2. ACYCLIC REORIENTATION LATTICES AND THEIR QUOTIENTS

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ISM Discovery School
QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS?

Björner-Edelman-Ziegler, Hyperplane arrangements with a lattice of regions ('90)
Reading, Lattice congruences, fans and Hopf algebras ('05)
Reading, Lattice theory of the poset of regions ('16)
Padrol–P.–Ritter, Shard polytopes ('23)
P., Acyclic reorientation lattices and their lattice quotients ('21+)
Dana–Hanson–Thomas, Shard polytopes via representation theory (24+)
QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS?

\( \mathcal{H} \) hyperplane arrangement in \( \mathbb{R}^n \)

\( B \) distinguished region of \( \mathbb{R}^n \setminus \mathcal{H} \)

inversion set of a region \( C = \) set of hyperplanes of \( \mathcal{H} \) that separate \( B \) and \( C \)

poset of regions \( \text{PR}(\mathcal{H}, B) = \) regions of \( \mathbb{R}^n \setminus \mathcal{H} \) ordered by inclusion of inversion sets

**THM.** The poset of regions \( \text{PR}(\mathcal{H}, B) \)

- is never a lattice when \( B \) is not a simple region,
- is always a lattice when \( \mathcal{H} \) is a simplicial arrangement.

[Björner-Edelman-Ziegler, Hyperplane arrangements with a lattice of regions (’90)]

**THM.** If \( \text{PR}(\mathcal{H}, B) \) is a lattice, and \( \equiv \) is a lattice congruence of \( \text{PR}(\mathcal{H}, B) \), the cones obtained by glueing together the regions of \( \mathbb{R}^n \setminus \mathcal{H} \) in the same congruence class form a complete fan.

[Reading, Lattice congruences, fans and Hopf algebras (’05)]

Is the quotient fan polytopal?
SHARDS FOR HYPERPLANE ARRANGEMENTS

shard = piece of hyperplane obtained after cutting all rank 2 subgroups
shard poset = (pre)poset of forcing relations among shards

Reading, Lattice and order properties of the poset of regions in a hyperplane arrangement (’03)
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Reading, Lattice and order properties of the poset of regions in a hyperplane arrangement ('03)
**SHARDS FOR HYPERPLANE ARRANGEMENTS**

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Reading, *Lattice and order properties of the poset of regions in a hyperplane arrangement* (‘03)
**SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?**

**shard** = piece of hyperplane obtained after cutting all rank 2 subgroups

**shard poset** = (pre)poset of forcing relations among shards

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**shard polytope** for a shard $\Sigma$ = polytope such that the union of walls of its normal fan

- contains the shard $\Sigma$,
- is contained in the union of the shards forcing $\Sigma$

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Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions
SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

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- contains the shard $\Sigma$,
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Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

If any shard $\Sigma$ admits a shard polytope $\mathcal{SP}(\Sigma)$, then
- for any lattice congruence $\equiv$ of $\text{PR}(\mathcal{H}, B)$, the quotient fan $\mathcal{F}_\equiv$ is the normal of the Minkowski sum of the shard polytopes $\mathcal{SP}(\Sigma)$ for $\Sigma$ in the shard ideal $\Sigma_\equiv$
- if the arrangement $\mathcal{H}$ is simplicial, then the shard polytopes $\mathcal{SP}(\Sigma)$ form a basis for the type cone of the fan defined by $\mathcal{H}$ (up to translation)

Padrol–P.–Ritter, Shard polytopes ('23)
**SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?**

**shard** = piece of hyperplane obtained after cutting all rank 2 subgroups

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Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

Partial answers: Shard polytopes exist for

- type $A$ and $B$ Coxeter arrangements  
- all graphical arrangements whose poset of regions is a semidistributive lattice
  P., *Acyclic reorientation lattices and their lattice quotients* (‘21+)
- Coxeter arrangements of simply laced types ($A$, $D$, $E$)  
  Dana–Hanson–Thomas, *Shard polytopes via representation theory* (24+)
ACYCLIC REORIENTATION LATTICES

