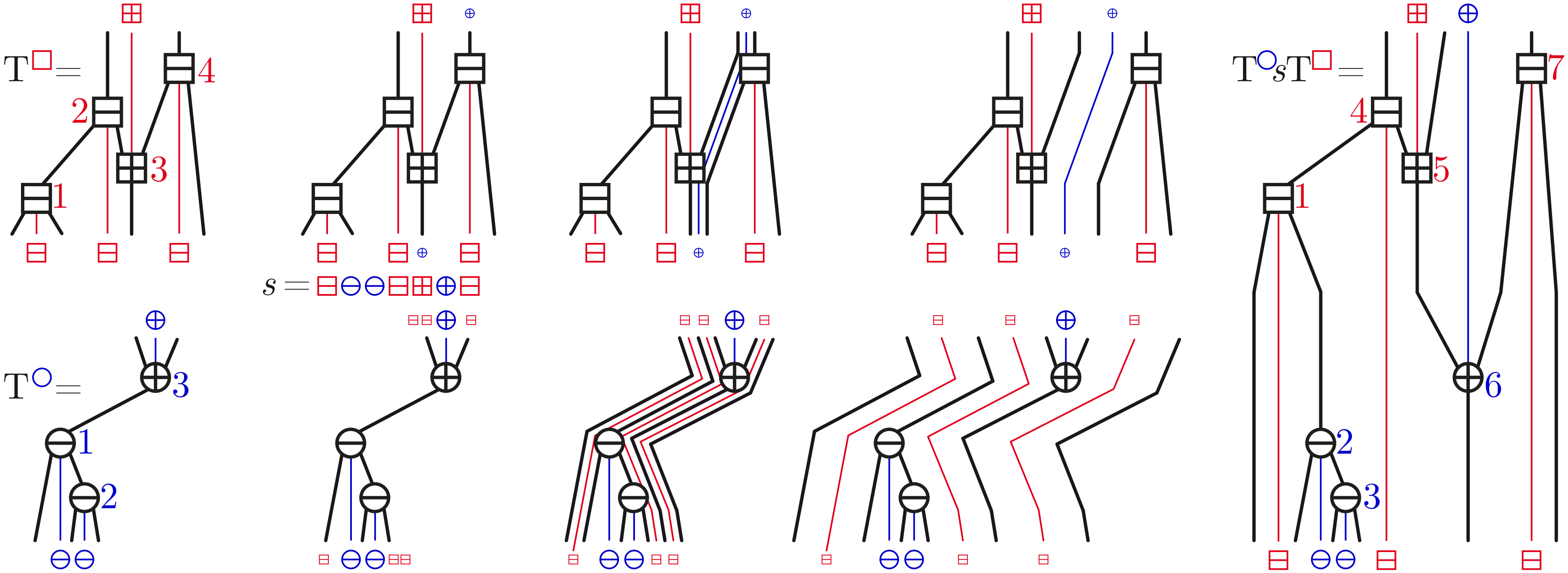
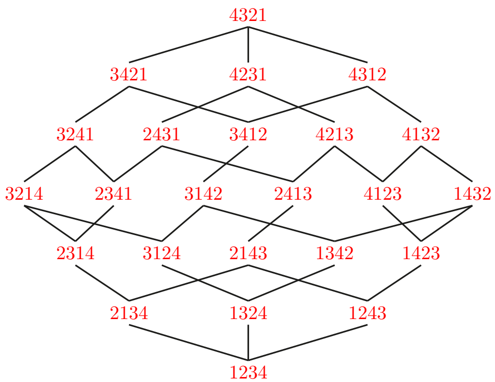
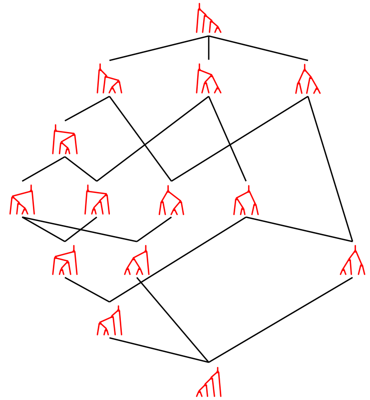
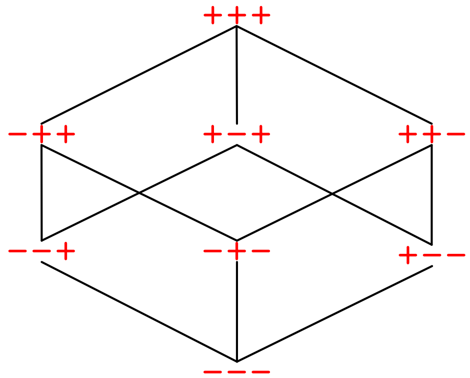
