# Brick polytopes — Lattices — Hopf algebras

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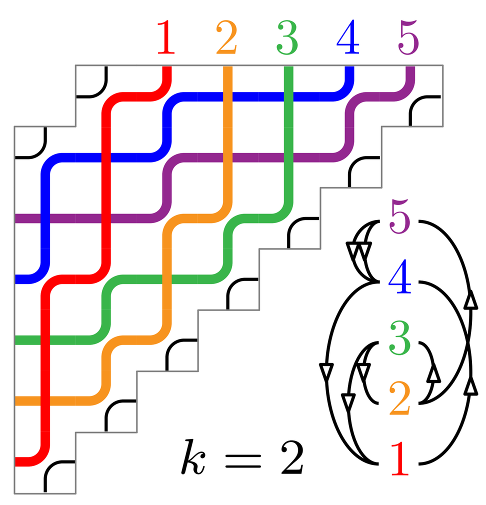
## Acyclic twists

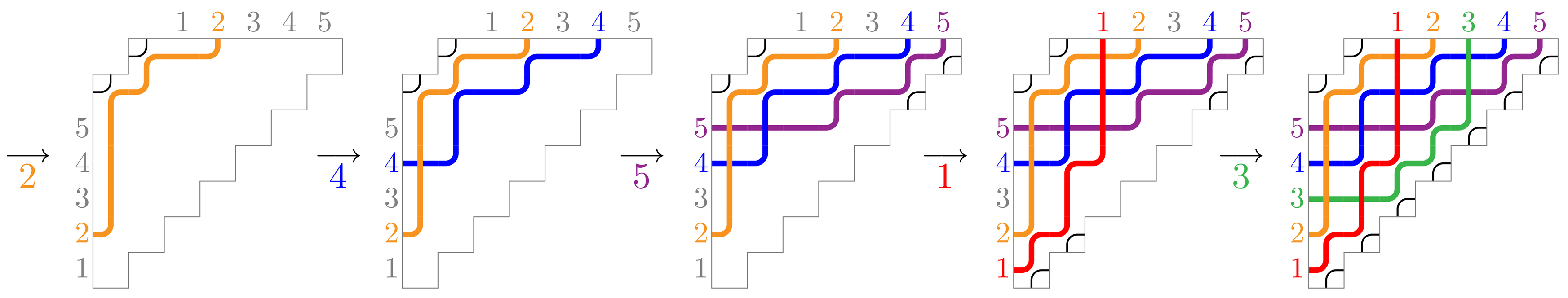
$(k,n)$-twist = pipe dream in the trapezoidal shape of height $n$ and width $k$

Contact graph of a twist $\mathrm{T}$ = graph $\mathrm{T}^\#$ with
- a vertex for each pipe of $\mathrm{T}$
- an arc for each contact of $\mathrm{T}$ from the SE pipe to the NW pipe




Acyclic twist = when its contact graph has no oriented cycle

Insertion algorithm
surjection $\mathrm{ins}^k$: permutations of $[n] \longrightarrow$ acyclic $(k,n)$-twists
algo: insert pipes from right to left as northwest as possible

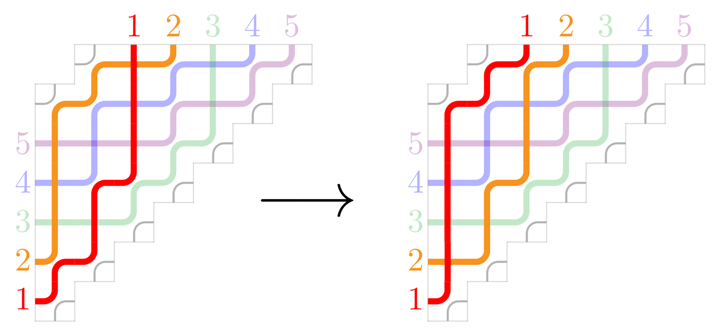


$\mathrm{ins}^2(31542)$

$k$-twist congruence = transitive closure of
$$UacV_1b_1V_2b_2\cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2\cdots V_kb_kW$$
where $a < b_i < c$ for all $i \in [k]$

PROP. $\tau \equiv^k \tau' \iff \mathrm{ins}^k(\tau) = \mathrm{ins}^k(\tau') = \mathrm{T} \iff \tau, \tau' \in \mathcal{L}(\mathrm{T}^\#)$

