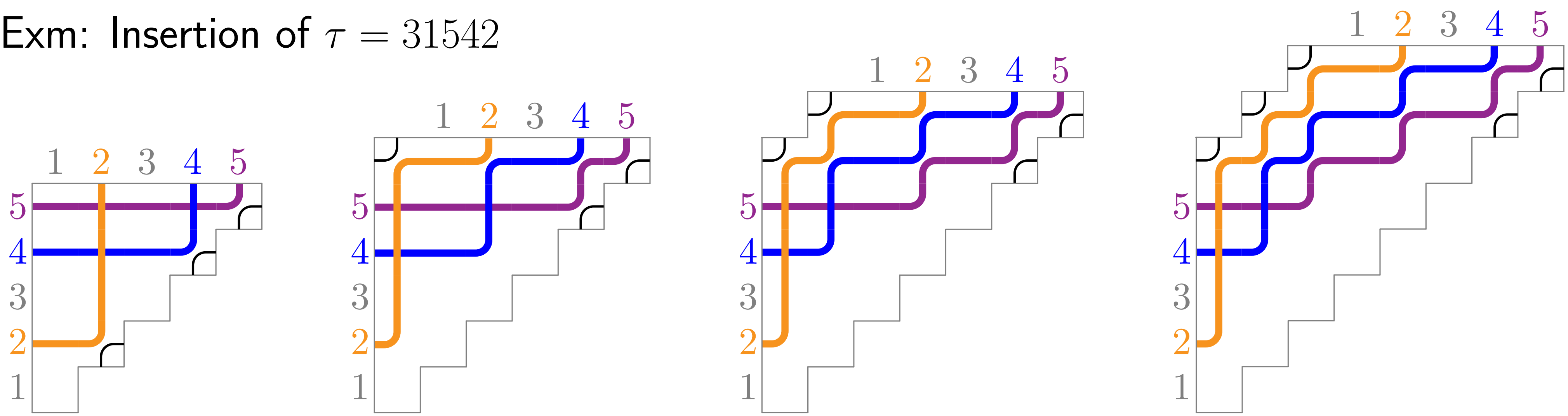

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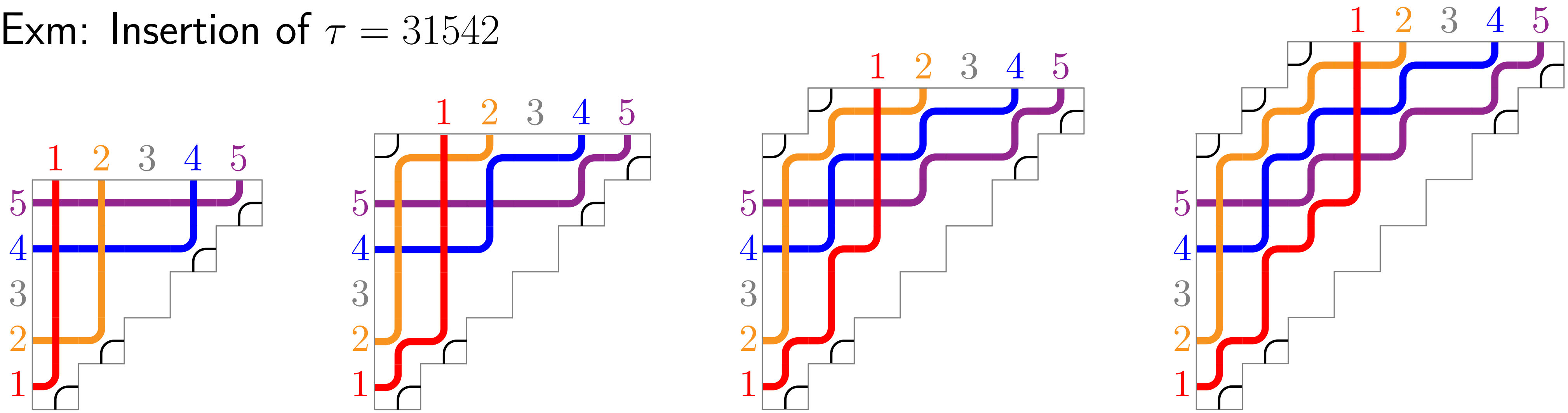

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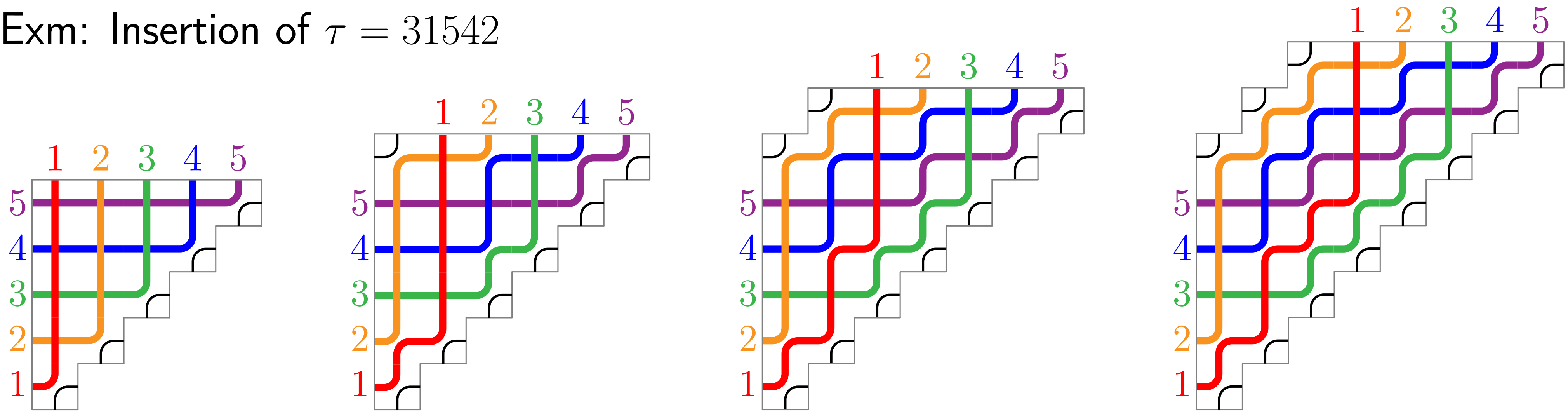

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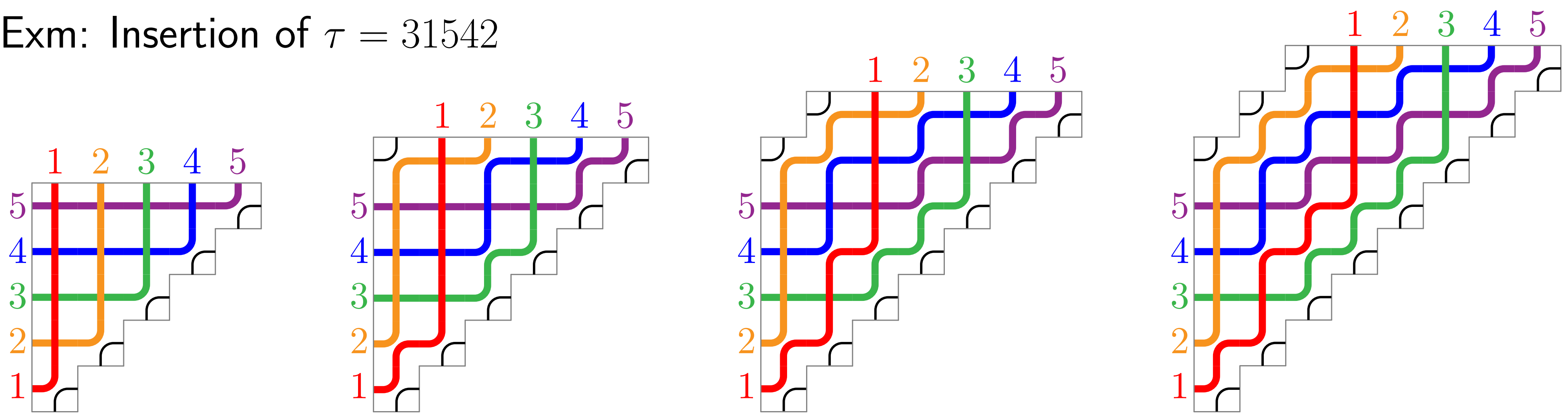


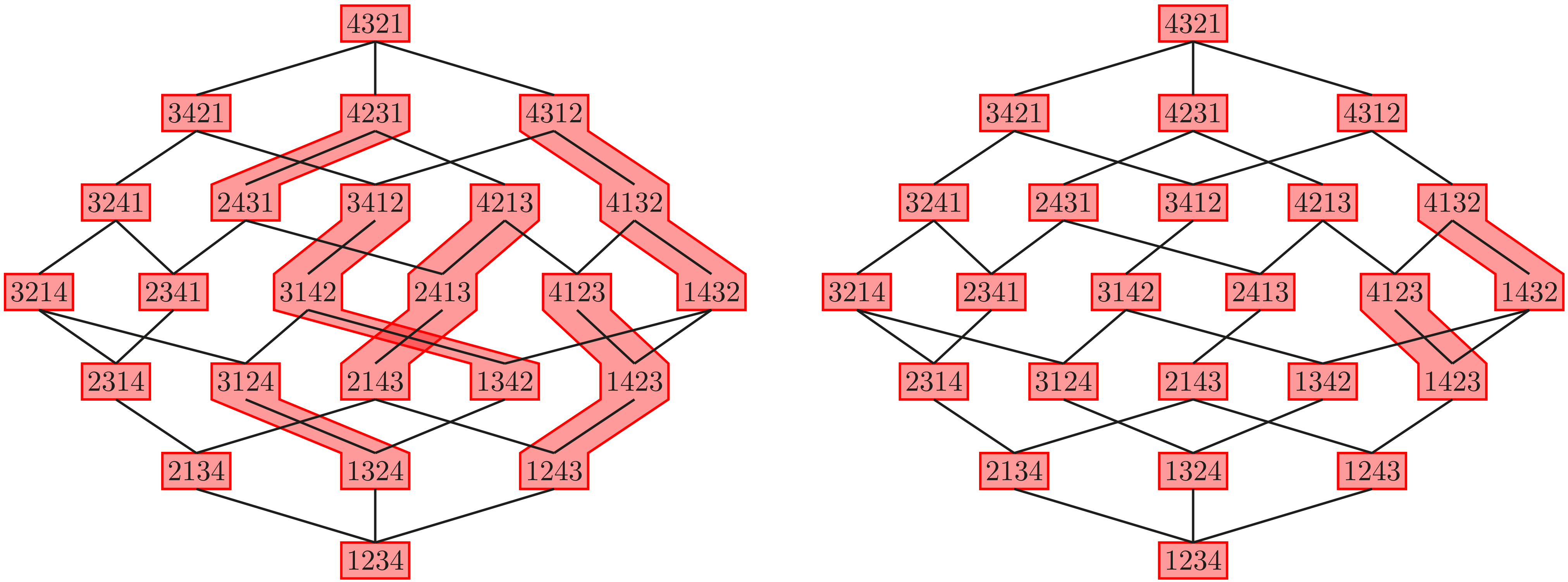
THM. ins^k is a surjection from permutations of $[n]$ to acyclic (k, n)-twists.
fiber of a (k, n)-twist T = linear extensions of its contact graph $\mathrm{T}^{\#}$.

Exm: insertion in binary search trees

k-TWIST CONGRUENCE

DEF. k-twist congruence $=$ equivalence relation $\equiv^k$ on $\mathfrak{S}_n$ defined as the transitive closure of the rewriting rule

$$U ac V_1 b_1 V_2 b_2 \cdots V_k b_k W \equiv^k U ca V_1 b_1 V_2 b_2 \cdots V_k b_k W \qquad \text{if } a < b_i < c \text{ for all } i \in [k].$$



PROP. For any $\tau, \tau' \in \mathfrak{S}_n$, we have $\tau \equiv^k \tau' \iff \text{ins}^k(\tau) = \text{ins}^k(\tau')$.

LATTICE CONGRUENCES

DEF. Order congruence = equivalence relation $\equiv$ on a poset P such that:

(i) Every equivalence class under $\equiv$ is an interval of P.

(ii) The projection $\pi_{\downarrow} : P \to P$ (resp. $\pi^{\uparrow} : P \to P$), which maps an element of P to the minimal (resp. maximal) element of its equivalence class, is order preserving.

poset quotient = $X \le Y$ in $P/\equiv \iff \exists\, x \in X,\ y \in Y$ such that $x \le y$ in P.

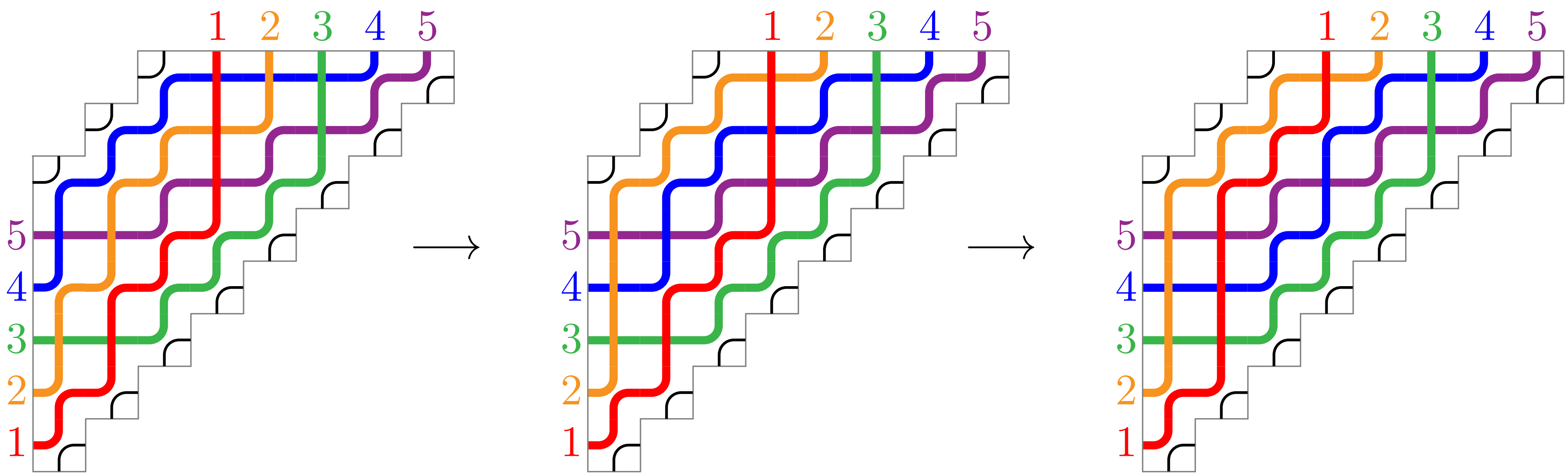
If moreover P is a lattice, $\equiv$ is automatically a lattice congruence, compatible with meets and joins: $x \equiv x'$ and $y \equiv y' \;\Rightarrow\; x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$.

lattice quotient = $X \wedge Y$ and $X \vee Y$ are the congruence classes of $x \wedge y$ and $x \vee y$ for arbitrary representatives $x \in X$ and $y \in Y$.

THM. The k-twist congruence is a lattice quotient of the weak order.

INCREASING FLIP LATTICE

flip in a k-twist $=$ exchange an elbow with the unique crossing between its two pipes
increasing flip $=$ the elbow is southwest of the crossing
increasing flip order $=$ transitive closure of the increasing flip graph



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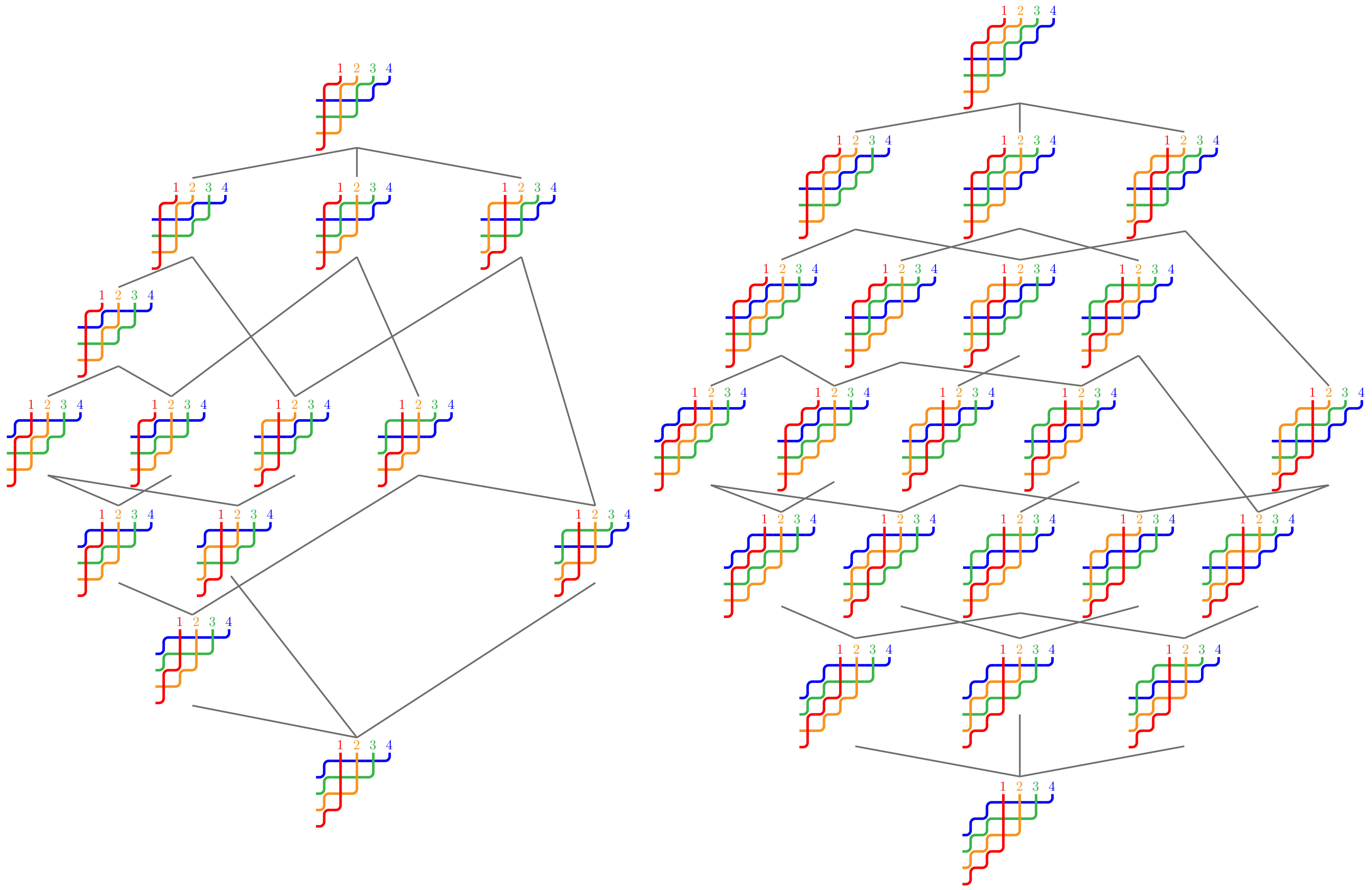

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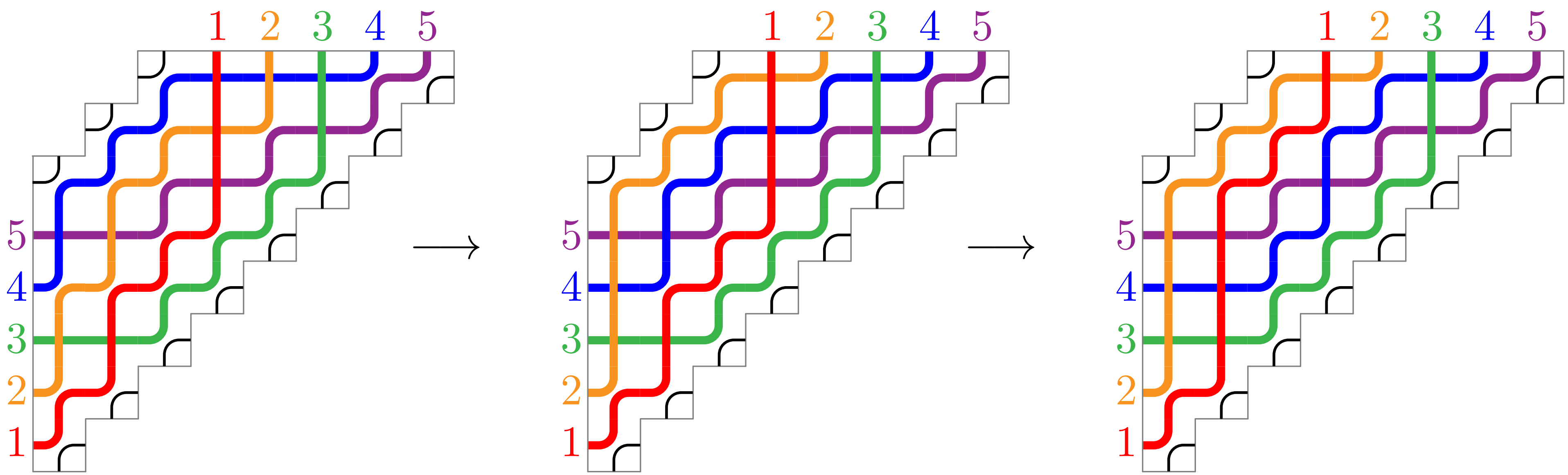
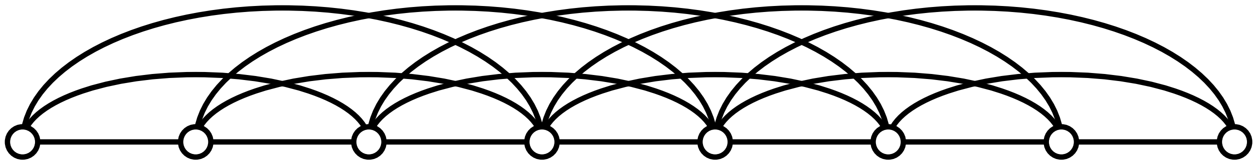


PROP. The increasing flip order on acyclic k-twists is isomorphic to:

- the quotient lattice of the weak order by the k-twist congruence $\equiv^k$,
- the sublattice of the weak order induced by the permutations of $\mathfrak{S}_n$ avoiding the pattern $1(k+2)-(\sigma_1+1)-\cdots-(\sigma_k+1)$ for all $\sigma \in \mathfrak{S}_k$.

k-RECOILS

$\mathrm{G}^k(n)$ = graph with vertex set $[n]$ and edge set $\big\{\{i,j\} \in [n]^2 \;\big|\; i < j \le i+k\big\}$

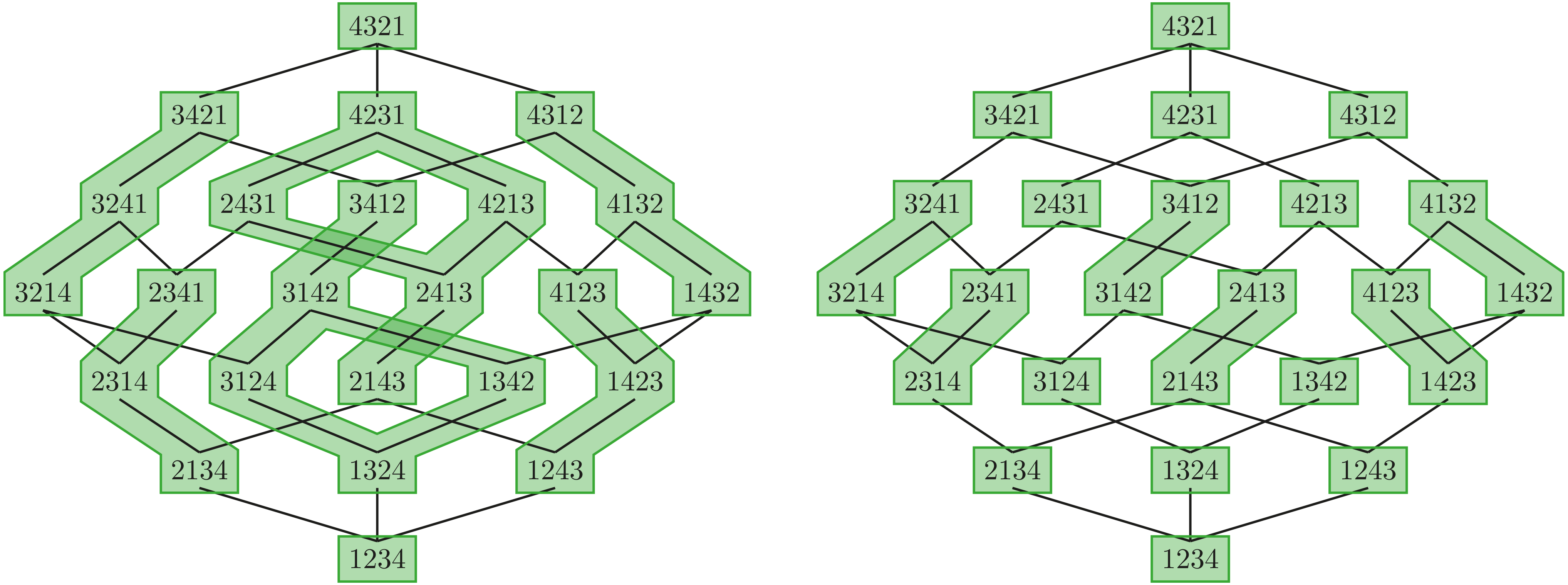


number of acyclic orientations of $\mathrm{G}^k(n) = \begin{cases} n! & \text{if } n \le k \\ k!\,(k+1)^{n-k} & \text{if } n \ge k \end{cases}$

k-recoils scheme of $\tau \in \mathfrak{S}_n$ = acyclic orientation $\mathrm{rec}^k(\tau)$ of $\mathrm{G}^k(n)$ with edge $i \to j$ for all $i, j \in [n]$ such that $|i - j| \le k$ and $\tau^{-1}(i) < \tau^{-1}(j)$

k-RECOILS

k-recoil congruence = equivalence relation $\approx^k$ on $\mathfrak{S}_n$ defined as the transitive closure of the rewriting rule $UijV \approx^k UjiV$ if $i + k < j$.



