Acyclic reorientation lattices and their lattice quotients

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Philippe Flajolet Seminar
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PERMUTAHEDRA & ASSOCIAHEDRA
**LATTICES: WEAK ORDER AND TAMARI LATTICE**

**lattice** = partially ordered set $L$ where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$

weak order = permutations of $[n]$

ordered by paths of simple transpositions

Tamari lattice = binary trees on $[n]$

ordered by paths of right rotations
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Tamari lattice = binary trees on $[n]$ ordered by paths of right rotations

sylvester congruence = equivalence classes are sets of linear extensions of binary trees
= equivalence classes are fibers of BST insertion
= rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$
LATTICES: WEAK ORDER AND TAMARI LATTICE

**lattice** = partially ordered set $L$ where any $X \subseteq L$ admits a **meet** $\bigwedge X$ and a **join** $\bigvee X$

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weak order = permutations of \([n]\) ordered by paths of simple transpositions
Tamari lattice = binary trees on \([n]\) ordered by paths of right rotations

lattice congruence = equivalence relation \( \equiv \) which respects meets and joins
\( x \equiv x' \) and \( y \equiv y' \) \( \implies \) \( x \wedge y \equiv x' \wedge y' \) and \( x \vee y \equiv x' \vee y' \)

quotient lattice = lattice on classes with \( X \leq Y \) \( \iff \exists x \in X, y \in Y, x \leq y \)
**FANS: BRAID FAN AND SYLVESTER FAN**

**polyhedral cone** = positive span of a finite set of vectors
= intersection of a finite set of linear half-spaces

fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face
fan = collection of polyhedral cones closed by faces and intersecting along faces

\[ \text{braid fan} = C(\sigma) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \]

\[ \text{sylvester fan} = C(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \rightarrow j \text{ in } T \} \]
fan = collection of polyhedral cones closed by faces and intersecting along faces

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sylvestor fan = \( \mathbb{C}(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \to j \text{ in } T \} \)

quotient fan = \( \mathbb{C}(T) \) is obtained by glueing \( \mathbb{C}(\sigma) \) for all linear extensions \( \sigma \) of \( T \)
polytope = convex hull of a finite set of points
= bounded intersection of a finite set of affine half-spaces

face = intersection with a supporting hyperplane
face lattice = all the faces with their inclusion relations
POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

polytope = convex hull of a finite set of points
= bounded intersection of a finite set of affine half-spaces

permutahedron $\text{Perm}(n)$
= $\text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \, | \, \sigma \in S_n \right\}$
= $H \cap \bigcap_{\emptyset \neq J \subseteq [n]} H_J$
where $H_J = \left\{ x \in \mathbb{R}^n \, | \, \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$

associahedron $\text{Asso}(n)$
= $\text{conv} \left\{ [\ell(T, i) \cdot r(T, i)]_{i \in [n]} \, | \, T \text{ binary tree} \right\}$
= $H \cap \bigcap_{1 \leq i < j \leq n} H_{[i,j]}$

Stasheff (’63)
Shnider–Sternberg (’93)
Loday (’04)
POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

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Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)
permutahedron $\text{Perm}(n) \quad \longrightarrow \quad \text{braid fan}$

associahedron $\text{Asso}(n) \quad \longrightarrow \quad \text{Sylvester fan}$

face $F$ of polytope $P$

normal cone of $F =$ positive span of the outer normal vectors of the facets containing $F$

normal fan of $P =$ \{ normal cone of $F$ \mid $F$ face of $P$ \}
**LATTICES – FANS – POLYTOPES**

**permutahedron** $\mathbb{Perm}(n)$
- $\Rightarrow$ braid fan
- $\Rightarrow$ weak order on permutations

**associahedron** $\mathbb{Asso}(n)$
- $\Rightarrow$ Sylvester fan
- $\Rightarrow$ Tamari lattice on binary trees
LATTICE THEORY OF THE WEAK ORDER
weak order = permutations of $[n]$ ordered by paths of simple transpositions
**INVERSION SETS**