P., Acyclic reorientation lattices and their lattice quotients ('21+)
ACYCLIC REORIENTATION POSETS

$D$ directed acyclic graph

$\mathcal{AR}_D = \text{all acyclic reorientations of } D$, ordered by inclusion of their sets of reversed arcs

minimal element $D$

maximal element $\bar{D}$

self-dual under reversing all arcs

cover relations = flipping a single arc

flippable arcs of $E = \text{transitive reduction of } E$

$= E \setminus \{(u, v) \in E \mid \exists \text{ directed path } u \rightsquigarrow v \text{ in } E\}$
**ACYCLIC REORIENTATION POSETS**

Let $D$ be a directed acyclic graph.

- $\mathcal{AR}_D = \text{all acyclic reorientations of } D$, ordered by inclusion of their sets of reversed arcs.

- **Minimal element** $D$
- **Maximal element** $\bar{D}$
- **Self-dual under reversing all arcs**
- **Cover relations** = flipping a single arc
- **Flippable arcs of** $E = \text{transitive reduction of } E$
  
  $= E \setminus \{(u, v) \in E \mid \exists \text{ directed path } u \rightsquigarrow v \text{ in } E\}$
ACYCLIC REORIENTATION POSETS

\[ D \] directed acyclic graph

\[ \mathcal{AR}_D = \text{all acyclic reorientations of } D, \text{ ordered by inclusion of their sets of reversed arcs} \]
\( D_{\text{vertebrate}} = \text{transitive reduction of any induced subgraph of } D \) is a forest

**THM.** \( \mathcal{AR}_D \) lattice \( \iff \) \( D_{\text{vertebrate}} \)

P. ('21+)
$D_{\text{vertebrate}} =$ transitive reduction of any induced subgraph of $D$ is a forest

**THM.** $\mathcal{AR}_D$ lattice $\iff D$ vertebrate

P. (‘21+)
ACYCLIC REORIENTATION LATTICES

$D$ vertebrate = transitive reduction of any induced subgraph of $D$ is a forest

**THM.** $\mathcal{AR}_D$ lattice $\iff$ $D$ vertebrate

$X$ subset of arcs of $D$ is
- **closed** if all arcs of $D$ in the transitive closure of $X$ also belong to $X$
- **coclosed** if its complement is closed
- **biclosed** if it is closed and coclosed

**PROP.** If $D$ vertebrate,

$X$ biclosed $\iff$ the reorientation of $X$ is acyclic
ACYCLIC REORIENTATION LATTICES

\[ D_{\text{vertebrate}} = \text{transitive reduction of any induced subgraph of } D \text{ is a forest} \]

**THM.** \( \mathcal{AR}_D \) lattice \( \iff \) \( D_{\text{vertebrate}} \)  

**PROP.** If \( D_{\text{vertebrate}} \),  

\[
\text{bwd}(E_1 \lor \ldots \lor E_k) = \text{transitive closure of } \text{bwd}(E_1) \cup \cdots \cup \text{bwd}(E_k)
\]

\[
\text{fwd}(E_1 \land \ldots \land E_k) = \text{transitive closure of } \text{fwd}(E_1) \cup \cdots \cup \text{fwd}(E_k)
\]
DISTRIBUTIVITY & SEMIDISTRIBUTIVITY

P., Acyclic reorientation lattices and their lattice quotients (’21+).
THM. $\mathcal{AR}_D$ distributive lattice $\iff$ $D$ forest $\iff$ $\mathcal{AR}_D$ boolean lattice

P. (‘21+)
$D$ skeletal =

- $D$ vertebrate = transitive reduction of any induced subgraph of $D$ is a forest
- $D$ filled = any directed path joining the endpoints of an arc in $D$ induces a tournament

**THM.** $\mathcal{AR}_D$ semidistributive lattice $\iff D$ is skeletal

P. ('21+)
ROPES & NON-CROSSING ROPE DIAGRAMS

P., Acyclic reorientation lattices and their lattice quotients (’21+)
rope of $D = \text{quadruple } \rho = (u, v, \nabla, \Delta)$ where