Flip = exchange an elbow with its corresponding crossing
Increasing flip = elbow SE crossing

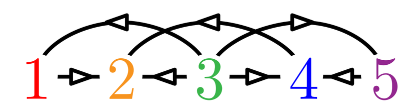


## Acyclic orientations

$\mathrm{G}^k(n)$ = graph with vertices $[n]$ & edges $\{i,j\}$ for $i < j \le i+k$

$k$-recoil scheme of $\tau \in \mathfrak{S}_n$ = acyclic orientation $\mathrm{rec}^k(\tau)$ of $\mathrm{G}^k(n)$ with edge $i \to j$ when $|i-j| \le k$ and $\tau^{-1}(i) < \tau^{-1}(j)$

$k$-recoil congruence = transitive closure of
$UacV \approx^k UcaV$ if $a + k < c$



$\mathrm{rec}^2(31542)$

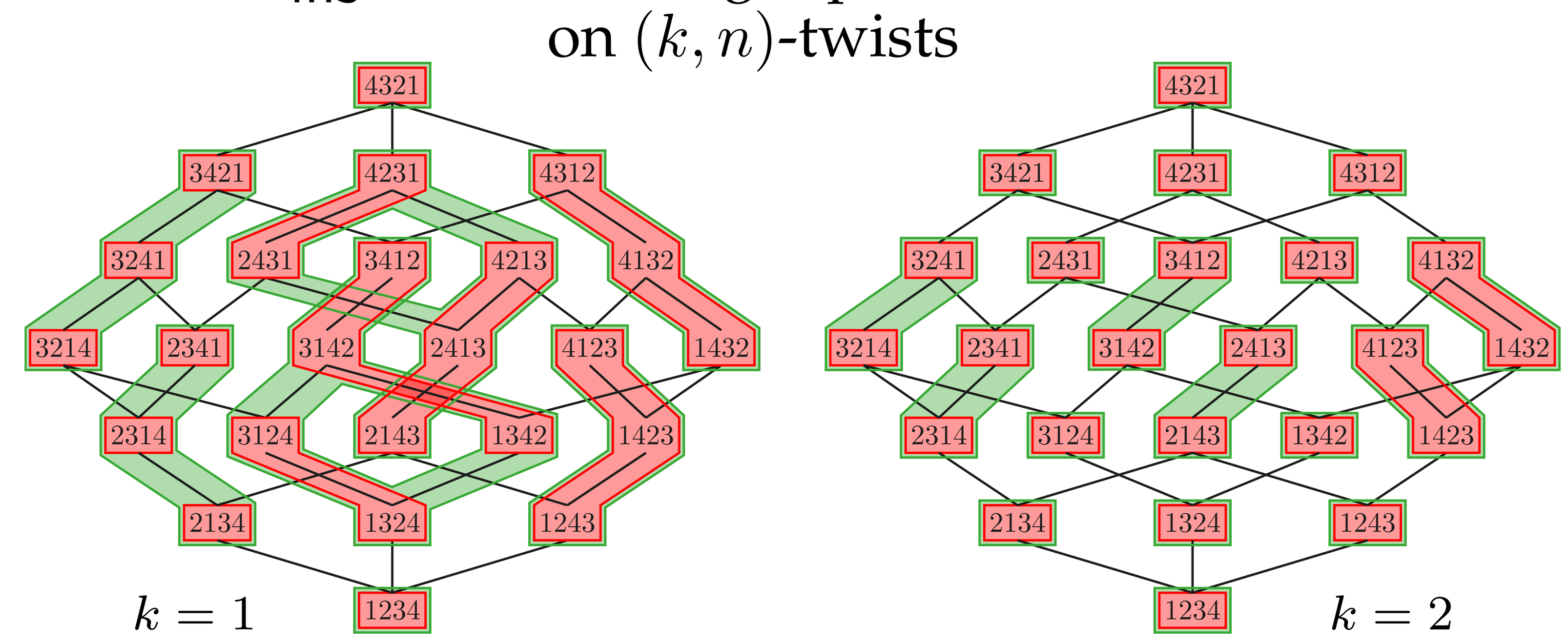
PROP. $\tau \approx^k \tau' \iff \mathrm{rec}^k(\tau) = \mathrm{rec}^k(\tau') = \mathrm{O} \iff \tau, \tau' \in \mathcal{L}(\mathrm{O})$