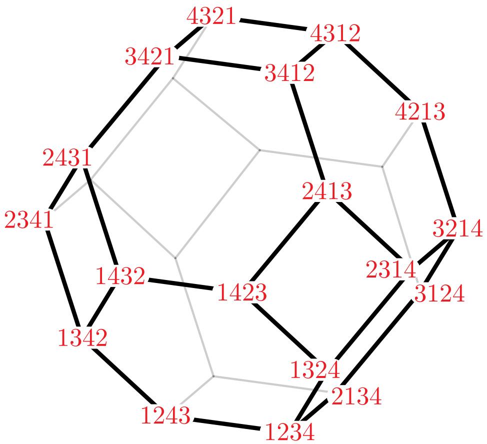
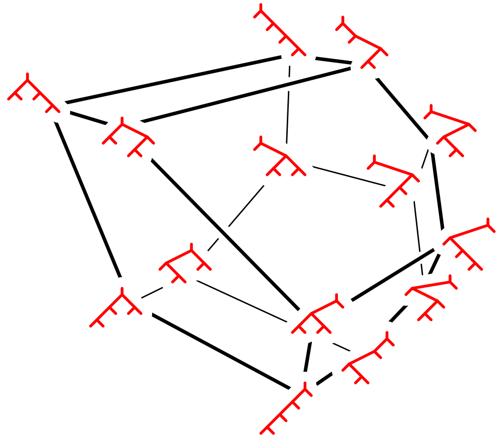
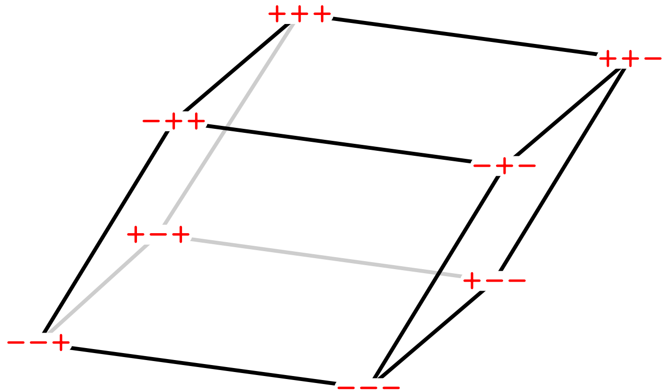
CAMBRIAN TREES