- $(u, v)$ is an arc of $D$
- $\nabla \sqcup \Delta$ partitions the transitive support of $(u, v)$ minus $\{u, v\}$
Rope of $D$ = quadruple $\rho = (u, v, \nabla, \triangle)$ where

- $(u, v)$ is an arc of $D$
- $\nabla \sqcup \triangle$ partitions the transitive support of $(u, v)$ minus \{u, v\}

THM. \hspace{1cm} join irreducibles of $\mathcal{AR}_D$ $\longleftrightarrow$ \hspace{1cm} ropes of $D$ \hspace{1cm} P. (’21+)
ropes of $D = \text{quadruple } \rho = (u, v, \nabla, \triangle)$ where

- $(u, v)$ is an arc of $D$
- $\nabla \sqcup \triangle$ partitions the transitive support of $(u, v)$ minus $\{u, v\}$

THM. \hspace{1cm} \text{join irreducibles of } \mathcal{AR}_D \quad \longleftrightarrow \quad \text{ropes of } D \hspace{1cm} \text{P. \textquoteleft21}^+$

\begin{align*}
\text{canonical join representations of } \mathcal{AR}_D & \quad \longleftrightarrow \quad \text{non-crossing rope diagrams of } \mathcal{AR}_D \\

(u, v, \nabla, \triangle) \text{ and } (u', v', \nabla', \triangle') \text{ are crossing if there are } w \neq w' \text{ such that} \\
w \in (\nabla \cup \{u, v\}) \cap (\triangle' \cup \{u', v'\}) \text{ and } w' \in (\triangle \cup \{u, v\}) \cap (\nabla' \cup \{u', v'\})
\end{align*}
CONGRUENCES & QUOTIENTS

P., Acyclic reorientation lattices and their lattice quotients (‘21+’)

(\(u, v, \nabla, \Delta\)) subrope of \((u', v', \nabla', \Delta')\) if \(u, v \in \{u', v'\} \cup \nabla' \cup \Delta'\) and \(\nabla \subseteq \nabla'\) and \(\Delta \subseteq \Delta'\)

**PROP.** congruence lattice of \(\mathcal{AR}_D\) \(\simeq\) lower ideal lattice of subrope order  

**CORO.** \(\equiv\) lattice congruence of \(\mathcal{AR}_D\)
- \(E\) minimal in its \(\equiv\)-class \(\iff\) \(\delta(E) \subseteq \mathcal{R}_\equiv\)
- quotient \(\mathcal{AR}_D/\equiv \simeq\) subposet of \(\mathcal{AR}_D\) induced by \(\{E \in \mathcal{AR}_D \mid \delta(E) \subseteq \mathcal{R}_\equiv\}\)
COHERENT CONGRUENCES

\((\mathcal{U}, \Omega)\) = two of arbitrary subsets of \(V\)
\(\mathcal{R}_{(\mathcal{U}, \Omega)}\) = lower ideal of ropes \((u, v, \triangledown, \triangle)\) of \(D\) such that \(\triangledown \subseteq \mathcal{U}\) and \(\triangle \subseteq \Omega\)
coherent congruence \(\equiv_{(\mathcal{U}, \Omega)}\) = congruence with subrope ideal \(\mathcal{R}_{(\mathcal{U}, \Omega)}\)

examples:
- sylvester congruence = subrope ideal contains only ropes \((u, v, \triangledown, \emptyset)\)

P.–Pons, Permutrees ('18)
COHERENT CONGRUENCES

\((\mathcal{U}, \Omega) = \text{two of arbitrary subsets of } V\)
\(\mathcal{R}(\mathcal{U}, \Omega) = \text{lower ideal of ropes } (u, v, \nabla, \triangle) \text{ of } D \text{ such that } \nabla \subseteq \mathcal{U} \text{ and } \triangle \subseteq \Omega\)

coherent congruence \(\equiv (\mathcal{U}, \Omega) = \text{congruence with subrope ideal } \mathcal{R}(\mathcal{U}, \Omega)\)

examples:

