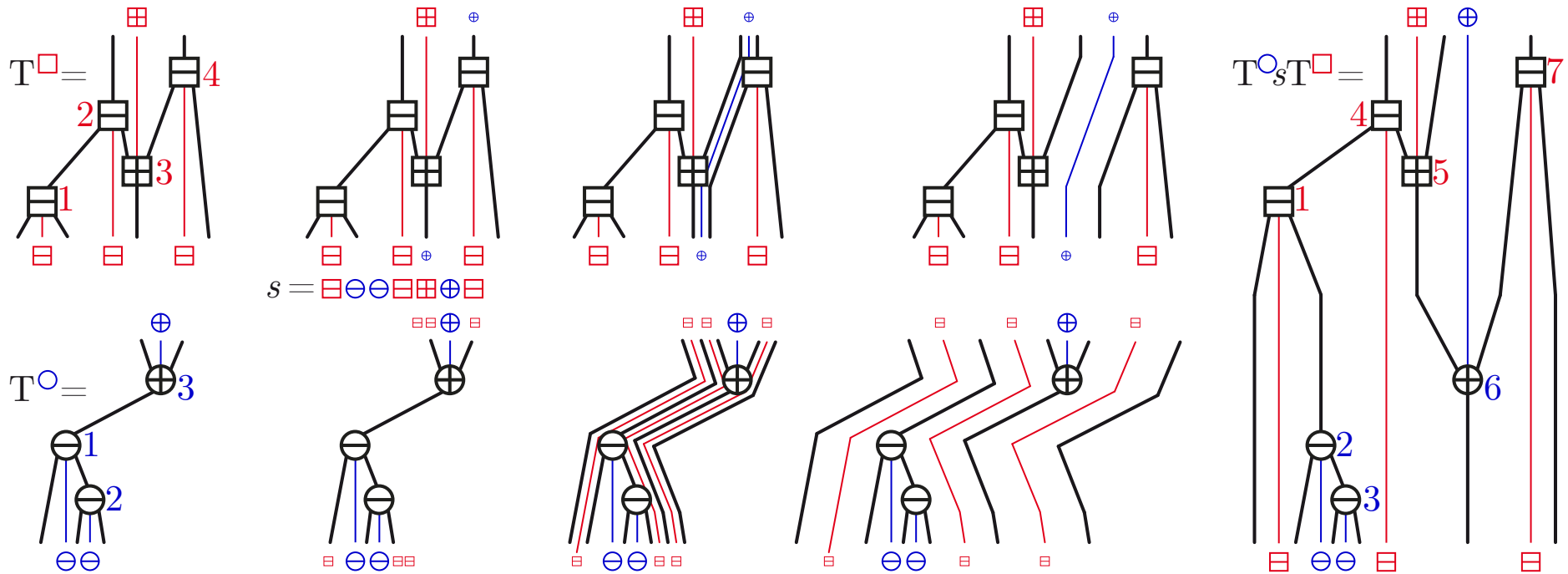


CAMBRIAN TREES



Grégory CHATEL
(Univ. MIV)

Carsten LANGE
(Univ. Munich)

Vincent PILAUD
(CNRS & LIX)

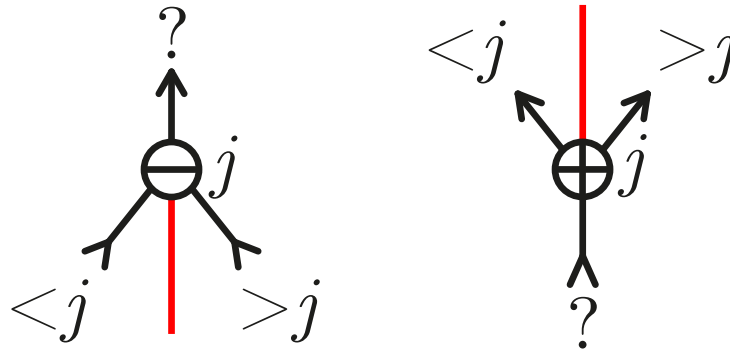
MOTIVATION

	permutations	binary trees	binary sequences
Combinatorics			
Geometry			
Algebra	<p>Malvenuto-Reutenauer algebra</p> $\text{FQSym} = \text{vect} \langle \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \rangle$ $\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} \mathbb{F}_\sigma$ $\Delta \mathbb{F}_\sigma = \sum_{\sigma \in \tau * \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$	<p>Loday-Ronco algebra</p> $\text{PBT} = \text{vect} \langle \mathbb{P}_T \mid T \in \mathcal{BT} \rangle$ $\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{T \nearrow^{T'} \leq T'' \leq T \nwarrow_{T'}} \mathbb{P}_{T''}$ $\Delta \mathbb{F}_\gamma = \sum_{\gamma \text{ cut}} B(T, \gamma) \otimes A(T, \gamma)$	<p>Solomon algebra</p> $\text{Rec} = \text{vect} \langle \mathbb{X}_\eta \mid \eta \in \pm^* \rangle$ $\mathbb{X}_\eta \cdot \mathbb{X}_{\eta'} = \mathbb{X}_{\eta+\eta'} + \mathbb{X}_{\eta-\eta'}$ $\Delta \mathbb{X}_\eta = \sum_{\gamma \text{ cut}} B(\eta, \gamma) \otimes A(\eta, \gamma)$

COMBINATORICS

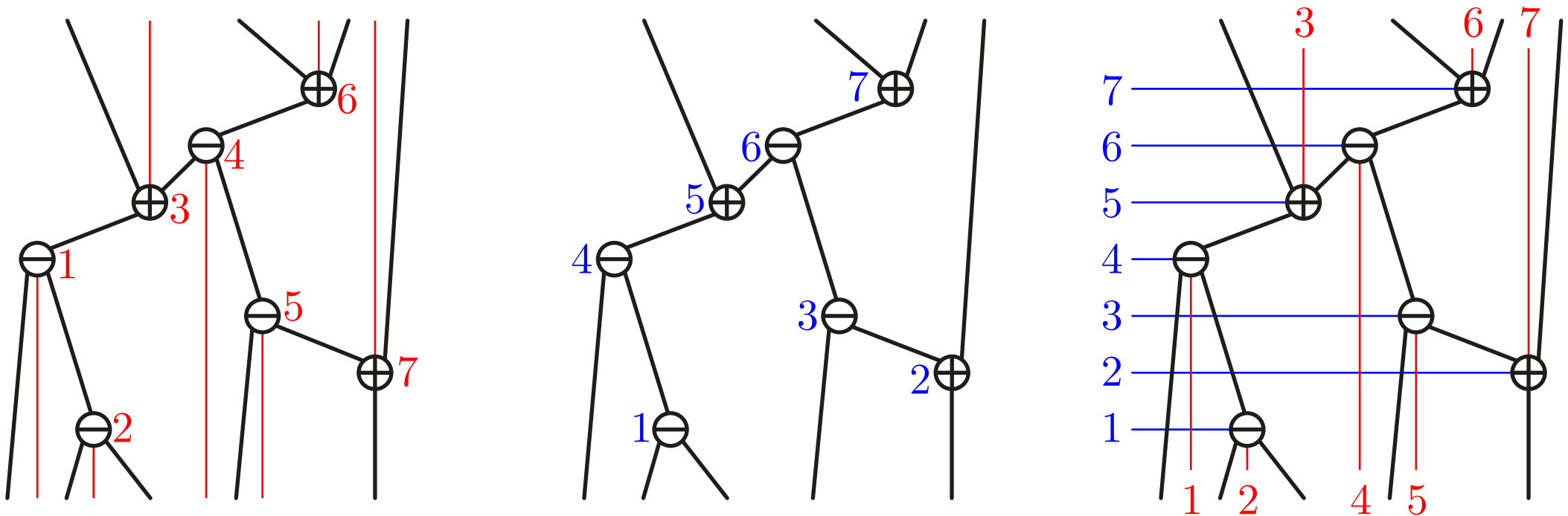
CAMBRIAN TREES

Cambrian tree = directed and labeled tree such that



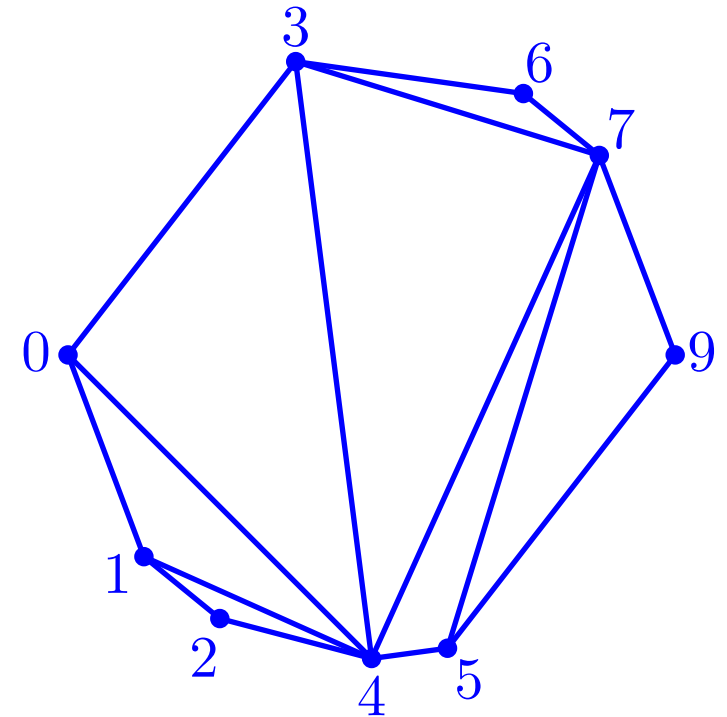
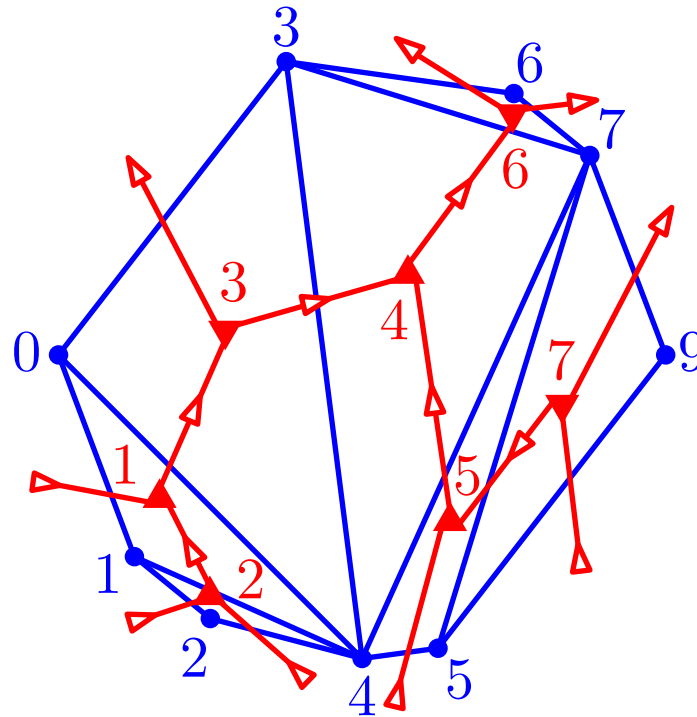
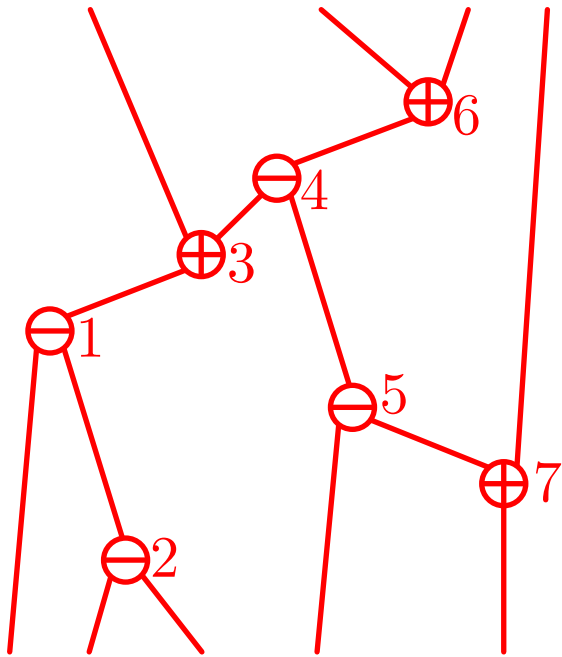
increasing tree = directed and labeled tree such that labels increase along arcs

leveled Cambrian tree = directed tree with a Cambrian labeling and an increasing labeling



CAMBRIAN TREES AND TRIANGULATIONS

Cambrian trees are dual to triangulations of polygons



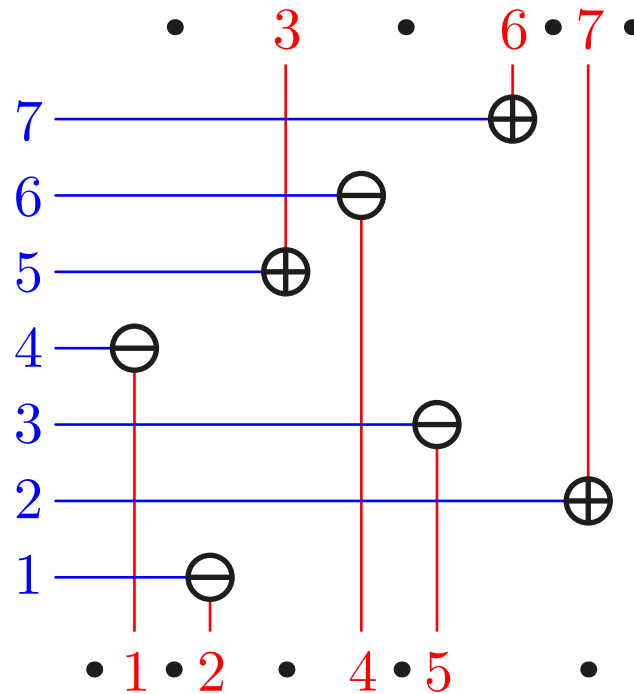
signature \longleftrightarrow vertices above or below $[0, 9]$
 node j \longleftrightarrow triangle $i < j < k$

For any signature ε , there are $C_n = \frac{1}{n+1} \binom{2n}{n}$ ε -Cambrian trees

CAMBRIAN CORRESPONDENCE

Cambrian correspondence = signed permutation \mapsto leveled Cambrian tree

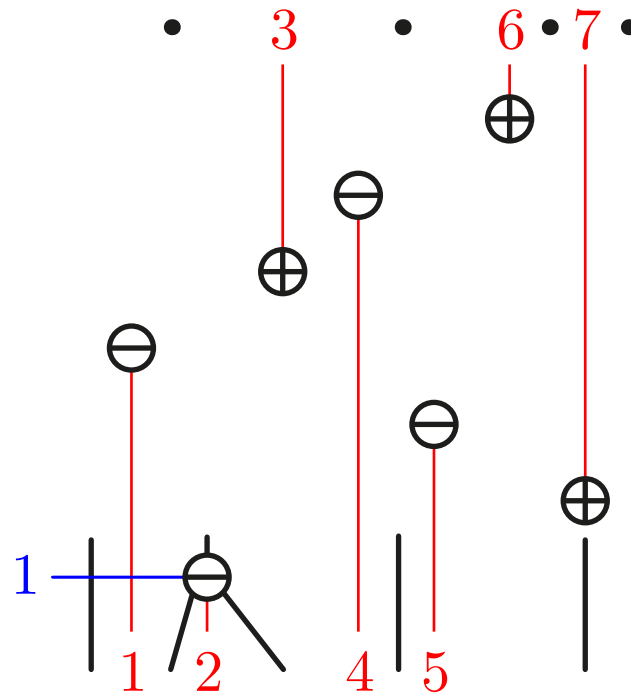
Exm: signed permutation $\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}$



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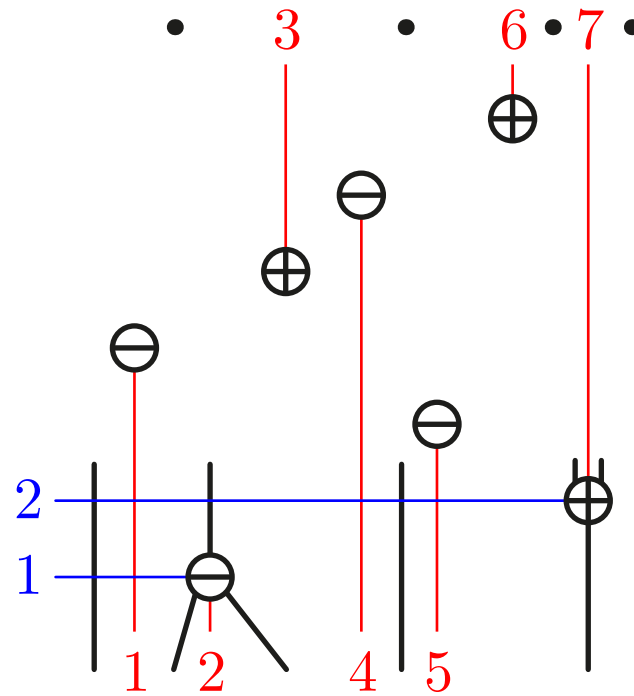
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CAMBRIAN CORRESPONDENCE

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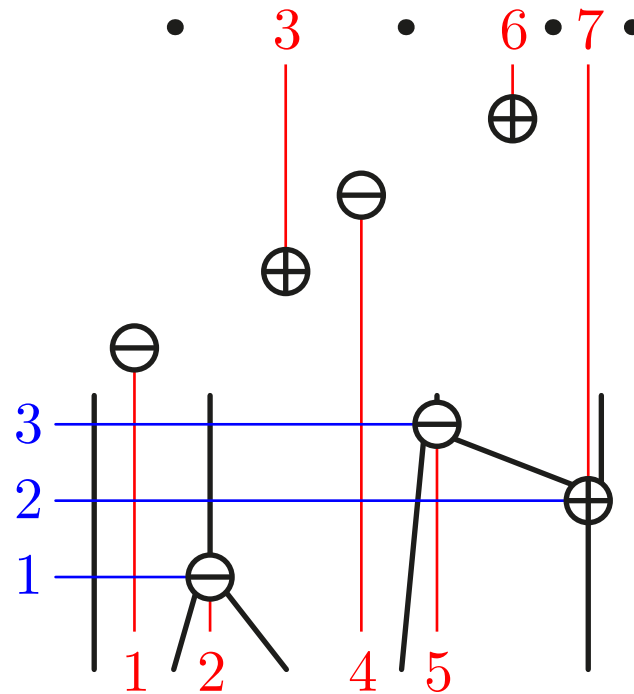
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CAMBRIAN CORRESPONDENCE

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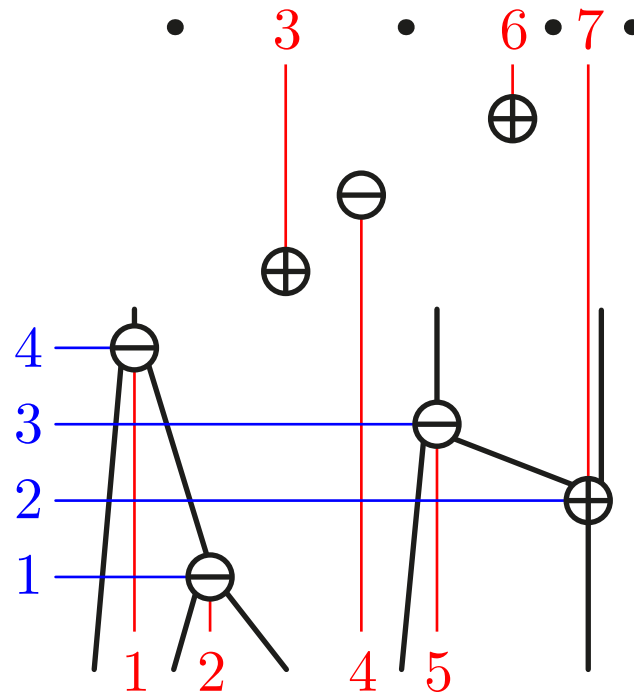
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CAMBRIAN CORRESPONDENCE

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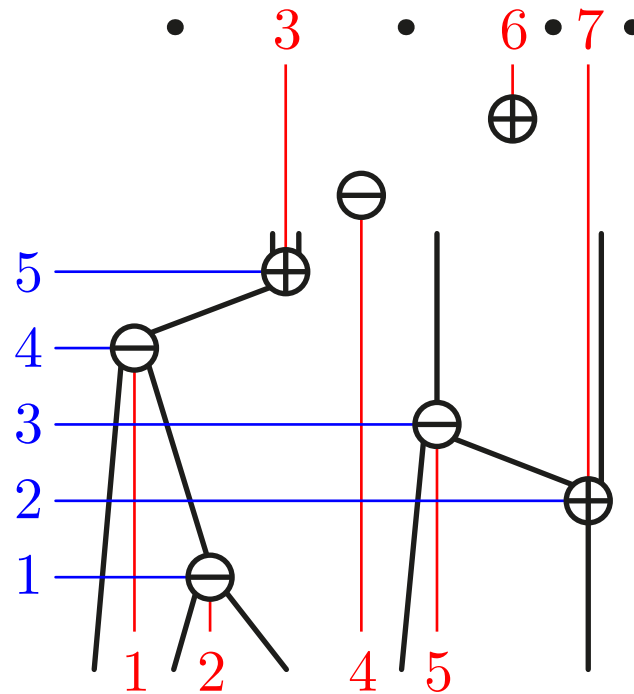
Exm: signed permutation $\underline{2}\bar{7}\underline{5}\underline{1}\bar{3}\underline{4}\bar{6}$



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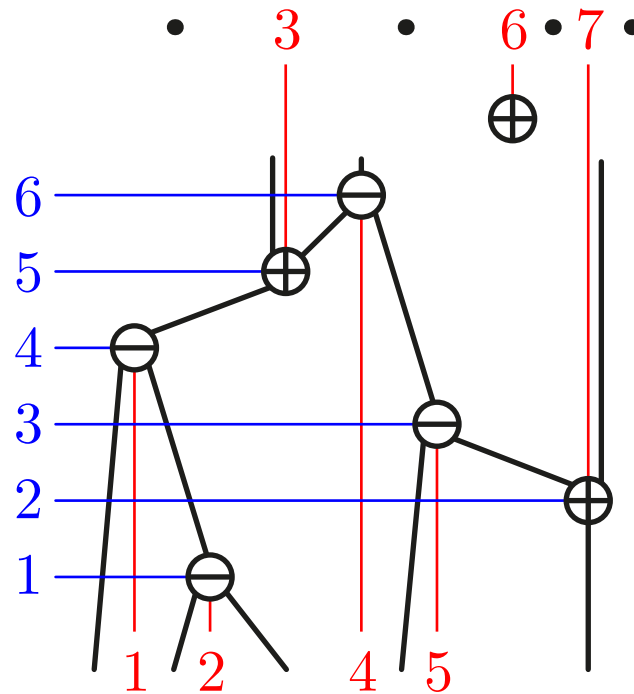
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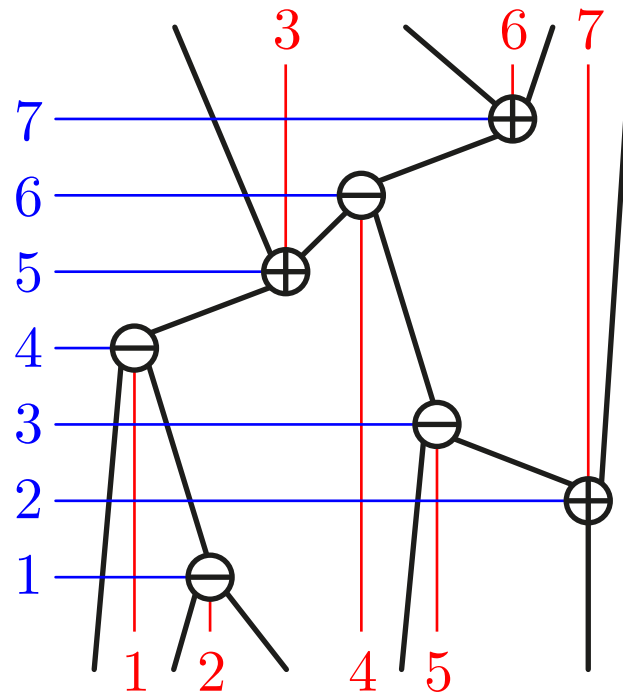
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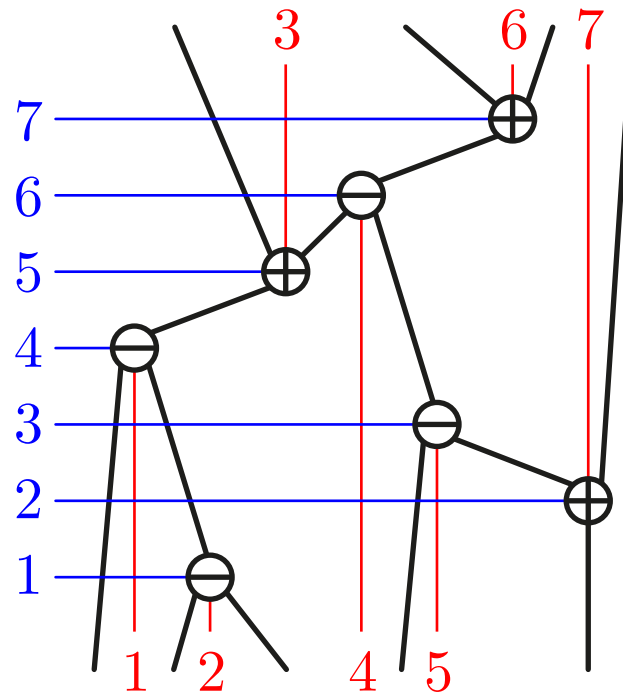
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$\mathbf{P}(\tau)$ = \mathbf{P} -symbol of τ = Cambrian tree produced by Cambrian correspondence