Canopy of $(k,n)$-twist $\mathrm{T}$ = acyclic orientation $\mathrm{can}^k(\mathrm{T})$ of $\mathrm{G}^k(n)$ with edge $i \to j$ when $|i-j| \le k$ and $i$ below $j$ in $\mathrm{T}^\#$

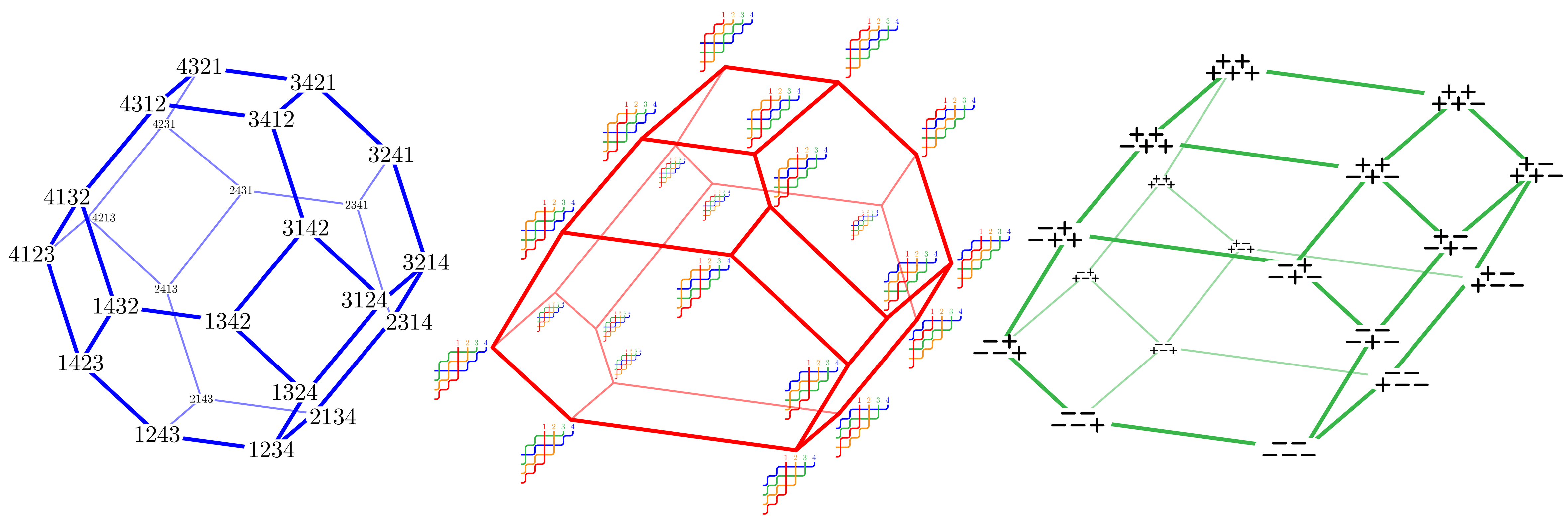
## Lattice homomorphisms

weak order on permutations of $[n]$ $\xrightarrow{\mathrm{rec}^k}$ boolean lattice on acyclic orient. of $\mathrm{G}^k(n)$

$\mathrm{ins}^k \searrow$ increasing flip order on $(k,n)$-twists $\nearrow \mathrm{can}^k$