**weak order** = permutations of \([n]\) ordered by paths of simple transpositions

permutations of \([n]\) ordered by inclusion of inversion sets

inversion of \(\sigma\) = pair \((\sigma_i, \sigma_j)\) such that \(i < j\) and \(\sigma_i > \sigma_j\)

**PROP.** inversion sets = transitive and cotransitive subsets of \(\{(b, a) \mid 1 \leq a < b \leq n\}\)

\(\text{inv}(\sigma_1 \lor \ldots \lor \sigma_k) = \text{transitive closure of } \text{inv}(\sigma_1) \cup \cdots \cup \text{inv}(\sigma_k)\)

\(\text{ninv}(\sigma_1 \land \ldots \land \sigma_k) = \text{transitive closure of } \text{ninv}(\sigma_1) \cup \cdots \cup \text{ninv}(\sigma_k)\)
join representation of \( y \in L = \text{subset } J \subseteq L \text{ such that } y = \bigvee J \)

\( y = \bigvee J \) irredundant if \( \nexists J' \subsetneq J \text{ with } y = \bigvee J' \)

ordered by containment of order ideals: \( J \leq J' \iff \forall z \in J, \exists z' \in J', z \leq z' \)

canonical join representation of \( y = \) minimal irredundant join representation of \( y \)

\( = \) lowest way to write \( y \) as a join

\( \implies \) a canonical join representation is an antichain of join irreducible elements of \( L \)
DISTRIBUTIVE AND SEMIDISTRIBUTIVE LATTICES

\((L, \leq, \land, \lor)\) finite lattice is

- **distributive** if \(x \lor (y \land z) = (x \lor y) \land (x \lor z)\) for any \(x, y, z \in L\)

- **join semidistributive** if \(x \lor y = x \lor z\) implies \(x \lor (y \land z) = x \lor y\) for any \(x, y, z \in L\)

- **semidistributive** if both join and meet semidistributive

\[
\begin{align*}
\text{distributive} & \quad \text{semidistributive} & \quad \text{not semidistributive} \\
\mathcal{L} & : & \\ 
\emptyset & \quad a & \quad b & \quad c & \quad d & \\
\emptyset & \quad a & \quad a \lor b \quad b \lor c & \\
\emptyset & \quad a & \quad a \lor b & \\
\emptyset & \quad a \lor b \quad b \lor c & \\
\emptyset & \quad a \lor b \quad a \lor c & \\
\emptyset & \quad a \lor c \quad b \lor c & \\
\emptyset & \quad c \quad d & \\
\emptyset & \quad c \quad d & \\
\emptyset & \quad c & \\
\emptyset & \quad d & \\
\emptyset & \\
\end{align*}
\]
(\(L, \leq, \wedge, \vee\)) finite lattice is

- **distributive** if \(x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\) for any \(x, y, z \in L\)

  \[ \implies \text{canonical join representations = antichains of join irreducibles} \]
  \[ \implies L \cong \text{inclusion poset of lower ideals of} \ JI(L) \]

- **join semidistributive** if \(x \vee y = x \vee z\) implies \(x \vee (y \wedge z) = x \vee y\) for any \(x, y, z \in L\)

  \[ \implies \text{any} \ y \in L \ \text{admits the canonical join representation} \ y = \bigvee_{x \leq y} k_\vee(x, y) \]
  \[ \text{where} \ k_\vee(x, y) \ \text{is the unique minimal element of} \ \{z \in L \mid x \vee z = y\} \]

- **semidistributive** if both join and meet semidistributive

\[
\begin{align*}
\text{distributive} & \quad \text{semidistributive} & \quad \text{not semidistributive} \\
\end{align*}
\]
FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS

draw all points \((\sigma_i, i)\) and all segments from \((\sigma_i, i)\) to \((\sigma_{i+1}, i + 1)\) with \(\sigma_i > \sigma_{i+1}\)
and project down to an horizontal line allowing arcs to bend but not to cross or pass points

arc = \(x\)-monotone curve joining two points and wiggling around the horizontal axis (up to deformations)

compatible arcs =

- left(\(\alpha\)) \(\neq\) left(\(\alpha'\)) and right(\(\alpha\)) \(\neq\) right(\(\alpha'\)),
- \(\alpha\) and \(\alpha'\) are not crossing.