PROP. For any $\tau, \tau' \in \mathfrak{S}_n$, we have $\tau \approx^k \tau' \iff \text{rec}^k(\tau) = \text{rec}^k(\tau')$.

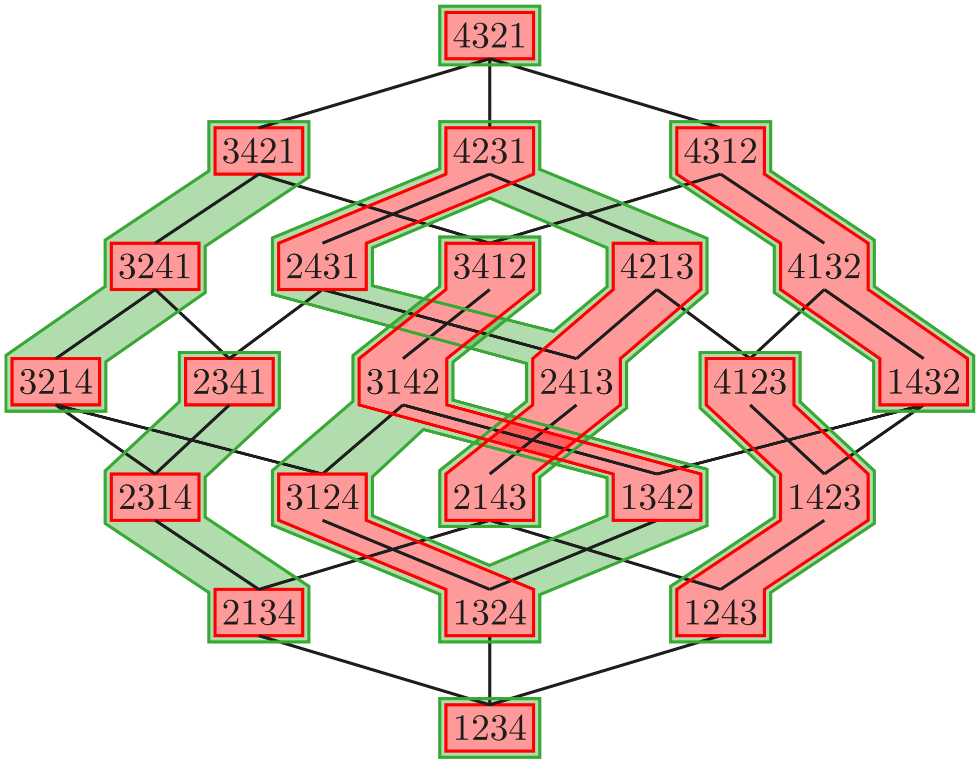
Novelli, Reutenauer, Thibon. Generalized descent patterns in permutations and associated Hopf Algebras. 2011

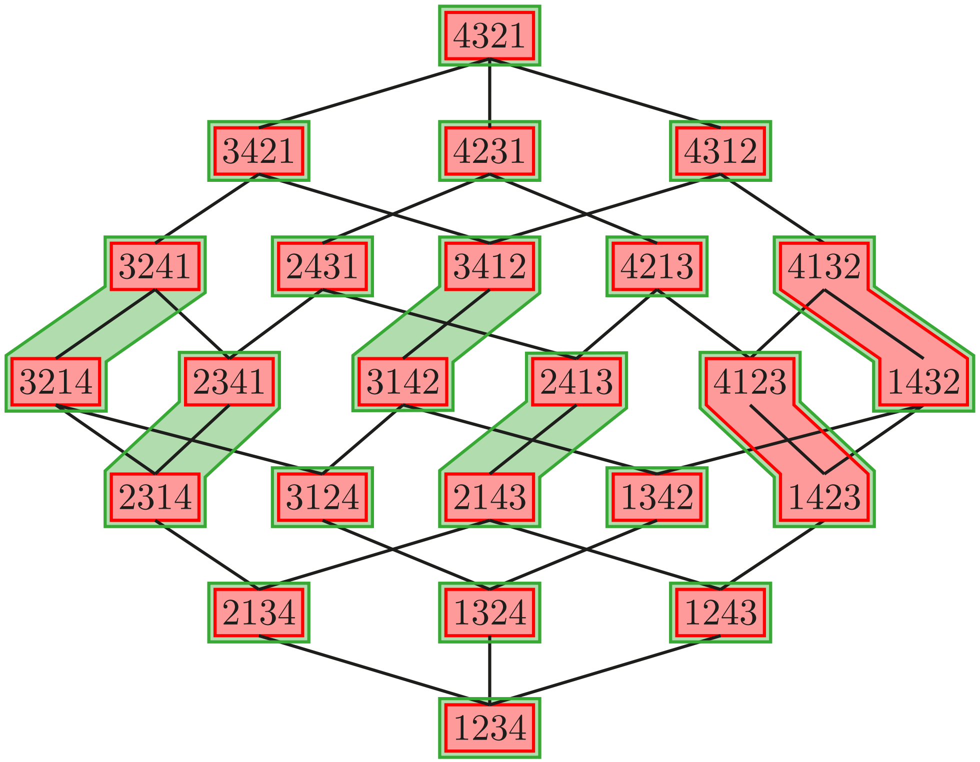
THM. The k-recoil congruence is a lattice quotient of the weak order.

k-CANOPY

The maps ins^k, can^k, and rec^k define a commutative diagram of lattice homomorphisms:

$$\begin{array}{ccc} \mathfrak{S}_n & \xrightarrow{\text{rec}^k} & \mathcal{AO}^k(n) \\ & \searrow^{\text{ins}^k} \quad\quad \nearrow_{\text{can}^k} & \\ & \mathcal{AT}^k(n) & \end{array}$$





ALGEBRA

SHUFFLE AND CONVOLUTION

For $n, n' \in \mathbb{N}$, consider the set of perms of $\mathfrak{S}_{n+n'}$ with at most one descent, at position n:

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n')\}$$

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

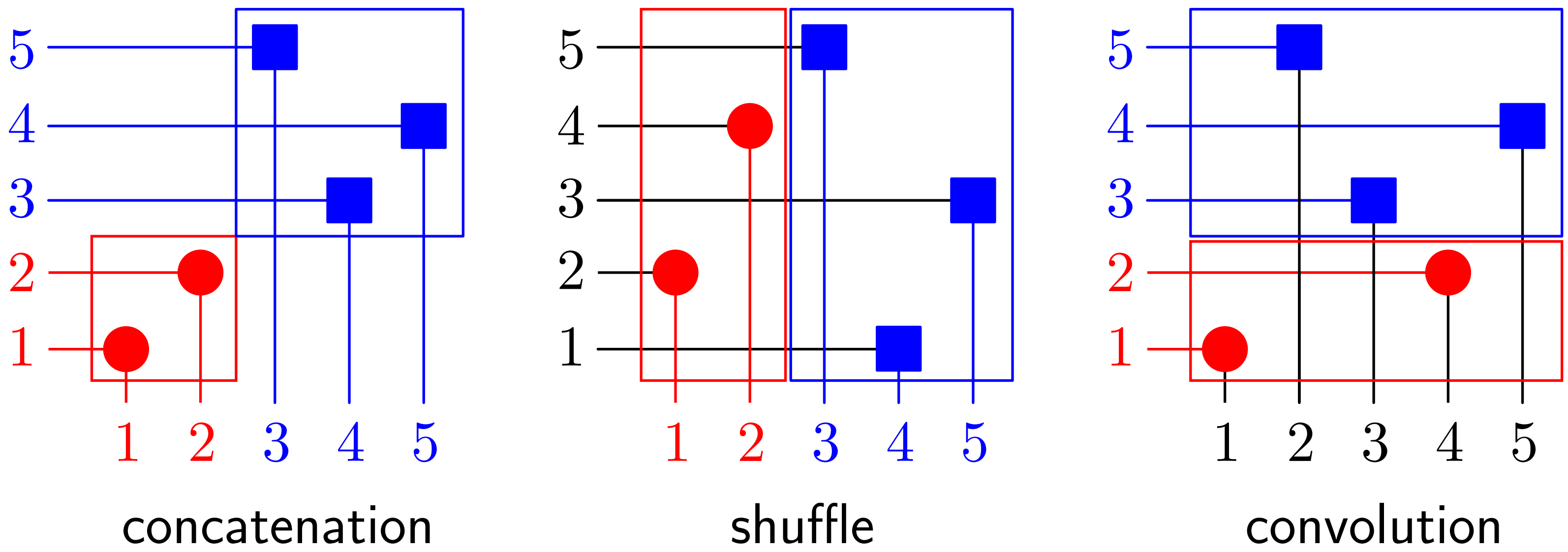
shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1)+n, \dots, \tau'(n')+n] \in \mathfrak{S}_{n+n'}$

shifted shuffle product $\tau \,\bar{\sqcup\!\sqcup}\, \tau' = \left\{(\tau\bar{\tau}') \circ \pi^{-1} \;\middle|\; \pi \in \mathfrak{S}^{(n,n')}\right\} \subset \mathfrak{S}_{n+n'}$

convolution product $\tau \star \tau' = \left\{\pi \circ (\tau\bar{\tau}') \;\middle|\; \pi \in \mathfrak{S}^{(n,n')}\right\} \subset \mathfrak{S}_{n+n'}$

Exm: $12 \,\bar{\sqcup\!\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



MALVENUTO & REUTENAUER'S HOPF ALGEBRA ON PERMUTATIONS

DEF. Combinatorial Hopf Algebra = combinatorial vector space $\mathcal{B}$ endowed with

$$\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$$

$$\text{coproduct } \triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

which are "compatible", ie.

$$\begin{array}{ccccc}
\mathcal{B} \otimes \mathcal{B} & \xrightarrow{\quad \cdot \quad} & \mathcal{B} & \xrightarrow{\quad \triangle \quad} & \mathcal{B} \otimes \mathcal{B} \\
\Big\downarrow {\scriptstyle \triangle \otimes \triangle} & & & & \Big\uparrow {\scriptstyle \cdot \otimes \cdot} \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & & \xrightarrow[\quad I \otimes \mathrm{swap} \otimes I \quad]{} & & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}
\end{array}$$

Malvenuto-Reutenauer algebra = Hopf algebra FQSym with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ and where

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

Malvenuto-Reutenauer. *Duality between quasi-symmetric functions and the Solomon descent algebra*. 1995

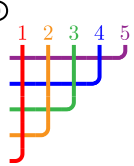
HOPF SUBALGEBRA

k-Twist algebra = vector subspace Twist^k of FQSym generated by

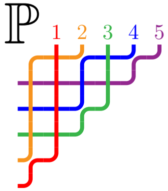
$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau \in \mathfrak{S} \\ \mathsf{ins}^k(\tau) = \mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T}^{\#})} \mathbb{F}_\tau,$$

for all acyclic k-twists T.

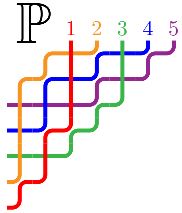
Exm:

$\mathbb{P}$ 

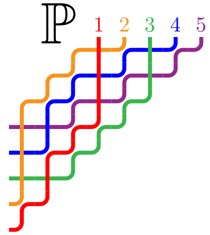
$= \sum_{\tau \in \mathfrak{S}_5} \mathbb{F}_\tau$

$\mathbb{P}$ 

$= \mathbb{F}_{13542} + \mathbb{F}_{15342} + \mathbb{F}_{31542} + \mathbb{F}_{51342} + \mathbb{F}_{35142} + \mathbb{F}_{53142} + \mathbb{F}_{35412} + \mathbb{F}_{53412}$

$\mathbb{P}$ 

$= \mathbb{F}_{31542} + \mathbb{F}_{35142}$

$\mathbb{P}$ 

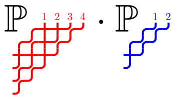
$= \mathbb{F}_{31542}$.

THEO. Twist^k is a subalgebra of FQSym

Loday-Ronco. *Hopf algebra of the planar binary trees*. 1998
Hivert-Novelli-Thibon. *The algebra of binary search trees*. 2005

GAME: Explain the product and coproduct directly on the k-twists...

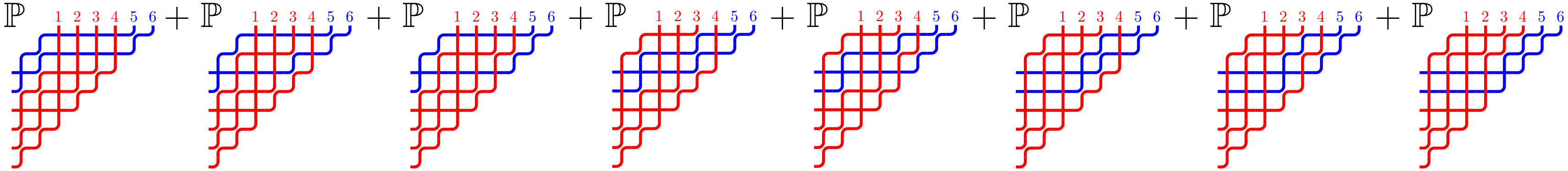
PRODUCT



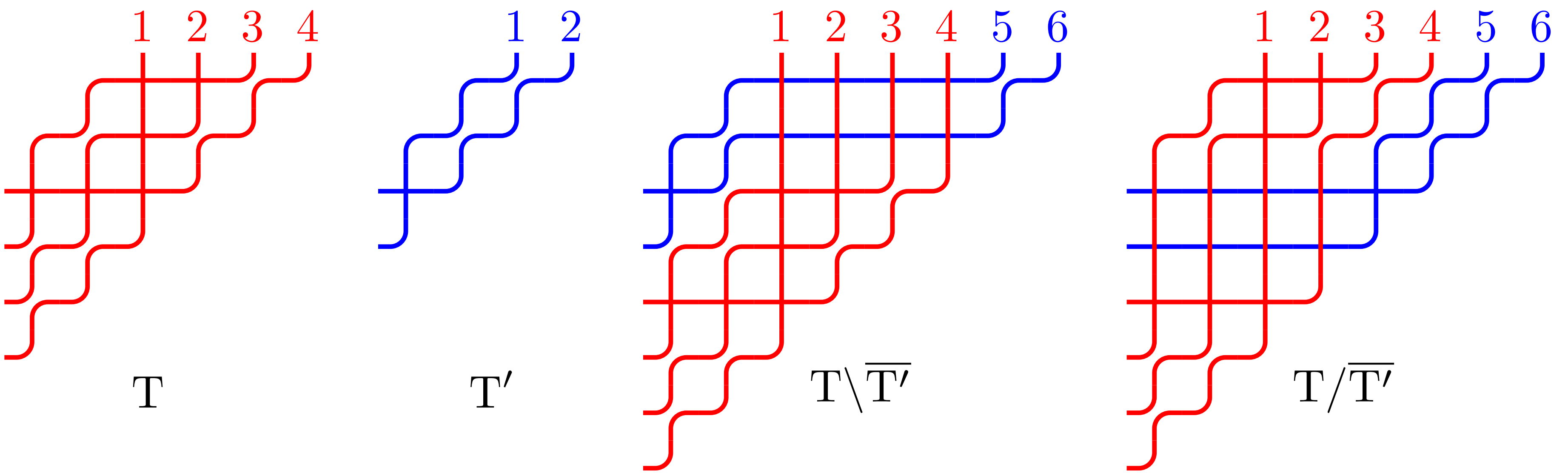
$$\mathbb{P} \cdot \mathbb{P} = (\mathbb{F}_{1423} + \mathbb{F}_{4123}) \cdot \mathbb{F}_{21}$$

$$= \begin{bmatrix} \mathbb{F}_{142365} \\ + \mathbb{F}_{412365} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{142635} \\ + \mathbb{F}_{146235} \\ + \mathbb{F}_{412635} \\ + \mathbb{F}_{416235} \\ + \mathbb{F}_{461235} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164235} \\ + \mathbb{F}_{614235} \\ + \mathbb{F}_{641235} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{142653} \\ + \mathbb{F}_{146253} \\ + \mathbb{F}_{412653} \\ + \mathbb{F}_{416253} \\ + \mathbb{F}_{461253} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164253} \\ + \mathbb{F}_{614253} \\ + \mathbb{F}_{641253} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{146523} \\ + \mathbb{F}_{416523} \\ + \mathbb{F}_{461523} \\ + \mathbb{F}_{465123} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{164523} \\ + \mathbb{F}_{614523} \\ + \mathbb{F}_{641523} \\ + \mathbb{F}_{645123} \end{bmatrix} + \begin{bmatrix} \mathbb{F}_{165423} \\ + \mathbb{F}_{615423} \\ + \mathbb{F}_{651423} \\ + \mathbb{F}_{654123} \end{bmatrix}$$

$$= \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P} + \mathbb{P}$$



PROP. For $\mathrm{T} \in \mathcal{AT}^k(n)$ and $\mathrm{T}' \in \mathcal{AT}^k(n')$ acyclic k-twists, $\mathbb{P}_\mathrm{T} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_\mathrm{S} \mathbb{P}_\mathrm{S}$, where S runs over the interval between $\mathrm{T} \backslash \mathrm{T}'$ and T/T' in the $(k, n+n')$-twist lattice.