Grégory CHATEL (Univ. MIV)
Carsten LANGE (Univ. Munich)
Vincent PILAUD (CNRS & LIX)

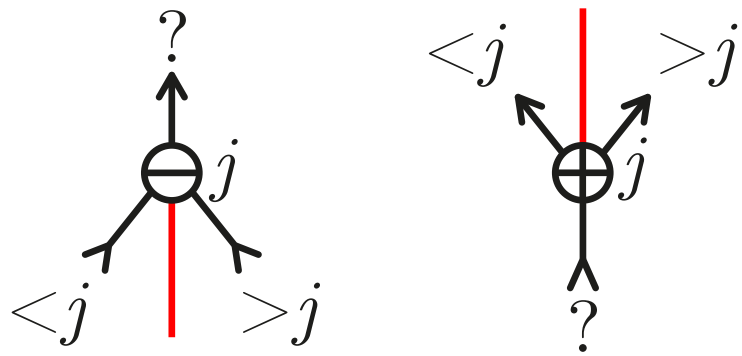
MOTIVATION

	permutations	binary trees	binary sequences
Combinatorics			
Geometry			
Algebra	Malvenuto-Reutenauer algebra FQSym $= \mathrm{vect}\langle\, \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \,\rangle$ $\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma$ $\triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$	Loday-Ronco algebra PBT $= \mathrm{vect}\langle\, \mathbb{P}_\mathrm{T} \mid \mathrm{T} \in \mathcal{BT} \,\rangle$ $\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{T \nearrow T' \le T'' \le T \nwarrow T'} \mathbb{P}_{T''}$ $\triangle \mathbb{F}_\gamma = \sum_{\gamma \text{ cut}} B(T,\gamma) \otimes A(T,\gamma)$	Solomon algebra Rec $= \mathrm{vect}\langle\, \mathbb{X}_\eta \mid \eta \in \pm^* \,\rangle$ $\mathbb{X}_\eta \cdot \mathbb{X}_{\eta'} = \mathbb{X}_{\eta+\eta'} + \mathbb{X}_{\eta-\eta'}$ $\triangle \mathbb{X}_\eta = \sum_{\gamma \text{ cut}} B(\eta,\gamma) \otimes A(\eta,\gamma)$