noncrossing arc diagrams = set of pairwise compatible arcs

\[ \text{THM. } \delta \text{ is a bijection from permutations to noncrossing arc diagrams} \]

permutation \(\sigma = 2537146\)

noncrossing arc diagram \(\delta(\sigma)\)
draw all points \((\sigma_i, i)\) and all segments from \((\sigma_i, i)\) to \((\sigma_{i+1}, i + 1)\) with \(\sigma_i > \sigma_{i+1}\)
and project down to an horizontal line allowing arcs to bend but not to cross or pass points

\[\text{arc} = x\text{-monotone curve joining two points and wiggling around the horizontal axis (up to deformations)}\]

\[\iff \text{quadruple } (a, b, A, B) \text{ with } a < b \text{ and } ]a, b[ = A \cup B\]

compatible arcs =

- \(\text{left}(\alpha) \neq \text{left}(\alpha')\) and \(\text{right}(\alpha) \neq \text{right}(\alpha')\),
- \(\alpha\) and \(\alpha'\) are not crossing.

\[\iff \alpha = (a, b, A, B) \text{ and } \alpha' = (a', b', A', B')\]

such that there is no \(x \neq x'\) with

\[x \in (A \cup \{a, b\}) \cap (B' \cup \{a', b'\}) \text{ and } x' \in (B \cup \{a, b\}) \cap (A' \cup \{a', b'\})\]

noncrossing arc diagrams = set of pairwise compatible arcs

\(\text{THM. } \delta\) is a bijection from permutations to noncrossing arc diagrams

\(\text{Reading ('15)}\)
WEAK ORDER ON NONCROSSING ARC DIAGRAMS
THM. $\sigma = \bigvee_{\alpha \in \delta(\sigma)} \delta^{-1}(\{\alpha\})$ is the canonical join representation
lattice congruence of $L = \equiv$ equivalence relation which respects meets and joins
\[ x \equiv x' \text{ and } y \equiv y' \implies x \land y \equiv x' \land y' \text{ and } x \lor y \equiv x' \lor y' \]

lattice quotient of $L/\equiv = L$ on equivalence classes of $L$ under $\equiv$ where
- $X \leq Y \iff \exists x \in X, y \in Y, x \leq y$
- $X \land Y = \text{equiv. class of } x \land y$ for any $x \in X$ and $y \in Y$
- $X \lor Y = \text{equiv. class of } x \lor y$ for any $x \in X$ and $y \in Y$
lattice congruence on $L$, then

- each class $X$ is an interval $[\pi_\downarrow(X), \pi_\uparrow(X)]$
- $L/\equiv$ is isomorphic (as poset) to the restriction of $L$ to the elements $x$ with $\pi_\downarrow(x) = x$
- $\pi_\downarrow(x) = x$ if and only if $\pi_\downarrow(j) = j$ for all canonical joinands $j$ of $x$
- canonical join representations in $L/\equiv$ are canonical join representations in $L$ that only involve join irreducibles $j$ with $\pi_\downarrow(j) = j$
THM. \equiv \text{lattice congruence of the weak order on } \mathcal{G}_n \\
\mathcal{A}_\equiv = \text{arcs corresponding to join irreducibles } \sigma \text{ with } \pi_{\downarrow}(\sigma) = \sigma \\
\mathcal{G}_n/\equiv \simeq \text{subposet induced by noncrossing arc diagrams with all arcs in } \mathcal{A}_\equiv
THM. \( \equiv \) lattice congruence of the weak order on \( \mathcal{S}_n \)
\( \mathcal{A}_\equiv = \) arcs corresponding to join irreducibles \( \sigma \) with \( \pi_\downarrow(\sigma) = \sigma \)
\( \mathcal{S}_n/\equiv \simeq \) subposet induced by noncrossing arc diagrams with all arcs in \( \mathcal{A}_\equiv \)