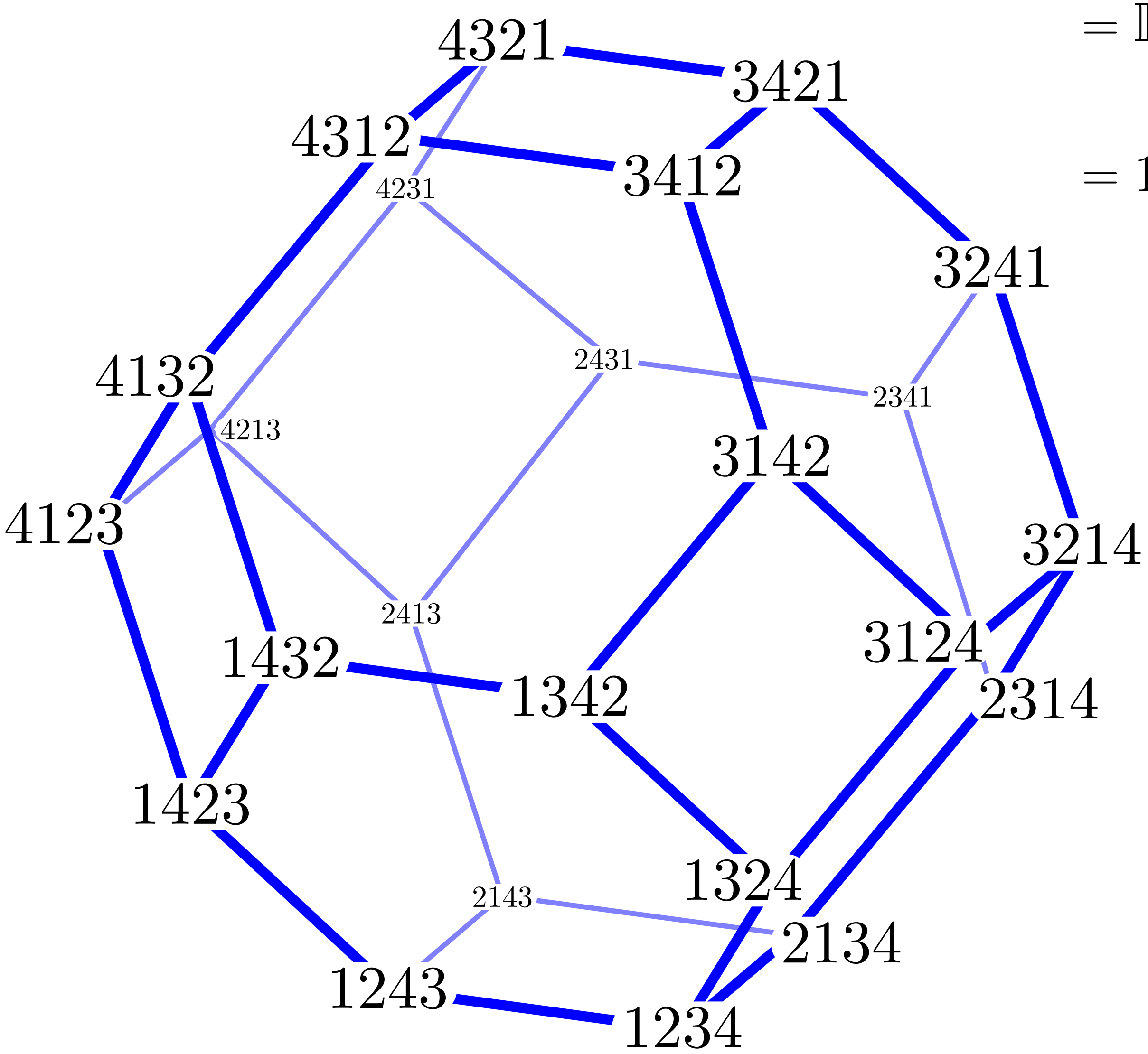
GEOMETRY

PERMUTAHEDRON

Permutahedron $\mathsf{Perm}^k(n) = \mathrm{conv}\left\{(\tau(1), \dots, \tau(n)) \mid \tau \in \mathfrak{S}_n\right\}$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

$$= \mathbb{1} + \sum_{1 \leq i < j \leq n} [\mathbf{e}_i, \mathbf{e}_j]$$



connections to

- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

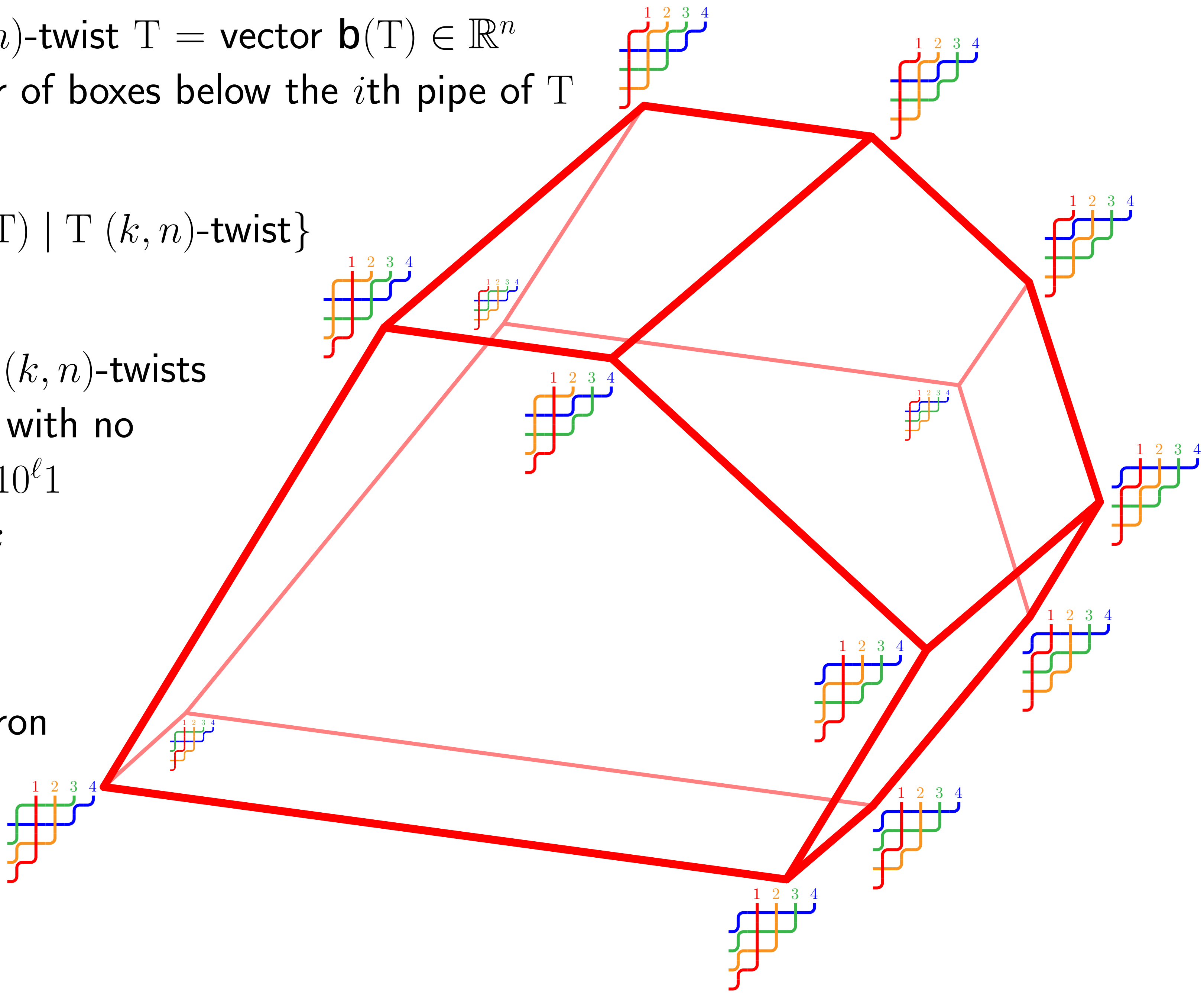
BRICK POLYTOPE

brick vector of a (k, n)-twist T = vector $\mathbf{b}(\mathrm{T}) \in \mathbb{R}^n$
with $\mathbf{b}(\mathrm{T})_i$ = number of boxes below the ith pipe of T

Brick polytope
$\mathrm{Brick}^k(n) = \mathrm{conv}\,\{\mathbf{b}(\mathrm{T}) \mid \mathrm{T}\ (k, n)\text{-twist}\}$

Vertices $\longleftrightarrow$ acyclic (k, n)-twists
Facets $\longleftrightarrow$ $0/1$-seqs with no subseqs $10^\ell 1$ for $\ell \geq k$

connections to
- Loday associahedron
- incidence cones of binary trees
- Tamari lattice



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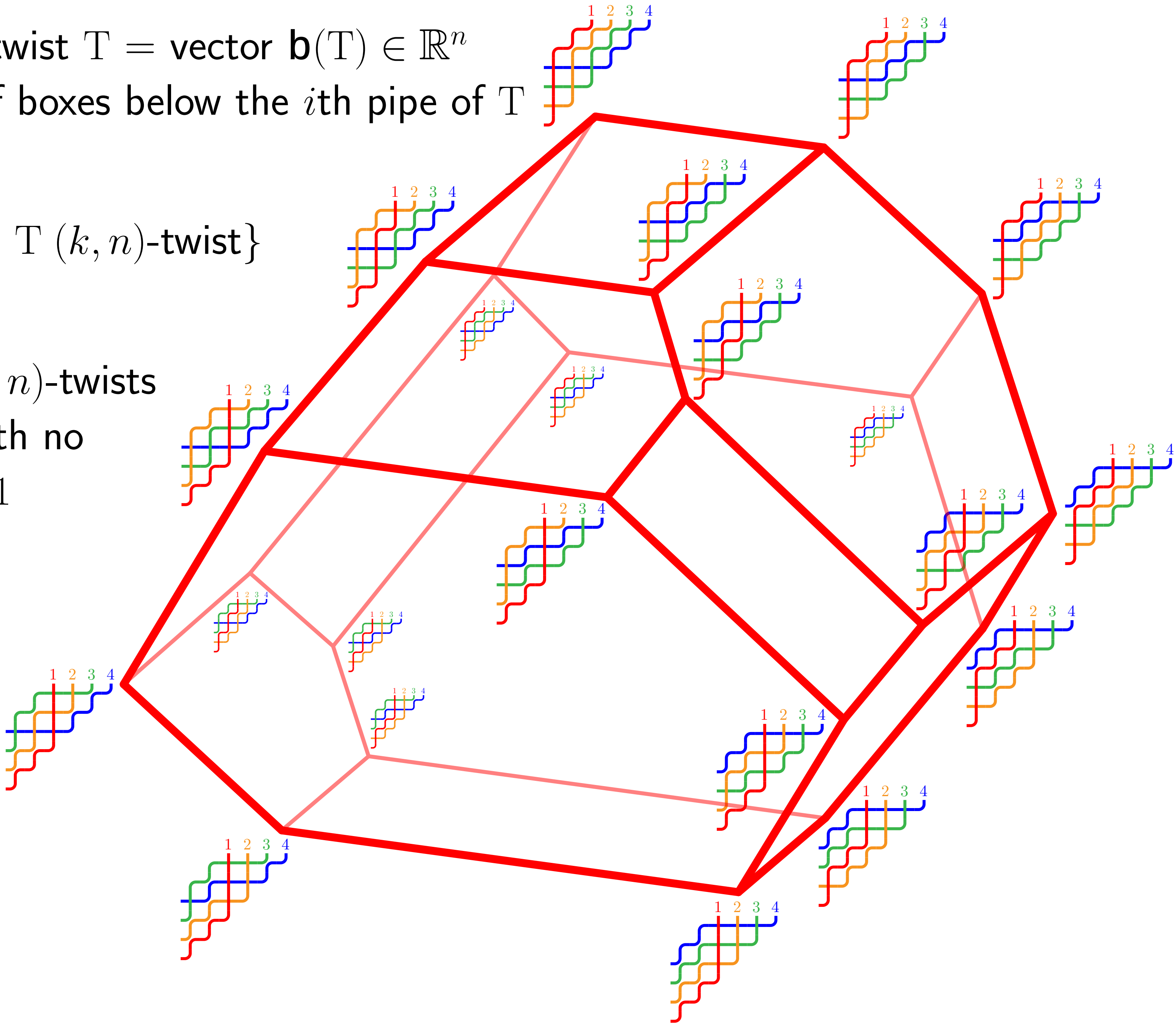

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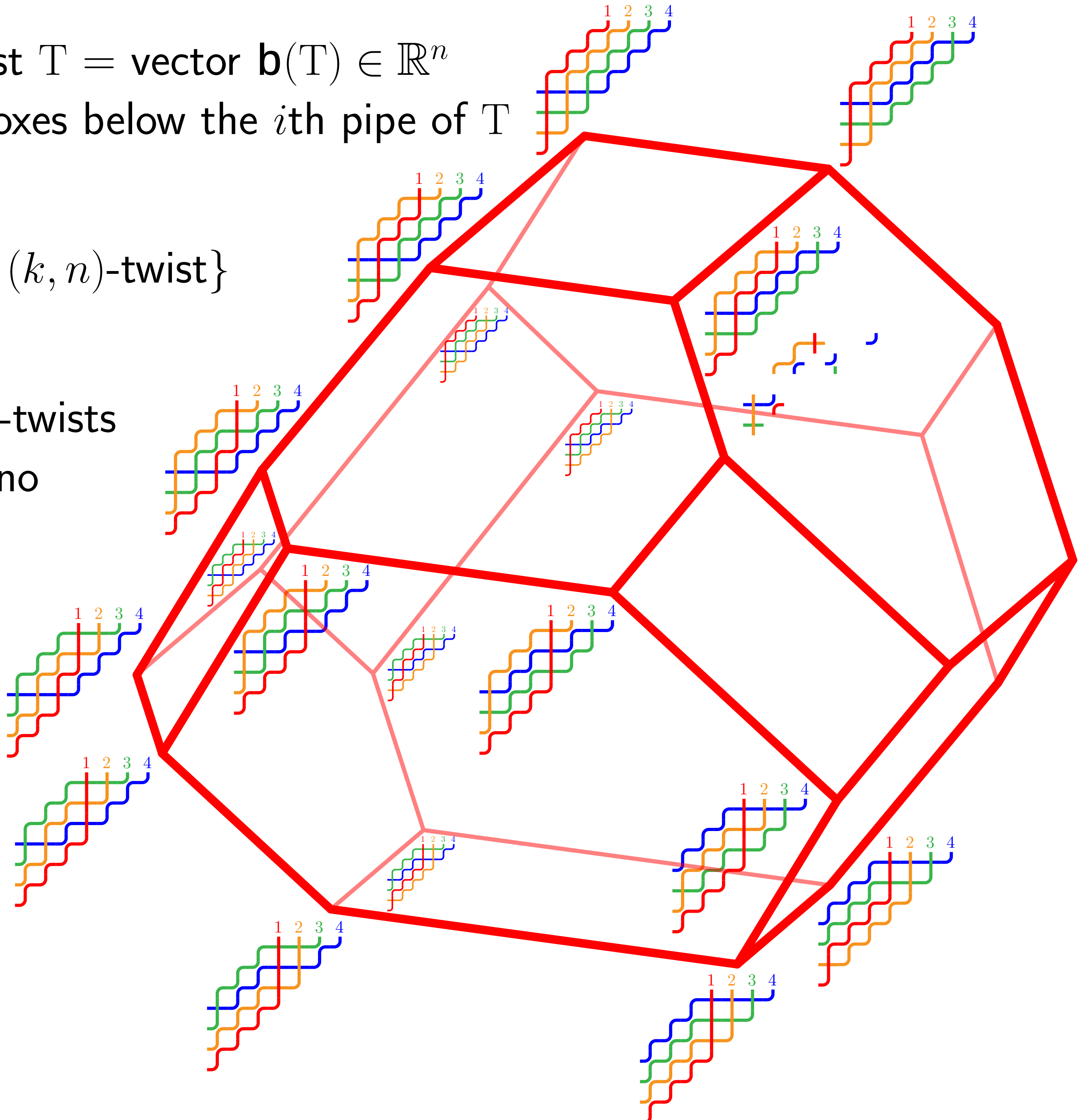
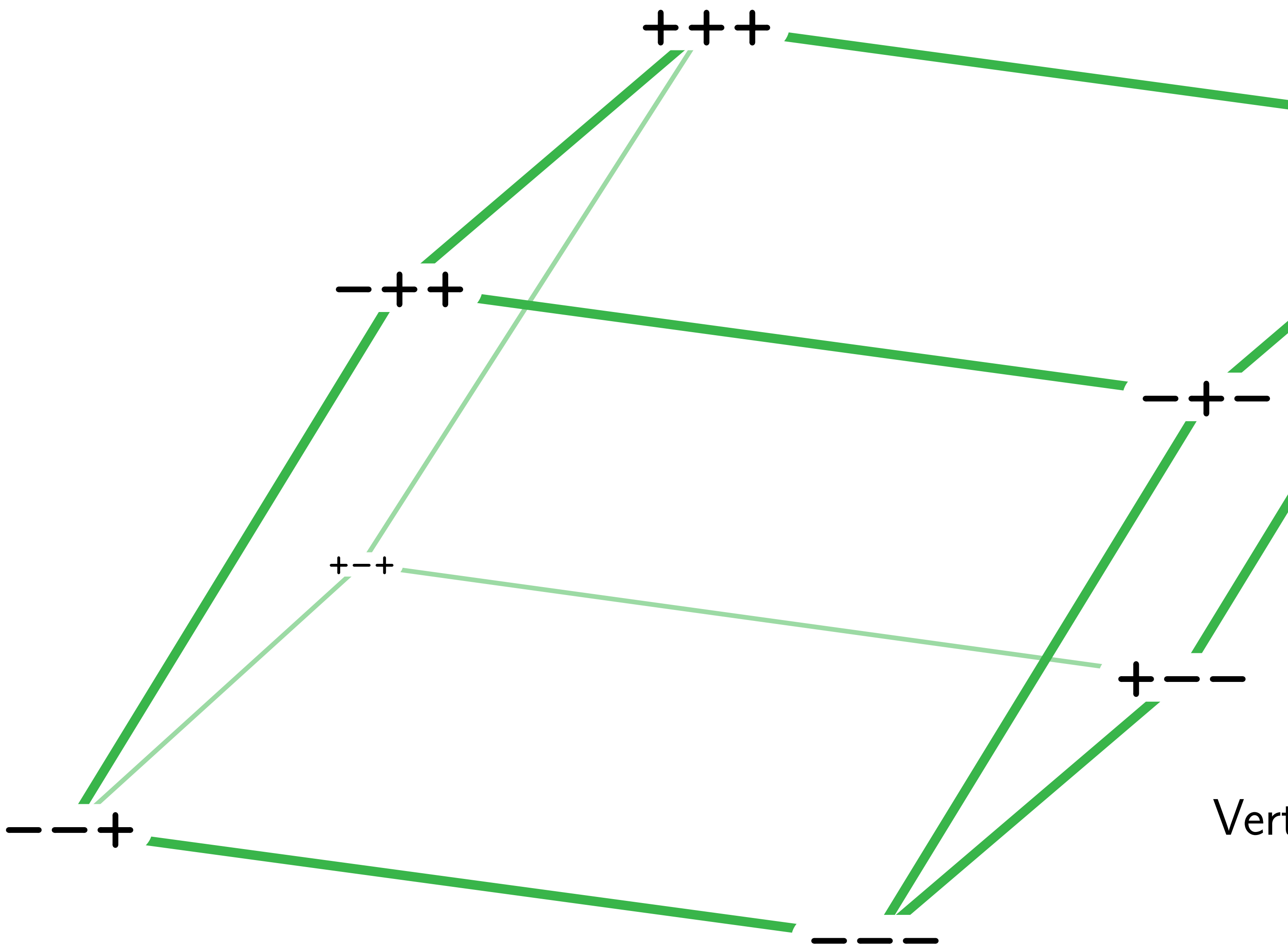


Vertices $\longleftrightarrow$ acyclic (k,n)-twists
Facets $\longleftrightarrow$ $0/1$-seqs with no subseqs $10^{\ell}1$ for $\ell \geq k$

connections to

- Loday associahedron
- incidence cones of binary trees
- Tamari lattice

ZONOTOPE



Zonotope

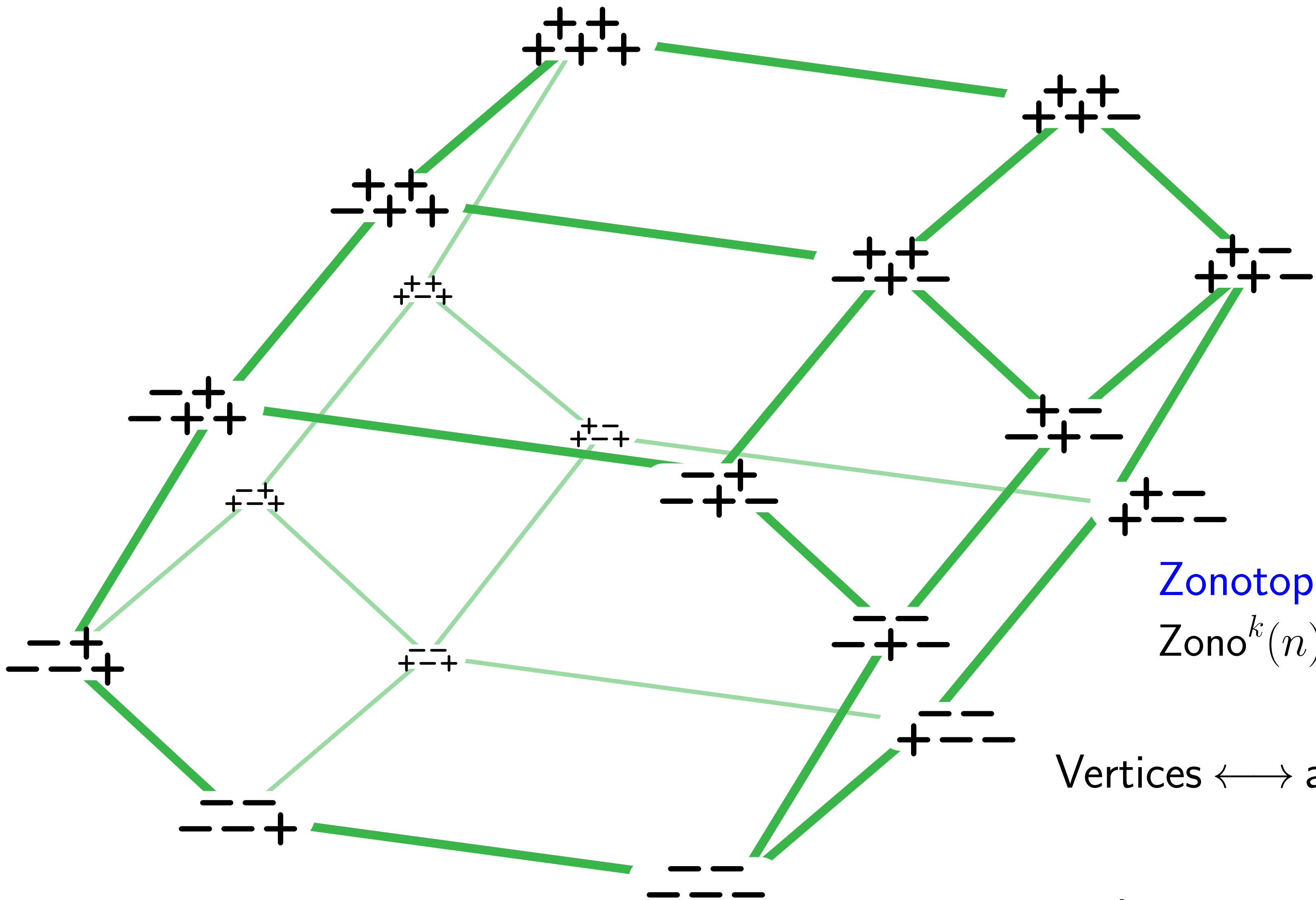
$$\mathsf{Zono}^k(n) = \sum_{|i-j|\le k} [\mathbf{e}_i, \mathbf{e}_j]$$

Vertices $\longleftrightarrow$ acyclic orientations of $\mathrm{G}^k(n)$

connections to

- matroids and oriented matroids
- hyperplane arrangements

ZONOTOPE



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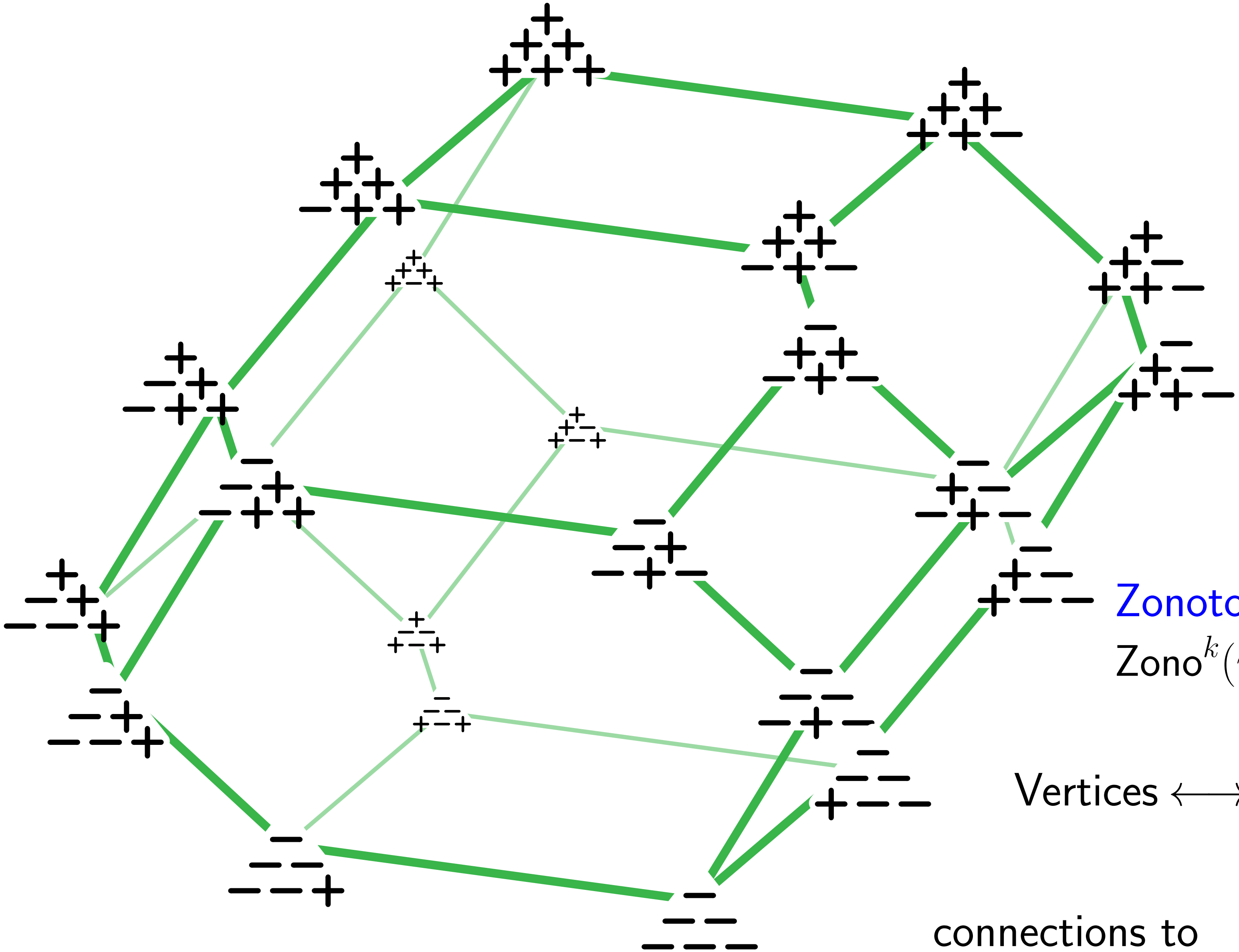