## Permutahedron — Brick polytope — Zonotope



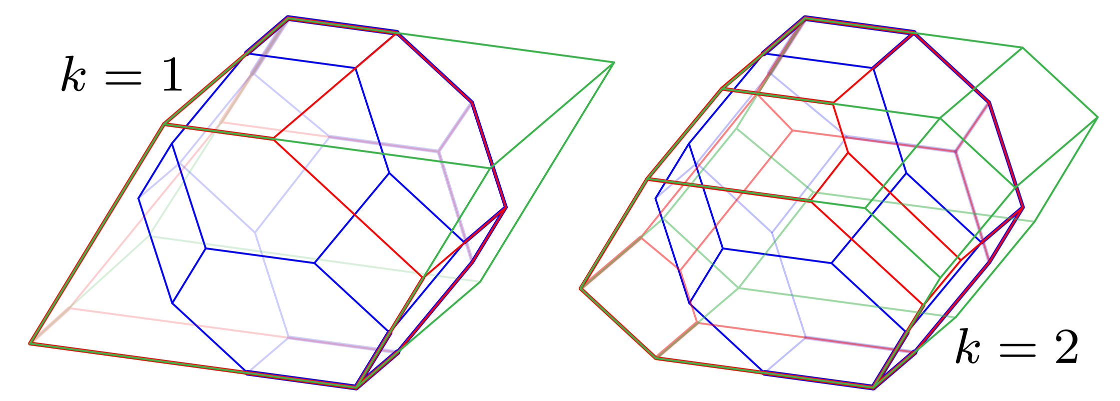
Perm$(n)$: $\mathrm{conv}\,\{\mathbf{x}(\tau) \mid \tau \in \mathfrak{S}_n\}$, $\mathbf{x}(\tau)_i = \tau(i)$

Brick$^k(n)$: $\mathrm{conv}\,\{\mathbf{b}(\mathrm{T}) \mid \mathrm{T}\ (k,n)\text{-twist}\}$, $\mathbf{b}(\mathrm{T})_i$ = nbr bricks below pipe $i$

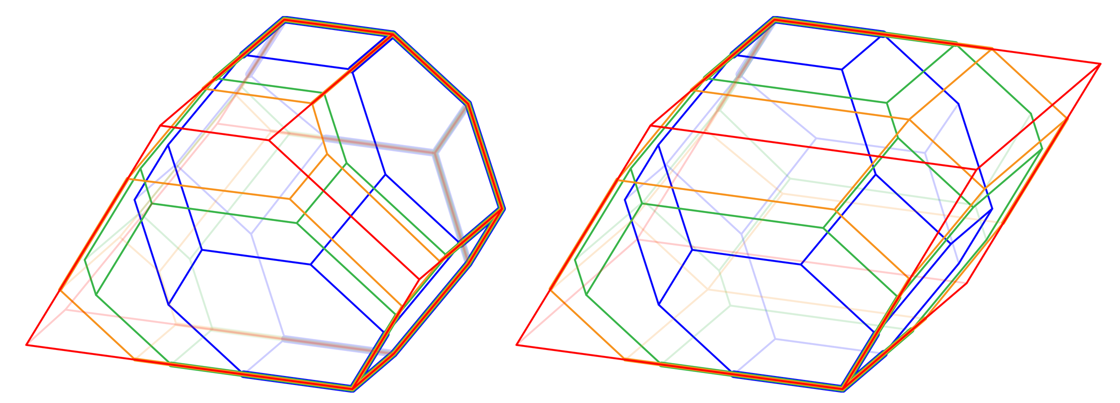
Zono$^k(n)$: $\sum_{|i-j|<k} [\mathbf{e}_i, \mathbf{e}_j]$

## Matriochka polytopes

Perm$(n)$ $\subset$ Brick$^k(n)$ $\subset$ Zono$^k(n)$




Brick$^1(n) \subset$ Brick$^2(n) \subset$ Brick$^3(n) \subset \cdots \subset$ Perm$^k(n)$



Zono$^1(n) \subset$ Zono$^2(n) \subset$ Zono$^3(n) \subset \cdots \subset$ Perm$^k(n)$

## Hopf algebras

Malvenuto-Reutenauer Hopf algebra = basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ and
$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup}\, \tau'} \mathbb{F}_\sigma \quad \text{and} \quad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

$12 \,\bar{\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$
$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$

$k$-twist algebra = subalgebra of MR algebra generated by
$\mathbb{P}_\mathrm{T} := \sum_{\mathrm{ins}^k(\tau)=\mathrm{T}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T}^\#)} \mathbb{F}_\tau$ for all acyclic $k$-twists $\mathrm{T}$