THM. The following are equivalent for a set of arcs \( \mathcal{A} \):
- there exists a lattice congruence \( \equiv \) on \( \mathcal{S}_n \) with \( \mathcal{A} = \mathcal{A}_\equiv \)
- \( \mathcal{A} \) is a lower ideal of the subarc order
THM. \[ \equiv \] lattice congruence of the weak order on \( \mathcal{S}_n \)
\[ \mathcal{A}_\equiv = \text{arcs corresponding to join irreducibles } \sigma \text{ with } \pi_\downarrow(\sigma) = \sigma \]
\[ \mathcal{S}_n/\equiv \cong \text{subposet induced by noncrossing arc diagrams with all arcs in } \mathcal{A}_\equiv \]

THM. The following are equivalent for a set of arcs \( \mathcal{A} \):
- there exists a lattice congruence \( \equiv \) on \( \mathcal{S}_n \) with \( \mathcal{A} = \mathcal{A}_\equiv \)
- \( \mathcal{A} \) is a lower ideal of the subarc order

\[(a, b, A, B) \text{ subarc of } (c, d, C, D) \iff c < a < b < d \text{ and } A \subseteq C \text{ and } B \subseteq D\]
arc ideal = lower ideal of the subarc order

essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687
\textit{arc ideal} = lower ideal of the subarc order

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quotient fan $\mathcal{F}_\equiv$ = chambers are obtained by glueing the chambers of the permutations $\sigma$ in the same congruence class of $\equiv$

quotientope = polytope with normal fan $\mathcal{F}_\equiv$

Reading ('05)
P.–Santos ('19)
Padrol–P.–Ritter ('20+)
quotient fan $\mathcal{F}_\equiv = \text{chambers are obtained by}\n\text{glueing the chambers of the permutations } \sigma\n\text{in the same congruence class of } \equiv$ 

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glueing the chambers of the permutations $\sigma$
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ACYCLIC REORIENTATION LATTICES
ACYCLIC REORIENTATION POSETS

$D$ directed acyclic graph

$\mathcal{AR}_D =$ all acyclic reorientations of $D$, ordered by inclusion of their sets of reversed arcs

- Minimal element $D$
- Maximal element $\bar{D}$
- Self-dual under reversing all arcs
- Cover relations = flipping a single arc

Flippable arcs of $E =$ transitive reduction of $E$

$$= E \setminus \{(u, v) \in E \mid \exists \text{ directed path } u \leadsto v \text{ in } E\}$$
ACYCLIC REORIENTATION POSETS

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$= E \setminus \{(u, v) \in E \mid \exists$ directed path $u \leadsto v$ in $E\}$
ACYCLIC REORIENTATION POSETS

$D$ directed acyclic graph

$\mathcal{AR}_D = \text{all acyclic reorientations of } D$, ordered by inclusion of their sets of reversed arcs
\( D_{\text{vertebrate}} = \text{transitive reduction of any induced subgraph of } D \text{ is a forest} \)

\textbf{THM.} \( \mathcal{AR}_D \text{ lattice } \iff D_{\text{vertebrate}} \)

\( \text{lattice} \quad \text{not lattice} \)
ACYCLIC REORIENTATION LATTICES

\[ D_{\text{vertebrate}} = \text{transitive reduction of any induced subgraph of } D \text{ is a forest} \]

**THM.** \( A\mathcal{R}_D \text{ lattice } \iff D \text{ vertebrate} \)

\[ \text{lattice} \quad \text{not lattice} \]
$D_{\text{vertebrate}}$ = transitive reduction of any induced subgraph of $D$ is a forest

**THM.** $\mathcal{AR}_D$ lattice $\iff$ $D$ vertebrate

$X$ subset of arcs of $D$ is
- closed if all arcs of $D$ in the transitive closure of $X$ also belong to $X$
- coclosed if its complement is closed
- biclosed if it is closed and coclosed

**PROP.** If $D$ vertebrate, $X$ biclosed $\iff$ the reorientation of $X$ is acyclic
ACYCLIC REORIENTATION LATTICES

$D\text{ vertebrate} =$ transitive reduction of any induced subgraph of $D$ is a forest