Zonotope

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Vertices $\longleftrightarrow$ acyclic orientations of $\mathrm{G}^k(n)$

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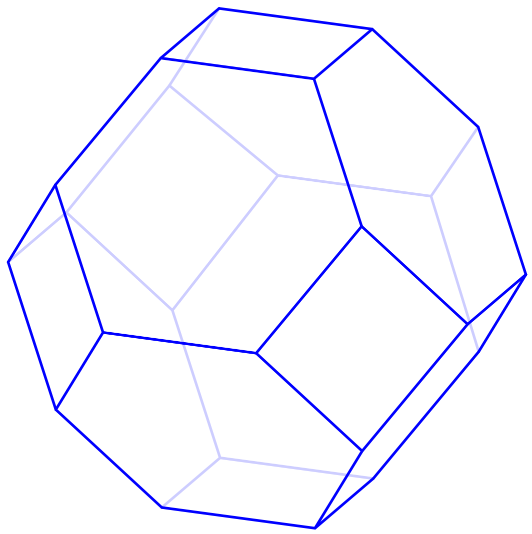
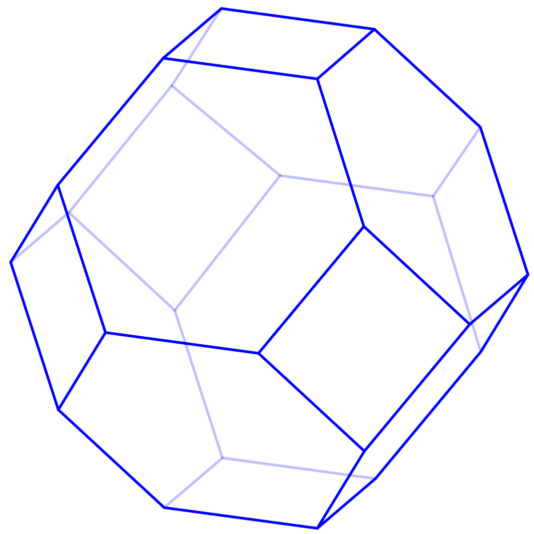
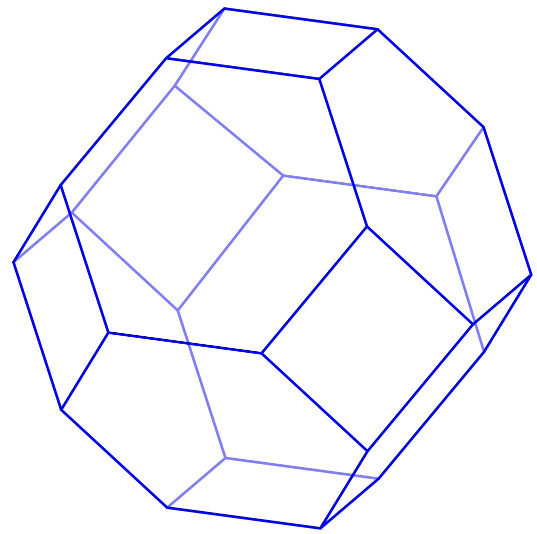
MATRIOCHKA POLYTOPES

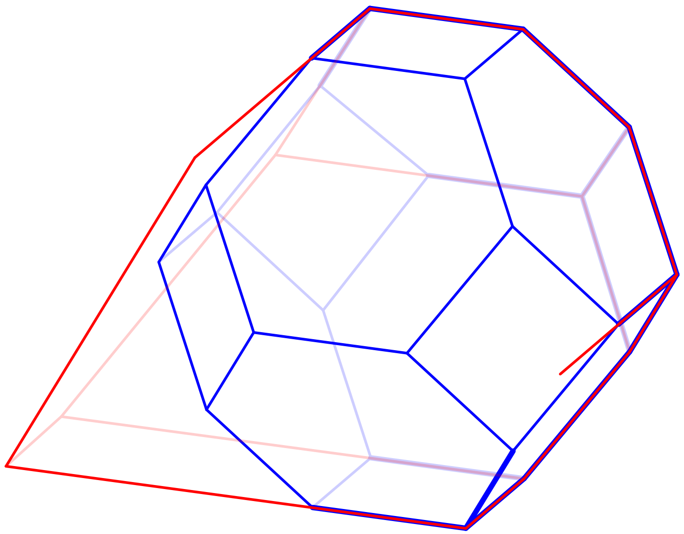






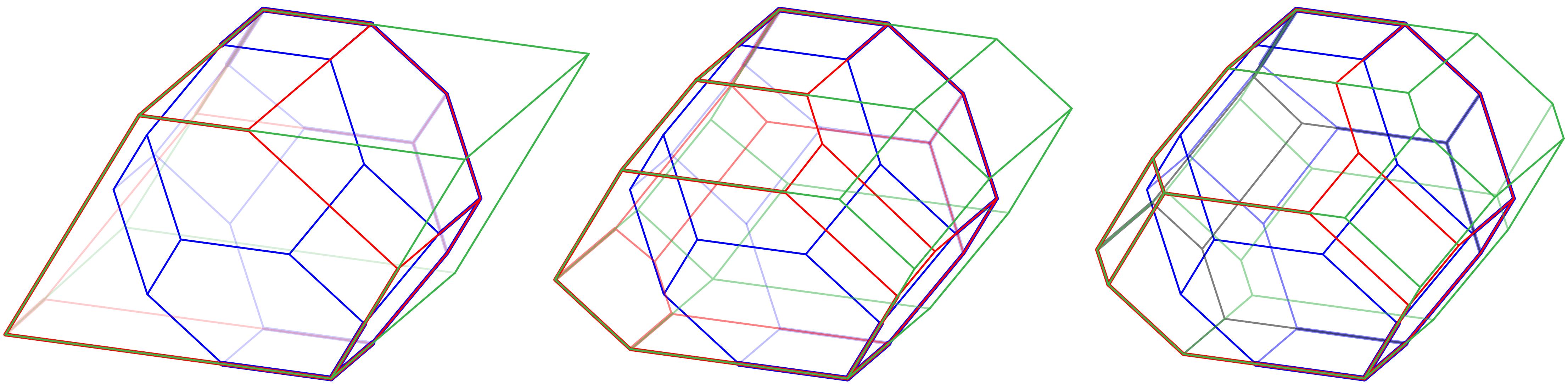
Permutahedron $\mathsf{Perm}^k(n)$

MATRIOCHKA POLYTOPES



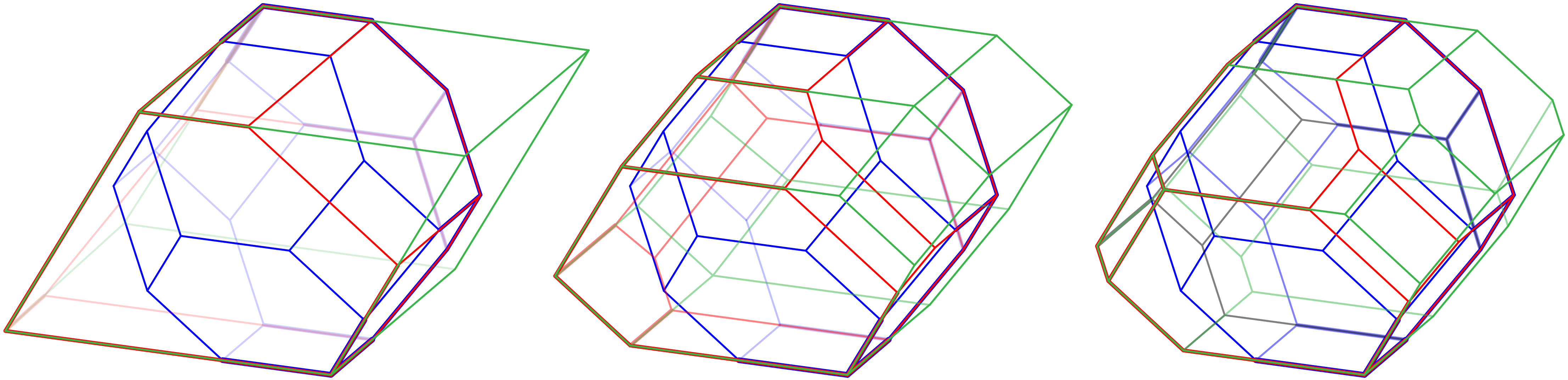
Permutahedron $\text{Perm}^k(n)$ $\subset$ Brick polytope $\text{Brick}^k(n)$

MATRIOCHKA POLYTOPES



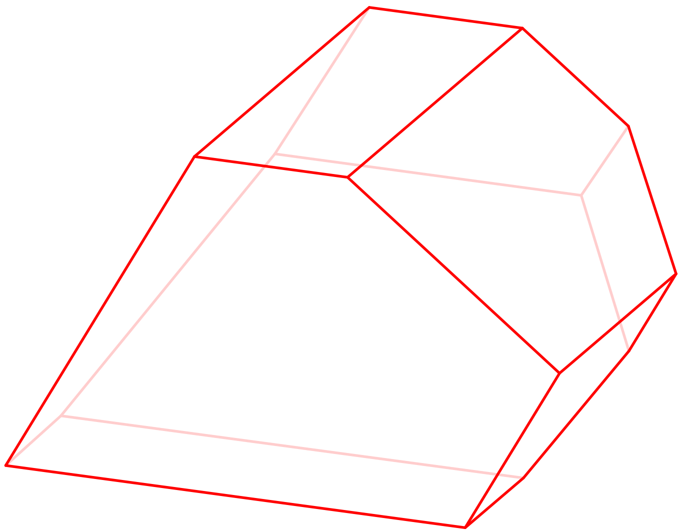
Permutahedron $\mathsf{Perm}^k(n)$ $\subset$ Brick polytope $\mathsf{Brick}^k(n)$ $\subset$ Zonotope $\mathsf{Zono}^k(n)$

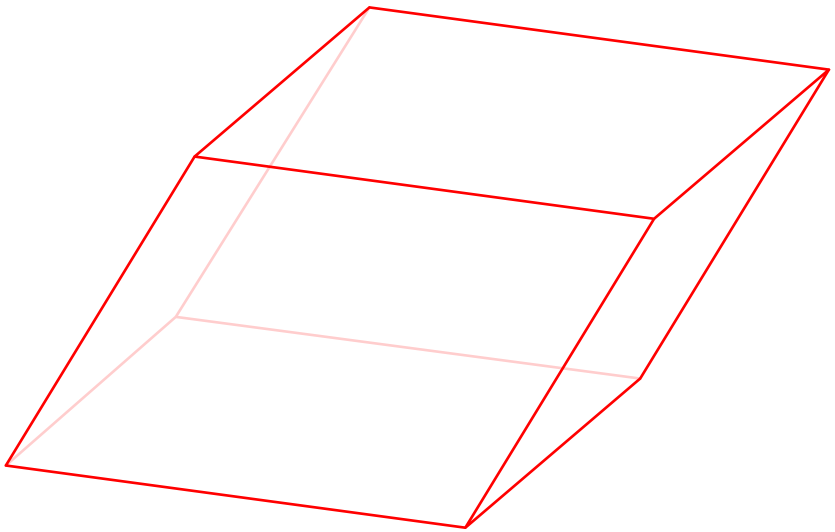
MATRIOCHKA POLYTOPES



Permutahedron $\mathsf{Perm}^k(n)$ $\subset$ Brick polytope $\mathsf{Brick}^k(n)$ $\subset$ Zonotope $\mathsf{Zono}^k(n)$

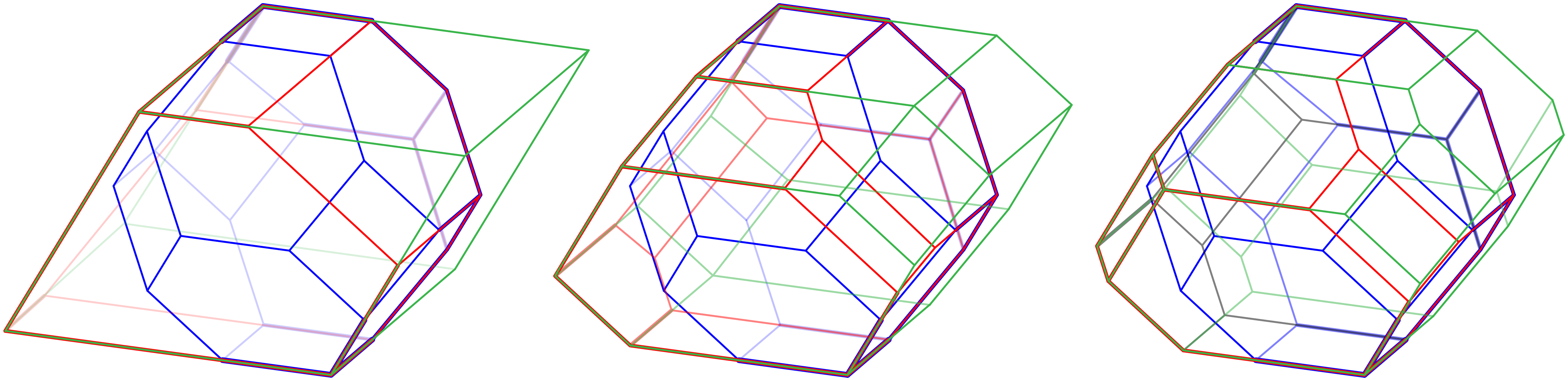
$\mathsf{Brick}^1(n)$





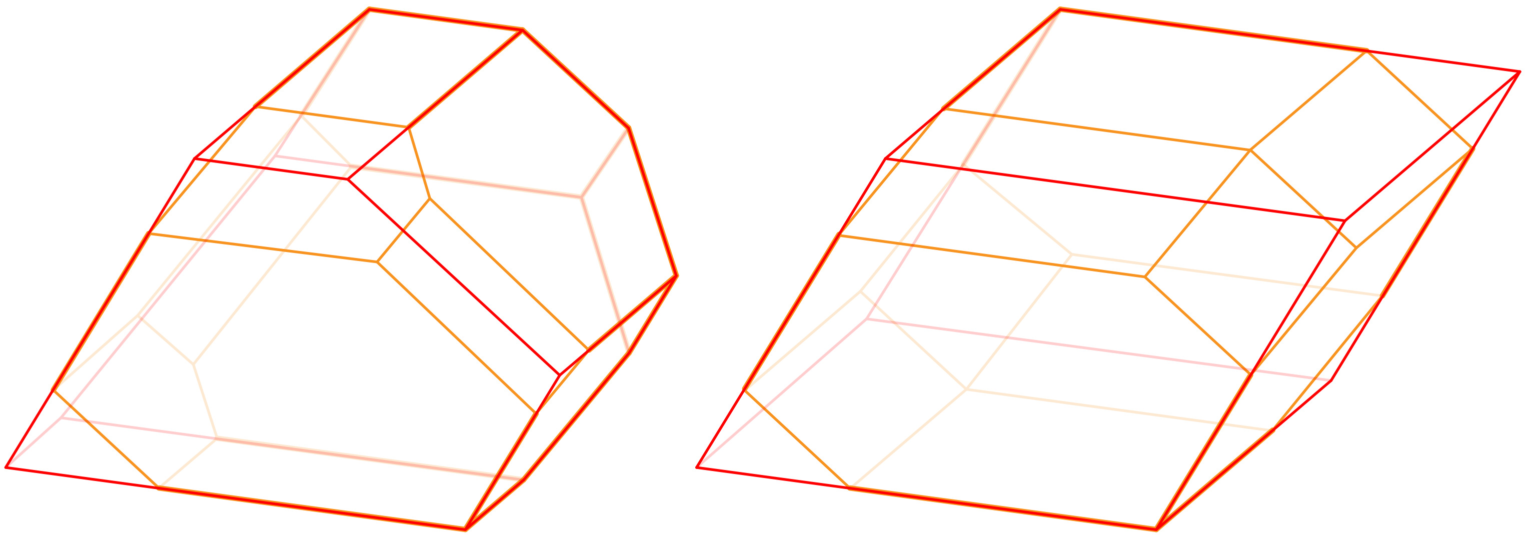
$\mathsf{Zono}^1(n)$

MATRIOCHKA POLYTOPES



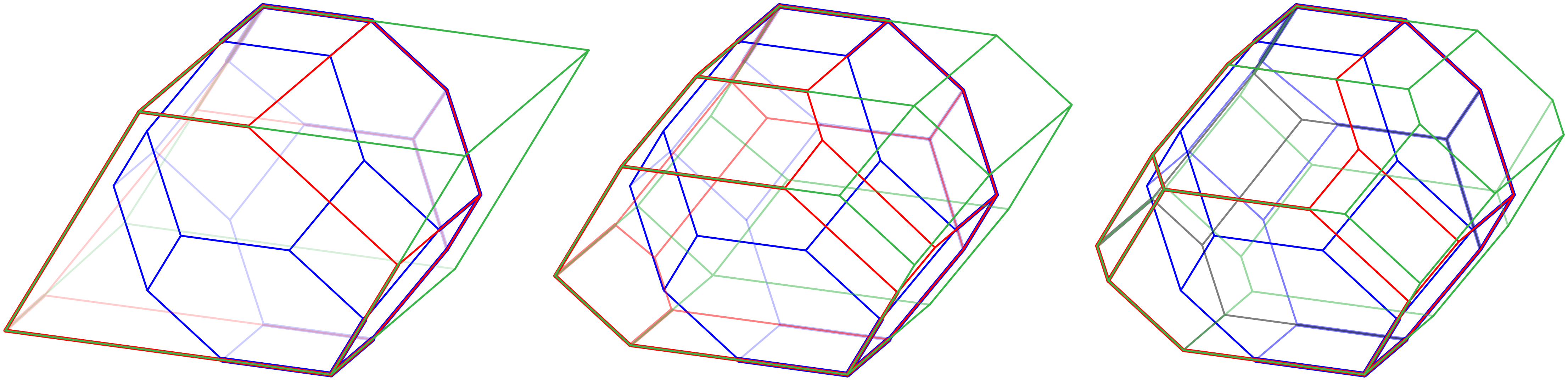
Permutahedron $\mathsf{Perm}^k(n)$ $\subset$ Brick polytope $\mathsf{Brick}^k(n)$ $\subset$ Zonotope $\mathsf{Zono}^k(n)$

$\mathsf{Brick}^1(n)$
$\cap$
$\mathsf{Brick}^2(n)$



$\mathsf{Zono}^1(n)$
$\cap$
$\mathsf{Zono}^2(n)$

MATRIOCHKA POLYTOPES



Permutahedron $\mathsf{Perm}^k(n)$ $\subset$ Brick polytope $\mathsf{Brick}^k(n)$ $\subset$ Zonotope $\mathsf{Zono}^k(n)$