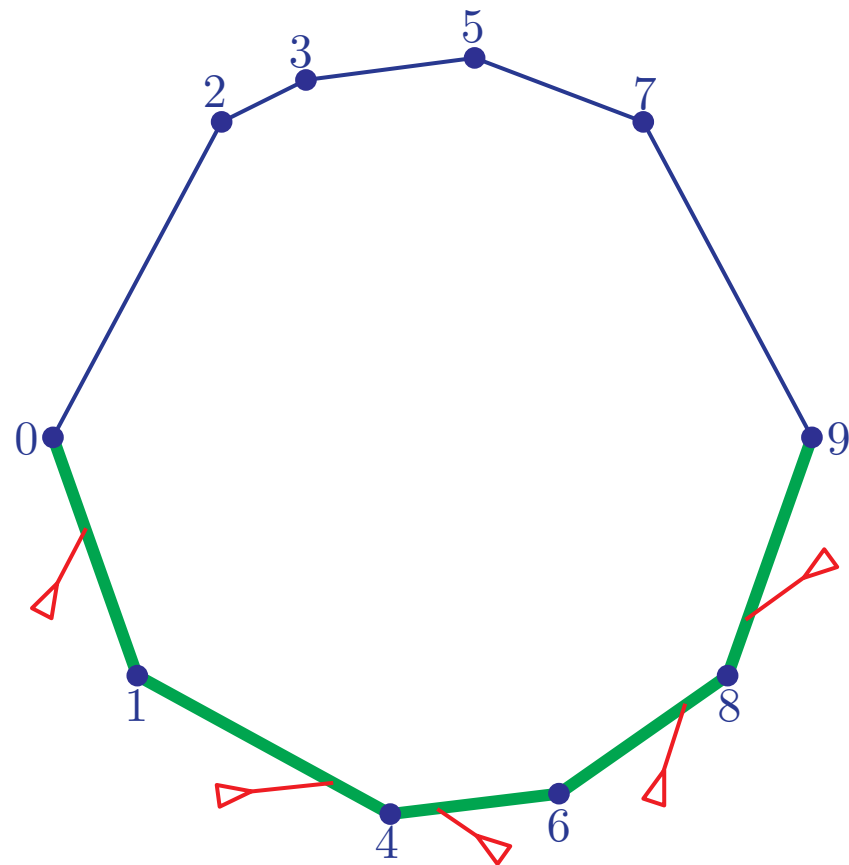
$\mathbf{Q}(\tau)$ = \mathbf{Q} -symbol of τ = increasing tree produced by Cambrian correspondence

(analogy to Robinson-Schensted algorithm)

CAMBRIAN CORRESPONDENCE AND TRIANGULATIONS

Cambrian map = signed permutation \mapsto triangulation

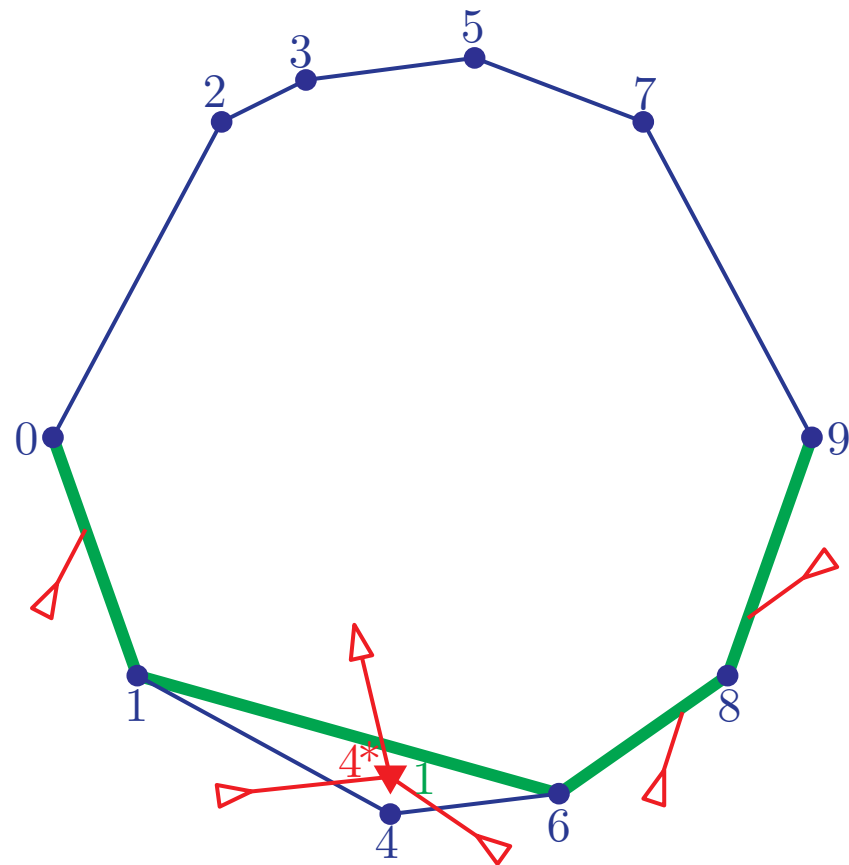
Exm: signed permutation $\underline{4}\overline{3}\overline{8}1\overline{6}\overline{2}\overline{5}\overline{7}$



CAMBRIAN CORRESPONDENCE AND TRIANGULATIONS

Cambrian map = signed permutation \mapsto triangulation

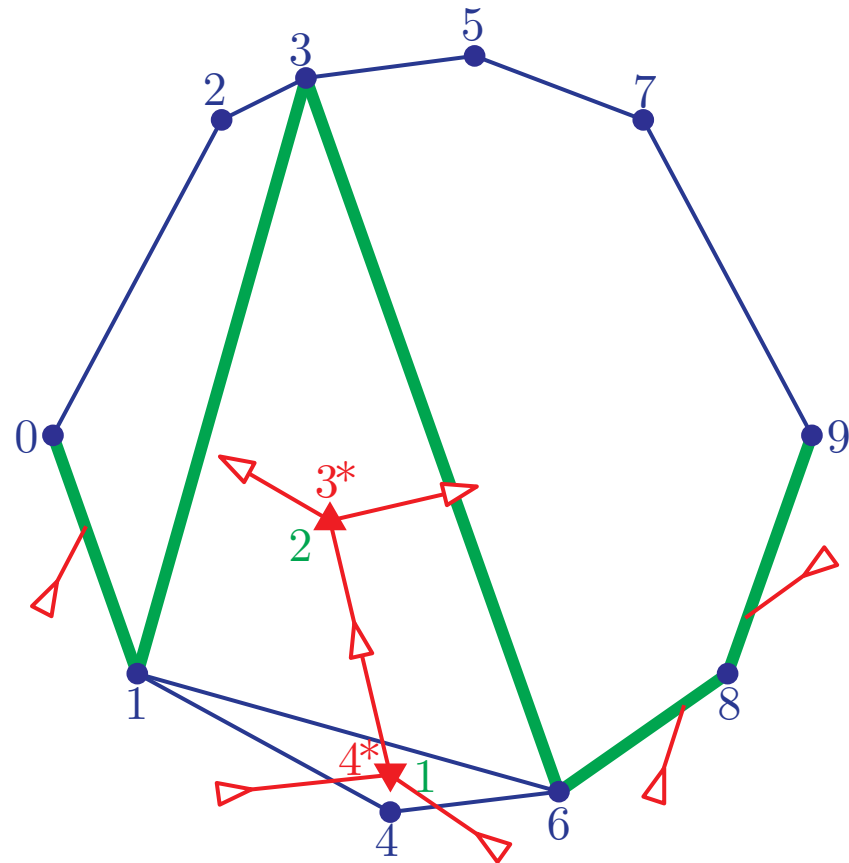
Exm: signed permutation $\underline{4}\overline{3}\overline{8}\underline{1}\overline{6}\overline{2}\overline{5}\overline{7}$



CAMBRIAN CORRESPONDENCE AND TRIANGULATIONS

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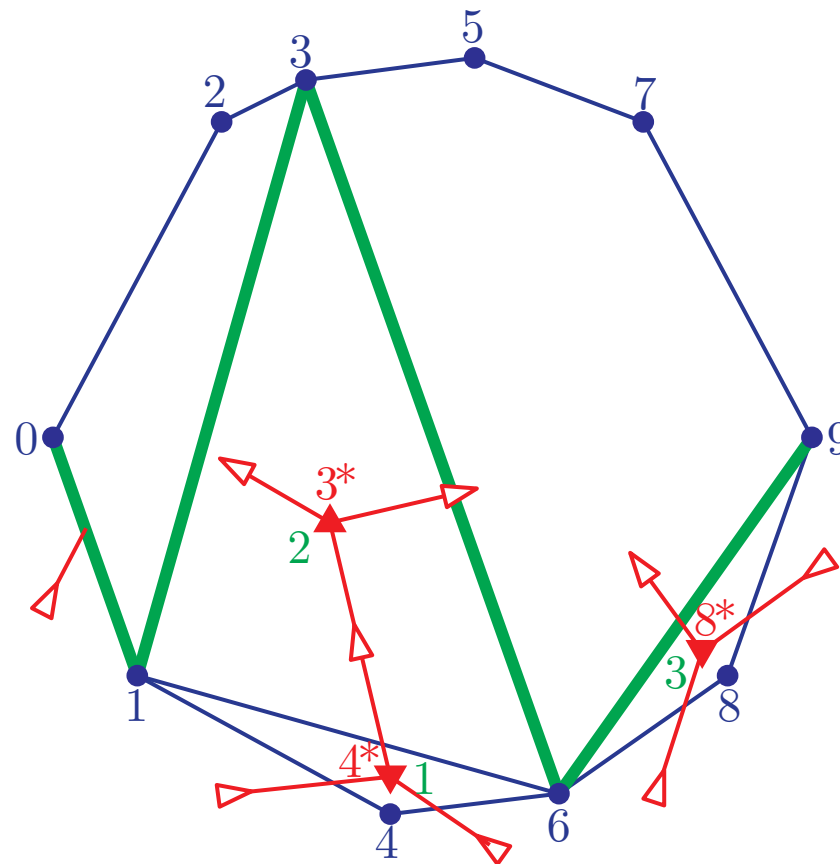
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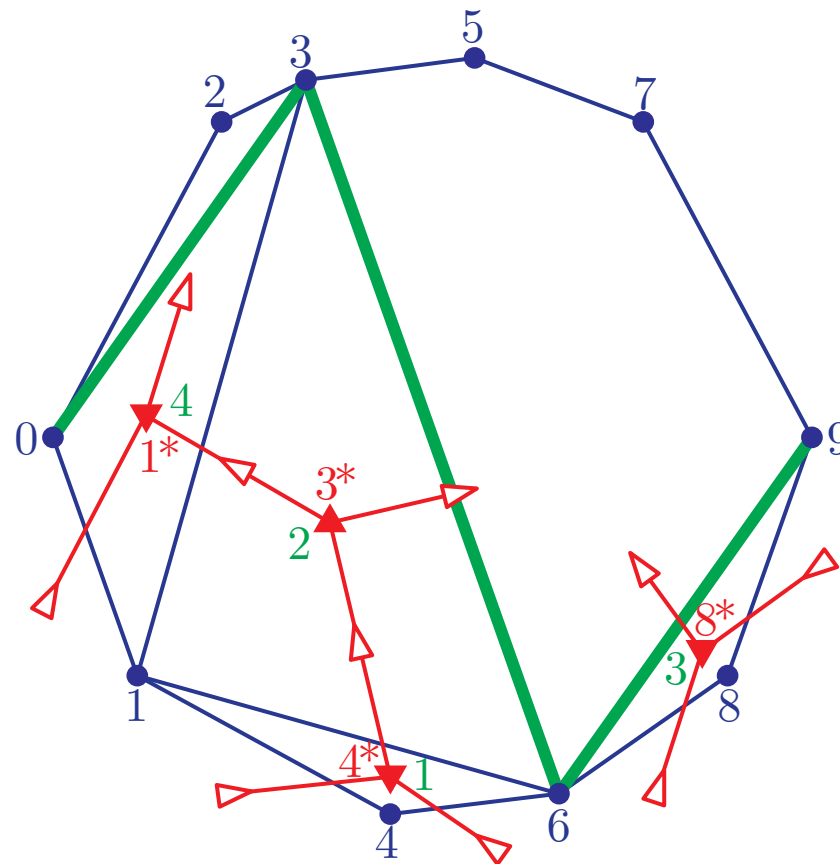
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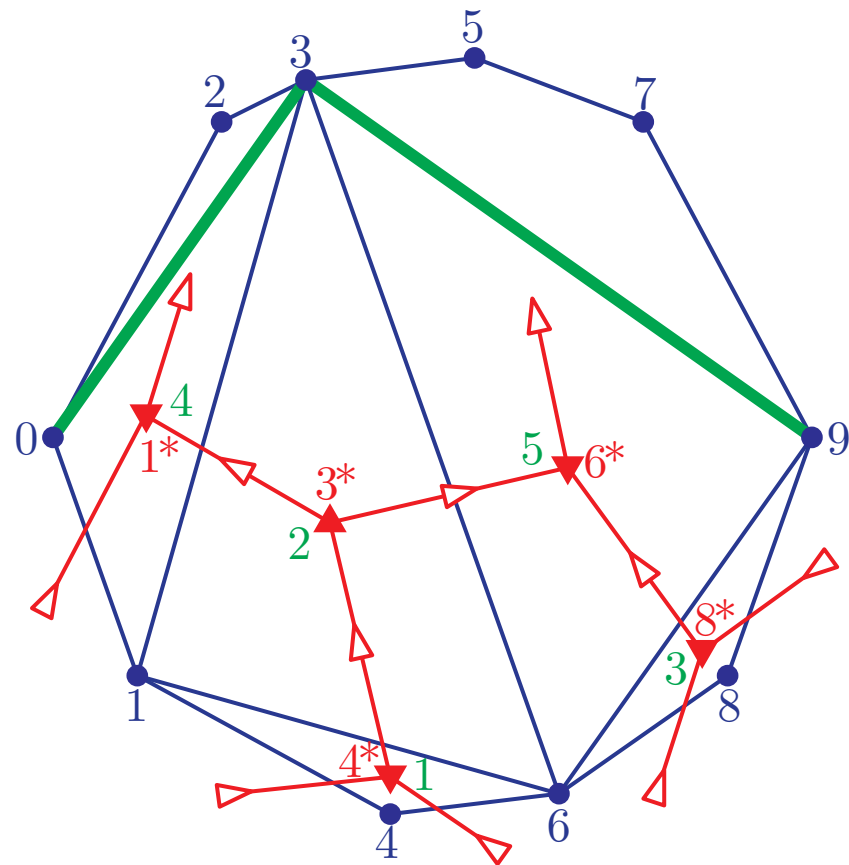
Exm: signed permutation $\underline{4}\overline{3}\overline{8}\underline{1}\overline{6}\overline{2}\overline{5}\overline{7}$



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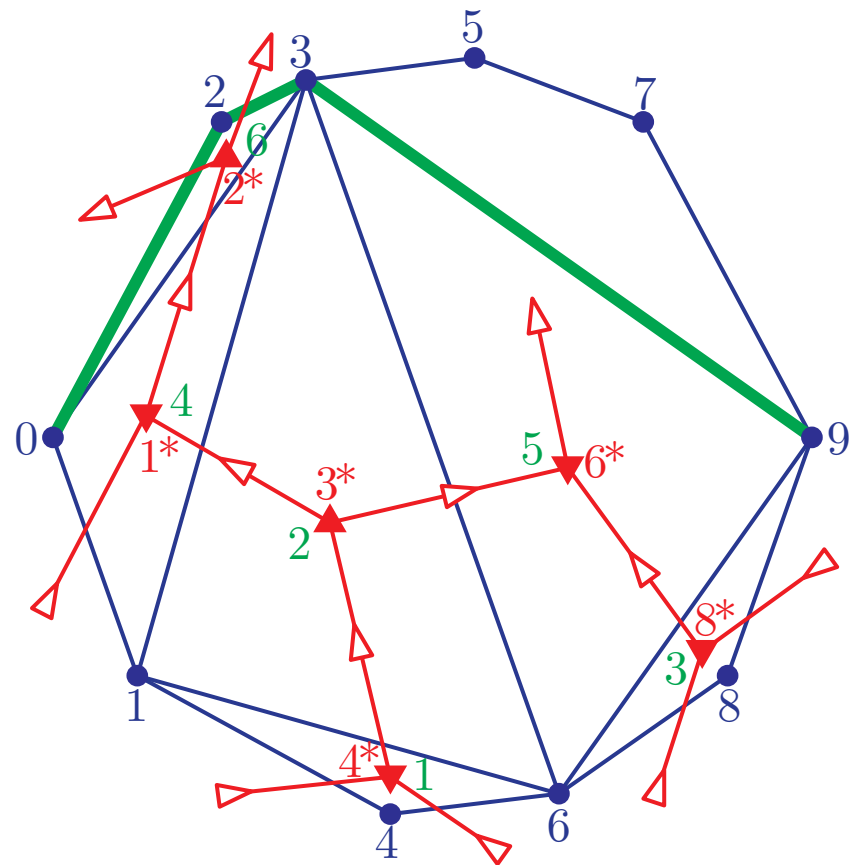
Exm: signed permutation $\underline{4}\overline{3}\overline{8}\underline{1}\overline{6}\overline{2}\overline{5}\overline{7}$



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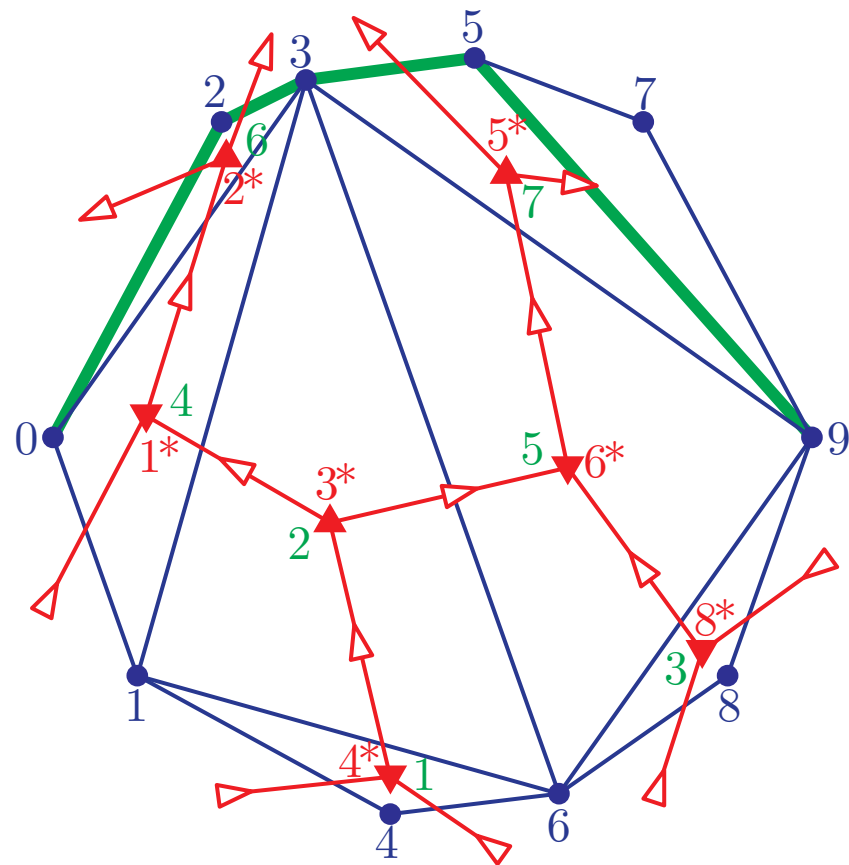
Exm: signed permutation $\underline{4}\overline{3}\underline{8}\underline{1}\underline{6}\overline{2}\overline{5}\overline{7}$



CAMBRIAN CORRESPONDENCE AND TRIANGULATIONS

Cambrian map = signed permutation \mapsto triangulation

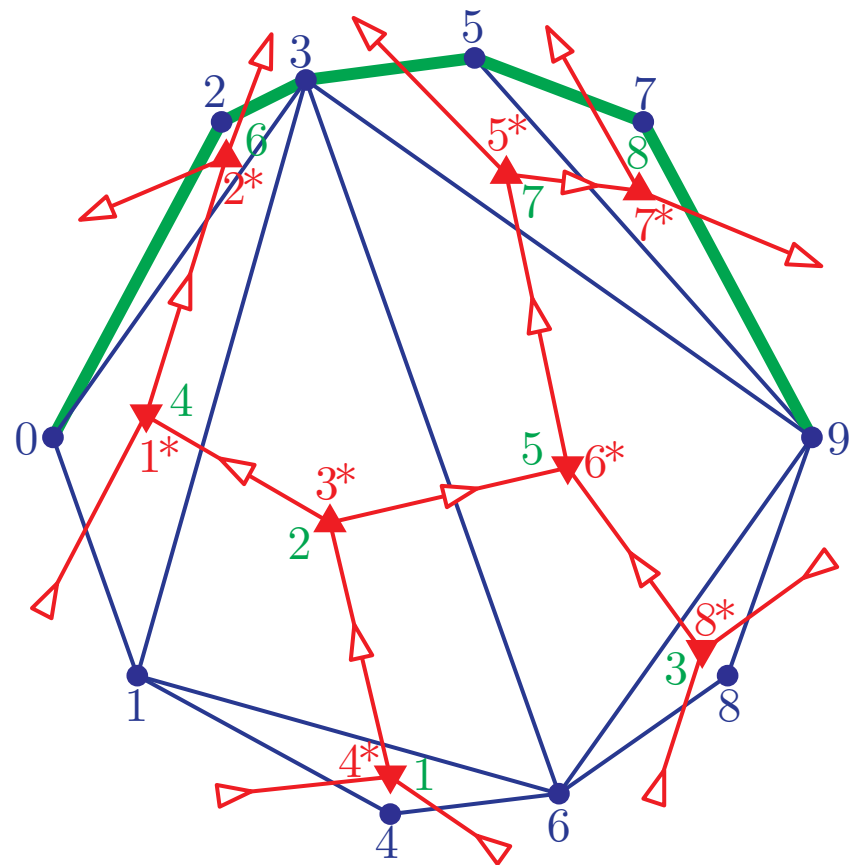
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CAMBRIAN CORRESPONDENCE AND TRIANGULATIONS

Cambrian map = signed permutation \mapsto triangulation

Exm: signed permutation $\underline{4}\overline{3}\overline{8}\underline{1}\overline{6}\underline{2}\overline{5}\overline{7}$