COMBINATORICS

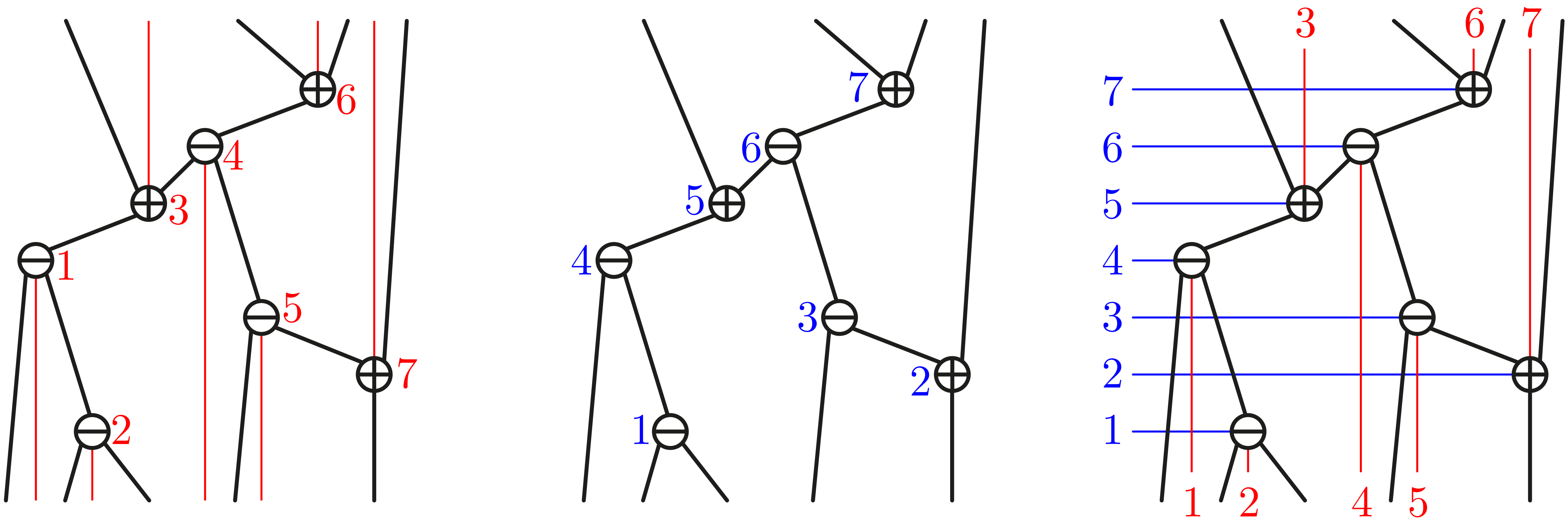
CAMBRIAN TREES

Cambrian tree = directed and labeled tree such that



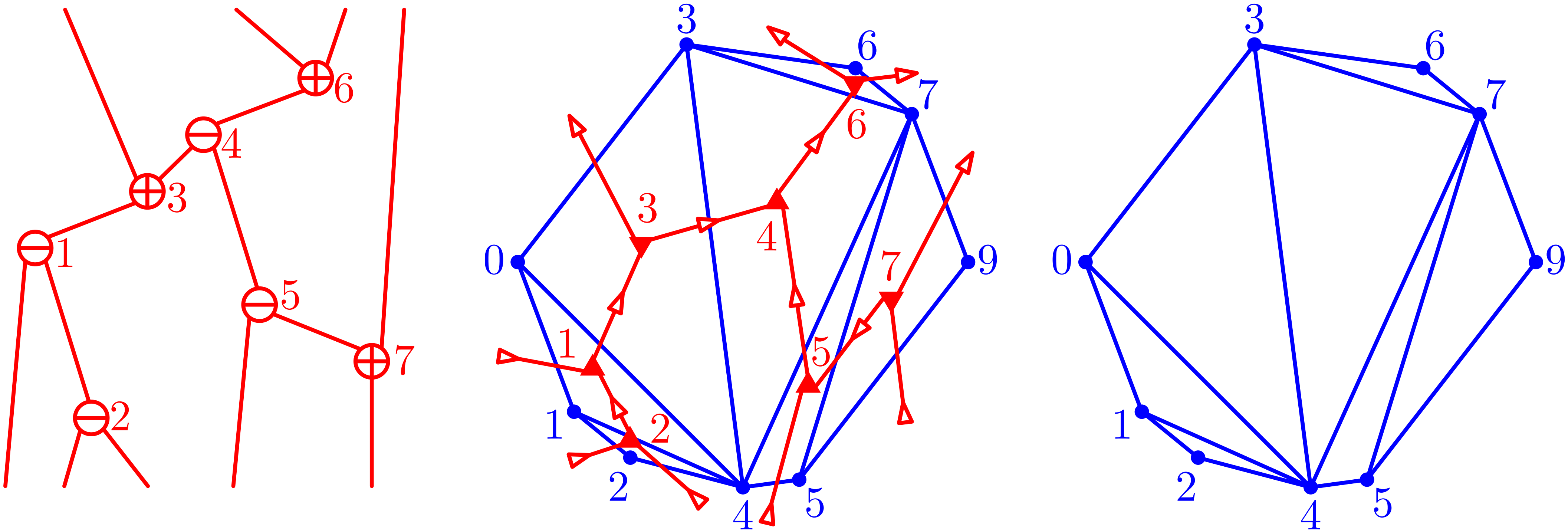
increasing tree = directed and labeled tree such that labels increase along arcs

leveled Cambrian tree = directed tree with a Cambrian labeling and an increasing labeling