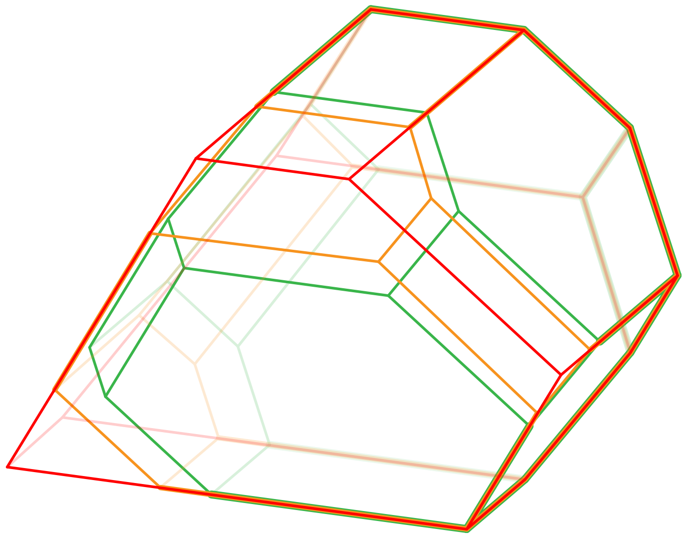
$\mathsf{Brick}^1(n)$

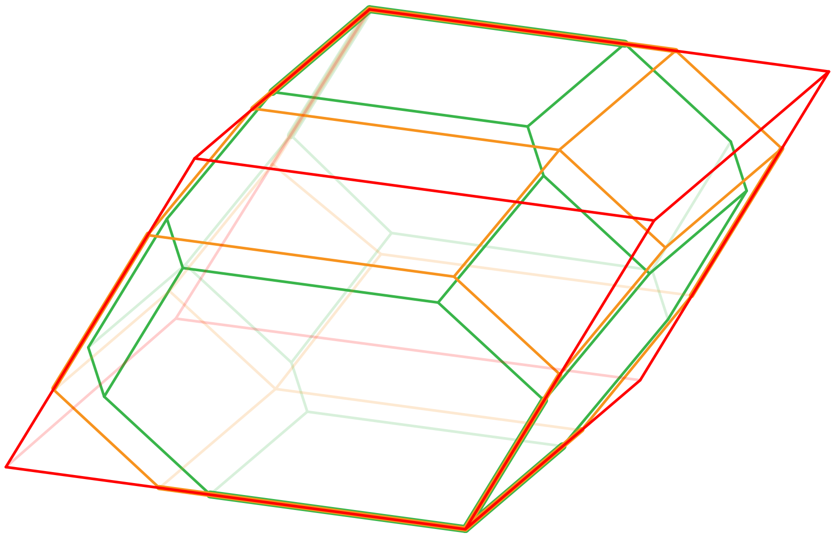
$\cap$

$\mathsf{Brick}^2(n)$

$\cap$

$\mathsf{Brick}^3(n)$





$\mathsf{Zono}^1(n)$

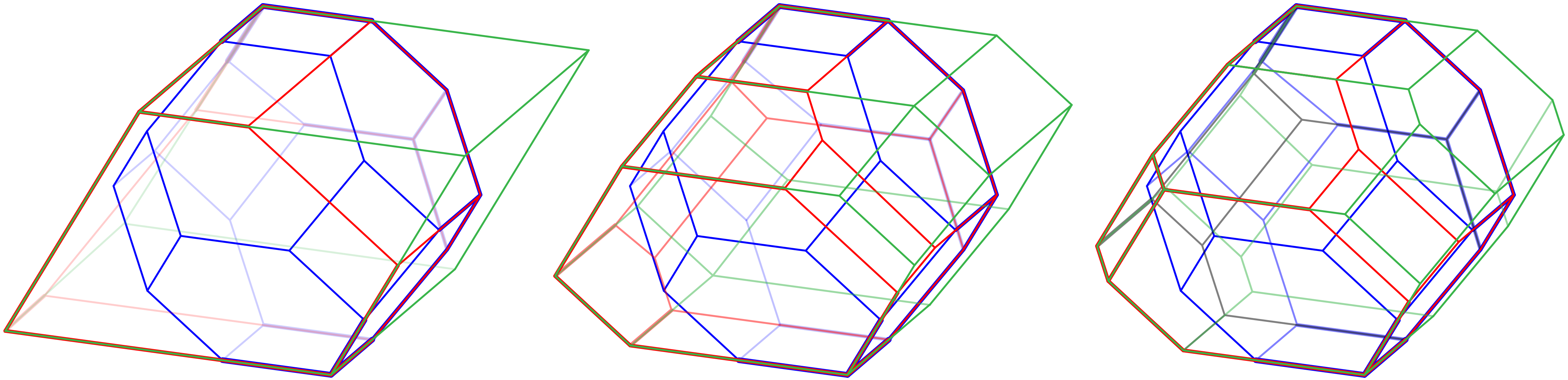
$\cap$

$\mathsf{Zono}^2(n)$

$\cap$

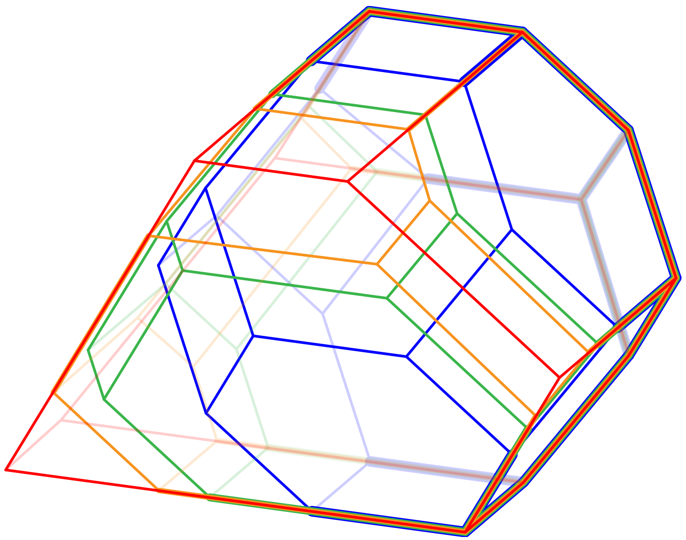
$\mathsf{Zono}^3(n)$

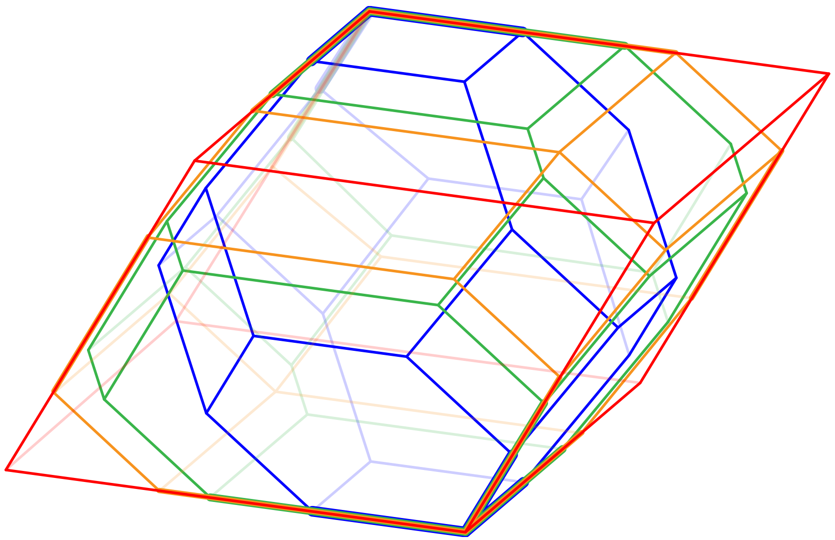
MATRIOCHKA POLYTOPES



Permutahedron $\mathrm{Perm}^k(n)$ $\subset$ Brick polytope $\mathrm{Brick}^k(n)$ $\subset$ Zonotope $\mathrm{Zono}^k(n)$

$\mathrm{Brick}^1(n)$
$\cap$
$\mathrm{Brick}^2(n)$
$\cap$
$\mathrm{Brick}^3(n)$
$\cap$
$\vdots$
$\cap$
$\mathrm{Perm}^k(n)$





$\mathrm{Zono}^1(n)$
$\cap$
$\mathrm{Zono}^2(n)$
$\cap$
$\mathrm{Zono}^3(n)$
$\cap$
$\vdots$
$\cap$
$\mathrm{Perm}^k(n)$

NORMAL CONES

For a poset $\triangleleft$, define $\mathrm{C}^\diamond(\triangleleft) = \{\mathbf{x} \in \mathbb{H} \mid x_i \le x_j \text{ for all } i \triangleleft j \text{ in } \mathrm{T}\}$.

PROP. The cones form complete simplicial fans:

$$\{\mathrm{C}^\diamond(\tau) \mid \tau \in \mathfrak{S}_n\} = \text{braid fan} = \text{normal fan of the permutahedron } \mathsf{Perm}^k(n),$$

$$\left\{\mathrm{C}^\diamond(\mathrm{T}) \;\middle|\; \mathrm{T} \in \mathcal{AT}^k(n)\right\} = \text{brick fan} = \text{normal fan of the brick polytope } \mathsf{Brick}^k(n),$$

$$\left\{\mathrm{C}^\diamond(\mathrm{O}) \;\middle|\; \mathrm{O} \in \mathcal{AO}^k(n)\right\} = \text{boolean fan} = \text{normal fan of the zonotope } \mathsf{Zono}^k(n).$$

PROP. The insertion map $\mathsf{ins}^k : \mathfrak{S}_n \to \mathcal{AT}^k(n)$, the k-canopy $\mathsf{can}^k : \mathcal{AT}^k(n) \to \mathcal{AO}^k(n)$ and the k-recoil map $\mathsf{rec}^k : \mathfrak{S}_n \to \mathcal{AO}^k(n)$ are characterized by:

$$\mathrm{T} = \mathsf{ins}^k(\tau) \iff \mathrm{C}(\mathrm{T}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\mathrm{T}) \supseteq \mathrm{C}^\diamond(\tau),$$

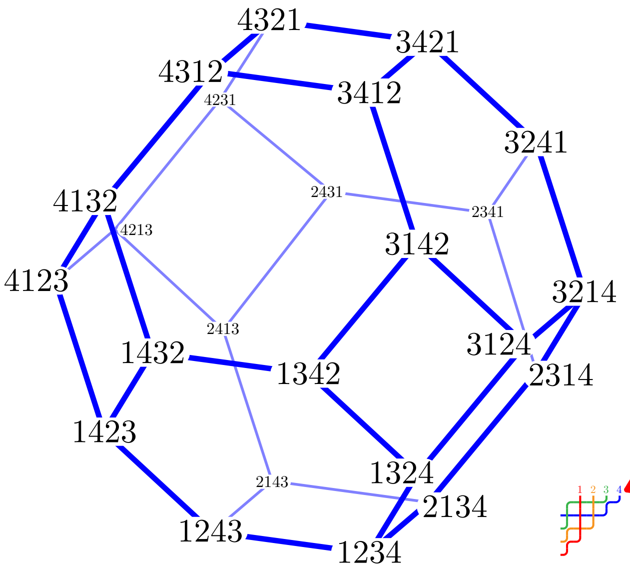
$$\mathrm{O} = \mathsf{can}^k(\mathrm{T}) \iff \mathrm{C}(\mathrm{O}) \subseteq \mathrm{C}(\mathrm{T}) \iff \mathrm{C}^\diamond(\mathrm{O}) \supseteq \mathrm{C}^\diamond(\mathrm{T}),$$

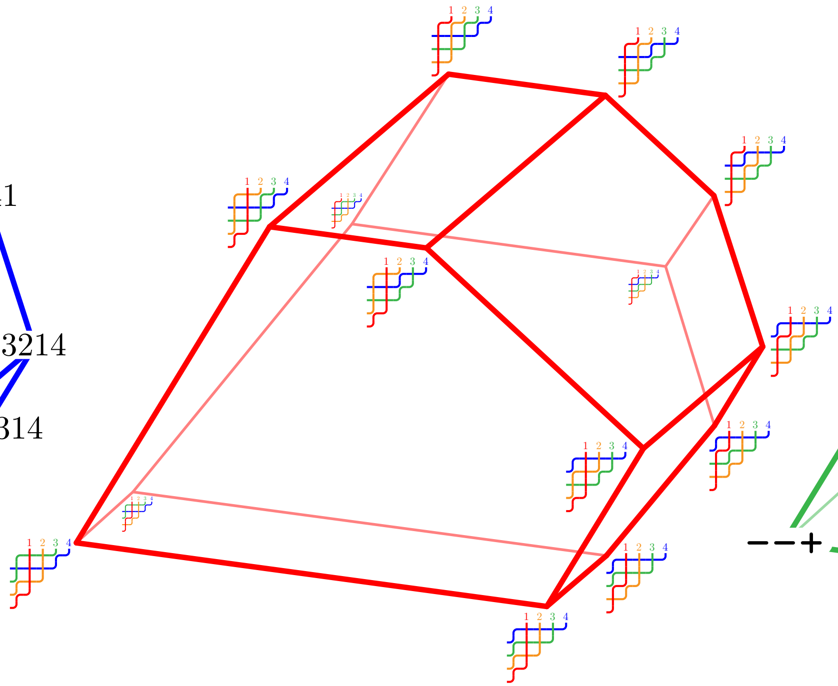
$$\mathrm{O} = \mathsf{rec}^k(\tau) \iff \mathrm{C}(\mathrm{O}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\mathrm{O}) \supseteq \mathrm{C}^\diamond(\tau).$$

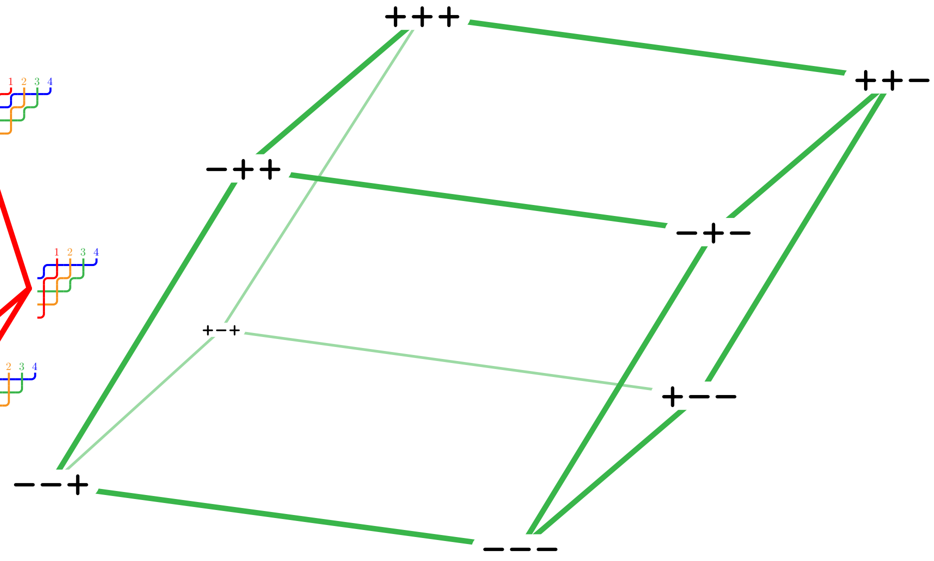
LINEAR ORIENTATION

Oriented in the direction $\sum_{i \in [n]} (n+1-2i)\, \mathbf{e}_i$, their graphs are Hasse diagrams of lattices:

permutahedron $\mathrm{Perm}^k(n)$ | brick polytope $\mathrm{Brick}^k(n)$ | zonotope $\mathrm{Zono}^k(n)$







weak order on $\mathfrak{S}_n$ | increasing flip lattice on acyclic (k,n)-twists | boolean lattice on acyclic orientations of $\mathrm{G}^k(n)$

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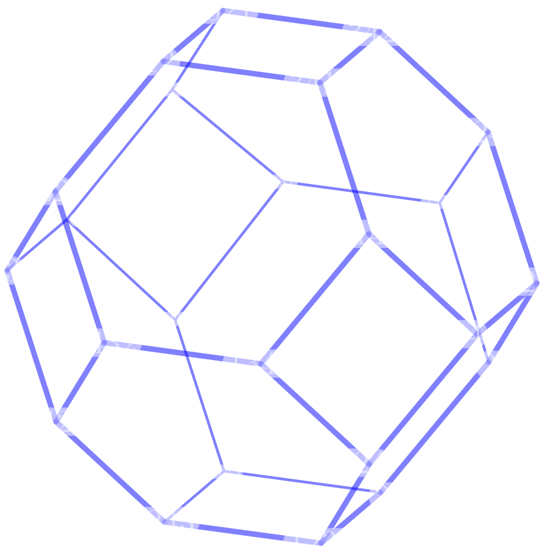
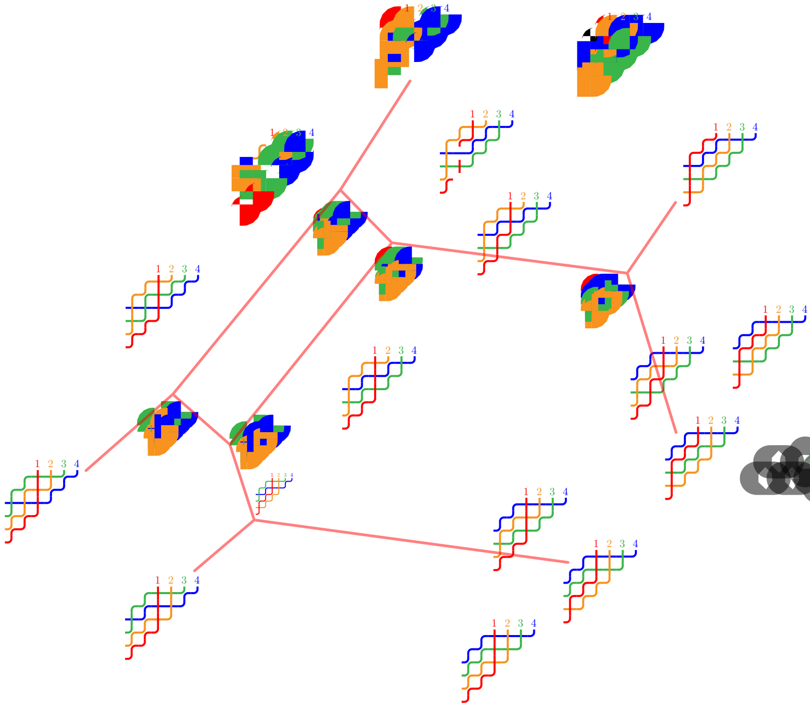


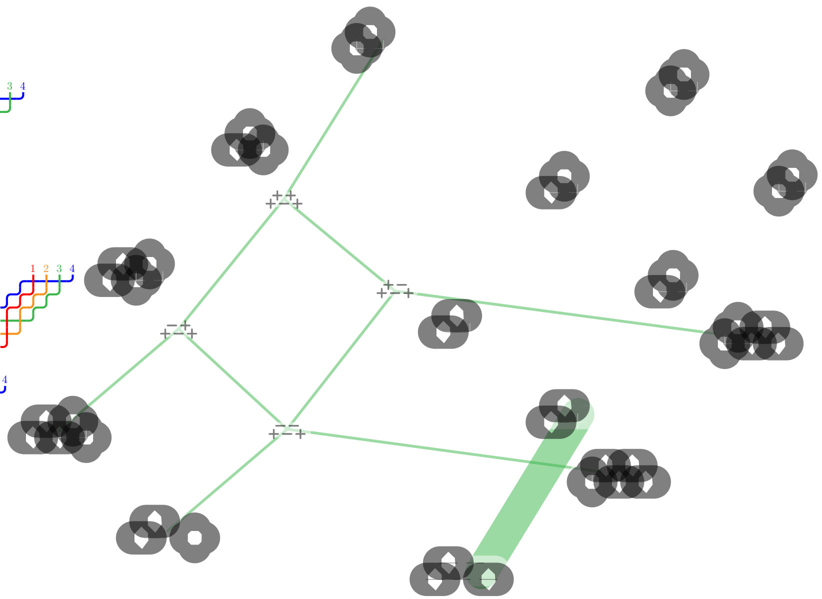
weak order on $\mathfrak{S}_n$

brick polytope $\mathsf{Brick}^k(n)$



increasing flip lattice
on acyclic (k,n)-twists

zonotope $\mathsf{Zono}^k(n)$

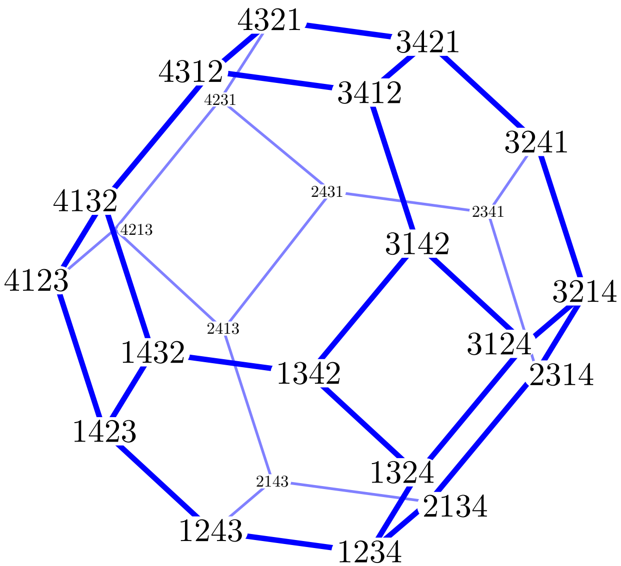


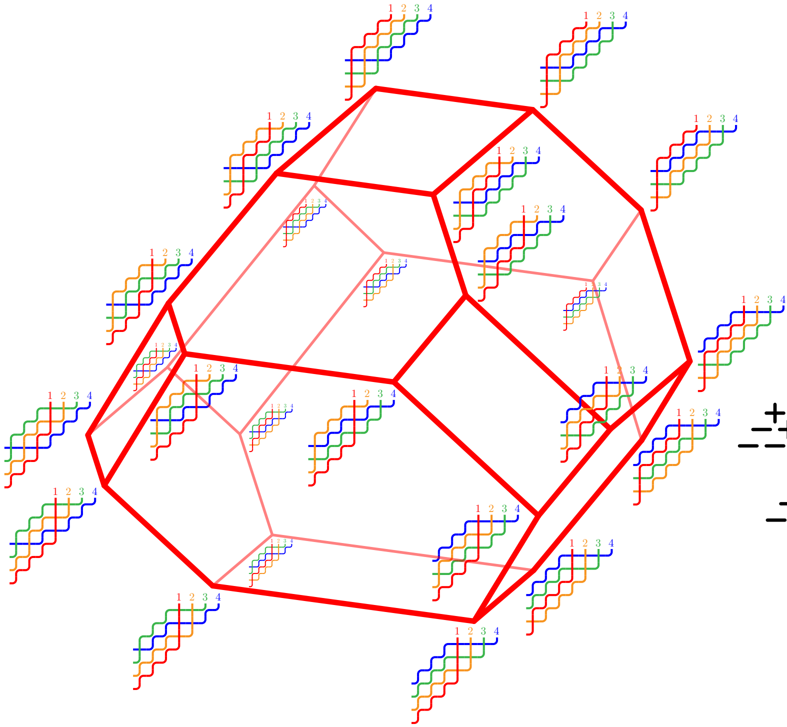
boolean lattice
on acyclic orientations of $\mathrm{G}^k(n)$

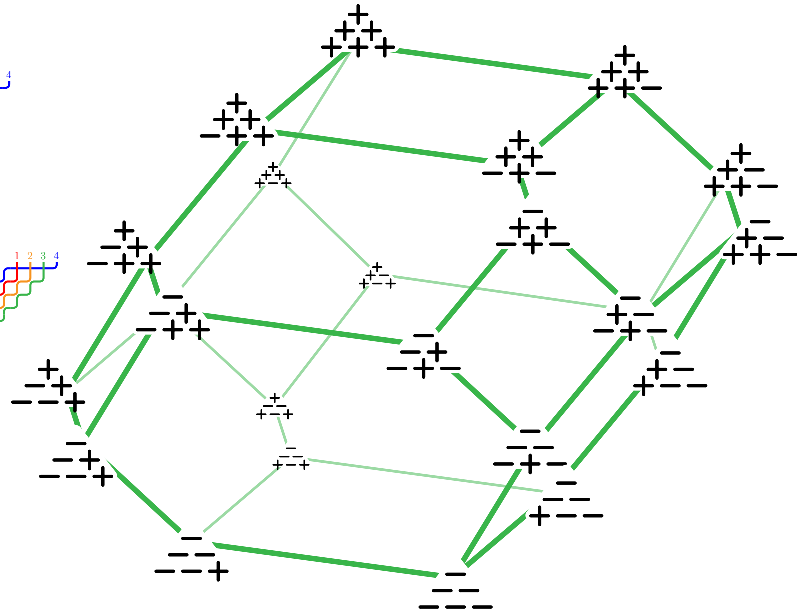
LINEAR ORIENTATION

Oriented in the direction $\sum_{i\in[n]} (n+1-2i)\,\mathbf{e}_i$, their graphs are Hasse diagrams of lattices:

permutahedron $\mathsf{Perm}^k(n)$ | brick polytope $\mathsf{Brick}^k(n)$ | zonotope $\mathsf{Zono}^k(n)$