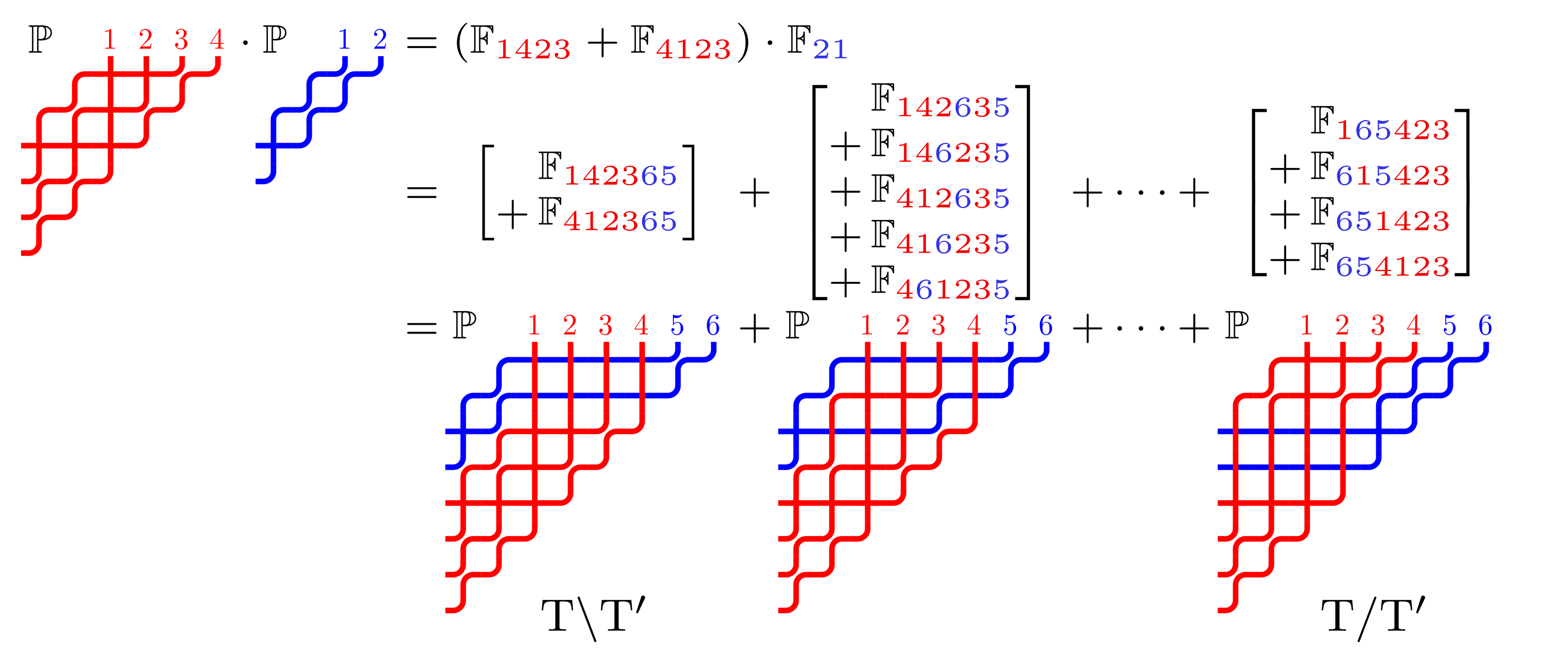
Exm: $k = 1 \implies$ Loday-Ronco Hopf algebra on binary trees

$k$-recoil algebra = subalgebra of MR algebra generated by
$\mathbb{X}_\mathrm{O} := \sum_{\mathrm{rec}^k(\tau)=\mathrm{O}} \mathbb{F}_\tau$ for all acyclic orientations $\mathrm{O}$ of $\mathrm{G}^k(n)$ for $n \in \mathbb{N}$

Exm: $k = 1 \implies$ Solomon descent algebra

Matriochkas: • $k$-recoil alg. $\hookrightarrow$ $k$-twist alg. $\hookrightarrow$ MR alg.
• if $\ell < k$, $\ell$-rec $\hookrightarrow$ $k$-rec and $\ell$-twist $\hookrightarrow$ $k$-twist

## Products




PROP. $\mathbb{P}_\mathrm{T} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_\mathrm{S} \mathbb{P}_\mathrm{S}$ where $\mathrm{S}$ runs over the interval between $\mathrm{T}\backslash\mathrm{T}'$ and $\mathrm{T}/\mathrm{T}'$ in the $(k, n+n')$-twist lattice

## Further topics...

arXiv:1505.07665

Multiplicative bases, $k$-twistiform algebras, ...
Cambrian, tuples, Schröder extensions