CAMBRIAN CONGRUENCE

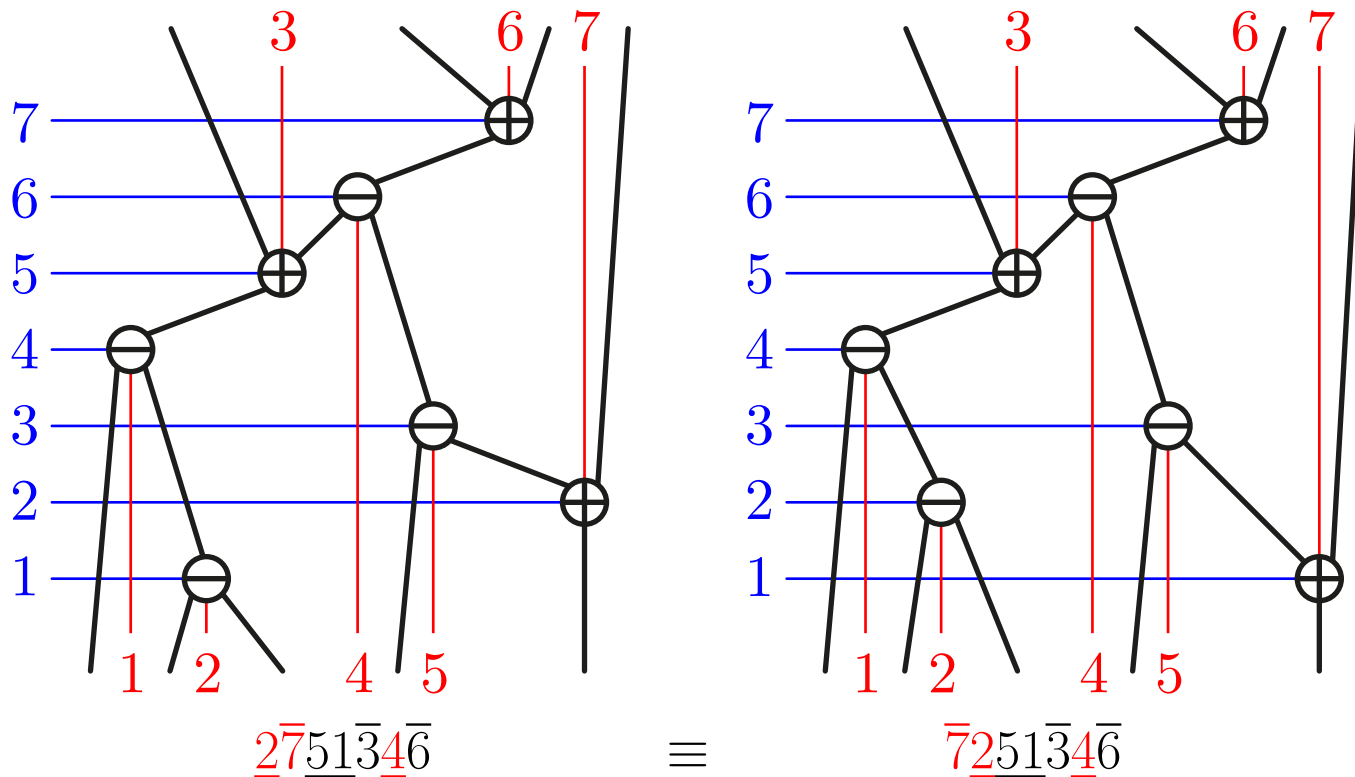
ε -Cambrian congruence = transitive closure of the rewriting rules

$$UacVbW \equiv_{\varepsilon} UcaVbW \quad \text{if } a < b < c \text{ and } \varepsilon_b = -$$

$$UbVacW \equiv_{\varepsilon} UbVcaW \quad \text{if } a < b < c \text{ and } \varepsilon_b = +$$

where a, b, c are elements of $[n]$ while U, V, W are words on $[n]$

PROP. $\tau \equiv_{\varepsilon} \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$



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PROP. Cambrian congruence class labeled by Cambrian tree T

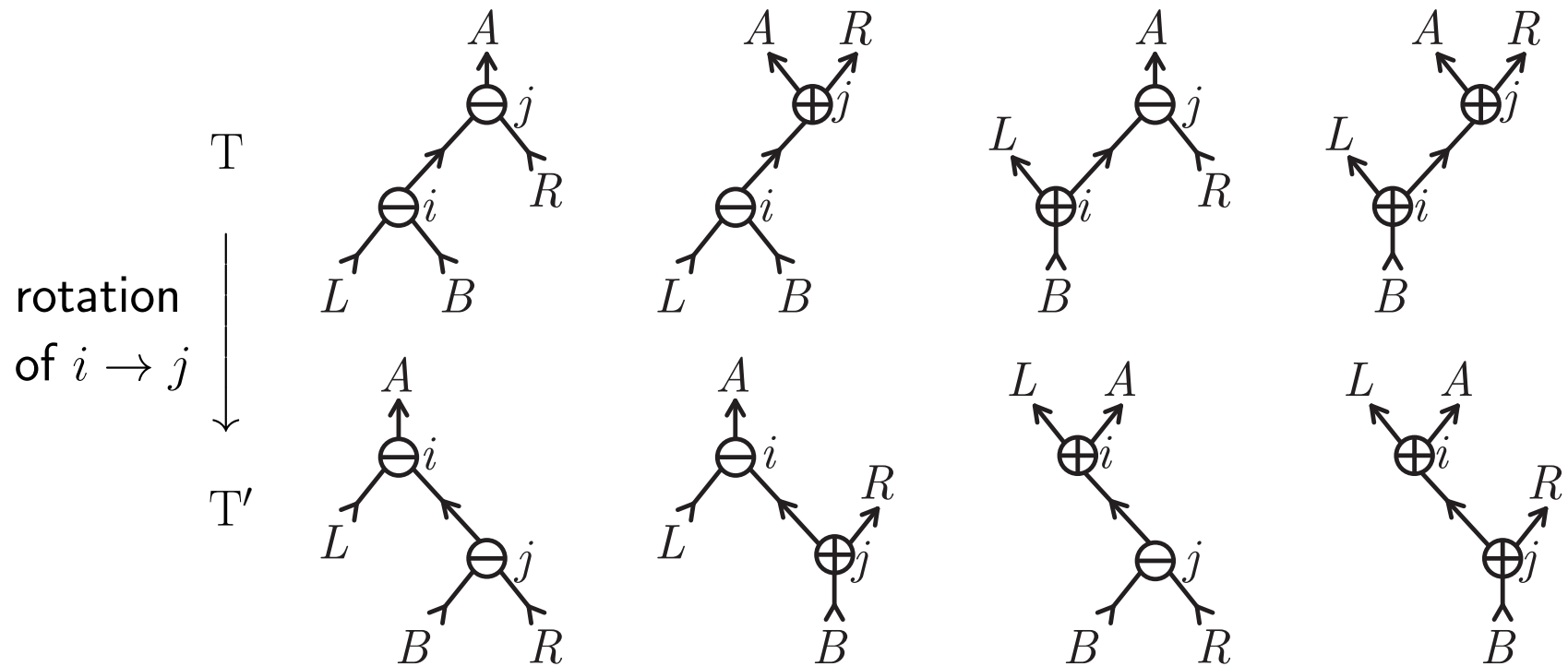
$$\{\tau \in \mathfrak{S}^{\varepsilon} \mid \mathbf{P}(\tau) = T\} = \{\text{linear extensions of } T\}$$

PROP. Cambrian classes are intervals of the weak order

minimums avoid $\bar{2}31$ and $31\underline{2}$ while maximums avoid $\bar{2}13$ and $13\underline{2}$

ROTATIONS AND CAMBRIAN LATTICES

Rotation operation preserves Cambrian trees:



increasing rotation = rotation of edge $i \rightarrow j$ where $i < j$

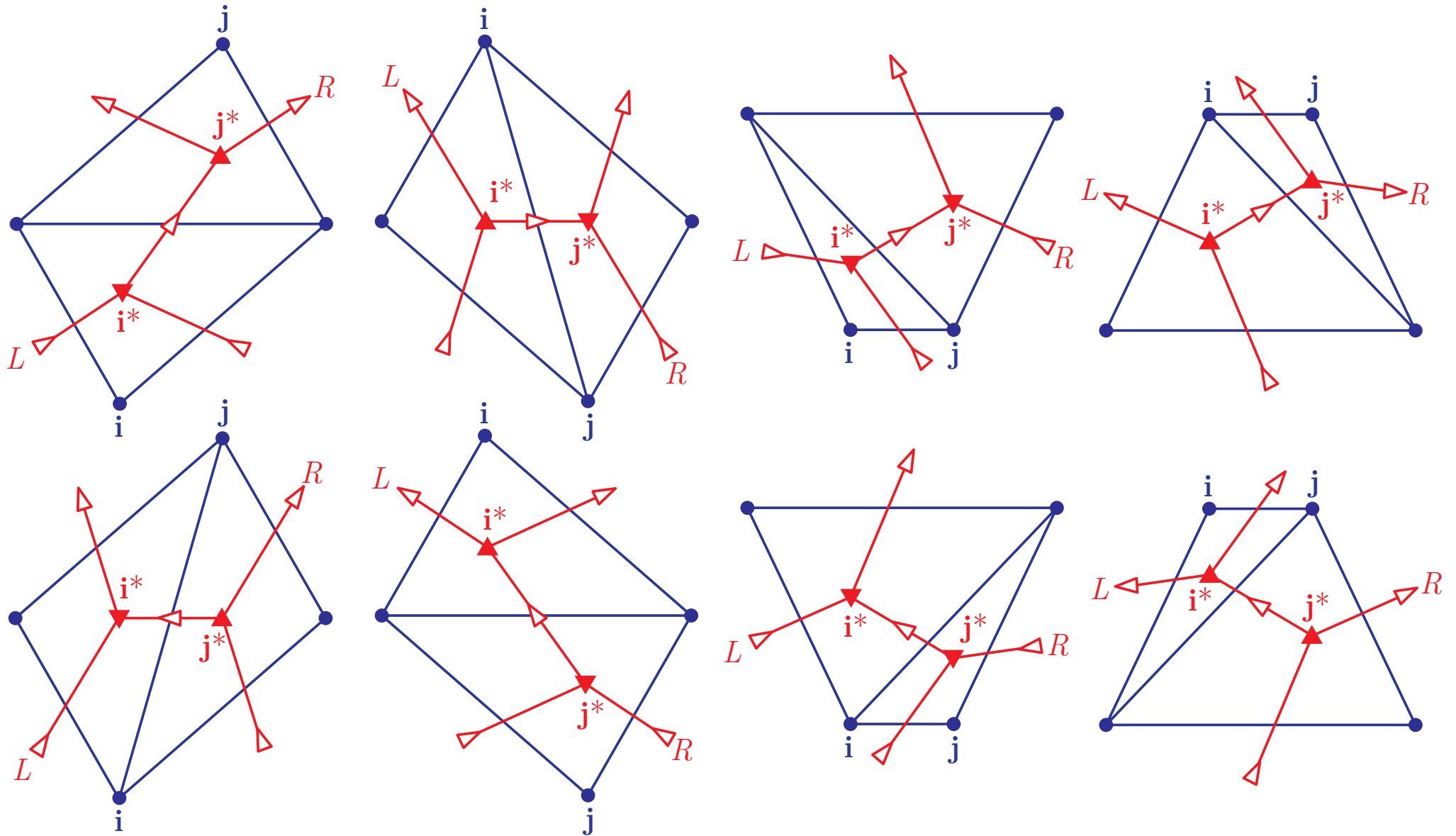
PROP. The transitive closure of the increasing rotation graph is the **Cambrian lattice**
 \mathbf{P} defines a lattice homomorphism from weak order to Cambrian lattice

Reading. Cambrian lattices. 2006

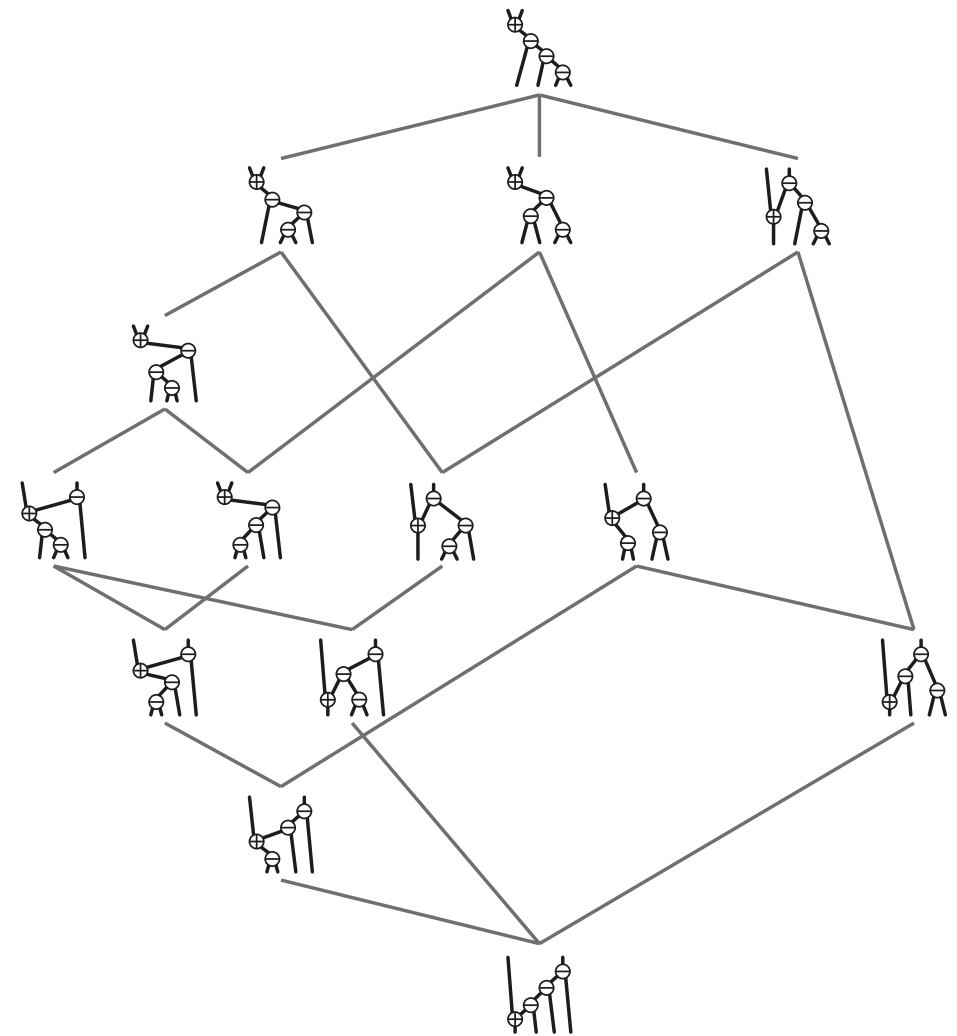
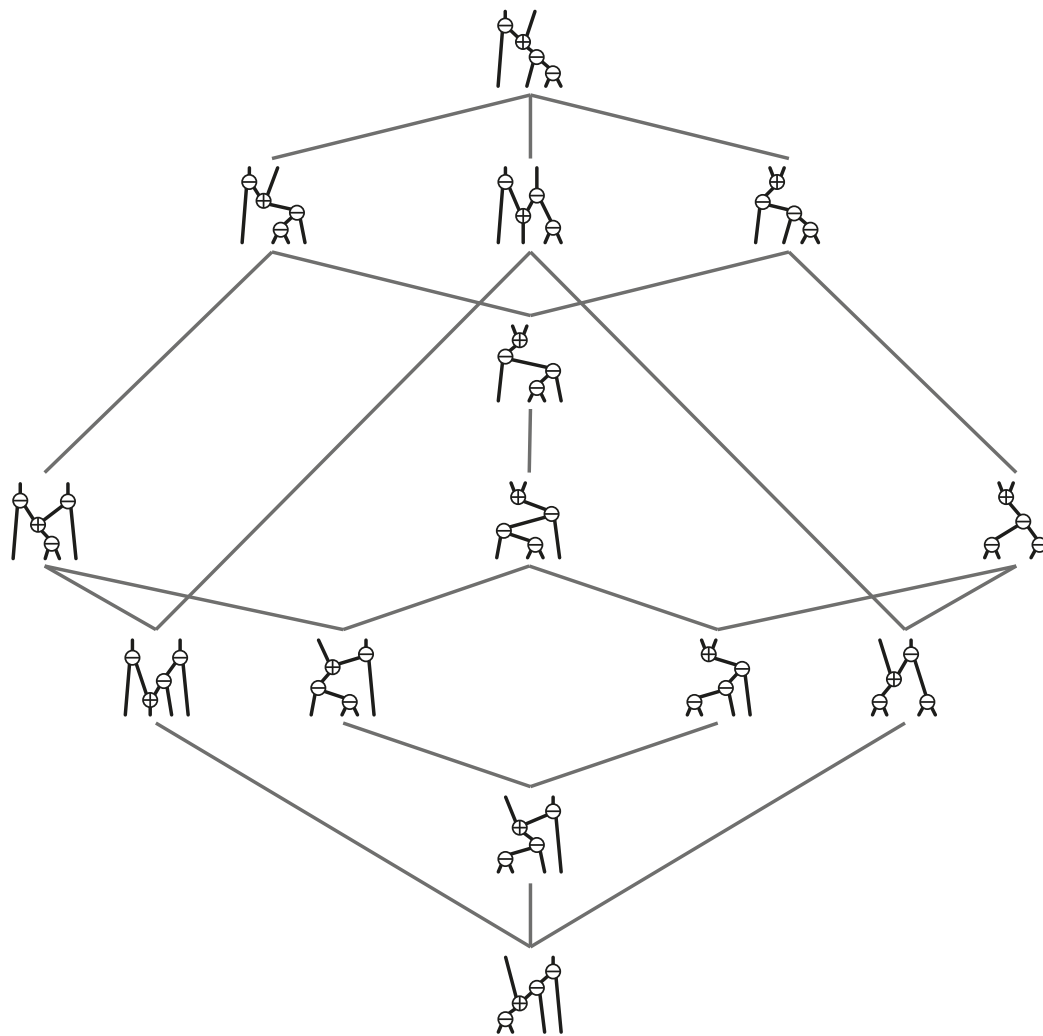
(rotation on Cambrian trees correspond to flips in triangulations)

ROTATIONS AND FLIPS

Rotation on Cambrian trees \longleftrightarrow flips on triangulations



ROTATIONS AND CAMBRIAN LATTICES



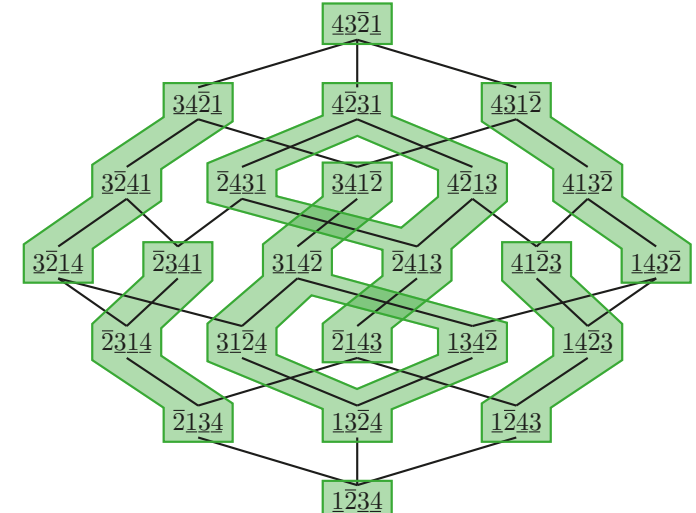
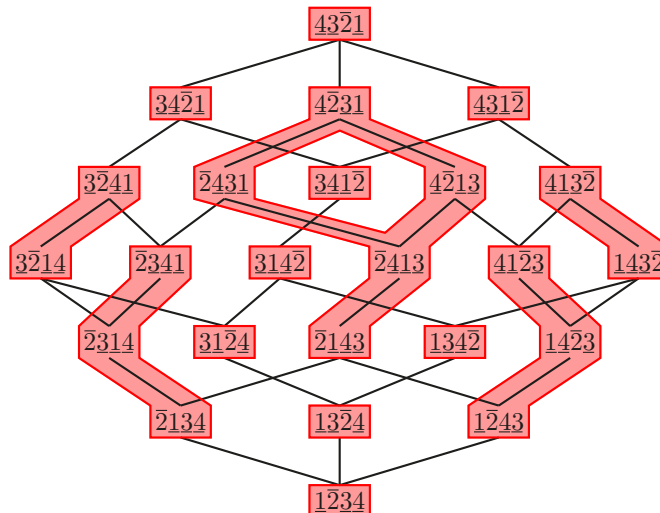
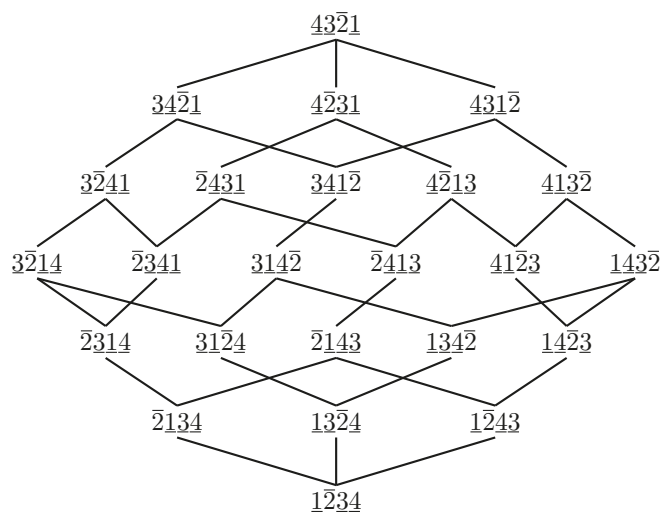
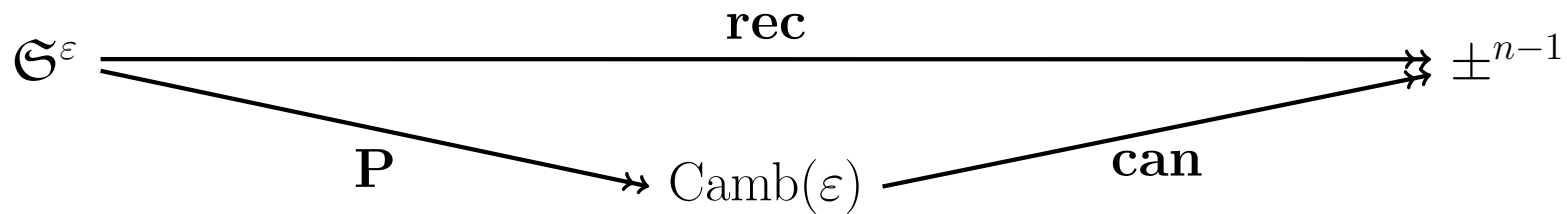
CANOPY

vertices i and $i + 1$ are always comparable in a Cambrian tree

Canopy of a Cambrian tree $T =$ sequence $\text{can}(T) \in \pm^{n-1}$ defined by

$$\text{can}(T)_i = \begin{cases} - & \text{if } i \text{ above } i + 1 \text{ in } T \\ + & \text{if } i \text{ below } i + 1 \text{ in } T \end{cases}$$

PROP. \mathbf{P} , can , and rec define lattice homomorphisms:



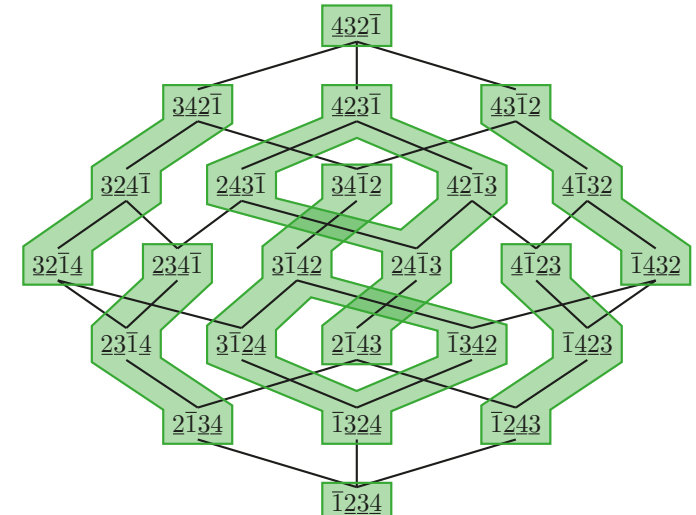
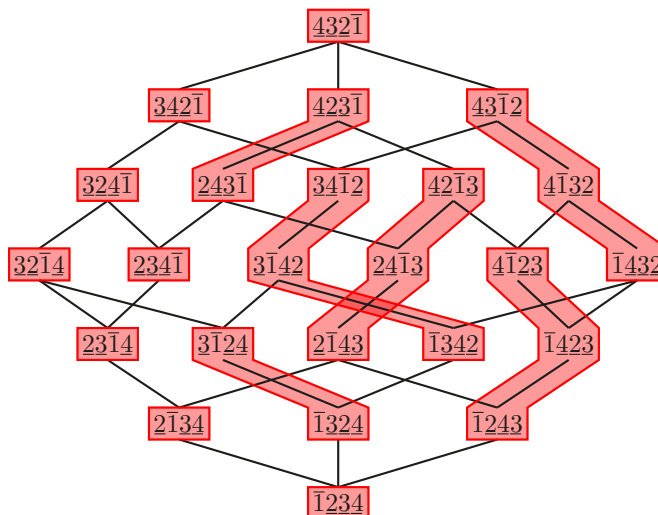
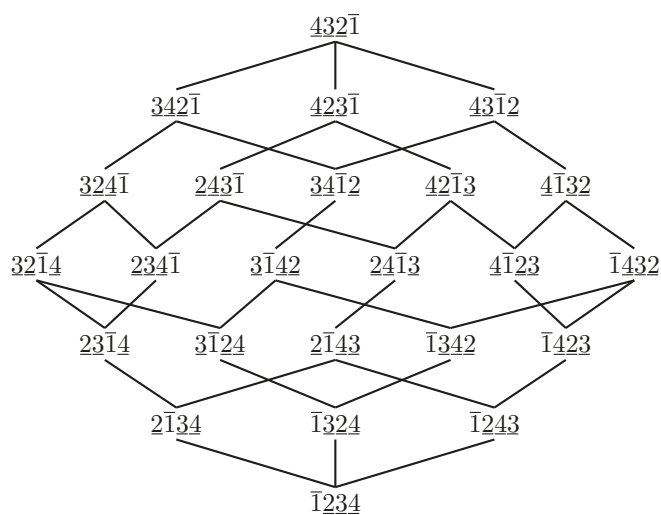
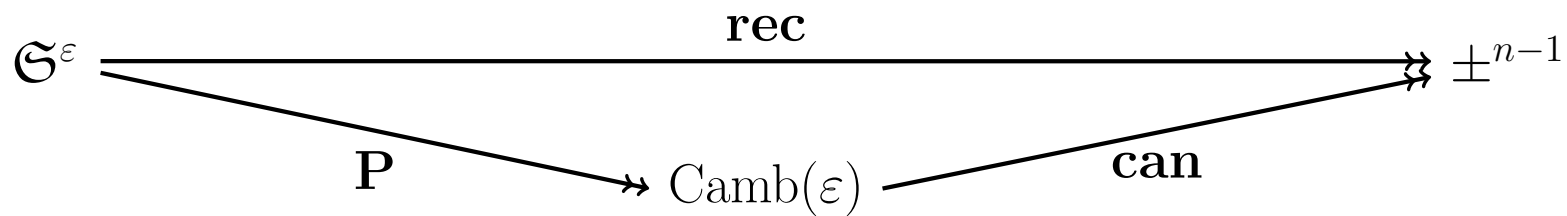
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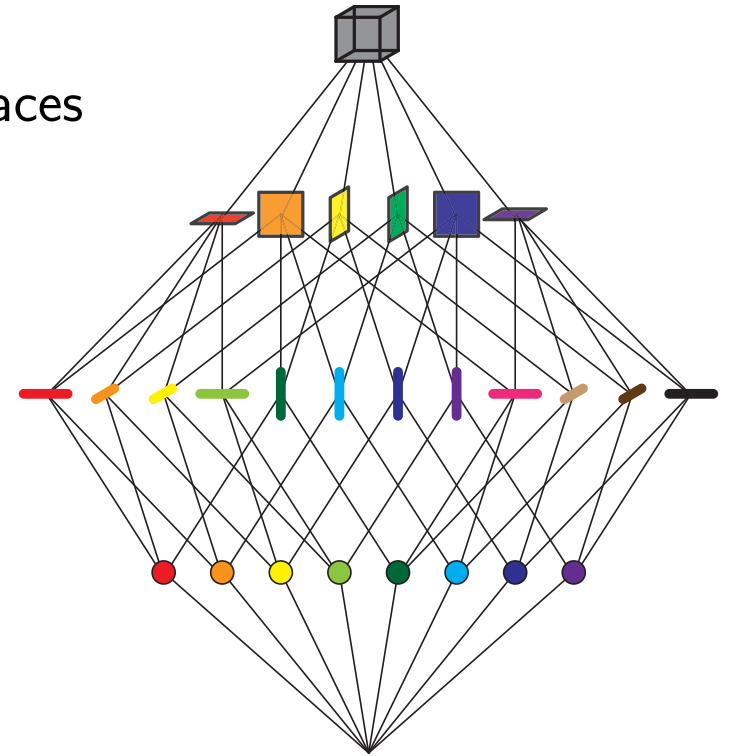
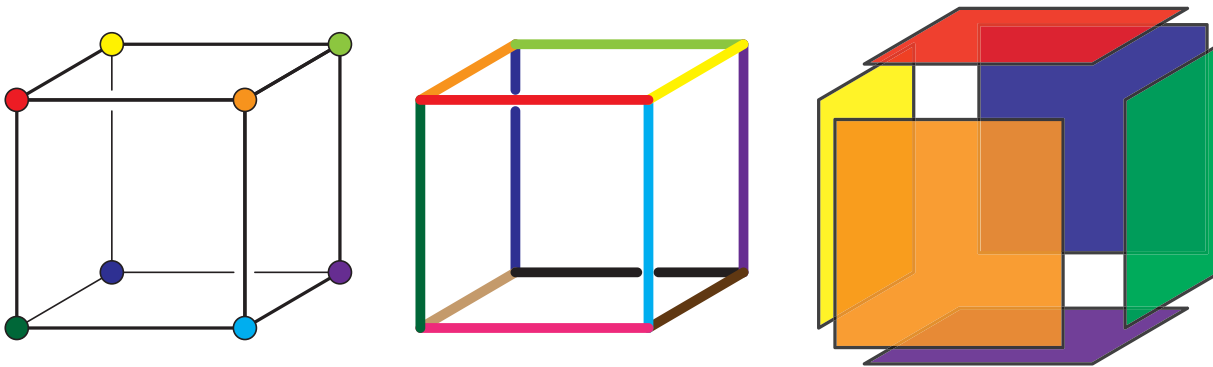
GEOMETRY

POLYTOPES & COMBINATORICS

polytope = convex hull of a finite set of \mathbb{R}^d
= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations



Given a set of points, determine the face lattice of its convex hull.

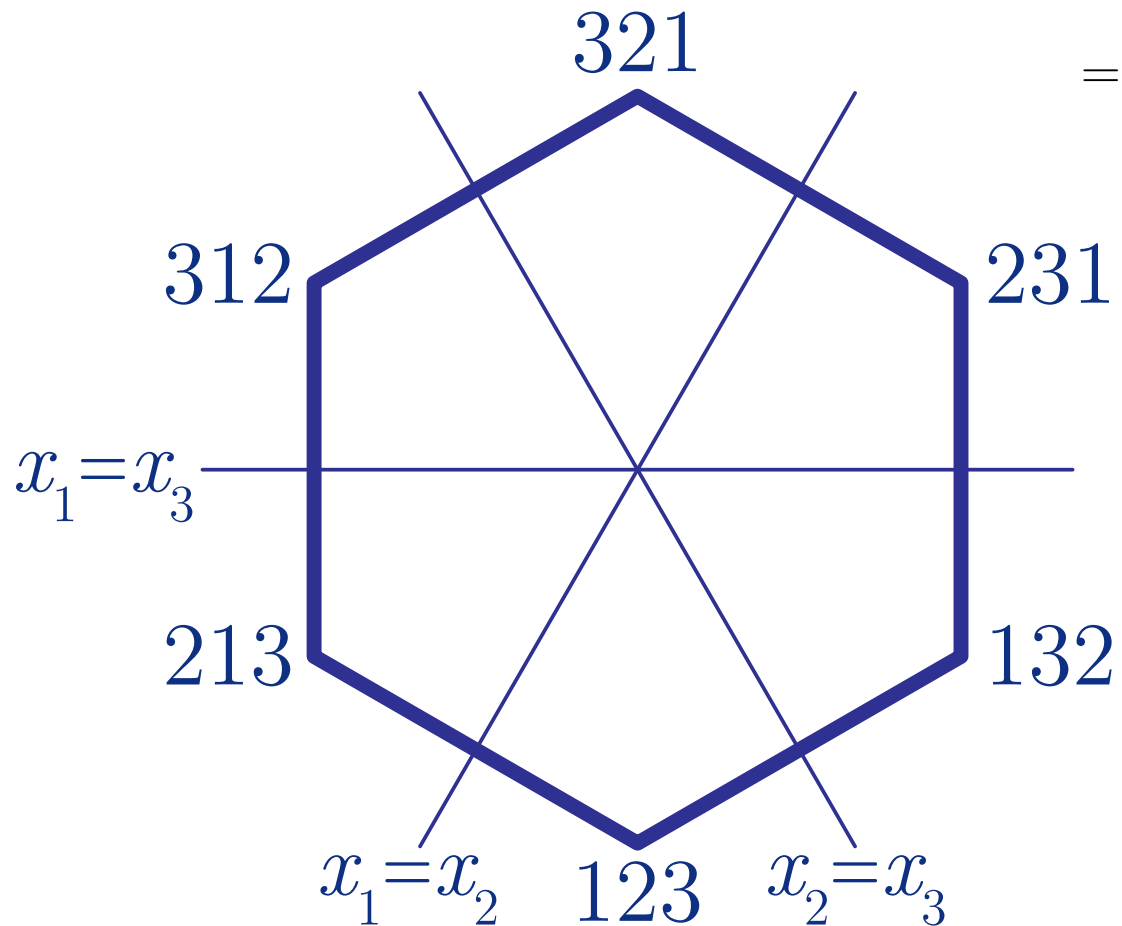
Given a lattice, is there a **polytope which realizes it?**

PERMUTAHEDRON

Permutohedron $\text{Perm}(n)$

$$= \text{conv} \{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\}$$

$$= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} \mathbf{H}^{\geq}(J)$$



PERMUTOHEDRON

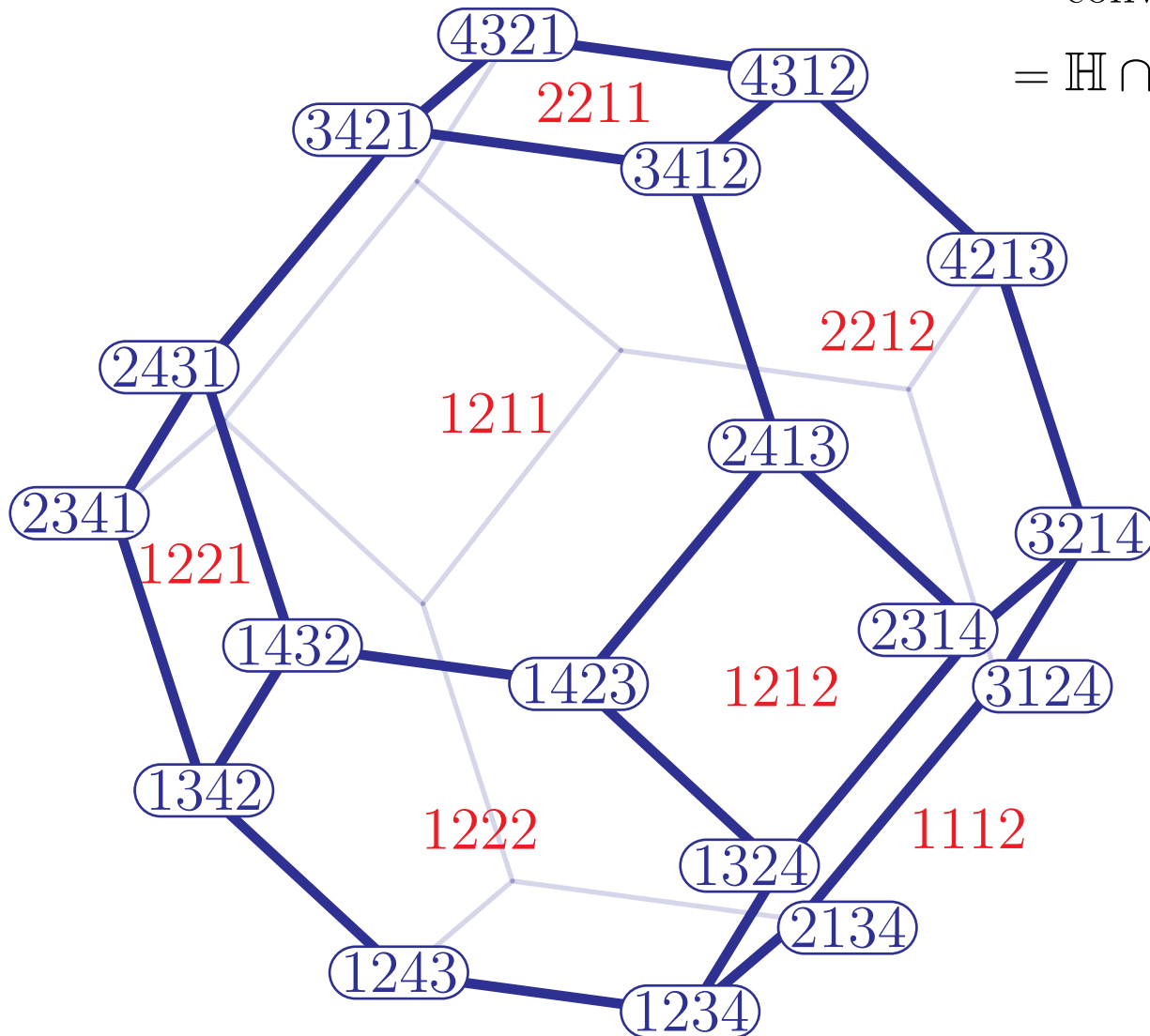
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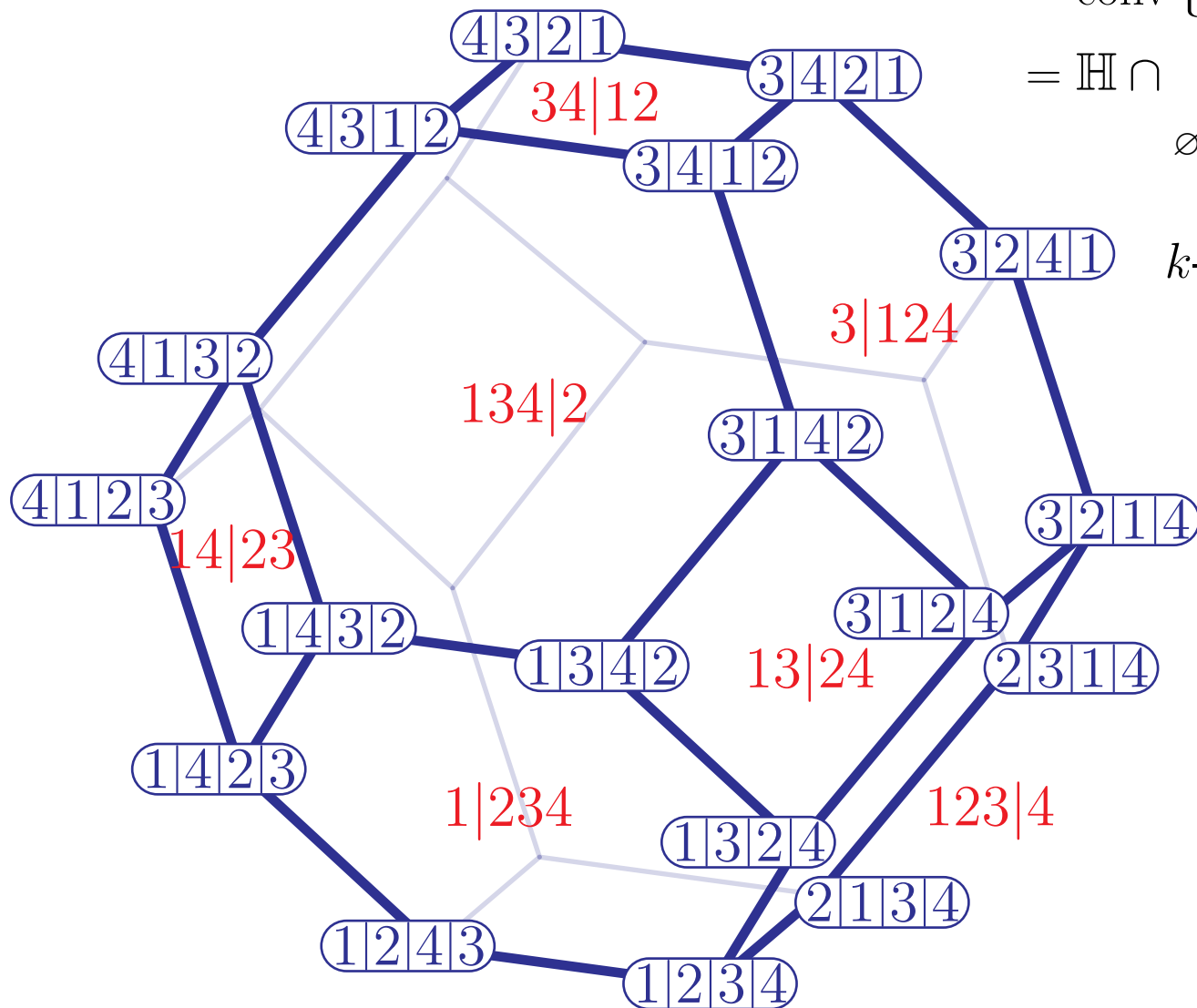
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k -faces of $\text{Perm}(n)$

\equiv surjections from $[n+1]$
to $[n+1-k]$



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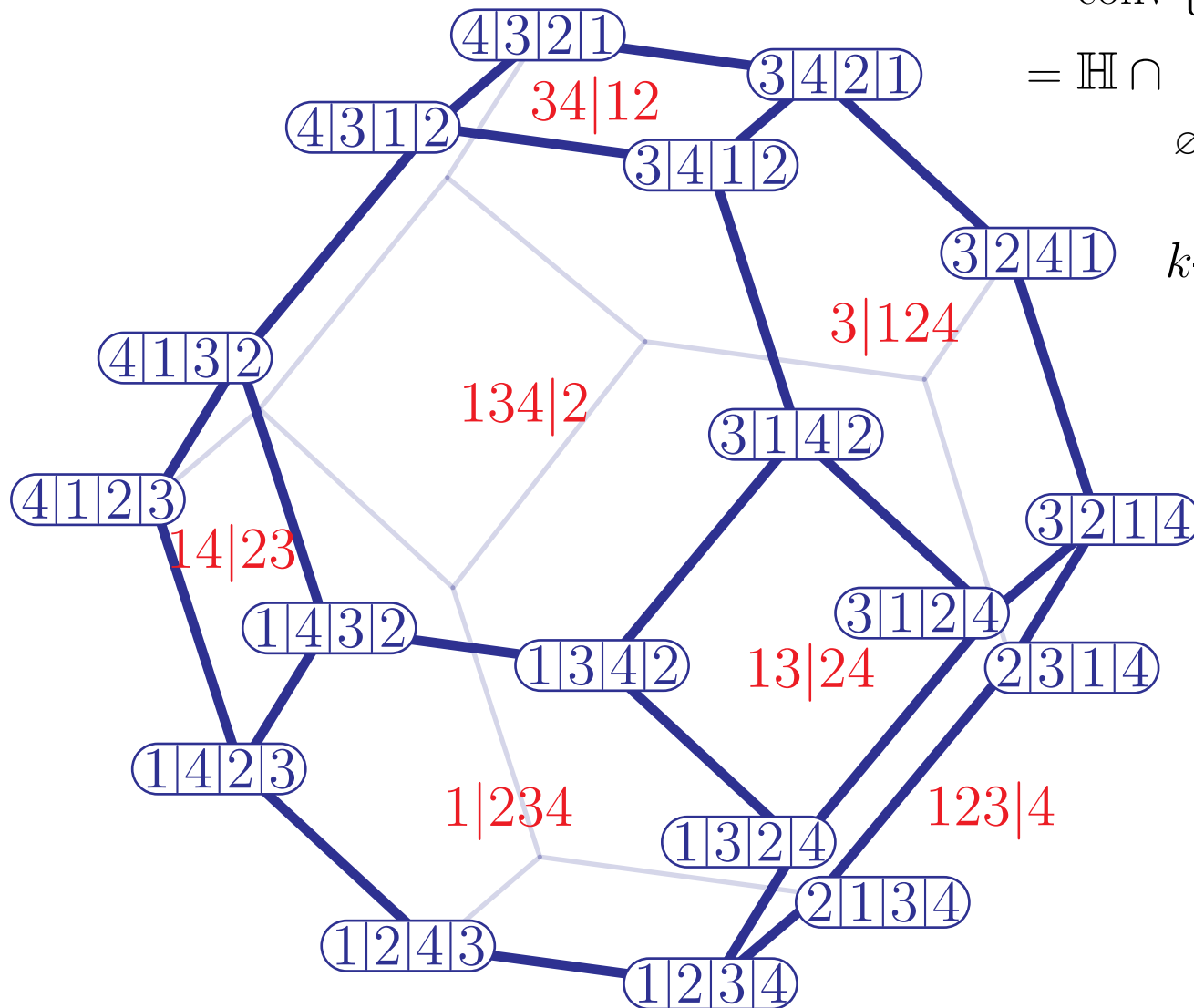
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\equiv ordered partitions of $[n+1]$
into $n+1-k$ parts