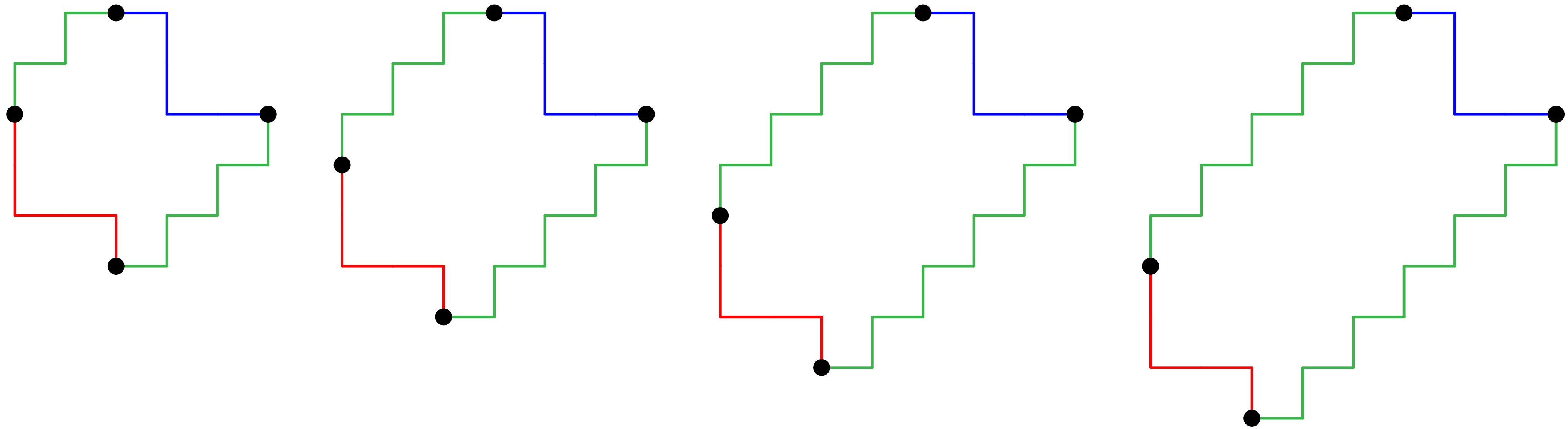
weak order on $\mathfrak{S}_n$

increasing flip lattice on acyclic (k,n)-twists

boolean lattice on acyclic orientations of $\mathrm{G}^k(n)$

THREE EXTENSIONS

CAMBRIANIZATION



$k = 0$ $\quad$ $k = 1$ $\quad$ $k = 2$ $\quad$ $k = 3$

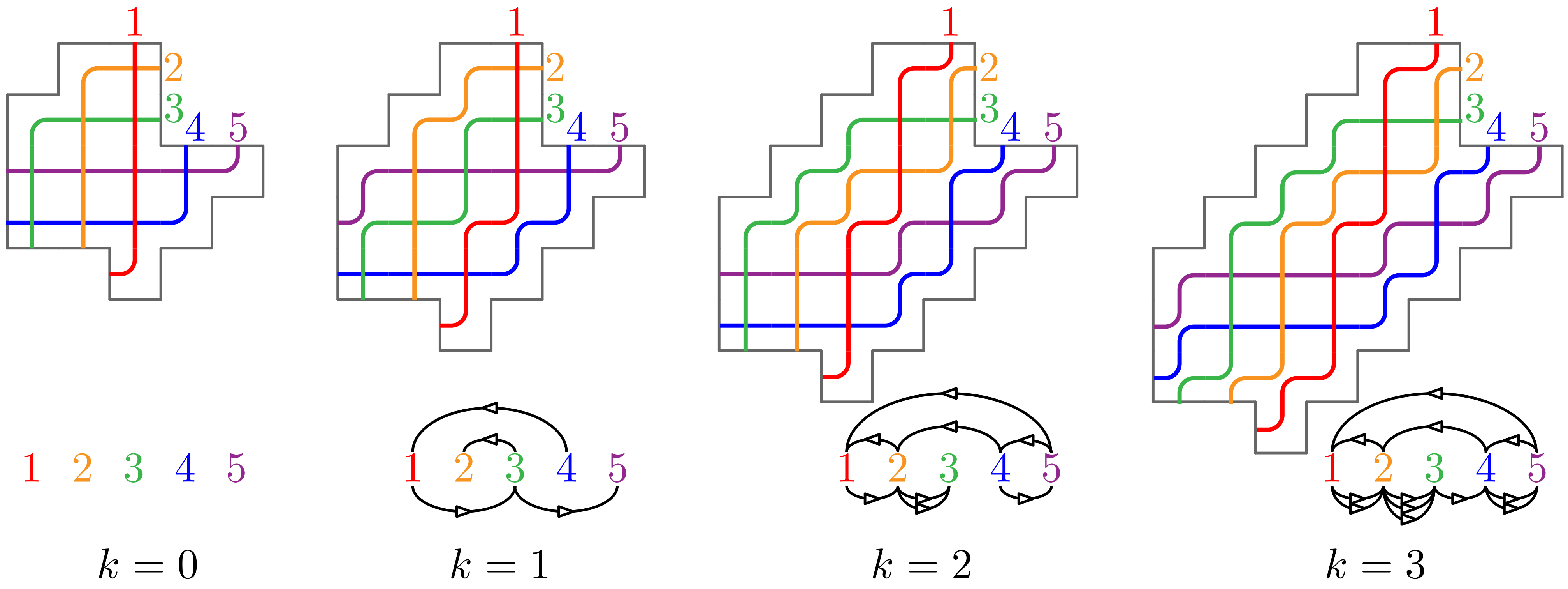
$k \in \mathbb{N}$ and $\varepsilon \in \pm^n$, define a shape $\mathrm{Sh}_\varepsilon^k$ formed by four monotone lattices paths:

(i) enter path: from $(|\varepsilon|_+, 0)$ to $(0, |\varepsilon|_-)$ with pth step north if $\varepsilon_p = -$ and west if $\varepsilon_p = +$,

(ii) exit path: from $(|\varepsilon|_+ + k, n + k)$ to $(n + k, |\varepsilon|_- + k)$ with pth step east if $\varepsilon_p = -$ and south if $\varepsilon_p = +$,

(iii) accordion paths: the path $(NE)^{|\varepsilon|_+ + k}$ from $(0, |\varepsilon|_-)$ to $(|\varepsilon|_+ + k, n + k)$ and the path $(EN)^{|\varepsilon|_- + k}$ from $(|\varepsilon|_+, 0)$ to $(n + k, |\varepsilon|_- + k)$.

CAMBRIANIZATION



Cambrian (k, ε)-twist $=$ pipe dream in $\mathrm{Sh}_{\varepsilon}^{k}$

contact graph of a twist T $=$ vertices are pipes of T and arcs are elbows of T

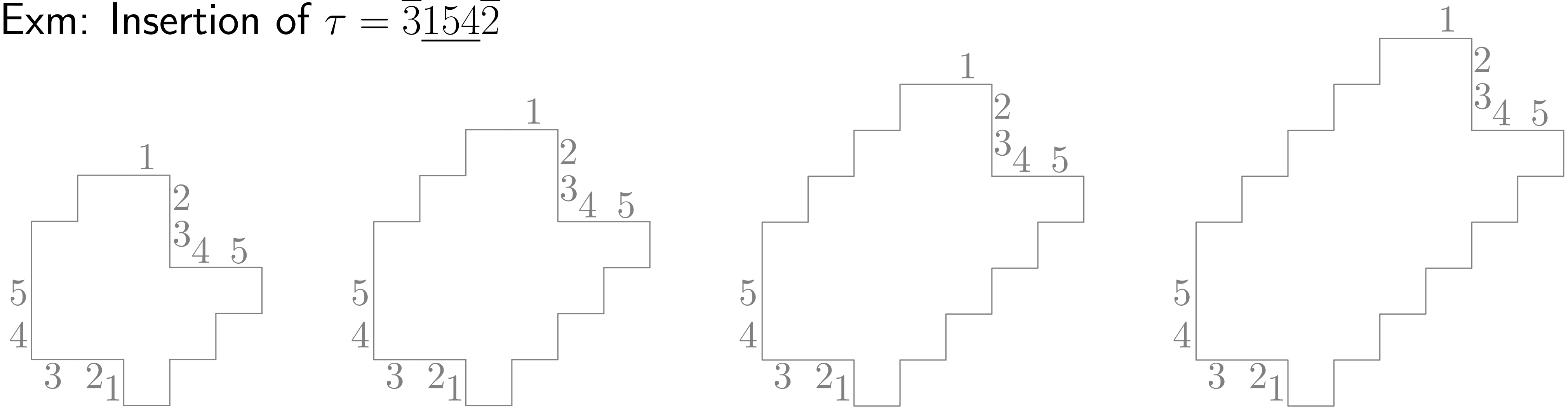
CAMBRIANIZATION

Input: a signed permutation $\tau = \tau_1 \cdots \tau_n$

Algo: Insert pipes one by one (from right to left) as northwest as possible

Output: an acyclic Cambrian (k, ε)-twist $\text{ins}^k(\tau)$

Exm: Insertion of $\tau = \bar{3}\underline{154}\bar{2}$



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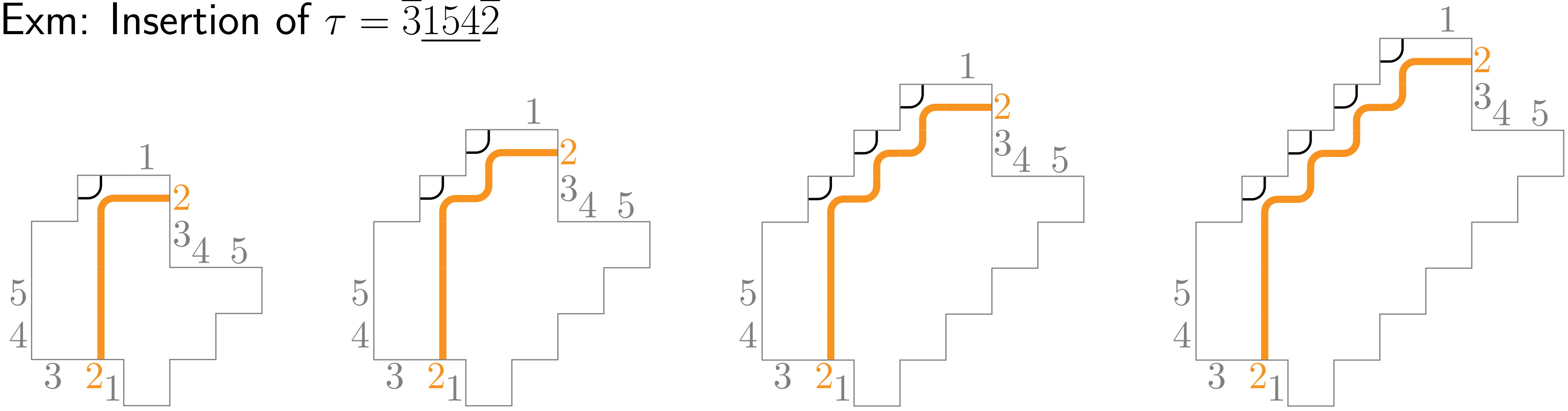

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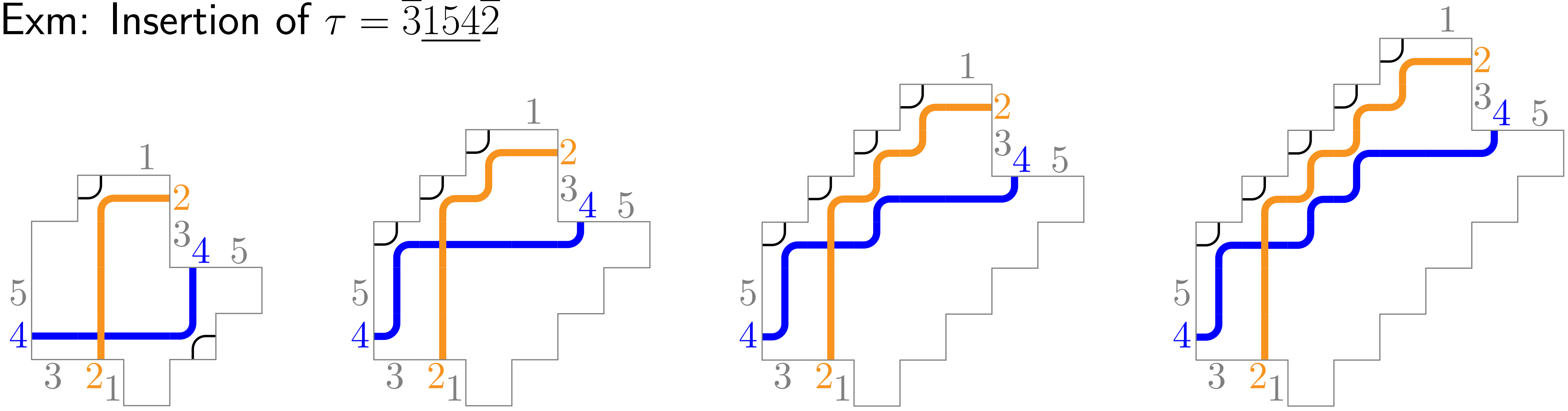

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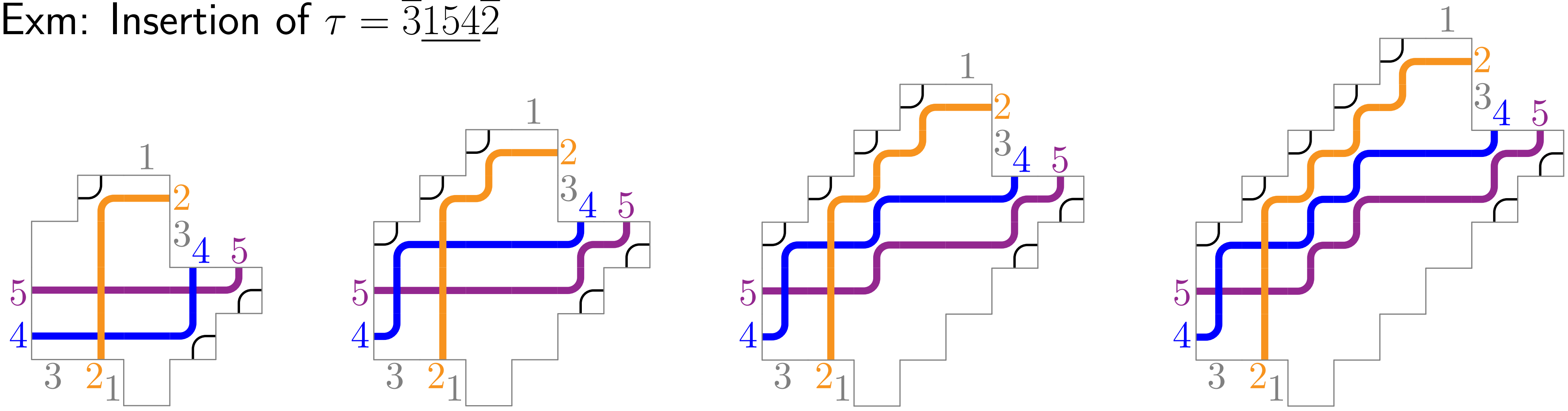

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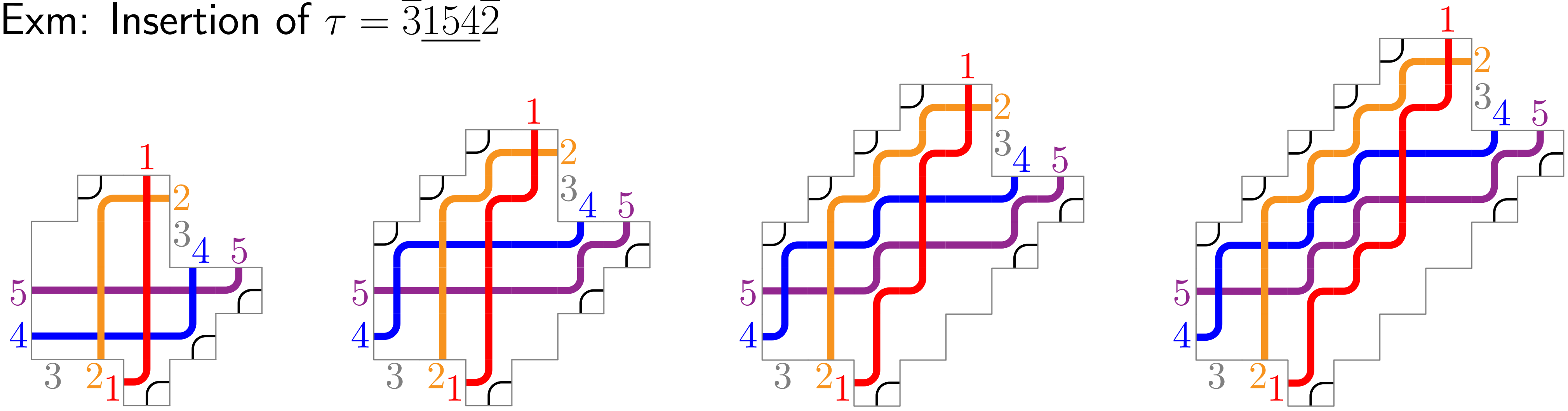

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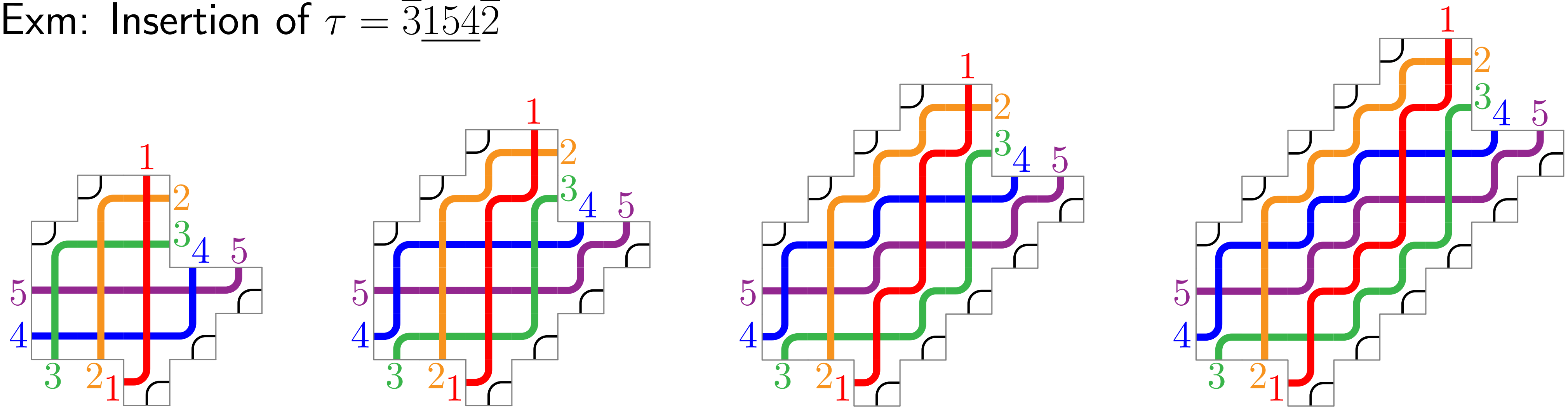

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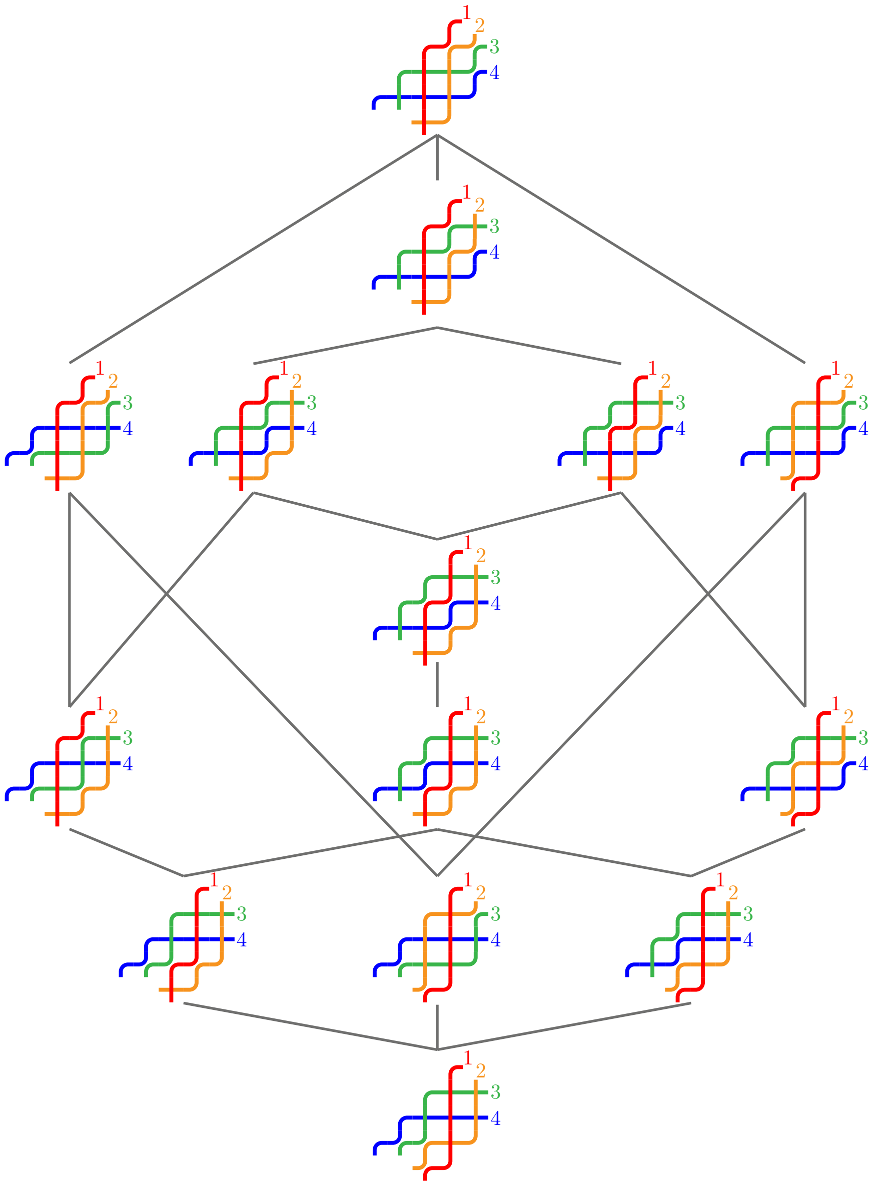
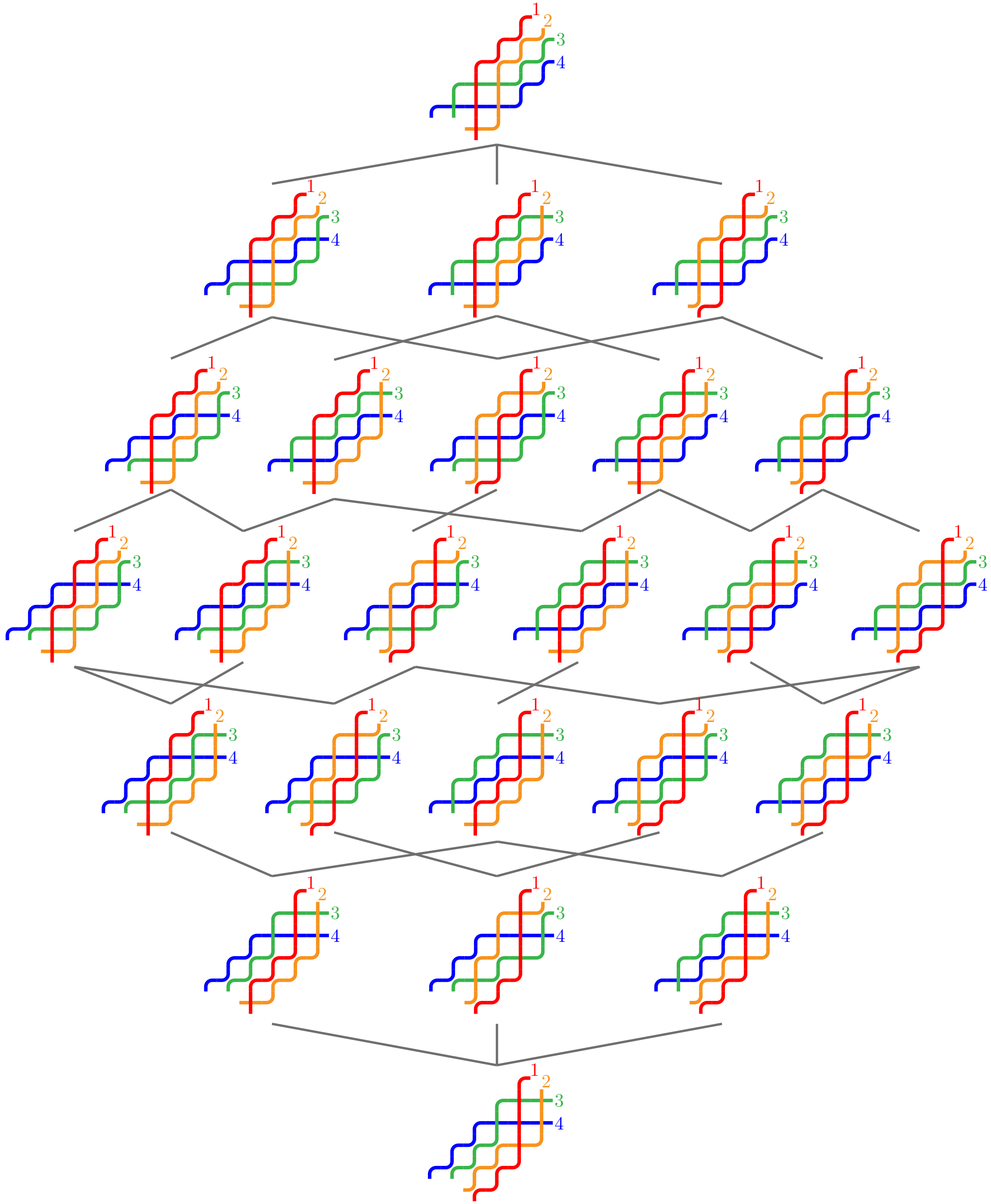
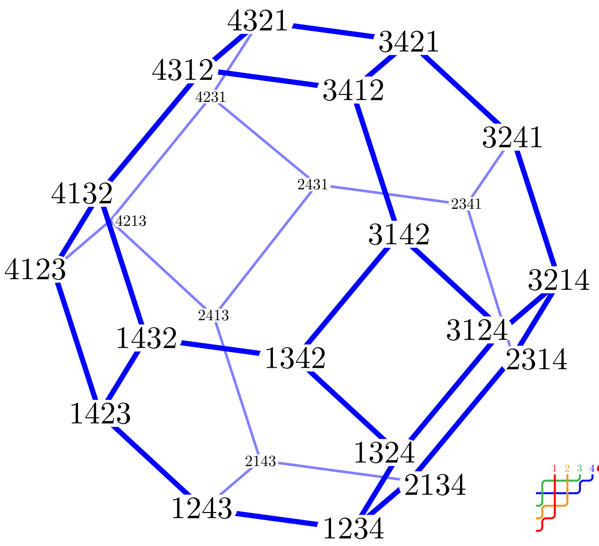
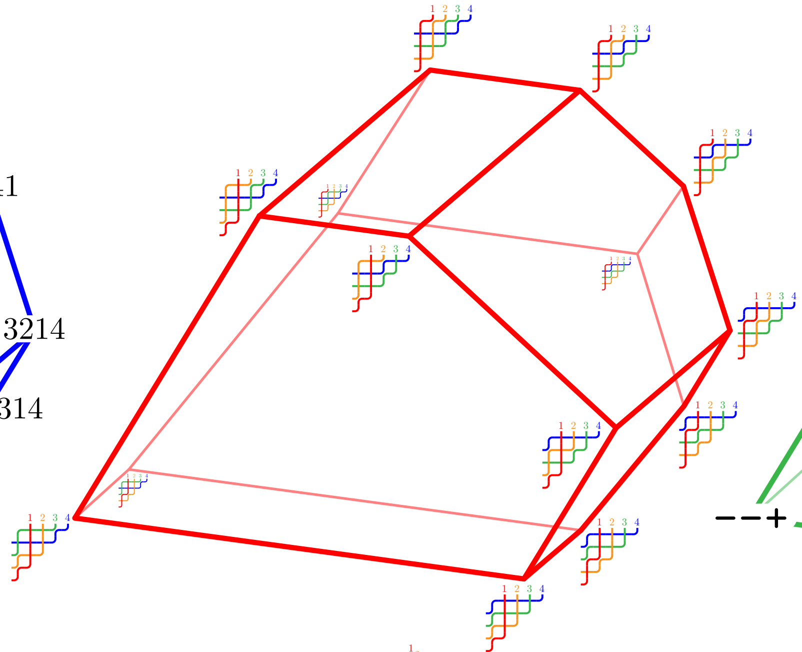