**THM.** $\mathcal{AR}_D$ lattice $\iff D$ vertebrate

**PROP.** If $D$ vertebrate,

\[
\text{bwd}(E_1 \lor \ldots \lor E_k) = \text{transitive closure of } \text{bwd}(E_1) \cup \cdots \cup \text{bwd}(E_k)
\]

\[
\text{fwd}(E_1 \land \ldots \land E_k) = \text{transitive closure of } \text{fwd}(E_1) \cup \cdots \cup \text{fwd}(E_k)
\]
THM. $\mathcal{AR}_D$ distributive lattice $\iff D$ forest $\iff \mathcal{AR}_D$ boolean lattice
**SEMIDISTRIBUTIVE ACYCLIC REORIENTATION LATTICES**

\( D \) skeletal =

- \( D \) vertebrate = transitive reduction of any induced subgraph of \( D \) is a forest
- \( D \) filled = any directed path joining the endpoints of an arc in \( D \) induces a tournament

**THM.** \( \mathcal{AR}_D \) semidistributive lattice \( \iff \) \( D \) is skeletal

![Diagram](image-url)
rope of $D = \text{quadruple } \rho = (u, v, \nabla, \triangle)$ where 
• $(u, v)$ is an arc of $D$
• $\nabla \sqcup \triangle$ partitions the transitive support of $(u, v)$ minus $\{u, v\}$
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- $(u, v)$ is an arc of $D$
- $\nabla \sqcup \triangle$ partitions the transitive support of $(u, v)$ minus $\{u, v\}$

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**THM.**  \( \text{join irreducibles of } \mathcal{AR}_D \leftrightarrow \text{ropes of } D \)
**ROPEs & NON-CROSSING ROPE DIAGRAMS**

**rope of** \( D = \text{quadruple} \ \rho = (u, v, \nabla, \triangle) \) where

- \( (u, v) \) is an arc of \( D \)
- \( \nabla \cup \triangle \) partitions the transitive support of \( (u, v) \) minus \( \{u, v\} \)

---

**THM.**

\[ \text{join irreducibles of } \mathcal{AR}_D \iff \text{ropes of } D \]

\[ \text{canonical join representations of } \mathcal{AR}_D \iff \text{non-crossing rope diagrams of } \mathcal{AR}_D \]

\( (u, v, \nabla, \triangle) \) and \( (u', v', \nabla', \triangle') \) are crossing if there are \( w \neq w' \) such that \( w \in (\nabla \cup \{u, v\}) \cap (\triangle' \cup \{u', v'\}) \) and \( w' \in (\triangle \cup \{u, v\}) \cap (\nabla' \cup \{u', v'\}) \)
CONGRUENCES & QUOTIENTS
(\(u, v, \nabla, \triangle\)) _subrope_ of \((u', v', \nabla', \triangle')\) if \(u, v \in \{u', v'\} \cup \nabla' \cup \triangle'\) and \(\nabla \subseteq \nabla'\) and \(\triangle \subseteq \triangle'\)

**PROP.** congruence lattice of \(\mathcal{AR}_D\) \(\simeq\) lower ideal lattice of subrope order

**CORO.** \(\equiv\) lattice congruence of \(\mathcal{AR}_D\)

- \(E\) minimal in its \(\equiv\)-class \(\iff\) \(\delta(E) \subseteq \mathcal{R}_\equiv\)
- quotient \(\mathcal{AR}_D/\equiv\) \(\simeq\) subposet of \(\mathcal{AR}_D\) induced by \(\{E \in \mathcal{AR}_D \mid \delta(E) \subseteq \mathcal{R}_\equiv\}\)
$$(\mathcal{U}, \Omega) = \text{two of arbitrary subsets of } V$$

$${\mathcal{R}}_{(\mathcal{U}, \Omega)} = \text{lower ideal of ropes } (u, v, \triangledown, \triangle) \text{ of } D \text{ such that } \triangledown \subseteq \mathcal{U} \text{ and } \triangle \subseteq \Omega$$

**coherent congruence** $\equiv (\mathcal{U}, \Omega) = \text{congruence with subrope ideal } {\mathcal{R}}_{(\mathcal{U}, \Omega)}$

**examples:**

- **sylvester congruence** = subrope ideal contains only ropes $(u, v, \triangledown, \emptyset)$