PERMUTOHEDRON



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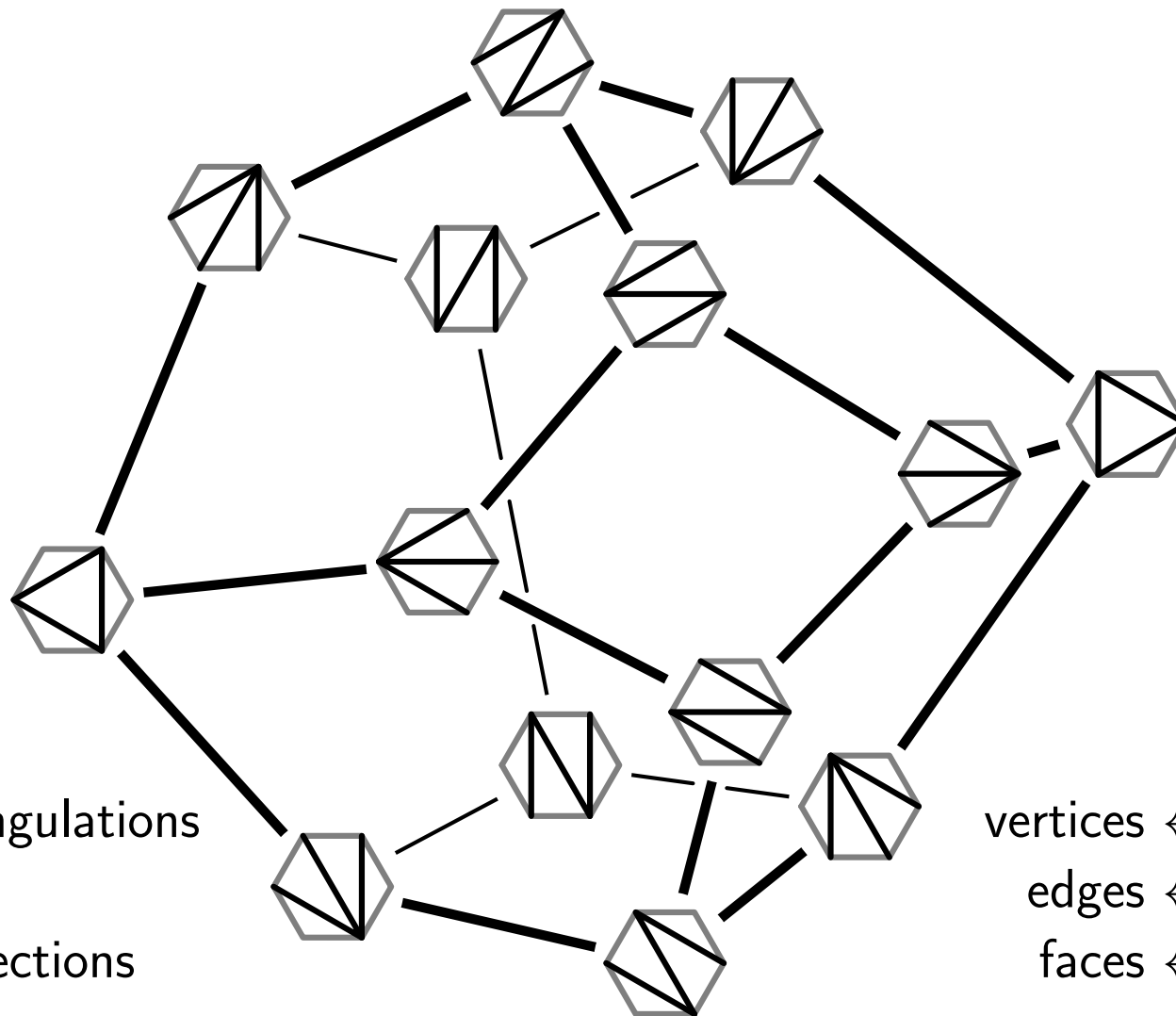
\equiv ordered partitions of $[n+1]$
into $n+1-k$ parts

connections to

- inversion sets
- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n + 3)$ -gon, ordered by reverse inclusion

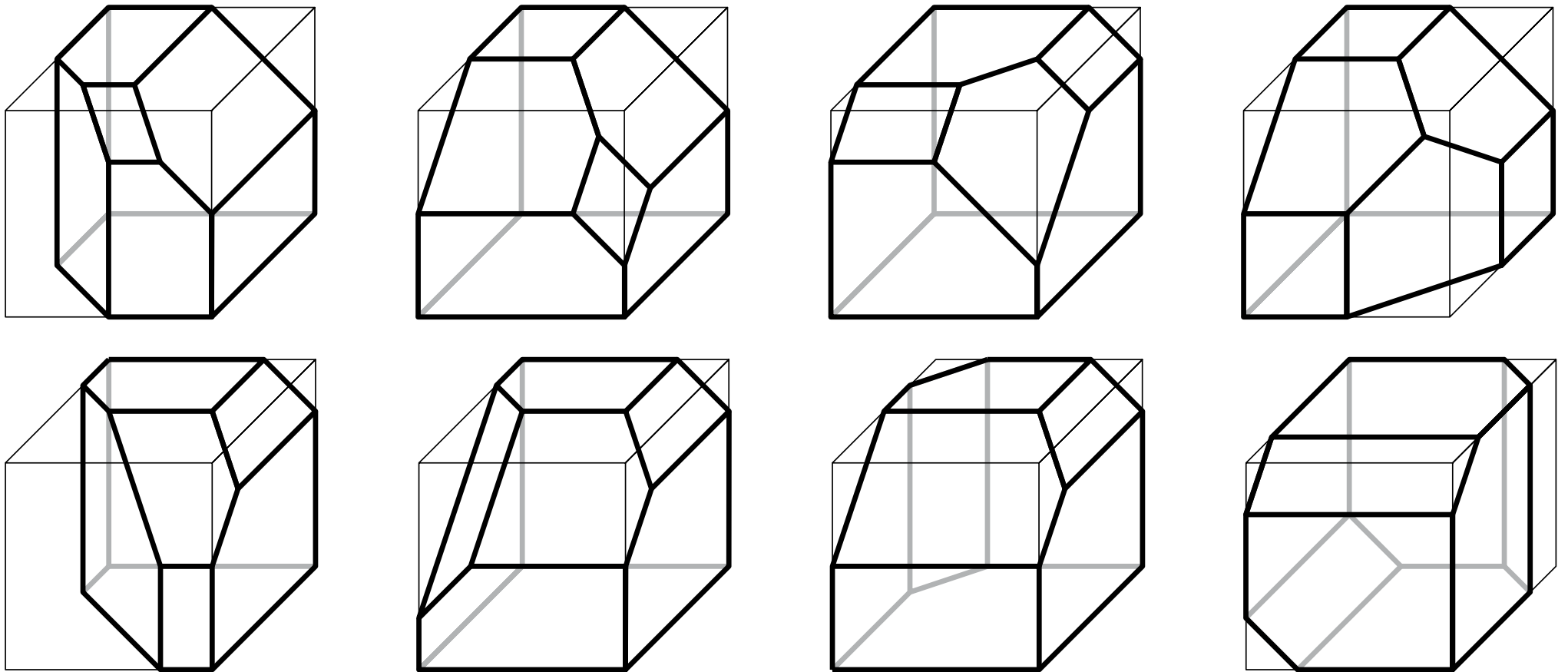


vertices \leftrightarrow triangulations
edges \leftrightarrow flips
faces \leftrightarrow dissections

vertices \leftrightarrow binary trees
edges \leftrightarrow rotations
faces \leftrightarrow Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n + 3)$ -gon, ordered by reverse inclusion



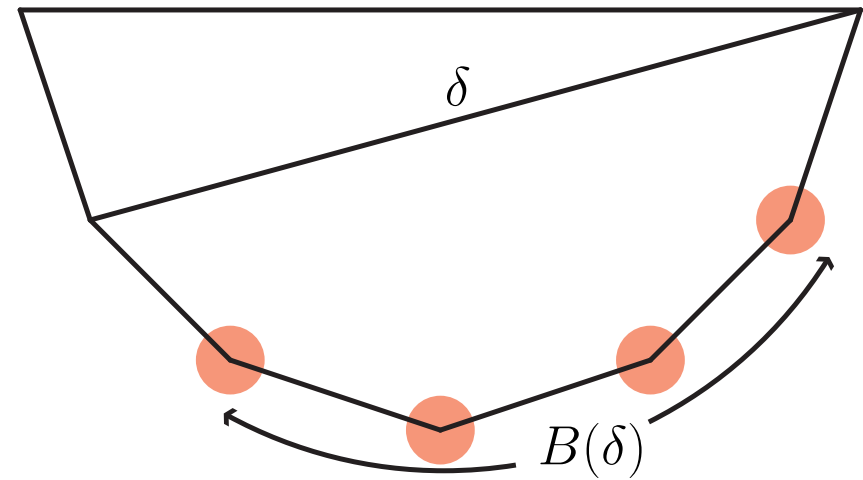
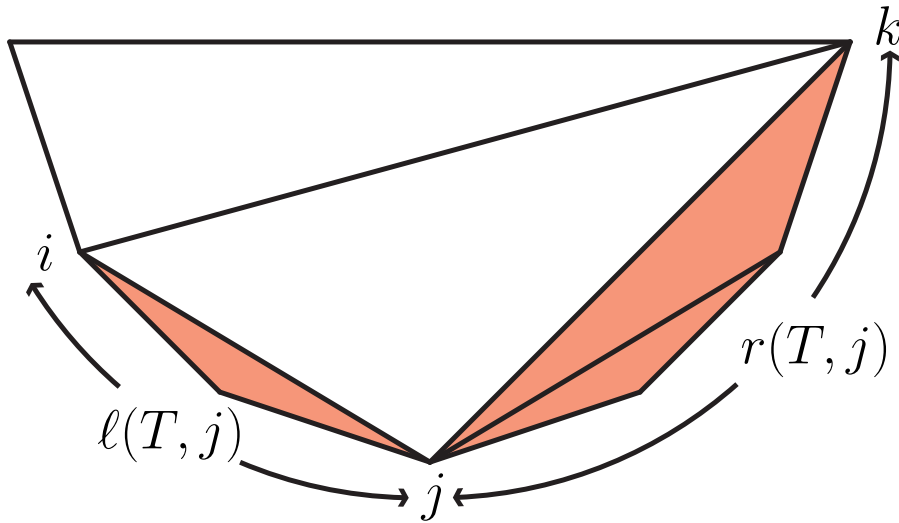
(Pictures by Ceballos-Santos-Ziegler)

Lee ('89), Gel'fand-Kapranov-Zelevinski ('94), Billera-Filliman-Sturmfels ('90), ..., Ceballos-Santos-Ziegler ('11)
Loday ('04), Hohlweg-Lange ('07), Hohlweg-Lange-Thomas ('12), P.-Santos ('12), P.-Stump ('12+), Lange-P. ('13+)

LODAY'S ASSOCIAHEDRON

Loday's associahedron = $\text{conv} \{L(T) \mid T \text{ triangulation of the } (n+3)\text{-gon}\}$

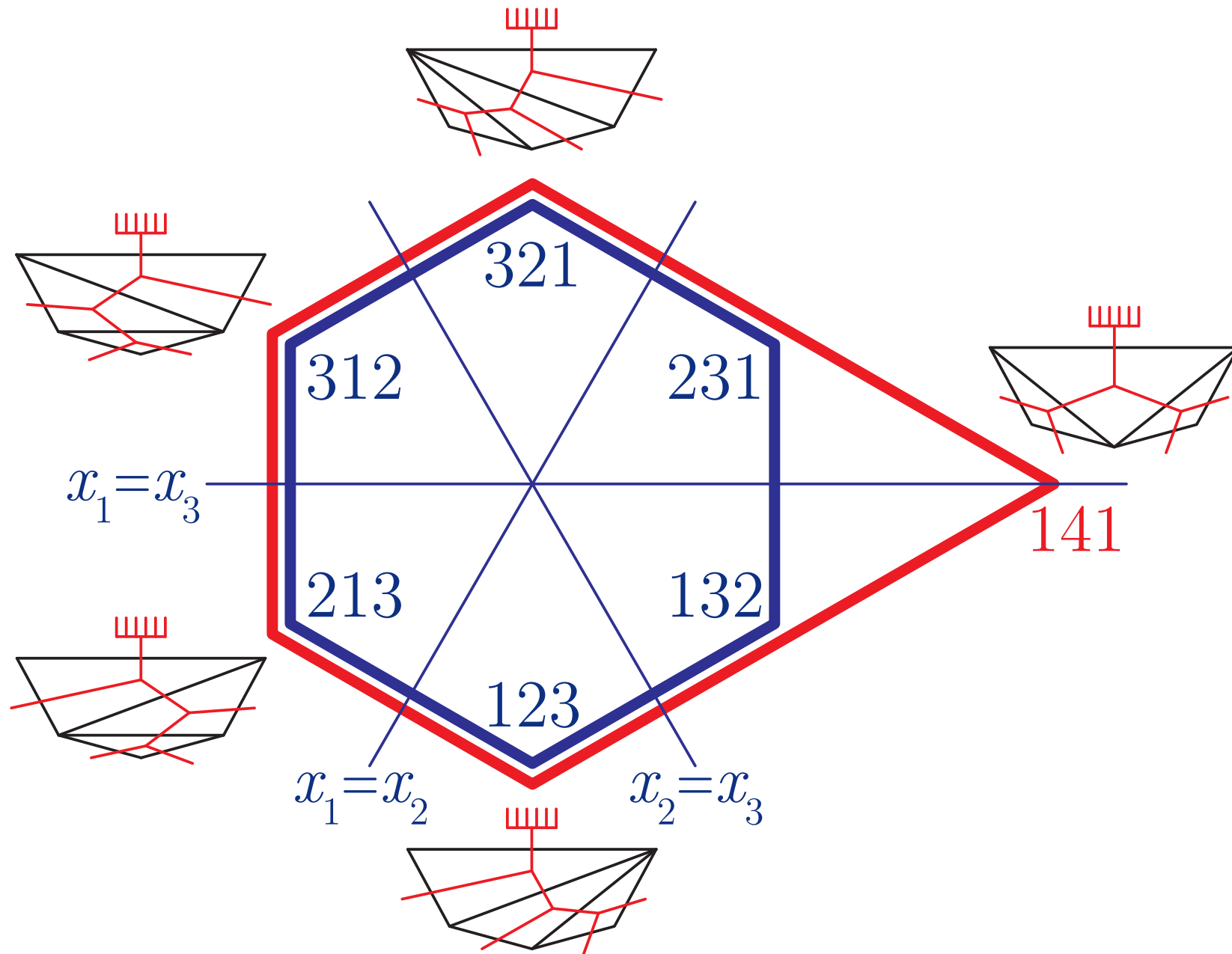
$$= \mathbb{H} \cap \bigcap_{\substack{\delta \text{ diagonal} \\ \text{of the } (n+3)\text{-gon}}} \mathbf{H}^{\geq}(\delta)$$



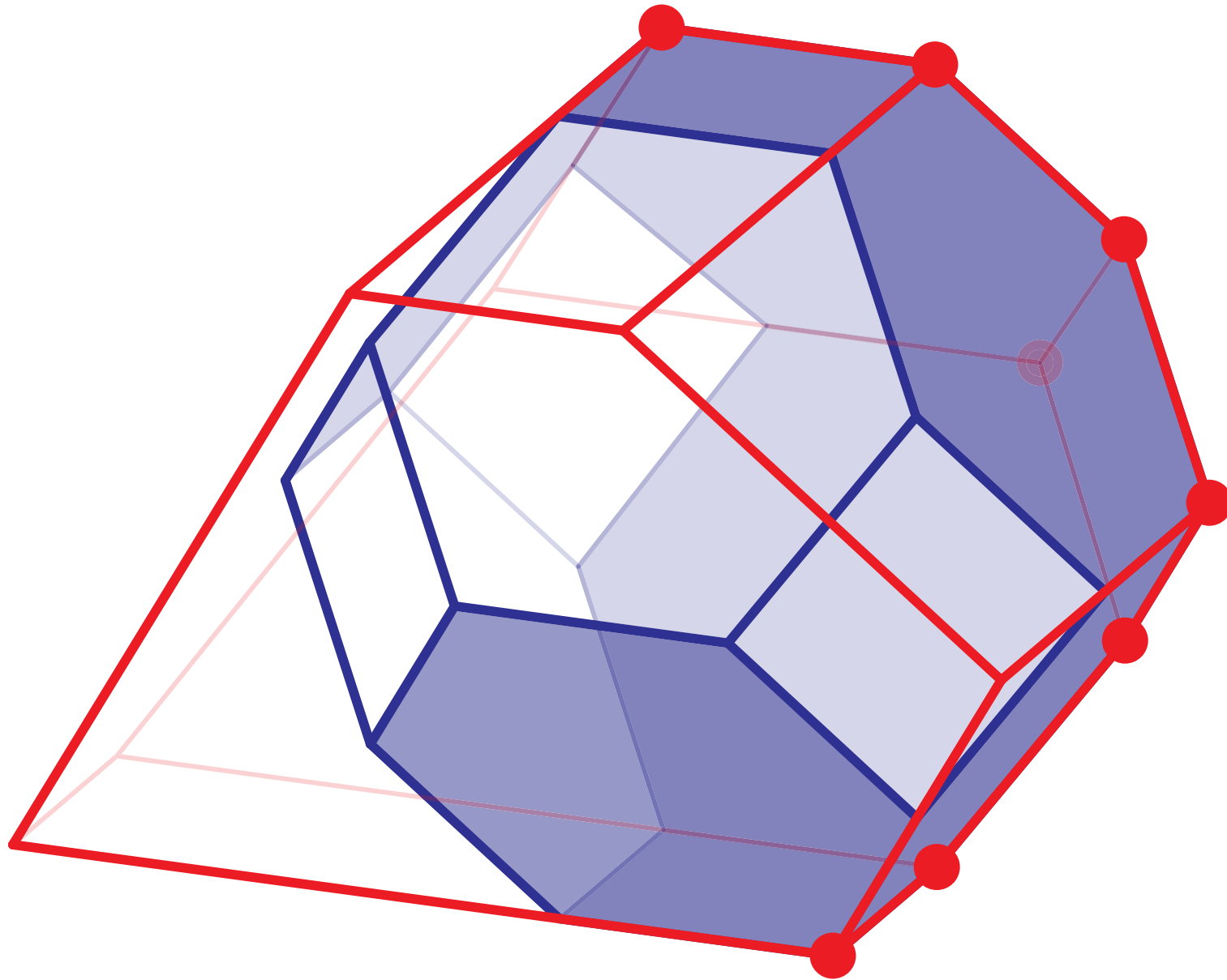
$$L(T) = (\ell(T, j) \cdot r(T, j))_{j \in [n+1]}$$

$$\mathbf{H}^{\geq}(\delta) = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid \sum_{j \in B(\delta)} x_j \geq \binom{|B(\delta)| + 1}{2} \right\}$$

LODAY'S ASSOCIAHEDRON AND PERMUTAHEDRON



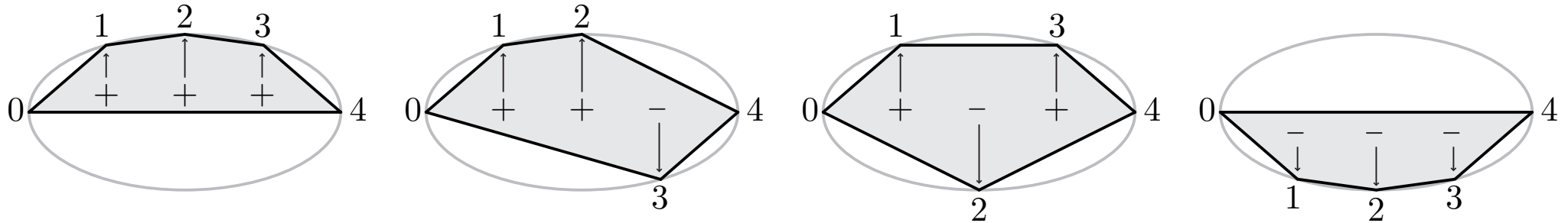
ASSOCIAHEDRON AND PERMUTAHEDRON



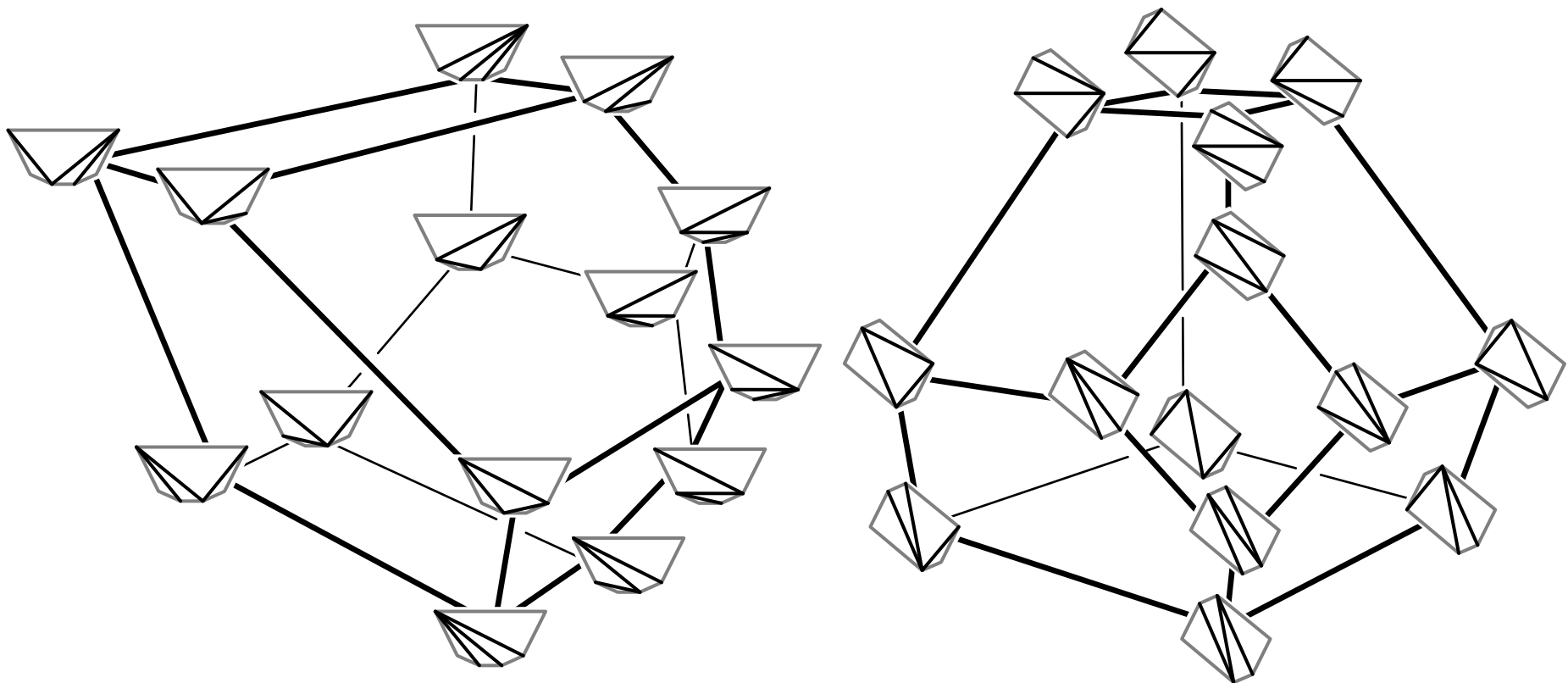
The associahedron is obtained from the permutahedron by removing facets

HOHLWEG & LANGE'S ASSOCIAHEDRA

Can also replace Loday's $(n + 3)$ -gon by others...