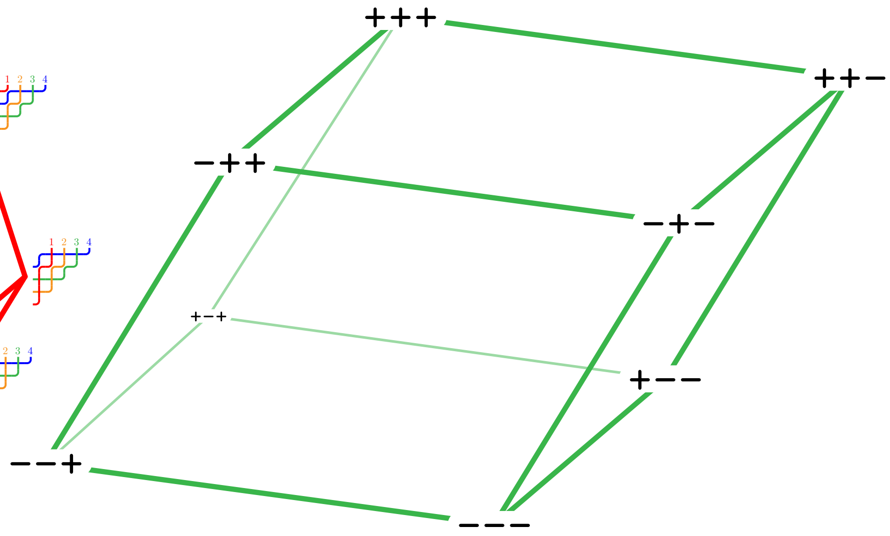


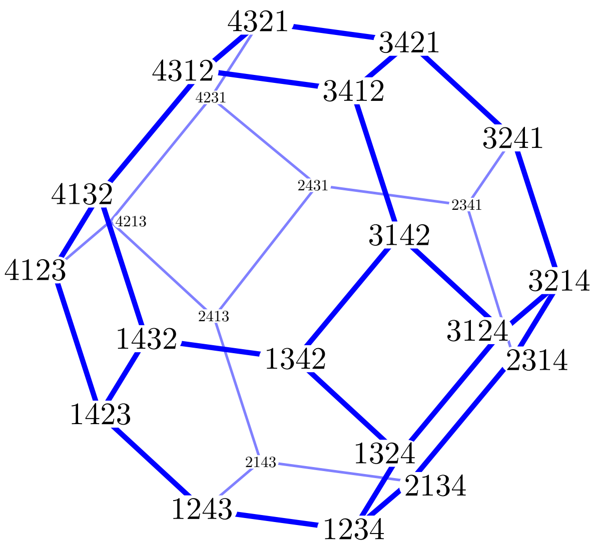
CAMBRIANIZATION

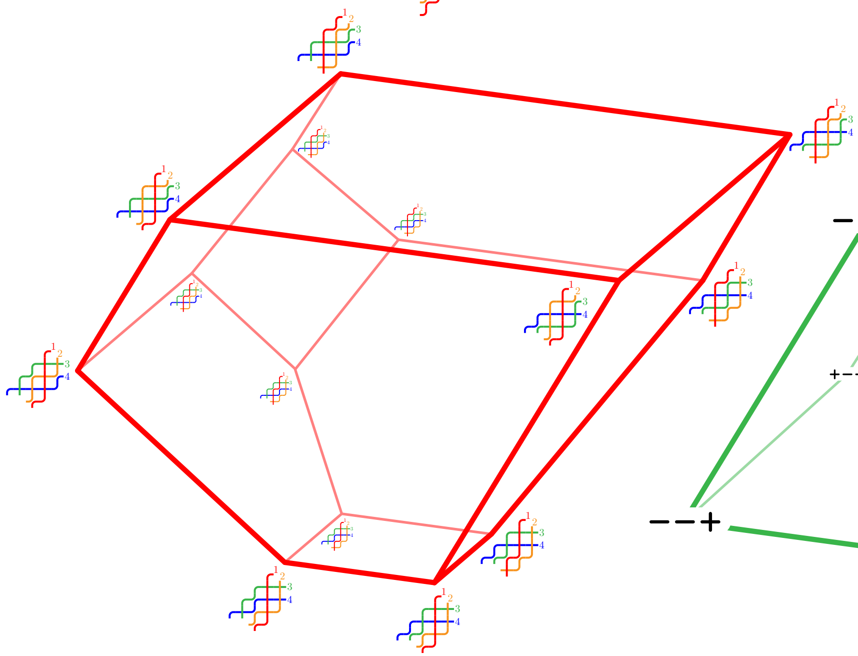
permutahedron $\mathrm{Perm}^k(n)$ | brick polytope $\mathrm{Brick}^k(\varepsilon)$ | zonotope $\mathrm{Zono}^k(n)$

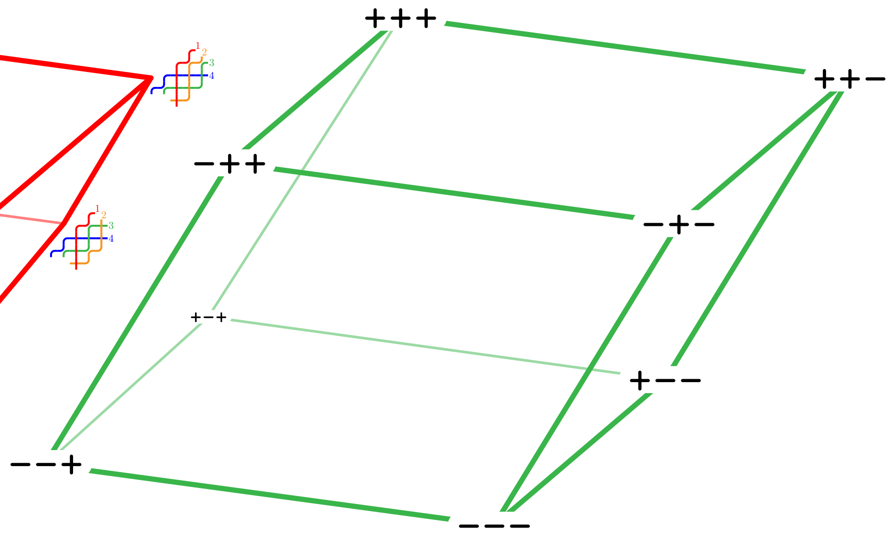






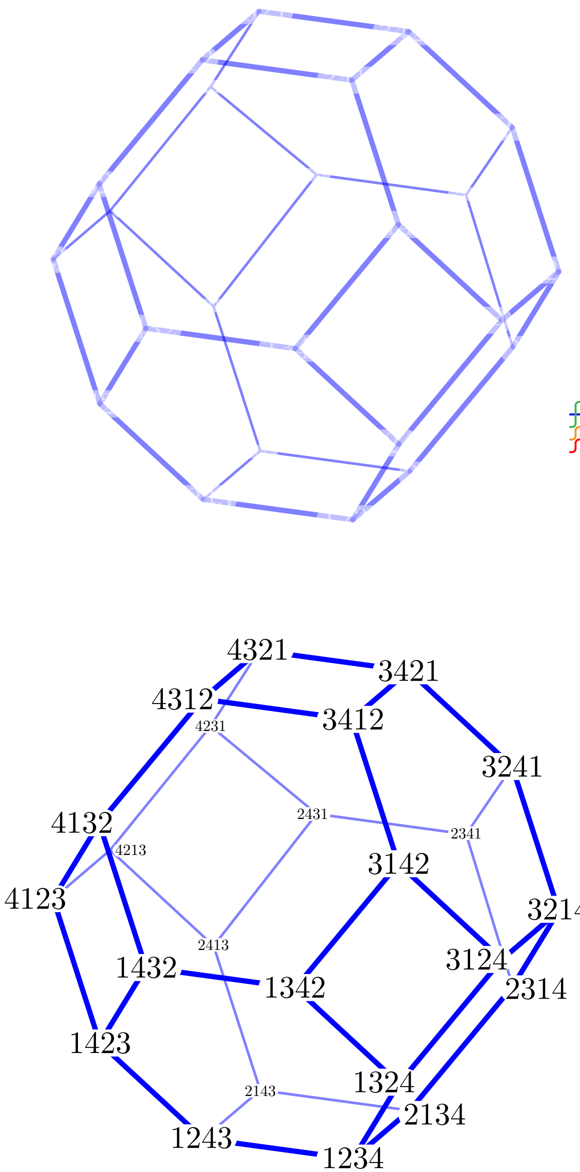




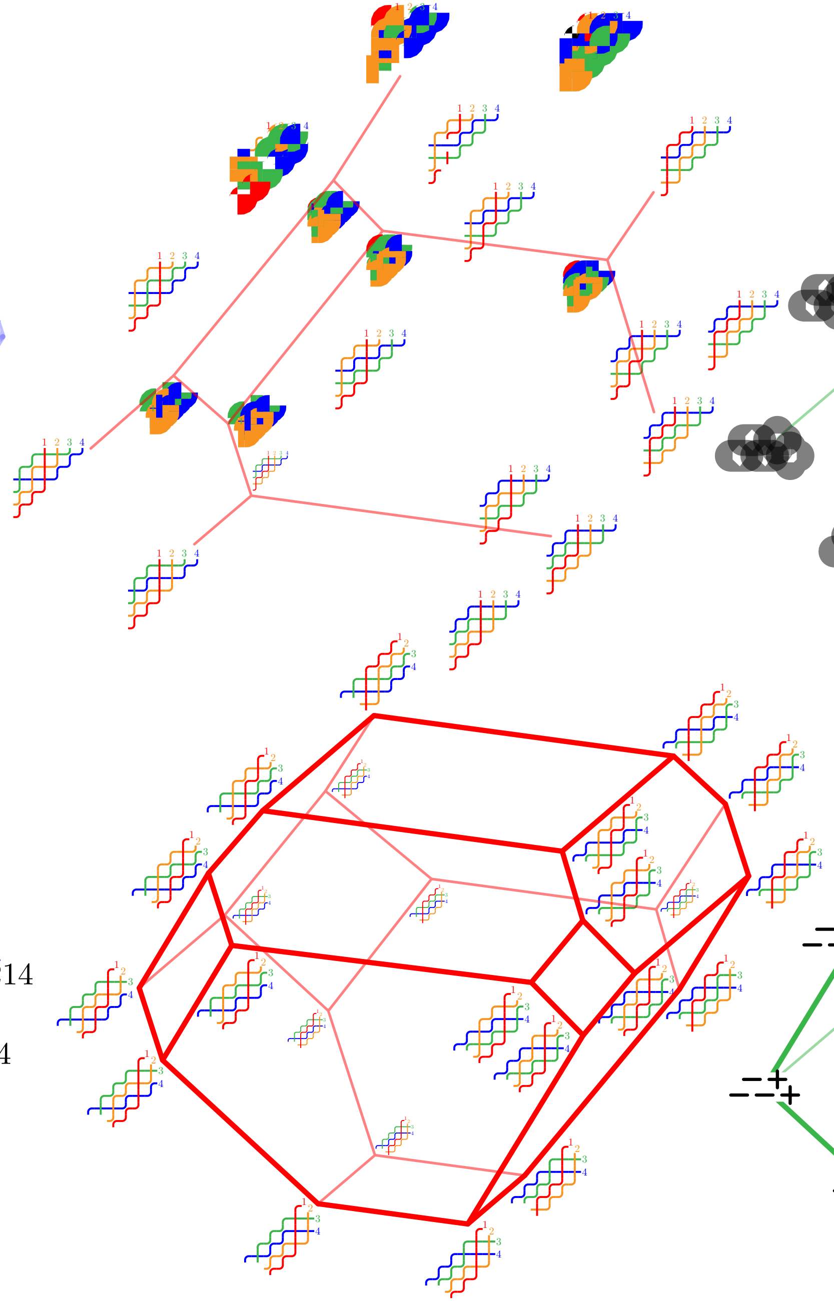


CAMBRIANIZATION

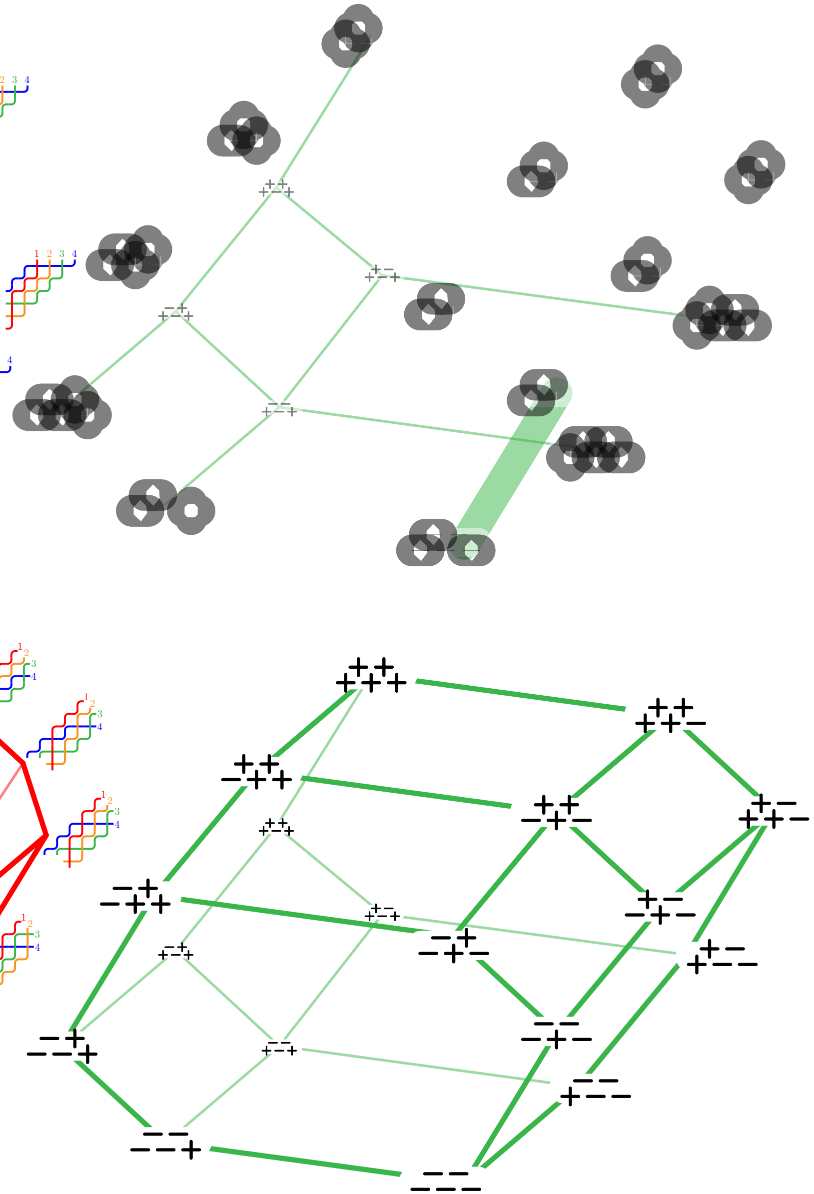
permutahedron $\mathsf{Perm}^k(n)$



brick polytope $\mathsf{Brick}^k(\varepsilon)$



zonotope $\mathsf{Zono}^k(n)$



CAMBRIANIZATION

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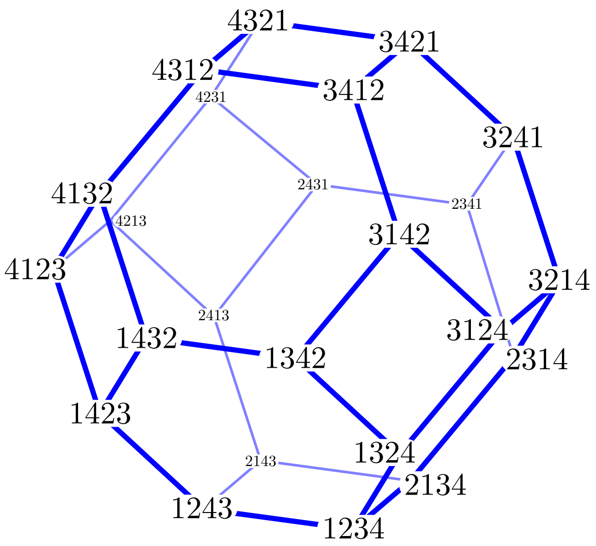
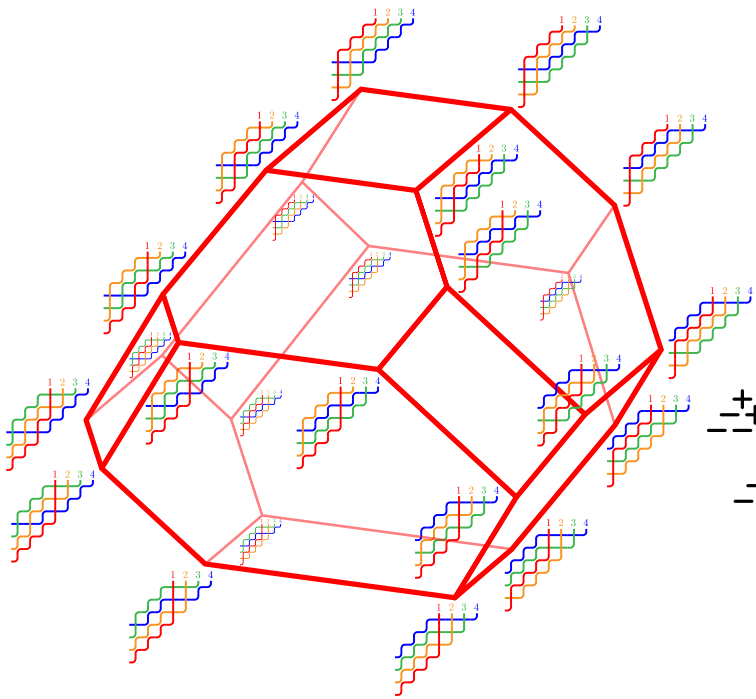
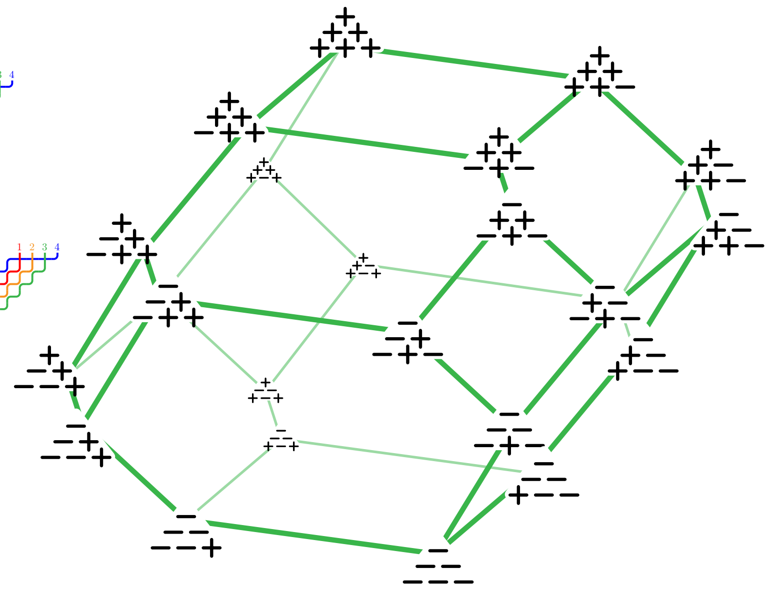
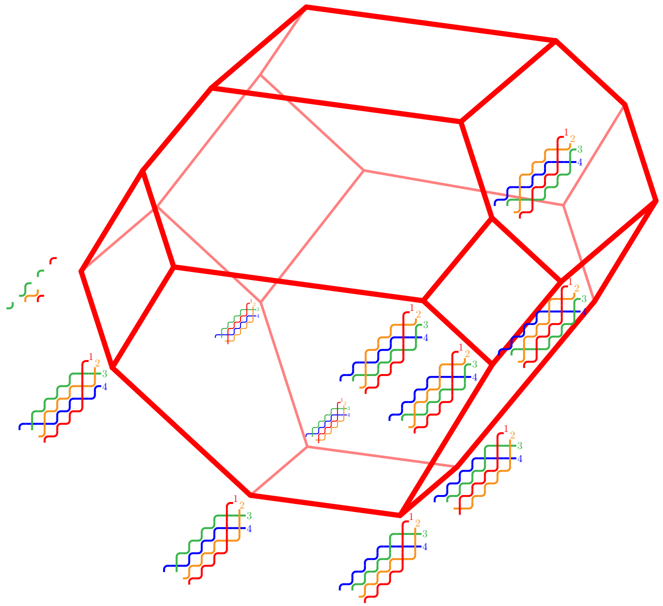


brick polytope Brick$^k(\varepsilon)$



zonotope Zono$^k(n)$



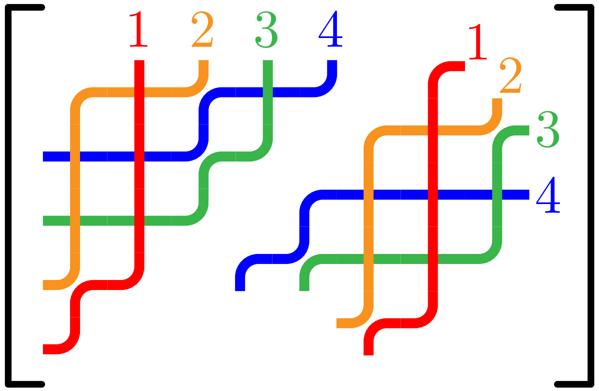


TUPLIZATION

$\mathcal{E} = [\varepsilon_1, \dots, \varepsilon_\ell]$ an ℓ-tuple of signatures