---

P.–Pons ('18)
COHERENT CONGRUENCES

\((\mathcal{U}, \Omega)\) = two of arbitrary subsets of \(V\)

\(\mathcal{R}_{(\mathcal{U}, \Omega)} = \) lower ideal of ropes \((u, v, \triangledown, \triangle)\) of \(D\) such that \(\triangledown \subseteq \mathcal{U}\) and \(\triangle \subseteq \Omega\)

coherent congruence \(\equiv (\mathcal{U}, \Omega) = \) congruence with subrope ideal \(\mathcal{R}_{(\mathcal{U}, \Omega)}\)

examples:
- **sylvester congruence** = subrope ideal contains only ropes \((u, v, \triangledown, \emptyset)\)
- **Cambrian congruences** = when \(\mathcal{U} \sqcup \Omega = V\)

P.–Pons ('18)

Reading ('06)
QUOTIENT FANS & QUOTIENTOPE S
$D$ directed acyclic graph

- **graphical arrangement** $\mathcal{H}_D = \text{arrangement of hyperplanes } x_u = x_v \text{ for all arcs } (u, v) \in D$
- **graphical zonotope** $Z_D = \text{Minkowski sum of } [e_u, e_v] \text{ for all arcs } (u, v) \in D$

| hyperplanes of $\mathcal{H}_D$ | $\leftrightarrow$ | summands of $Z_D$ | $\leftrightarrow$ | arcs of $D$
<table>
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<tbody>
<tr>
<td>regions of $\mathcal{H}_D$</td>
<td>$\leftrightarrow$</td>
<td>vertices of $Z_D$</td>
<td>$\leftrightarrow$</td>
</tr>
<tr>
<td>poset of regions of $\mathcal{H}_D$</td>
<td>$\leftrightarrow$</td>
<td>oriented graph of $Z_D$</td>
<td>$\leftrightarrow$</td>
</tr>
</tbody>
</table>
A lattice congruence \( \equiv \) of \( \mathcal{AR}_D \) defines a quotient fan \( \mathcal{F}_\equiv \) where the chambers of \( \mathcal{F}_\equiv \) are obtained by glueing the chambers of \( \mathcal{H}_D \) corresponding to acyclic reorientations in the same equivalence class of \( \equiv \).
THM. A lattice congruence \( \equiv \) of \( \mathcal{AR}_D \) defines a quotient fan \( \mathcal{F}_\equiv \) where the chambers of \( \mathcal{F}_\equiv \) are obtained by glueing the chambers of \( \mathcal{H}_D \) corresponding to acyclic reorientations in the same equivalence class of \( \equiv \)
The quotient fan $\mathcal{F}_\equiv$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of
• a Minkowski sum of associahedra of Hohlweg – Lange, and
• a Minkowski sum of shard polytopes of Padrol – P. – Ritter

$\rho$-alternating matching = pair $(M_\triangledown, M_\triangle)$ with $M_\triangledown \subseteq \{u\} \cup \triangledown$ and $M_\triangle \subseteq \triangle \cup \{v\}$ s.t. $M_\triangledown$ and $M_\triangle$ are alternating along the transitive reduction of $D$

shard polytope of $\rho = $ convex hull of signed character vectors of $\rho$-alternating matchings
THM. The quotient fan $\mathcal{F}_\equiv$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of
- a Minkowski sum of associahedra of Hohlweg – Lange, and
- a Minkowski sum of shard polytopes of Padrol – P. – Ritter

PROP. For the sylvester congruence, all facets defining inequalities of the associahedron of $D$ are facet defining inequalities of the graphical zonotope of $D$
SOME OPEN PROBLEMS
SIMPLE ASSOCIAHEDRA

CONJ. \( D \) has no induced subgraph isomorphic to \( \square \) or \( \square \) \( \iff \) the Hasse diagram of the \( D \)-Tamari lattice is regular \( \iff \) the \( D \)-associahedron is a simple polytope

regular \hspace{5cm} \text{non regular}
CONJ.  $D$ has no induced subgraph isomorphic to $\square$

$\iff$ all Cambrian associahedra of $D$ have the same number of vertices

$\iff$ all Cambrian associahedra of $D$ have isomorphic 1-skeleta

$\iff$ all Cambrian associahedra of $D$ have isomorphic face lattices
**PROP.** For the sylvester congruence, all facets defining inequalities of the associahedron of \( D \) are facet defining inequalities of the graphical zonotope of \( D \).