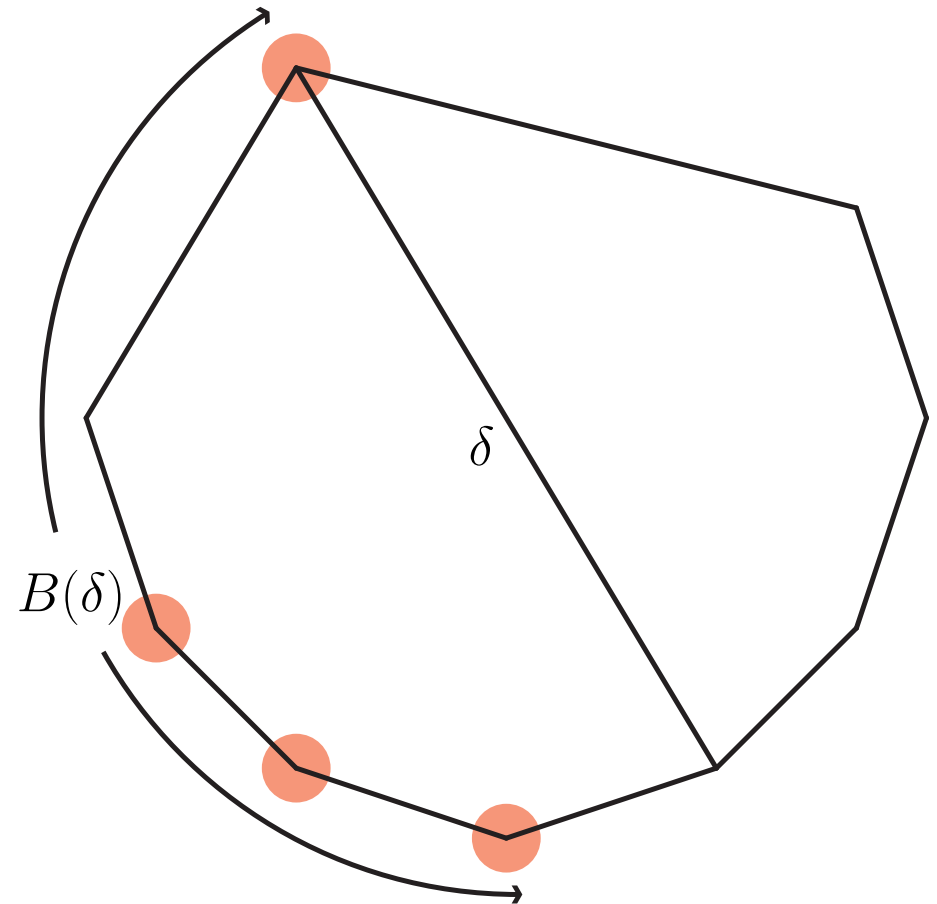
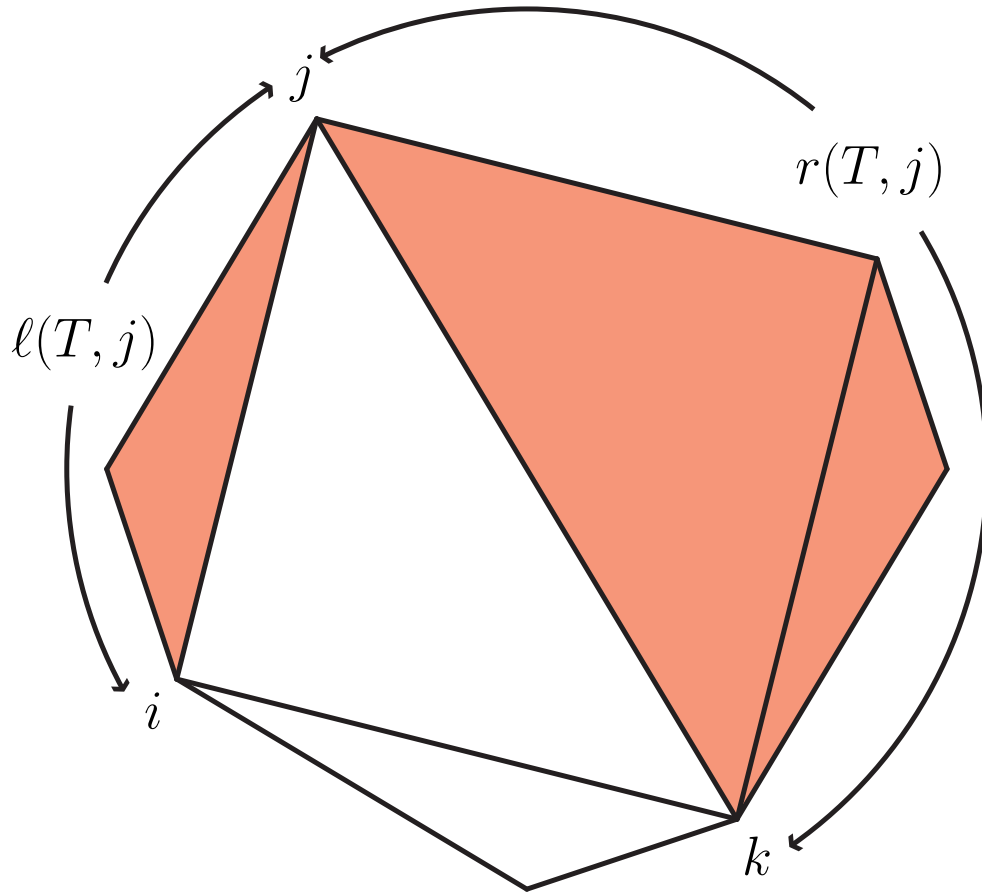


... to obtain different realizations of the associahedron



HOHLWEG & LANGE'S ASSOCIAHEDRA

$$\text{Asso}(P) = \text{conv} \{HL(T) \mid T \text{ triangulation of } P\} = \mathbb{H} \cap \bigcap_{\delta \text{ diagonal of } P} \mathbf{H}^{\geq}(\delta)$$

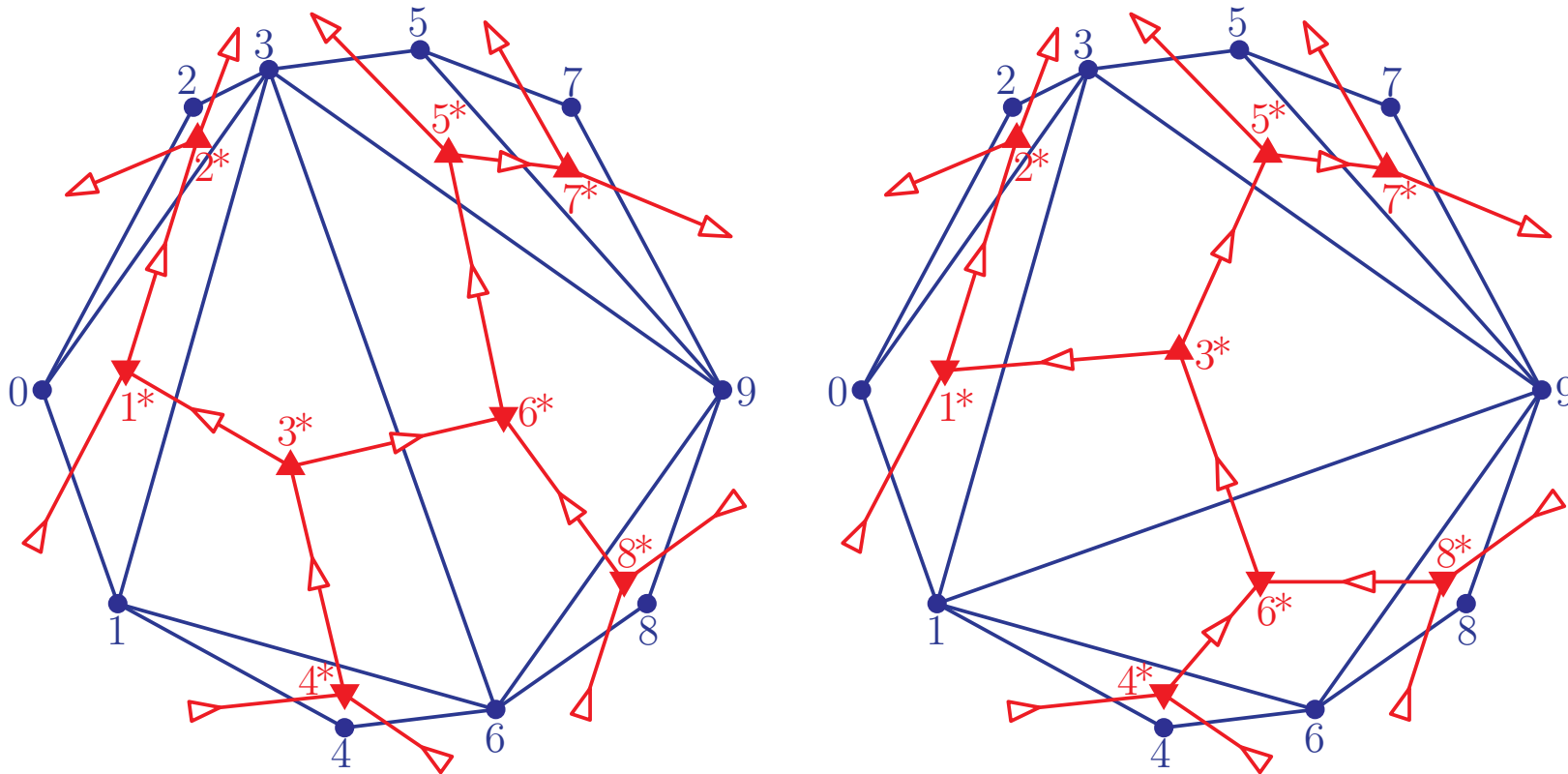


$$HL(T)_j = \begin{cases} l(T, j) \cdot r(T, j) & \text{if } j \text{ down} \\ n + 2 - l(T, j) \cdot r(T, j) & \text{if } j \text{ up} \end{cases}$$

$$\mathbf{H}^{\geq}(\delta) = \left\{ \mathbf{x} \mid \sum_{j \in B(\delta)} x_j \geq \binom{|B(\delta)| + 1}{2} \right\}$$

ASSOCIAHEDRA AND CAMBRIAN TREES

Lange-P., *Using spines to revisit a construction of the associahedron* ('13⁺)



Cambrian trees = labeled and oriented dual binary trees

Alternative vertex description of Hohlweg-Lange's associahedra:

$$HL(T)_j = \begin{cases} |\{\pi \text{ maximal path in } T \text{ with 2 incoming arcs at } j\}| & \text{if } j \text{ down} \\ n + 2 - |\{\pi \text{ maximal path in } T \text{ with 2 outgoing arcs at } j\}| & \text{if } j \text{ up} \end{cases}$$

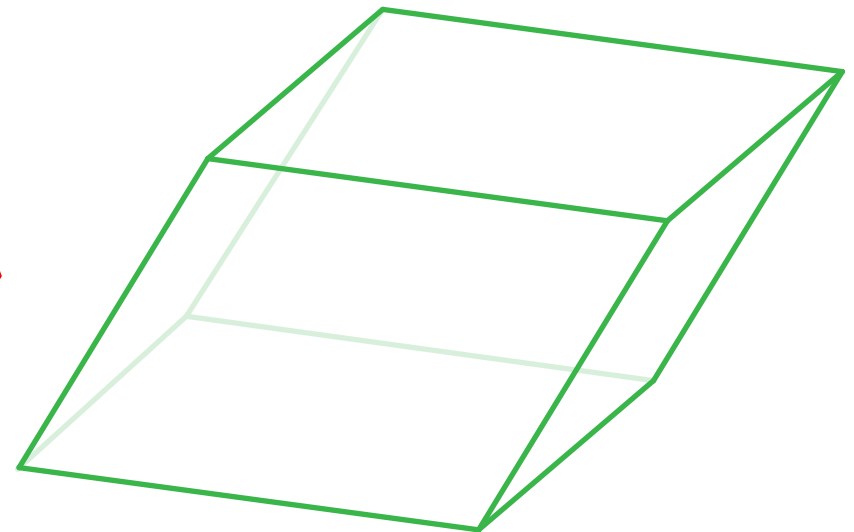
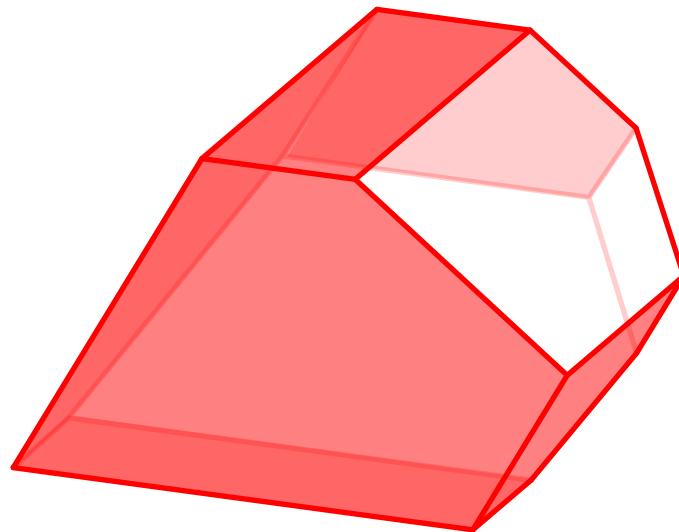
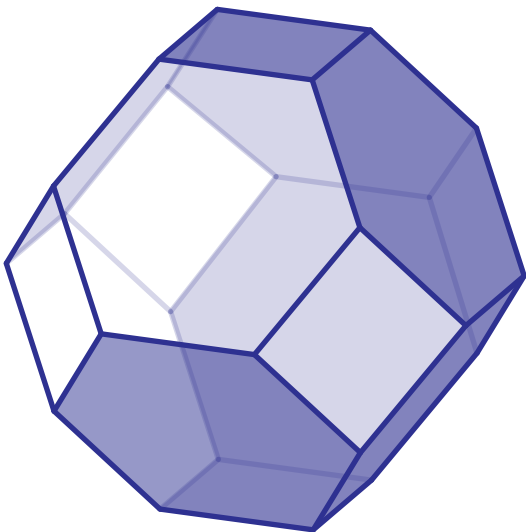
CAMBRIAN TREES AND NORMAL CONES

Incidence cone $C(T) = \text{cone} \{e_i - e_j \mid \text{for all } i \rightarrow j \text{ in } T\}$

Braid cone $C^\diamond(T) = \{\mathbf{x} \in \mathbb{H} \mid x_i \leq x_j \text{ for all } i \rightarrow j \text{ in } T\}$

THEO. The cones form complete simplicial fans:

- (i) $\{C^\diamond(\tau) \mid \tau \in \mathfrak{S}_n\} = \text{braid fan} = \text{normal fan of the permutahedron}$
- (ii) $\{C^\diamond(T) \mid T \in \text{Camb}(\varepsilon)\} = \varepsilon\text{-Cambrian fan} = \text{normal fan of the } \varepsilon\text{-associahedron}$
- (iii) $\{C^\diamond(\chi) \mid \chi \in \pm^{n-1}\} = \text{boolean fan} = \text{normal fan of the parallelepiped}$



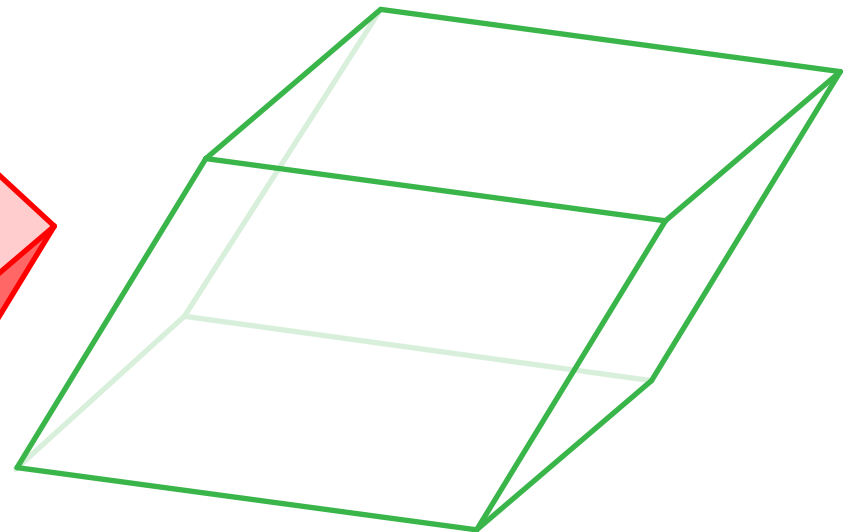
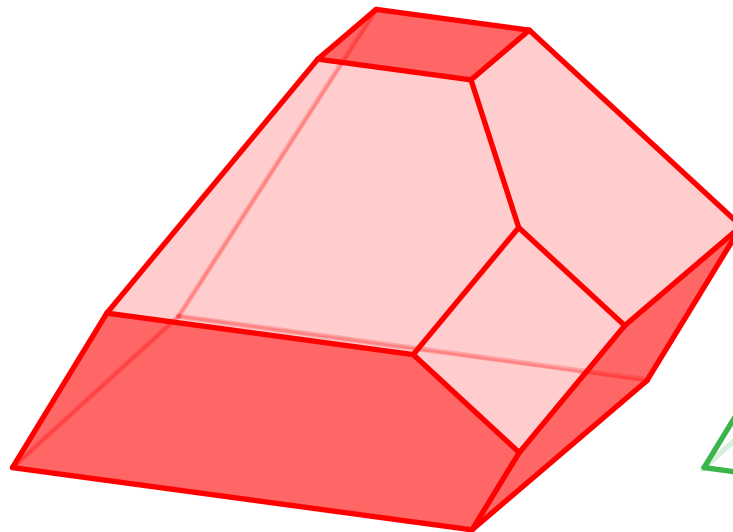
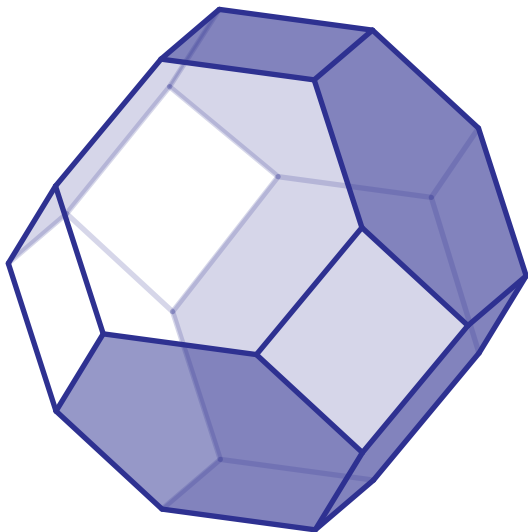
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CAMBRIAN TREES AND NORMAL CONES

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Characterization of fibers in terms of cones:

$$\begin{aligned} T = \mathbf{P}(\tau) &\iff C(T) \subseteq C(\tau) \iff C^\diamond(T) \supseteq C^\diamond(\tau), \\ \chi = \mathbf{can}(T) &\iff C(\chi) \subseteq C(T) \iff C^\diamond(\chi) \supseteq C^\diamond(T), \\ \chi = \mathbf{rec}(\tau) &\iff C(\chi) \subseteq C(\tau) \iff C^\diamond(\chi) \supseteq C^\diamond(\tau). \end{aligned}$$

ALGEBRA

SHUFFLE AND CONVOLUTION

For $n, n' \in \mathbb{N}$, consider the set of perms of $\mathfrak{S}_{n+n'}$ with at most one descent, at position n :

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau(1) < \dots < \tau(n) \text{ and } \tau(n+1) < \dots < \tau(n+n')\}$$

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

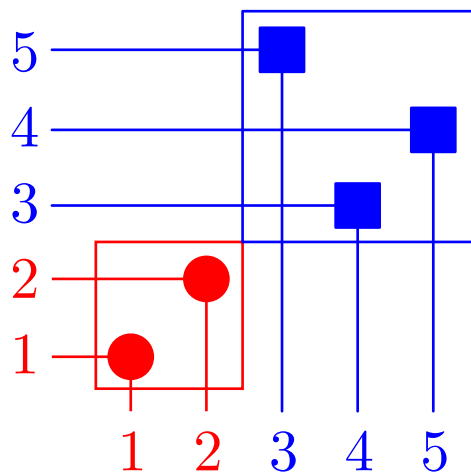
shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1) + n, \dots, \tau'(n') + n] \in \mathfrak{S}_{n+n'}$

shifted shuffle product $\tau\bar{\sqcup}\tau' = \{(\tau\bar{\tau}') \circ \pi^{-1} \mid \pi \in \mathfrak{S}^{(n,n')}\} \subset \mathfrak{S}_{n+n'}$

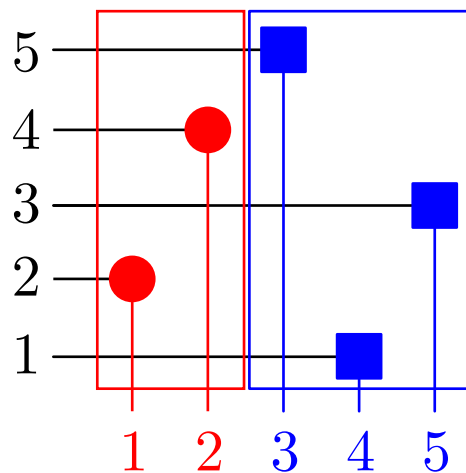
convolution product $\tau\star\tau' = \{\pi \circ (\tau\bar{\tau}') \mid \pi \in \mathfrak{S}^{(n,n')}\} \subset \mathfrak{S}_{n+n'}$

Exm: $12\bar{\sqcup}231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

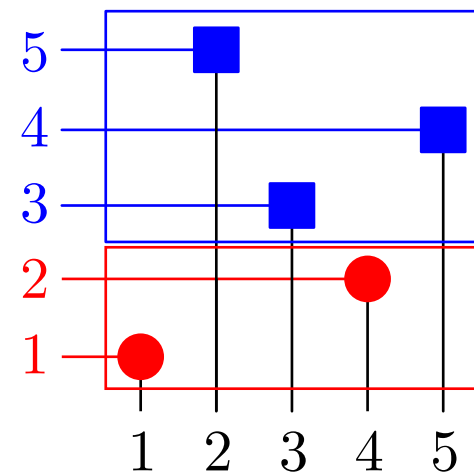
$12\star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



concatenation



shuffle



convolution

MALVENUTO-REUTENAUER ALGEBRA

DEF. Combinatorial Hopf Algebra = combinatorial vector space \mathcal{B} endowed with

$$\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$$

$$\text{coproduct } \Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$$

which are “compatible”, ie.

$$\begin{array}{ccccc}
 \mathcal{B} \otimes \mathcal{B} & \xrightarrow{\cdot} & \mathcal{B} & \xrightarrow{\Delta} & \mathcal{B} \otimes \mathcal{B} \\
 \Delta \otimes \Delta \downarrow & & & & \uparrow \cdot \otimes \cdot \\
 \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & \xrightarrow{I \otimes \text{swap} \otimes I} & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & &
 \end{array}$$

Malvenuto-Reteunauer algebra = Hopf algebra FQSym with basis $(\mathbb{F}_\tau)_{\tau \in \mathcal{G}}$ and where

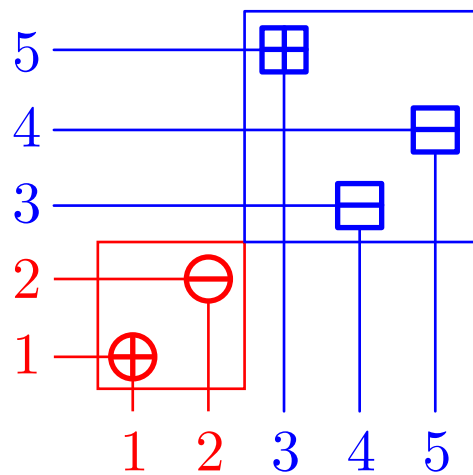
$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} \mathbb{F}_\sigma \quad \text{and} \quad \Delta \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

SIGNED VERSION

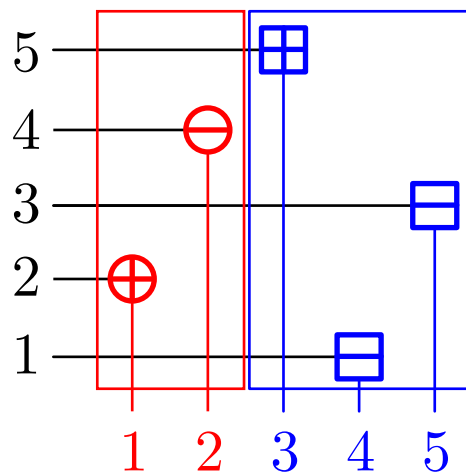
For signed permutations:

- signs are attached to values in the shuffle product
- signs are attached to positions in the convolution product

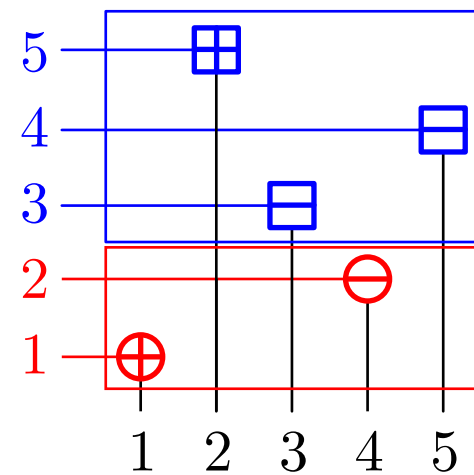
Exm: $\bar{1}\underline{2} \sqcup \underline{2}\bar{3}\bar{1} = \{\bar{1}\underline{2}\underline{4}\bar{5}\bar{3}, \bar{1}\underline{4}\underline{2}\bar{5}\bar{3}, \bar{1}\underline{4}\bar{5}\underline{2}\bar{3}, \bar{1}\underline{4}\bar{5}\bar{3}\underline{2}, \underline{4}\bar{1}\underline{2}\bar{5}\bar{3}, \underline{4}\bar{1}\bar{5}\underline{2}\bar{3}, \underline{4}\bar{1}\bar{5}\bar{3}\underline{2}, \underline{4}\bar{5}\bar{1}\underline{2}\bar{3}, \underline{4}\bar{5}\bar{1}\bar{3}\underline{2}, \underline{4}\bar{5}\bar{3}\bar{1}\underline{2}\}$,
 $\bar{1}\underline{2} \star \underline{2}\bar{3}\bar{1} = \{\bar{1}\underline{2}\underline{4}\bar{5}\bar{3}, \bar{1}\underline{3}\underline{4}\bar{5}\bar{2}, \bar{1}\underline{4}\underline{3}\bar{5}\bar{2}, \bar{1}\underline{5}\underline{3}\bar{4}\bar{2}, \underline{2}\bar{3}\underline{4}\bar{5}\bar{1}, \underline{2}\bar{4}\underline{3}\bar{5}\bar{1}, \underline{2}\bar{5}\underline{3}\bar{4}\bar{1}, \underline{3}\bar{4}\underline{2}\bar{5}\bar{1}, \underline{3}\bar{5}\underline{2}\bar{4}\bar{1}, \underline{4}\bar{5}\underline{2}\bar{3}\bar{1}\}$.



concatenation



shuffle



convolution

FQSym_{\pm} = Hopf algebra with basis $(\mathbb{F}_{\tau})_{\tau \in \mathcal{S}_{\pm}}$ and where

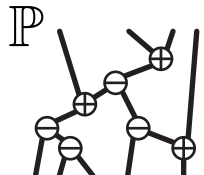
$$\mathbb{F}_{\tau} \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} \mathbb{F}_{\sigma} \quad \text{and} \quad \Delta \mathbb{F}_{\sigma} = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_{\tau} \otimes \mathbb{F}_{\tau'}$$

CAMBRIAN ALGEBRA AS SUBALGEBRA OF FQSym_{\pm}

Cambrian algebra = vector subspace Camb of FQSym_{\pm} generated by

$$\mathbb{P}_T := \sum_{\substack{\tau \in \mathfrak{G}_{\pm} \\ \mathbf{P}(\tau) = T}} \mathbb{F}_{\tau} = \sum_{\tau \in \mathcal{L}(T)} \mathbb{F}_{\tau},$$

for all Cambrian trees T .