- **sylvester congruence** = subrope ideal contains only ropes \((u, v, \nabla, \emptyset)\)
- **Cambrian congruences** = when \(\mathcal{U} \sqcup \Omega = V\)

P.–Pons, *Permutrees* ('18)

Reading, *Cambrian lattices* ('06)
QUOTIENT FANS & QUOTIENTOTOPES

P., Acyclic reorientation lattices and their lattice quotients (‘21+)
$D$ directed acyclic graph

**graphical arrangement** $\mathcal{H}_D = \text{arrangement of hyperplanes } x_u = x_v \text{ for all arcs } (u, v) \in D$

**graphical zonotope** $\mathcal{Z}_D = \text{Minkowski sum of } [e_u, e_v] \text{ for all arcs } (u, v) \in D$

### Diagram:

- **hyperplanes of** $\mathcal{H}_D$ ↔ summands of $\mathcal{Z}_D$ ↔ arcs of $D$
- **regions of** $\mathcal{H}_D$ ↔ vertices of $\mathcal{Z}_D$ ↔ acyclic reorientations of $D$
- **poset of regions of** $\mathcal{H}_D$ ↔ oriented graph of $\mathcal{Z}_D$ ↔ acyclic reorientation poset of $D$
THM. A lattice congruence $\equiv$ of $\mathcal{AR}_D$ defines a quotient fan $\mathcal{F}_\equiv$ where the chambers of $\mathcal{F}_\equiv$ are obtained by glueing the chambers of $\mathcal{H}_D$ corresponding to acyclic reorientations in the same equivalence class of $\equiv$
THM. A lattice congruence $\equiv$ of $\mathcal{AR}_D$ defines a quotient fan $\mathcal{F}_\equiv$ where the chambers of $\mathcal{F}_\equiv$ are obtained by glueing the chambers of $\mathcal{H}_D$ corresponding to acyclic reorientations in the same equivalence class of $\equiv$.
**THM.** The quotient fan $\mathcal{F}_\equiv$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of

- a Minkowski sum of associahedra of Hohlweg – Lange, and
- a Minkowski sum of shard polytopes of Padrol – P. – Ritter

$\rho$-alternating matching $= \text{pair } (M_\triangledown, M_\triangle)$ with $M_\triangledown \subseteq \{u\} \cup \triangledown$ and $M_\triangle \subseteq \triangle \cup \{v\}$ s.t. $M_\triangledown$ and $M_\triangle$ are alternating along the transitive reduction of $D$ shard polytope of $\rho =$ convex hull of signed charact. vectors of $\rho$-alternating matchings
**THM.** The quotient fan $\mathcal{F}_\equiv$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of
- a Minkowski sum of associahedra of Hohlweg – Lange, and
- a Minkowski sum of shard polytopes of Padrol – P. – Ritter

**PROP.** For the sylvester congruence, all facets defining inequalities of the associahedron of $D$ are facet defining inequalities of the graphical zonotope of $D
SOME OPEN PROBLEMS
CONJ. $D$ has no induced subgraph isomorphic to $\square$ or $\blacklozenge$

$\iff$ the Hasse diagram of the $D$-Tamari lattice is regular

$\iff$ the $D$-associahedron is a simple polytope
CONJ. $D$ has no induced subgraph isomorphic to $\square$

$\iff$ all Cambrian associahedra of $D$ have the same number of vertices

$\iff$ all Cambrian associahedra of $D$ have isomorphic 1-skeleta

$\iff$ all Cambrian associahedra of $D$ have isomorphic face lattices
**PROP.** For the Sylvester congruence, all facets defining inequalities of the associahedron of $D$ are facet defining inequalities of the graphical zonotope of $D$

**CONJ.** For any $\mathcal{U}, \Omega \subseteq V$, the quotient fan $\mathcal{F}_{(\mathcal{U}, \Omega)}$ is the normal fan of the polytope obtained by deleting inequalities of the graphical zonotope of $D$
THANKS