**CONJ.** For any \( \mathcal{U}, \Omega \subseteq V \), the quotient fan \( \mathcal{F}_{(\mathcal{U},\Omega)} \) is the normal fan of the polytope obtained by deleting inequalities of the graphical zonotope of \( D \).
Not all acyclic reorientation flip graphs admit a Hamiltonian cycle
**THM [SSW '93].** For $D$ chordal, the acyclic reorientation flip graph is Hamiltonian

**CONJ.** When $D$ is skeletal, all quotientopes admit a Hamiltonian cycle

... checked for all quotients, for all skeletal acyclic directed graphs up to 5 vertices ...
\( \mathcal{H} \) hyperplane arrangement in \( \mathbb{R}^n \)

base region \( B = \) distinguished region of \( \mathbb{R}^n \setminus \mathcal{H} \)

inversion set of a region \( C = \) set of hyperplanes of \( \mathcal{H} \) that separate \( B \) and \( C \)

poset of regions \( PR(\mathcal{H}, B) = \) regions of \( \mathbb{R}^n \setminus \mathcal{H} \) ordered by inclusion of inversion sets

**QU.** For which \( (\mathcal{H}, B) \) is the poset of regions \( PR \) a lattice?
LATTICE OF REGIONS OF HYPERPLANE ARRANGEMENTS

\( \mathcal{H} \) hyperplane arrangement in \( \mathbb{R}^n \)
base region \( B = \) distinguished region of \( \mathbb{R}^n \setminus \mathcal{H} \)
inversion set of a region \( C = \) set of hyperplanes of \( \mathcal{H} \) that separate \( B \) and \( C \)
poset of regions \( \mathrm{PR}(\mathcal{H}, B) = \) regions of \( \mathbb{R}^n \setminus \mathcal{H} \) ordered by inclusion of inversion sets

**QU.** For which \( (\mathcal{H}, B) \) is the poset of regions \( \mathrm{PR} \) a lattice?

**THM.** The poset of regions \( \mathrm{PR}(\mathcal{H}, B) \)
- is never a lattice when \( B \) is not a simplicial region
- is always a lattice when \( \mathcal{H} \) is a simplicial arrangement

**THM.** The poset of regions \( \mathrm{PR}(\mathcal{H}, B) \) is a semidistributive lattice
\[ \iff \mathcal{H} \text{ is tight with respect to } B \]

Björner–Edelman–Ziegler (’90)
Reading (’16)
**QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS**

\( \mathcal{H} \) hyperplane arrangement in \( \mathbb{R}^n \)

_base region_ \( B = \) distinguished region of \( \mathbb{R}^n \setminus \mathcal{H} \)

_inversion set of a region_ \( C = \) set of hyperplanes of \( \mathcal{H} \) that separate \( B \) and \( C \)

_poset of regions_ \( \text{PR}(\mathcal{H}, B) = \) regions of \( \mathbb{R}^n \setminus \mathcal{H} \) ordered by inclusion of inversion sets

**THM.** If \( \text{PR}(\mathcal{H}, B) \) is a lattice, and \( \equiv \) is a congruence of \( \text{PR}(\mathcal{H}, B) \), the cones obtained by gluing the regions of \( \mathbb{R}^n \setminus \mathcal{H} \) in the same congruence class form a complete fan \( \mathcal{F}_{\equiv} \)

**Reading (’05)**

**QU.** Is the quotient fan \( \mathcal{F}_{\equiv} \) always polytopal?
hyperoctahedral group = isometry group of the hypercube (or of its dual cross-polytope)

**THM.** The quotient fan of any lattice congruence of the type $B$ weak order is polytopal

*Padrol–P.–Ritter ('20+)*

Type $B$ quotientopes are obtained

- not as removahedra,
- not as Minkowski sum of cyclohedra,
- but as Minkowski sum of shard polytopes (but this is another story...).
THANK YOU