Exm:  = $\mathbb{F}_{\underline{2137546}} + \mathbb{F}_{\underline{2173546}} + \mathbb{F}_{\underline{2175346}} + \mathbb{F}_{\underline{2713546}} + \mathbb{F}_{\underline{2715346}}$
 $+ \mathbb{F}_{\underline{2751346}} + \mathbb{F}_{\underline{7213546}} + \mathbb{F}_{\underline{7215346}} + \mathbb{F}_{\underline{7251346}} + \mathbb{F}_{\underline{7521346}}$

THEO. Camb is a subalgebra of FQSym_{\pm}

(ie. the Cambrian congruence is “compatible” with the product and coproduct in FQSym_{\pm})

GAME: Explain the product and coproduct directly on the Cambrian trees...

PRODUCT IN CAMBRIAN ALGEBRA

$$\begin{aligned}
 \mathbb{P} \cdot \mathbb{P} &= \mathbb{F}_{\underline{12}} \cdot (\mathbb{F}_{\underline{213}} + \mathbb{F}_{\underline{231}}) \\
 &= \left(\begin{array}{l} \mathbb{F}_{\underline{12435}} + \mathbb{F}_{\underline{12453}} + \mathbb{F}_{\underline{14235}} \\ + \mathbb{F}_{\underline{14253}} + \mathbb{F}_{\underline{14523}} + \mathbb{F}_{\underline{41235}} \\ + \mathbb{F}_{\underline{41253}} + \mathbb{F}_{\underline{41523}} + \mathbb{F}_{\underline{45123}} \end{array} \right) + \left(\begin{array}{l} \mathbb{F}_{\underline{14325}} + \mathbb{F}_{\underline{14352}} \\ + \mathbb{F}_{\underline{14532}} + \mathbb{F}_{\underline{41325}} \\ + \mathbb{F}_{\underline{41352}} + \mathbb{F}_{\underline{41532}} \\ + \mathbb{F}_{\underline{45132}} \end{array} \right) + \left(\begin{array}{l} \mathbb{F}_{\underline{43125}} + \mathbb{F}_{\underline{43152}} \\ + \mathbb{F}_{\underline{43512}} + \mathbb{F}_{\underline{45312}} \end{array} \right) \\
 &= \mathbb{P} + \mathbb{P} + \mathbb{P}
 \end{aligned}$$

PROP. For any Cambrian trees T and T' ,

$$\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_S \mathbb{P}_S$$

where S runs over the interval $\left[T \nearrow \bar{T}', T \nwarrow \bar{T}' \right]$ in the $\varepsilon(T)\varepsilon(T')$ -Cambrian lattice

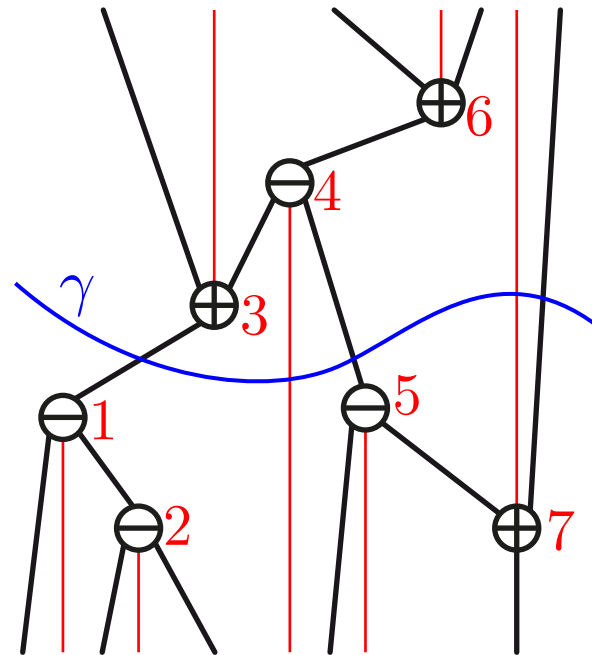
COPRODUCT IN CAMBRIAN ALGEBRA

$$\begin{aligned}
 \Delta \mathbb{P} &= \Delta(\mathbb{F}_{\overline{213}} + \mathbb{F}_{\overline{231}}) \\
 &= 1 \otimes (\mathbb{F}_{\overline{213}} + \mathbb{F}_{\overline{231}}) + \mathbb{F}_{\overline{1}} \otimes \mathbb{F}_{\overline{12}} + \mathbb{F}_{\overline{1}} \otimes \mathbb{F}_{\overline{21}} + \mathbb{F}_{\overline{21}} \otimes \mathbb{F}_{\overline{1}} + \mathbb{F}_{\overline{12}} \otimes \mathbb{F}_{\overline{1}} + (\mathbb{F}_{\overline{213}} + \mathbb{F}_{\overline{231}}) \otimes 1 \\
 &= 1 \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1 \\
 &= 1 \otimes \mathbb{P} + \mathbb{P} \otimes (\mathbb{P} \cdot \mathbb{P}) + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1.
 \end{aligned}$$

PROP. For any Cambrian tree S ,

$$\Delta \mathbb{P}_S = \sum_{\gamma} \left(\prod_{T \in B(S, \gamma)} \mathbb{P}_T \right) \otimes \left(\prod_{T' \in A(S, \gamma)} \mathbb{P}_{T'} \right)$$

where γ runs over all cuts of S , and $A(S, \gamma)$ and $B(S, \gamma)$ denote the Cambrian forests above and below γ respectively



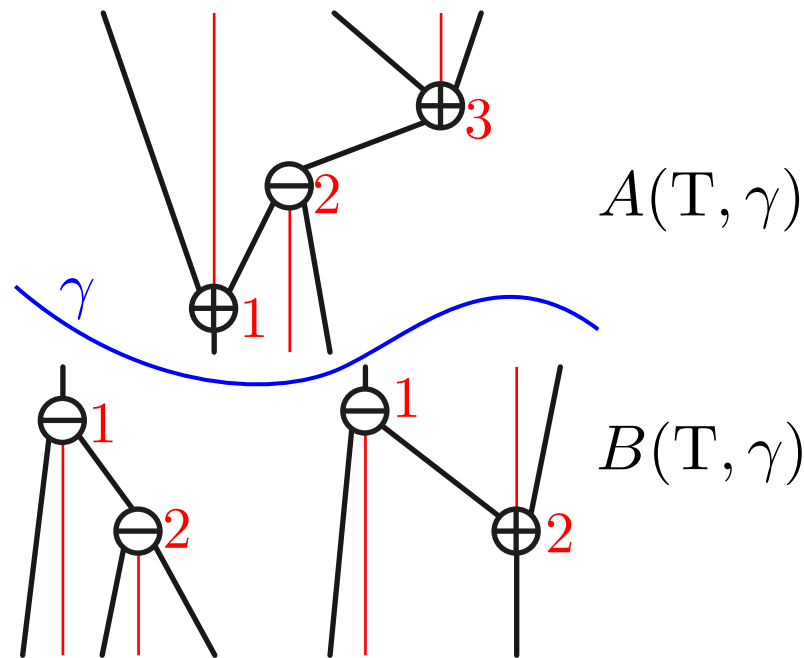
COPRODUCT IN CAMBRIAN ALGEBRA

$$\begin{aligned}
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 &= 1 \otimes (\mathbb{F}_{\overline{213}} + \mathbb{F}_{\overline{231}}) + \mathbb{F}_{\overline{1}} \otimes \mathbb{F}_{\overline{12}} + \mathbb{F}_{\overline{1}} \otimes \mathbb{F}_{\overline{21}} + \mathbb{F}_{\overline{21}} \otimes \mathbb{F}_{\overline{1}} + \mathbb{F}_{\overline{12}} \otimes \mathbb{F}_{\overline{1}} + (\mathbb{F}_{\overline{213}} + \mathbb{F}_{\overline{231}}) \otimes 1 \\
 &= 1 \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1 \\
 &= 1 \otimes \mathbb{P} + \mathbb{P} \otimes (\mathbb{P} \cdot \mathbb{P}) + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1.
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where γ runs over all cuts of S , and $A(S, \gamma)$ and $B(S, \gamma)$ denote the Cambrian forests above and below γ respectively



DUAL CAMBRIAN ALGEBRA AS QUOTIENT OF FQSym_{\pm}^*

FQSym_{\pm}^* = dual Hopf algebra with basis $(\mathbb{G}_{\tau})_{\tau \in \mathcal{S}_{\pm}}$ and where

$$\mathbb{G}_{\tau} \cdot \mathbb{G}_{\tau'} = \sum_{\sigma \in \tau \star \tau'} \mathbb{G}_{\sigma} \quad \text{and} \quad \Delta \mathbb{G}_{\sigma} = \sum_{\sigma \in \tau \sqcup \tau'} \mathbb{G}_{\tau} \otimes \mathbb{G}_{\tau'}$$

PROP. The graded dual Camb^* of the Cambrian algebra is isomorphic to the image of FQSym_{\pm}^* under the canonical projection

$$\pi : \mathbb{C}\langle A \rangle \longrightarrow \mathbb{C}\langle A \rangle / \equiv,$$

where \equiv denotes the Cambrian congruence. The dual basis $\mathbb{Q}_{\mathbb{T}}$ of $\mathbb{P}_{\mathbb{T}}$ is expressed as $\mathbb{Q}_{\mathbb{T}} = \pi(\mathbb{G}_{\tau})$, where τ is any linear extension of \mathbb{T}

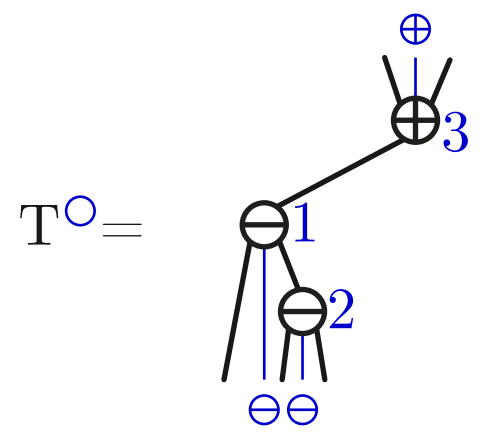
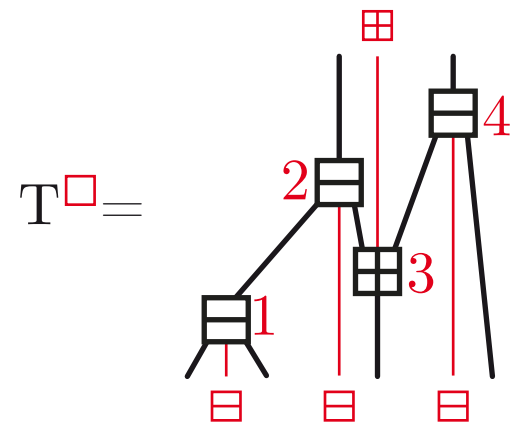
PRODUCT IN DUAL CAMBRIAN ALGEBRA

$$\begin{aligned}
 Q_{\text{tree}_1} \cdot Q_{\text{tree}_2} &= G_{\underline{12}} \cdot G_{\overline{213}} \\
 &= G_{\underline{12435}} + G_{\underline{13425}} + G_{\underline{14325}} + G_{\underline{15324}} + G_{\underline{23415}} + G_{\underline{24315}} + G_{\underline{25314}} + G_{\underline{34215}} + G_{\underline{35214}} + G_{\underline{45213}} \\
 &= Q_{\text{tree}_1} + Q_{\text{tree}_2} + Q_{\text{tree}_3} + Q_{\text{tree}_4} + Q_{\text{tree}_5} + Q_{\text{tree}_6} + Q_{\text{tree}_7} + Q_{\text{tree}_8} + Q_{\text{tree}_9} + Q_{\text{tree}_{10}}
 \end{aligned}$$

PROP. For any Cambrian trees T and T' ,

$$Q_T \cdot Q_{T'} = \sum_s Q_{T_s T'}$$

where s runs over all shuffles of $\varepsilon(T)$ and $\varepsilon(T')$



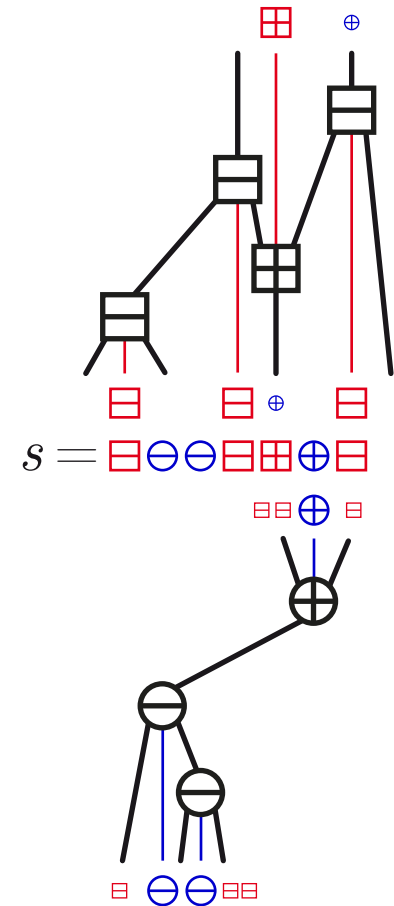
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 &= Q_{\text{tree}_1} + Q_{\text{tree}_2} + Q_{\text{tree}_3} + Q_{\text{tree}_4} + Q_{\text{tree}_5} + Q_{\text{tree}_6} + Q_{\text{tree}_7} + Q_{\text{tree}_8} + Q_{\text{tree}_9} + Q_{\text{tree}_{10}}
 \end{aligned}$$

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$$Q_T \cdot Q_{T'} = \sum_s Q_{T_s T'}$$

where s runs over all shuffles of $\varepsilon(T)$ and $\varepsilon(T')$



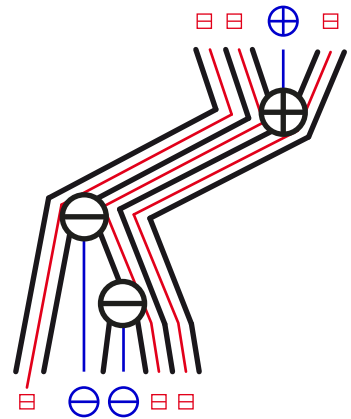
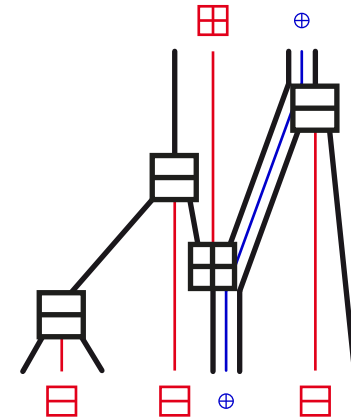
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 &= Q_{\text{tree}_1} + Q_{\text{tree}_2} + \dots + Q_{\text{tree}_{10}}
 \end{aligned}$$

PROP. For any Cambrian trees T and T' ,

$$Q_T \cdot Q_{T'} = \sum_s Q_{TsT'}$$

where s runs over all shuffles of $\varepsilon(T)$ and $\varepsilon(T')$



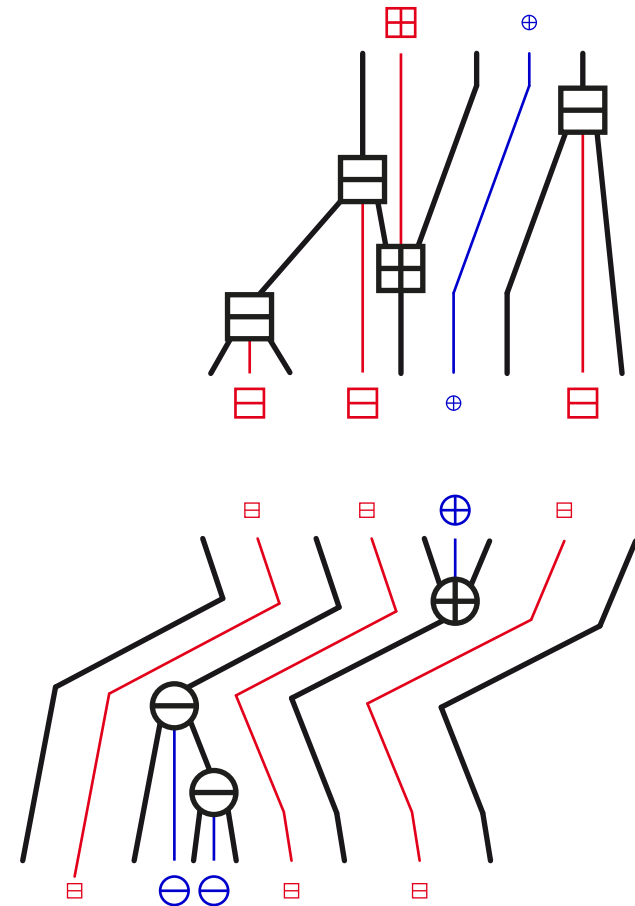
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$$\begin{aligned}
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 &= Q_{\text{tree}_1} + Q_{\text{tree}_2} + \dots + Q_{\text{tree}_{10}}
 \end{aligned}$$

PROP. For any Cambrian trees T and T' ,

$$Q_T \cdot Q_{T'} = \sum_s Q_{TsT'}$$

where s runs over all shuffles of $\varepsilon(T)$ and $\varepsilon(T')$



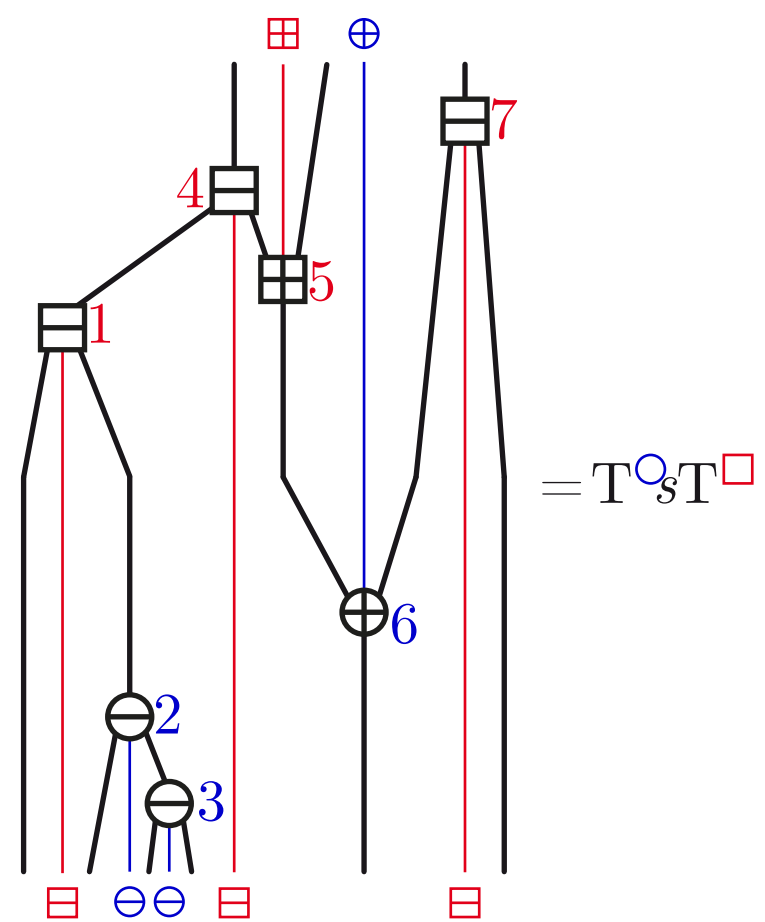
PRODUCT IN DUAL CAMBRIAN ALGEBRA

$$\begin{aligned}
 Q_{\text{tree}_1} \cdot Q_{\text{tree}_2} &= G_{\underline{12}} \cdot G_{\underline{213}} \\
 &= G_{\underline{12435}} + G_{\underline{13425}} + G_{\underline{14325}} + G_{\underline{15324}} + G_{\underline{23415}} + G_{\underline{24315}} + G_{\underline{25314}} + G_{\underline{34215}} + G_{\underline{35214}} + G_{\underline{45213}} \\
 &= Q_{\text{tree}_1} + Q_{\text{tree}_2} + \dots + Q_{\text{tree}_{10}}
 \end{aligned}$$

PROP. For any Cambrian trees T and T' ,

$$Q_T \cdot Q_{T'} = \sum_s Q_{T \circ_s T'}$$

where s runs over all shuffles of $\varepsilon(T)$ and $\varepsilon(T')$



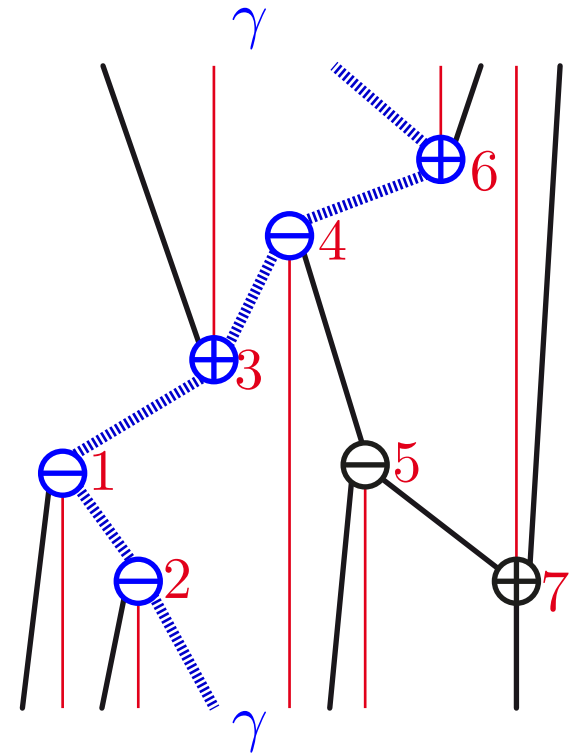
COPRODUCT IN DUAL CAMBRIAN ALGEBRA

$$\begin{aligned}
 \Delta Q_{\text{tree}} &= \Delta G_{\bar{2}1\bar{3}} \\
 &= 1 \otimes G_{\bar{2}1\bar{3}} + G_{\bar{1}} \otimes G_{\bar{1}\bar{2}} + G_{\bar{2}\bar{1}} \otimes G_{\bar{1}} + G_{\bar{2}1\bar{3}} \otimes 1 \\
 &= 1 \otimes Q_{\text{tree}} + Q_{\text{tree}_1} \otimes Q_{\text{tree}_2} + Q_{\text{tree}_3} \otimes Q_{\text{tree}_4} + Q_{\text{tree}} \otimes 1.
 \end{aligned}$$