$(k, \mathcal{E})$-twist tuple $=$ an ℓ-tuple $[\mathrm{T}_1, \dots, \mathrm{T}_\ell]$ where

- T_i is a (k, ε_i)-twist
- the union of the contact graphs $\mathrm{T}_1^\# \cup \dots \cup \mathrm{T}_\ell^\#$ is acyclic



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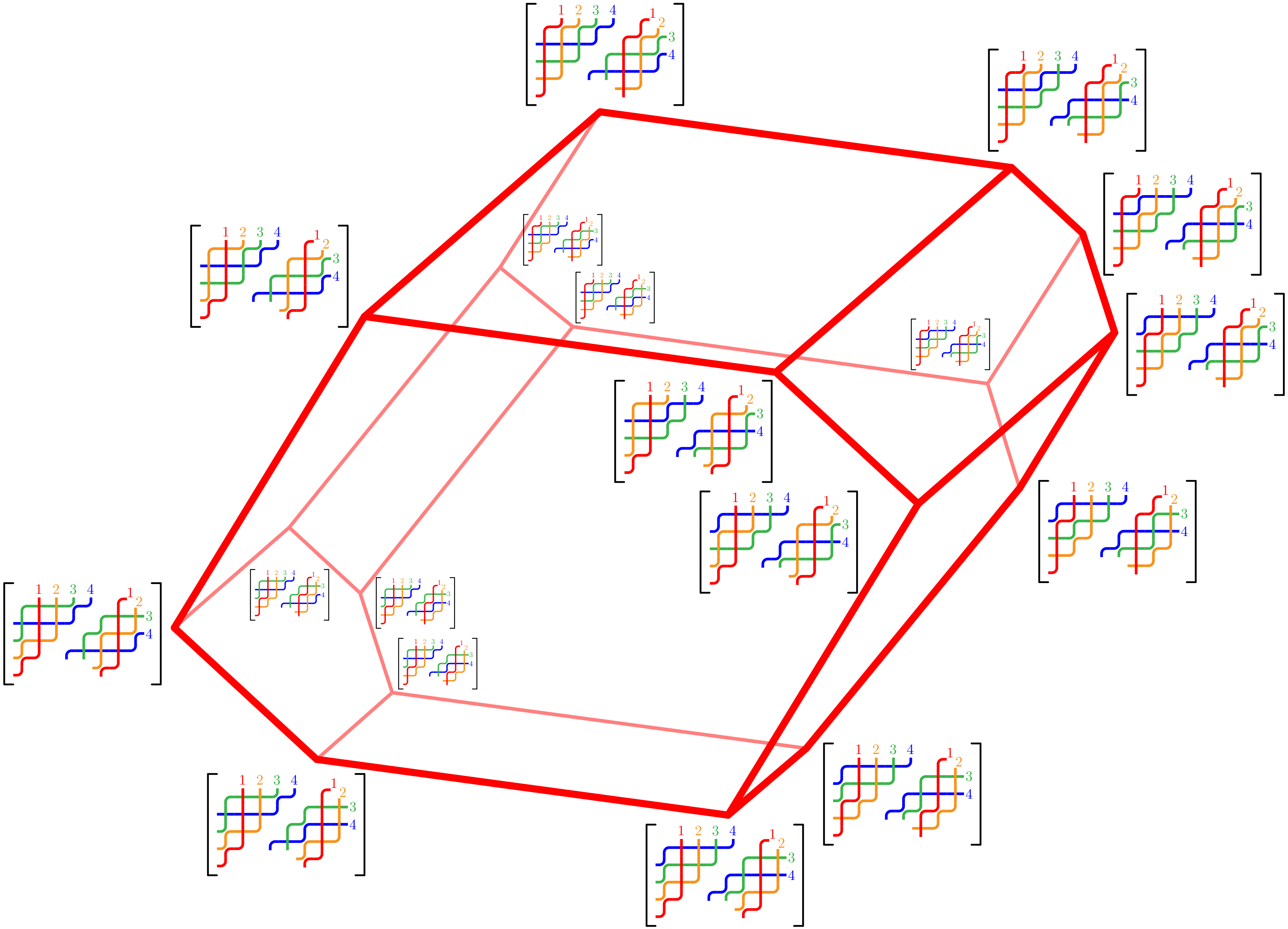
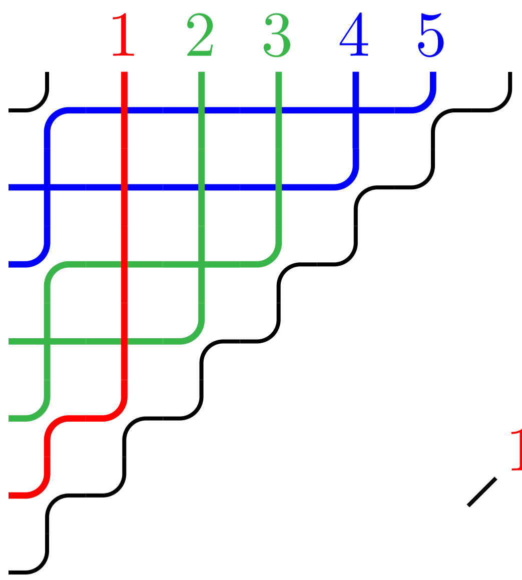
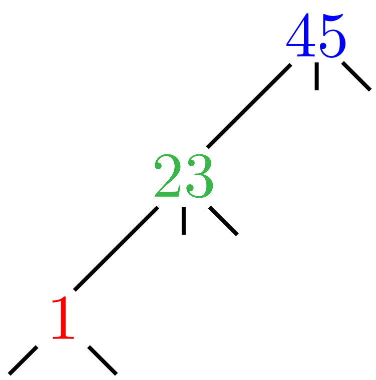


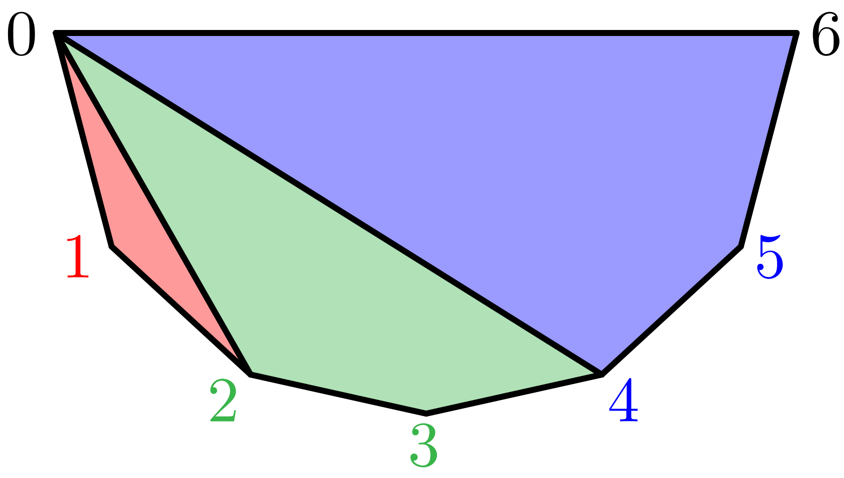
SCHRODERIZATION

hyperpipe $=$ union of pipes whose common elbows are changed to crossings

(k,n)-hypertwist $=$ collection of hyperpipes obtained from a (k,n)-twist T by merging subsets of pipes inducing connected subgraphs of $\mathrm{T}^{\#}$







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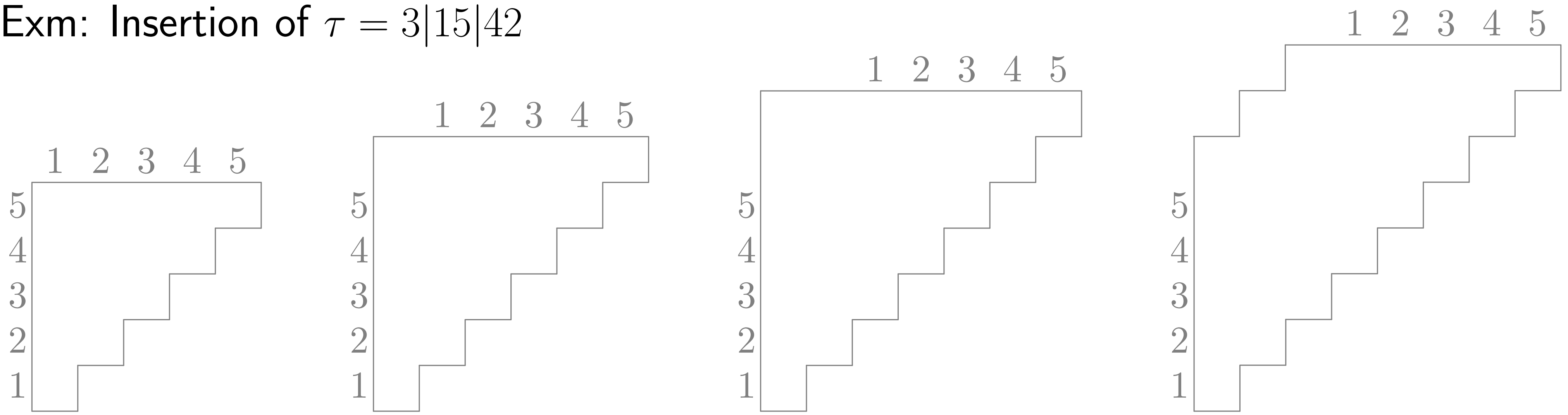
Exm: Insertion of $\tau = 3|15|42$



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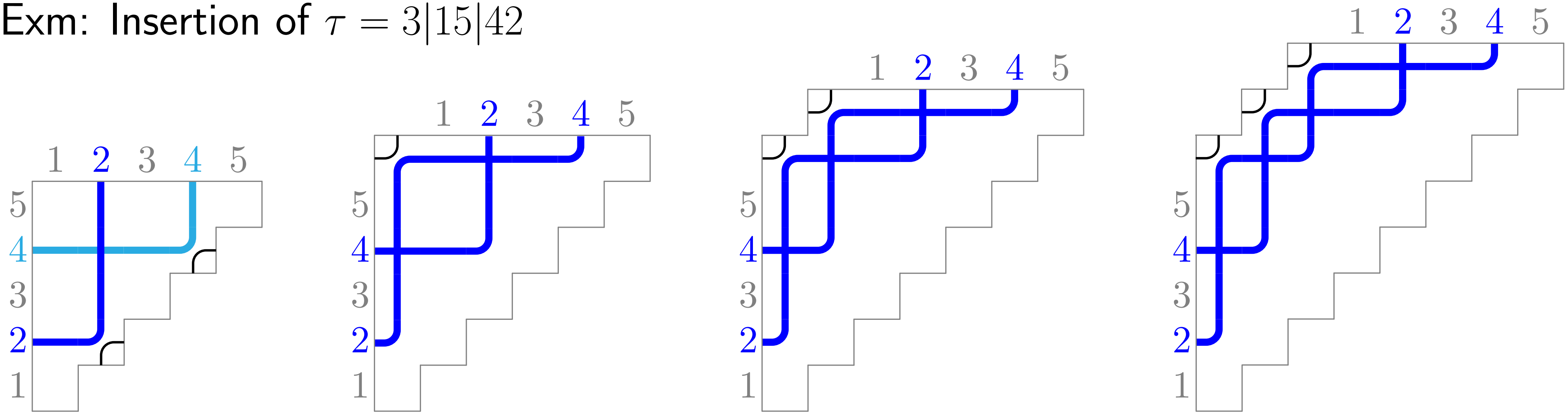

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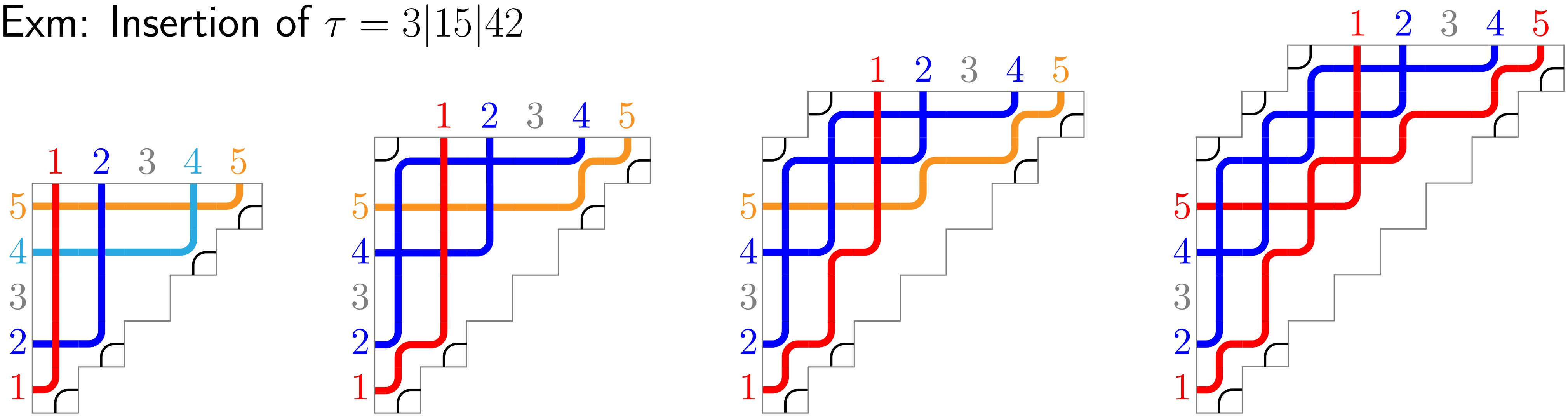

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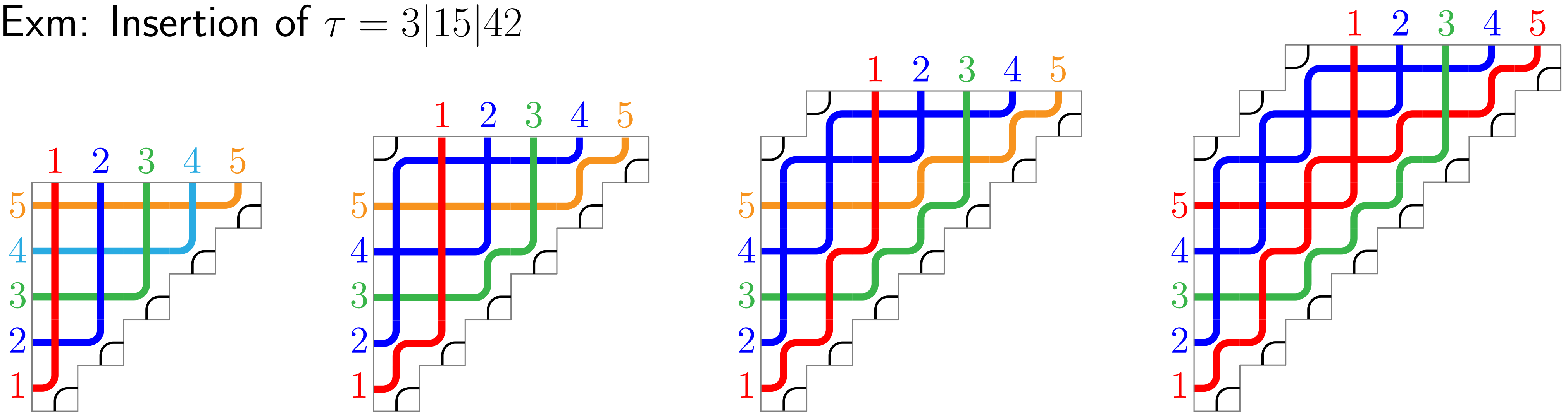

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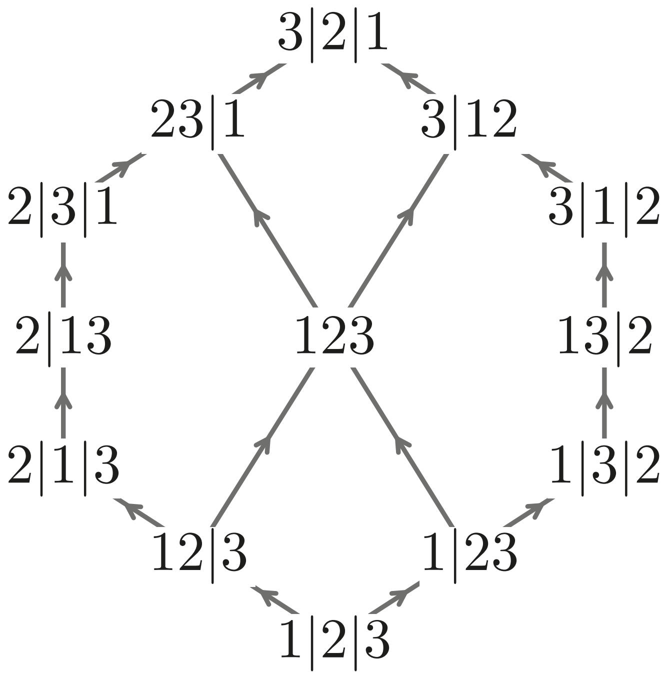

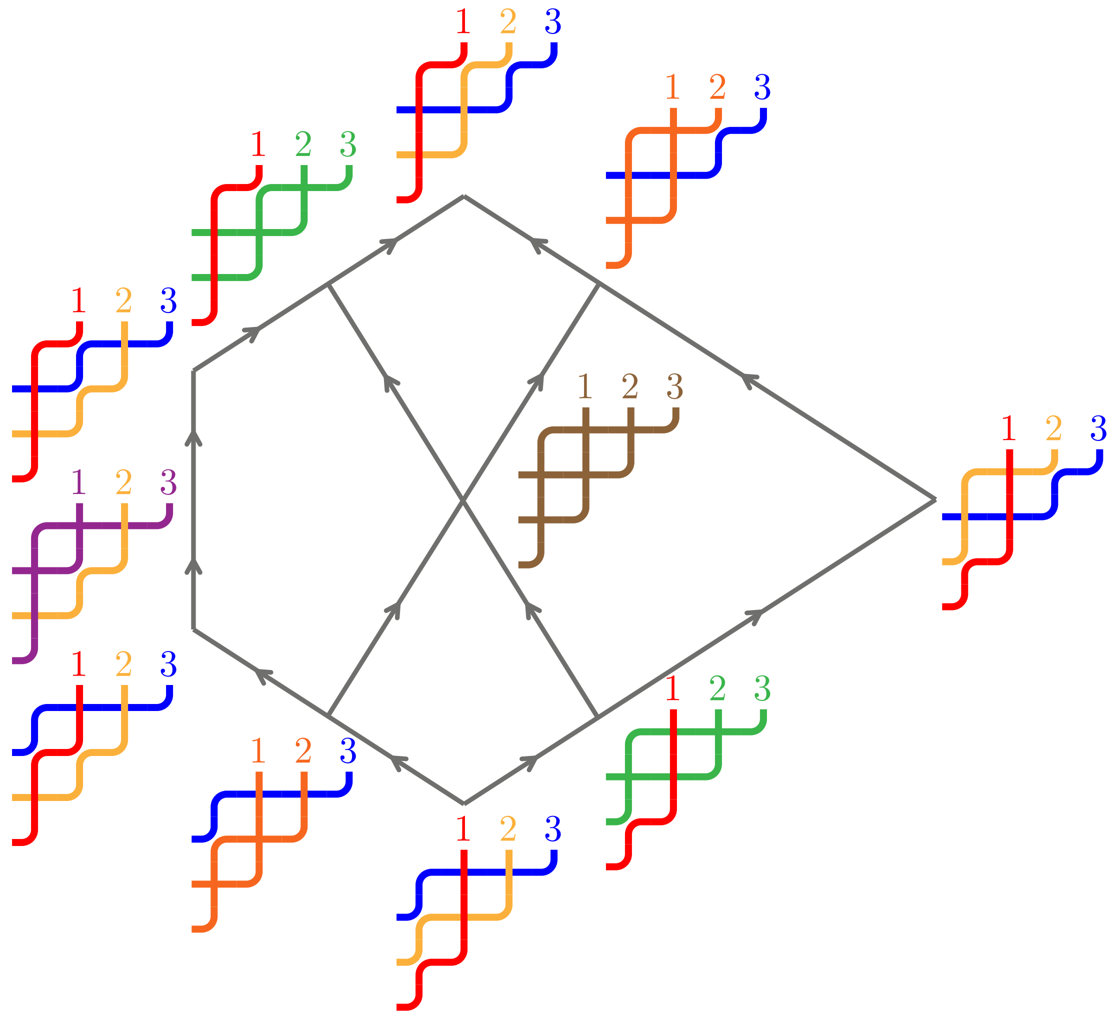

THANK YOU