PROP. For any Cambrian tree S ,

$$\Delta Q_S = \sum_{\gamma} Q_{L(S,\gamma)} \otimes Q_{R(S,\gamma)}$$

where γ runs over all gaps between vertices of S , and $L(S,\gamma)$ and $R(S,\gamma)$ denote the Cambrian trees left and right to γ respectively



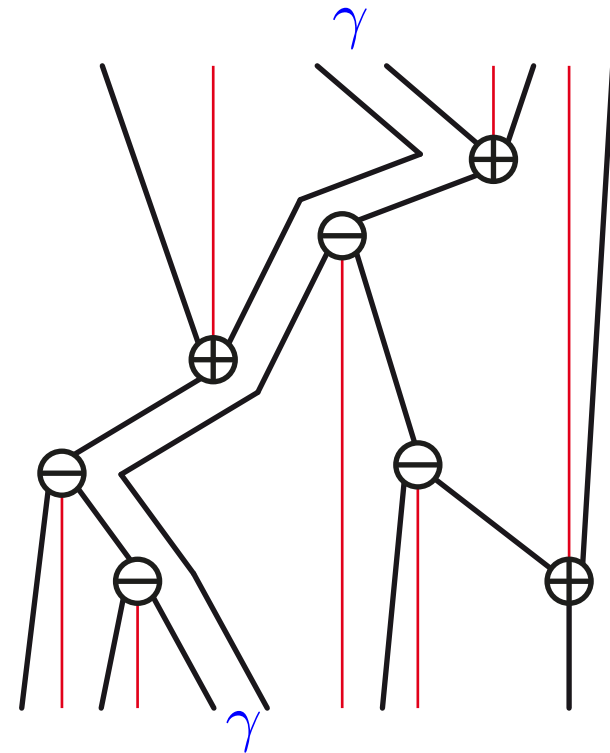
COPRODUCT IN DUAL CAMBRIAN ALGEBRA

$$\begin{aligned}
 \Delta Q_{\text{tree}} &= \Delta G_{\bar{2}1\bar{3}} \\
 &= 1 \otimes G_{\bar{2}1\bar{3}} + G_{\bar{1}} \otimes G_{\bar{1}\bar{2}} + G_{\bar{2}\bar{1}} \otimes G_{\bar{1}} + G_{\bar{2}1\bar{3}} \otimes 1 \\
 &= 1 \otimes Q_{\text{tree}} + Q_{\text{tree}_1} \otimes Q_{\text{tree}_2} + Q_{\text{tree}_3} \otimes Q_{\text{tree}_4} + Q_{\text{tree}} \otimes 1.
 \end{aligned}$$

PROP. For any Cambrian tree S ,

$$\Delta Q_S = \sum_{\gamma} Q_{L(S,\gamma)} \otimes Q_{R(S,\gamma)}$$

where γ runs over all gaps between vertices of S , and $L(S, \gamma)$ and $R(S, \gamma)$ denote the Cambrian trees left and right to γ respectively



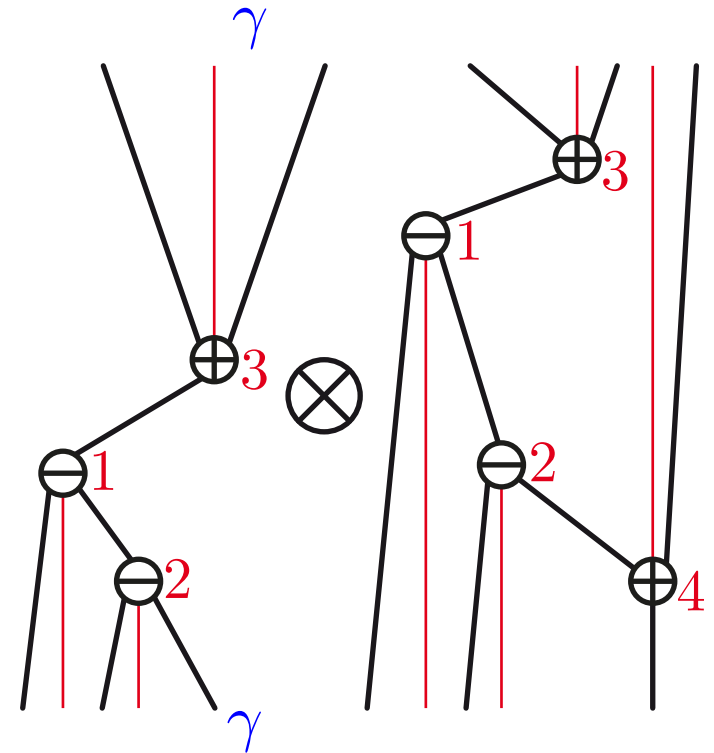
COPRODUCT IN DUAL CAMBRIAN ALGEBRA

$$\begin{aligned}
 \Delta Q_{\text{tree}} &= \Delta G_{\underline{213}} \\
 &= 1 \otimes G_{\underline{213}} + G_{\underline{1}} \otimes G_{\underline{12}} + G_{\underline{21}} \otimes G_{\underline{1}} + G_{\underline{213}} \otimes 1 \\
 &= 1 \otimes Q_{\text{tree}} + Q_{\text{tree}_1} \otimes Q_{\text{tree}_2} + Q_{\text{tree}_3} \otimes Q_{\text{tree}_4} + Q_{\text{tree}} \otimes 1.
 \end{aligned}$$

PROP. For any Cambrian tree S ,

$$\Delta Q_S = \sum_{\gamma} Q_{L(S,\gamma)} \otimes Q_{R(S,\gamma)}$$

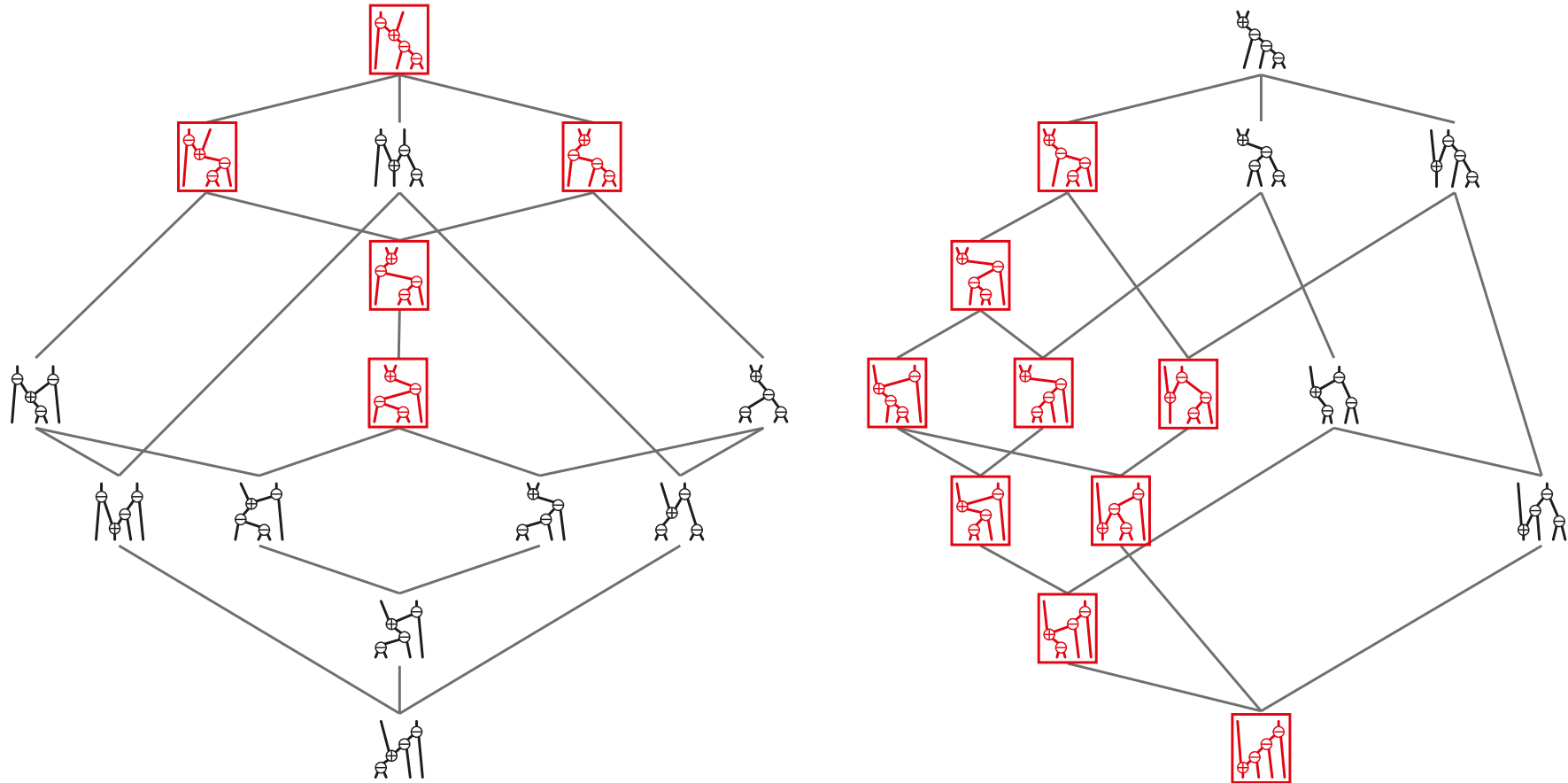
where γ runs over all gaps between vertices of S , and $L(S, \gamma)$ and $R(S, \gamma)$ denote the Cambrian trees left and right to γ respectively



MULTIPLICATIVE BASES

Define

$$\mathbb{E}^T := \sum_{T' \leq T} \mathbb{P}_{T'} \quad \text{and} \quad \mathbb{H}^T := \sum_{T' \leq T} \mathbb{P}_{T'}$$



PROP. $(\mathbb{E}^T)_{T \in \text{Camb}}$ and $(\mathbb{H}^T)_{T \in \text{Camb}}$ are multiplicative bases of Camb, ie.

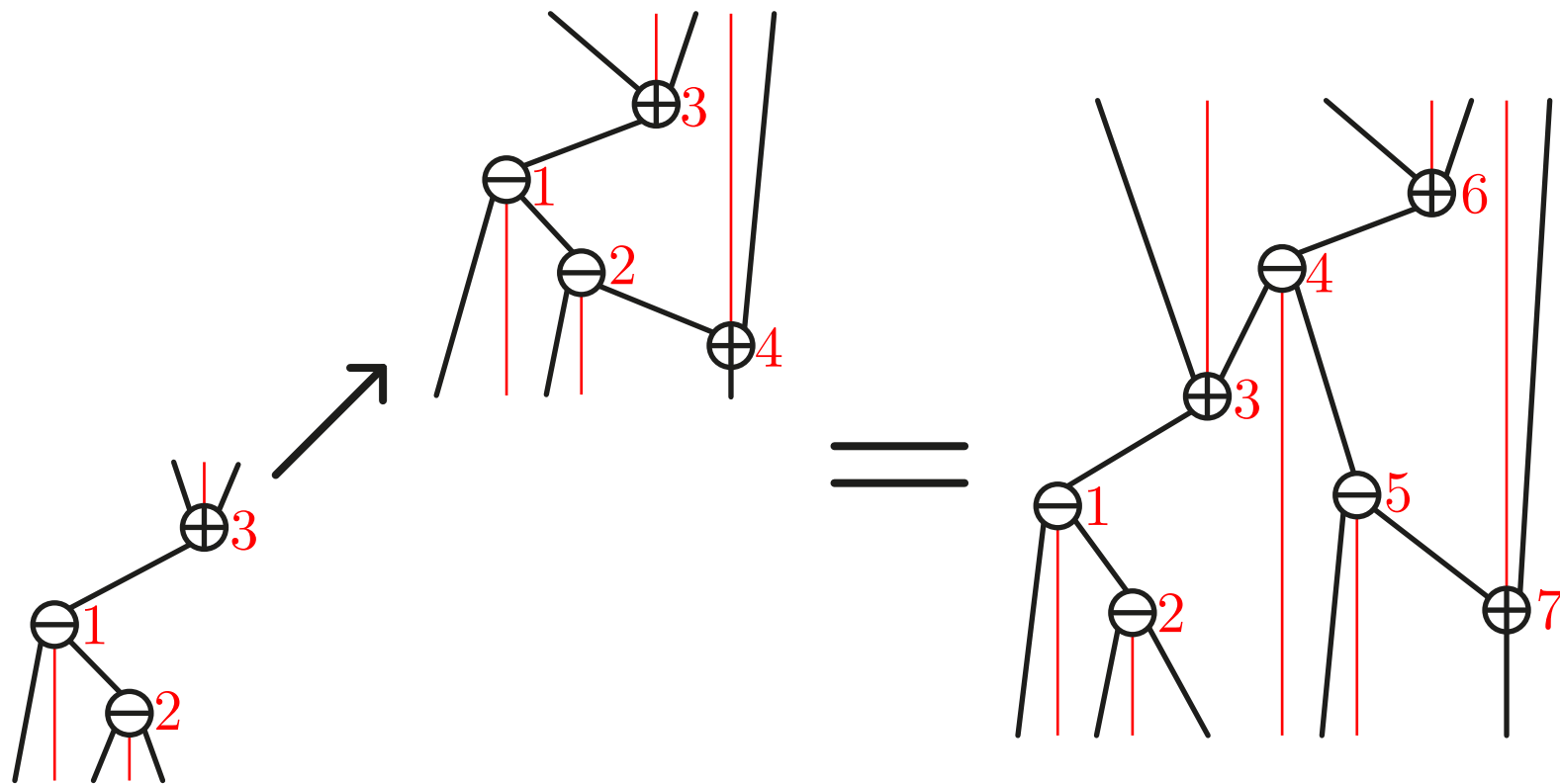
$$\mathbb{E}^T \cdot \mathbb{E}^{T'} = \mathbb{E}^{T \nearrow T'} \quad \text{and} \quad \mathbb{H}^T \cdot \mathbb{H}^{T'} = \mathbb{H}^{T \nwarrow T'}$$

INDECOMPOSABLE ELEMENTS

PROP. The following properties are equivalent for a Cambrian tree S :

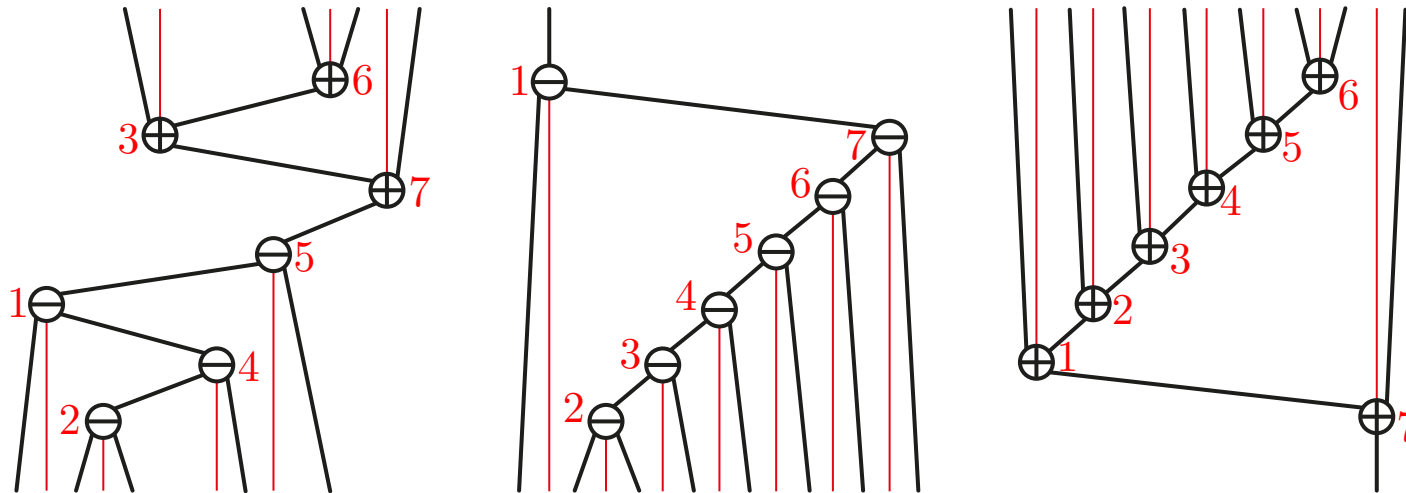
- \mathbb{E}^S can be decomposed into a product $\mathbb{E}^S = \mathbb{E}^T \cdot \mathbb{E}^{T'}$ for non-empty T, T'
- $([k] \parallel [n] \setminus [k])$ is an edge cut of S for some $k \in [n]$
- at least one linear extension τ of S is decomposable, ie. $\tau([k]) = [k]$ for some $k \in [n]$

The tree S is then called **\mathbb{E} -decomposable**

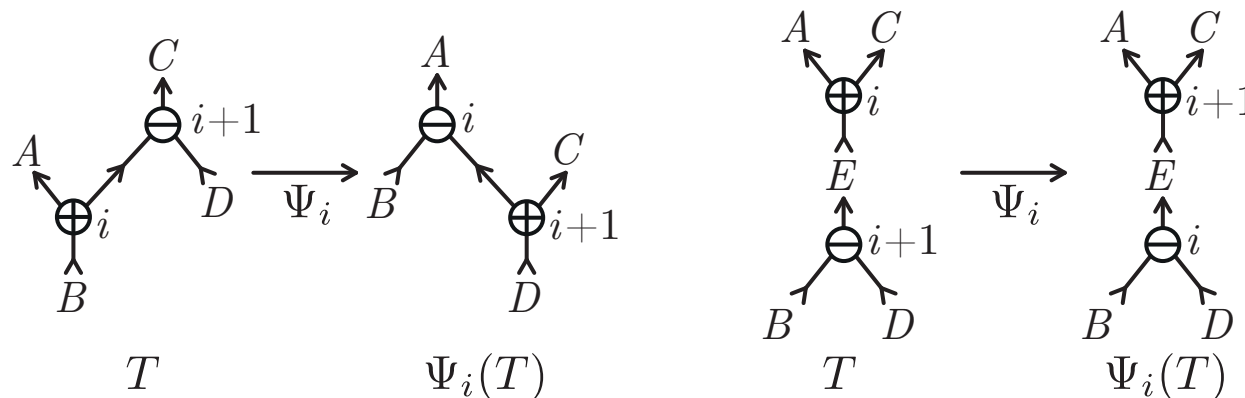


INDECOMPOSABLE ELEMENTS

PROP. For any signature $\varepsilon \in \pm^n$, the set of \mathbb{E} -indecomposable ε -Cambrian trees forms a principal upper ideal of the ε -Cambrian lattice



PROP. For any signature $\varepsilon \in \pm^n$, there are C_{n-1} \mathbb{E} -indecomposable ε -Cambrian trees. Therefore, there are $2^n C_{n-1}$ \mathbb{E} -indecomposable Cambrian trees on n vertices

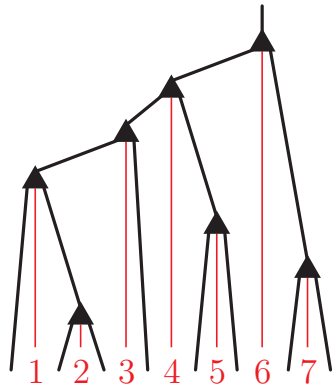


PERSPECTIVES

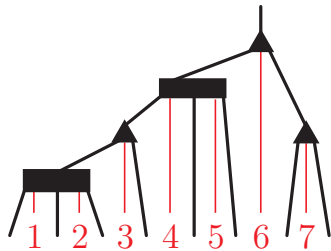
PERSPECTIVES

Extend combinatorial, geometric and algebraic properties of binary trees to further families of trees...

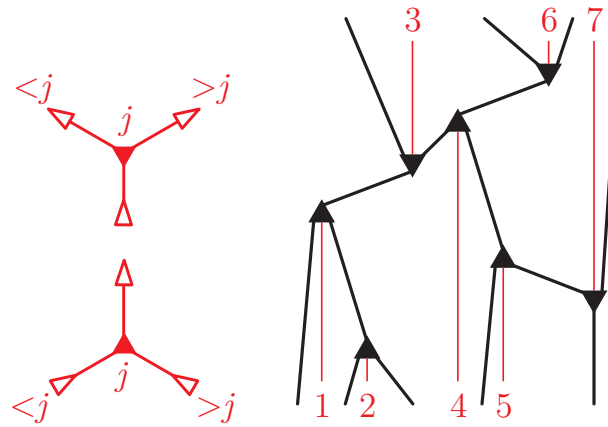
Binary trees



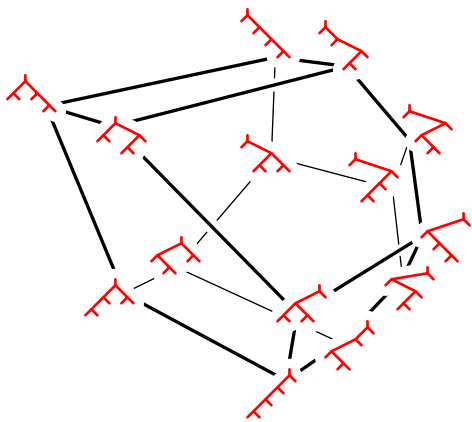
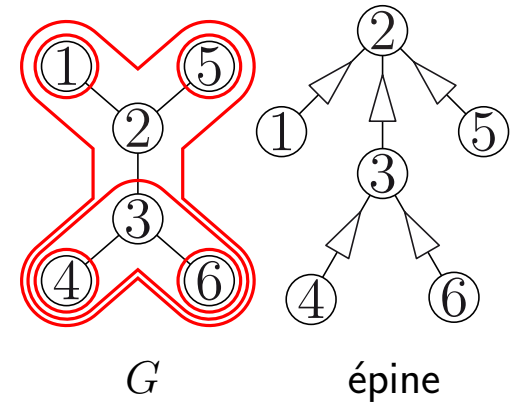
Schröder trees



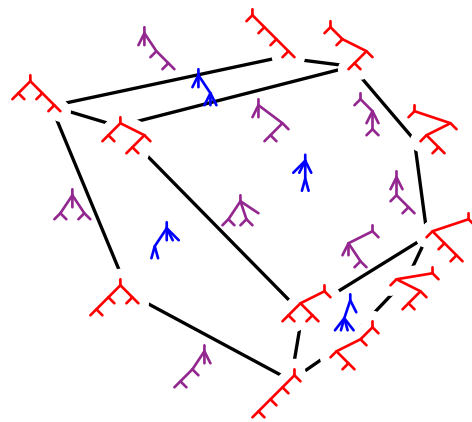
Cambrian trees



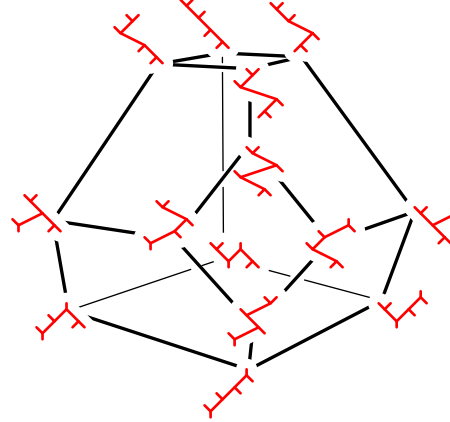
Spines of a graph



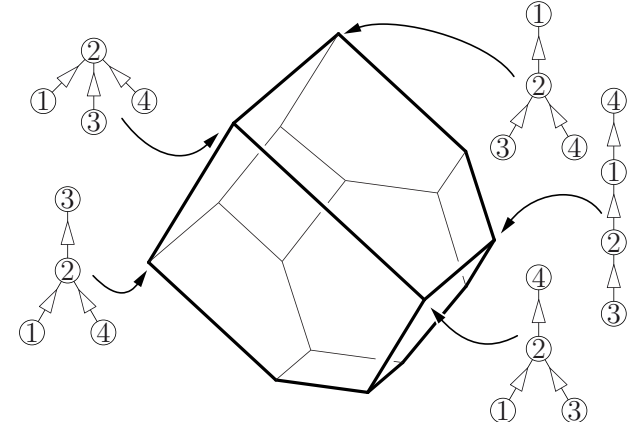
Loday-Ronco algebra



Packed words algebra



Cambrian algebra



Spine algebra ???

THANK YOU