CAMBRIAN TREES AND TRIANGULATIONS

Cambrian trees are dual to triangulations of polygons



signature $\longleftrightarrow$ vertices above or below $[0, 9]$

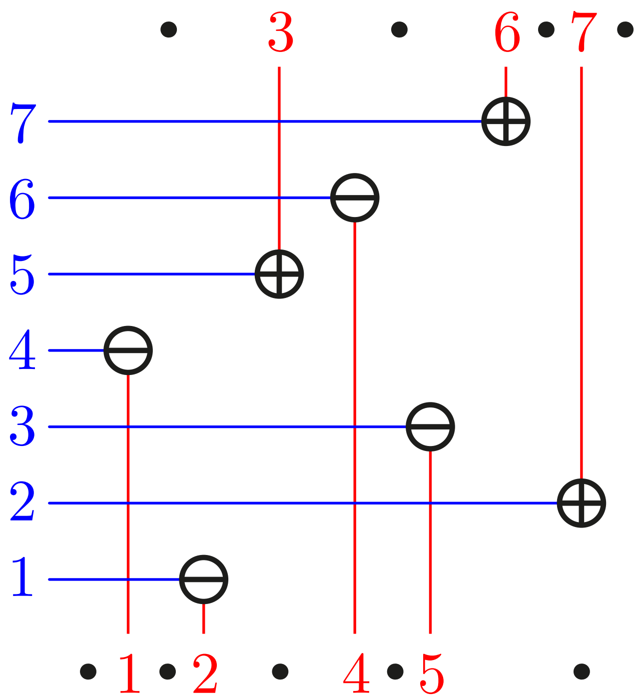
node j $\longleftrightarrow$ triangle $i < j < k$

For any signature ε, there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ ε-Cambrian trees

CAMBRIAN CORRESPONDENCE

Cambrian correspondence = signed permutation $\longmapsto$ leveled Cambrian tree

Exm: signed permutation $\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}$



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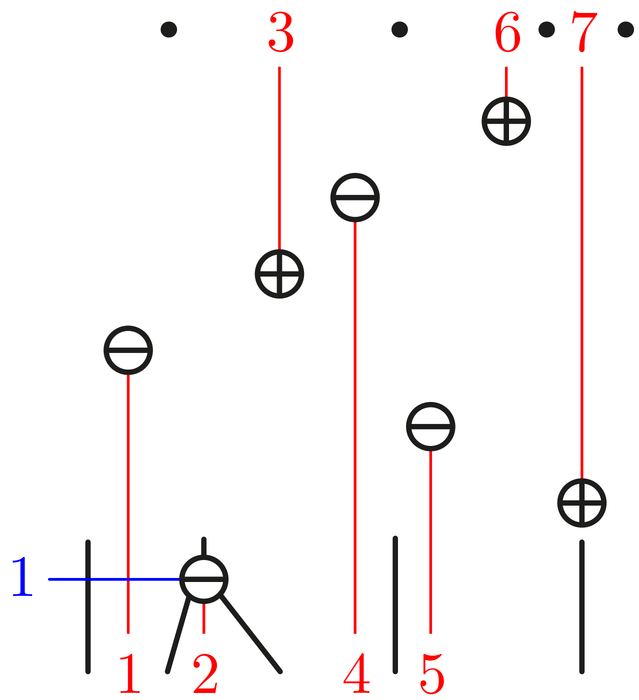

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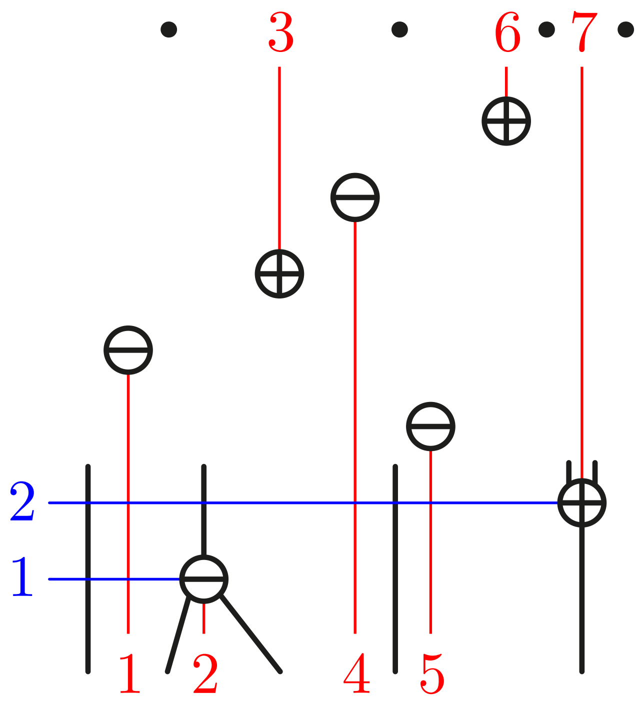

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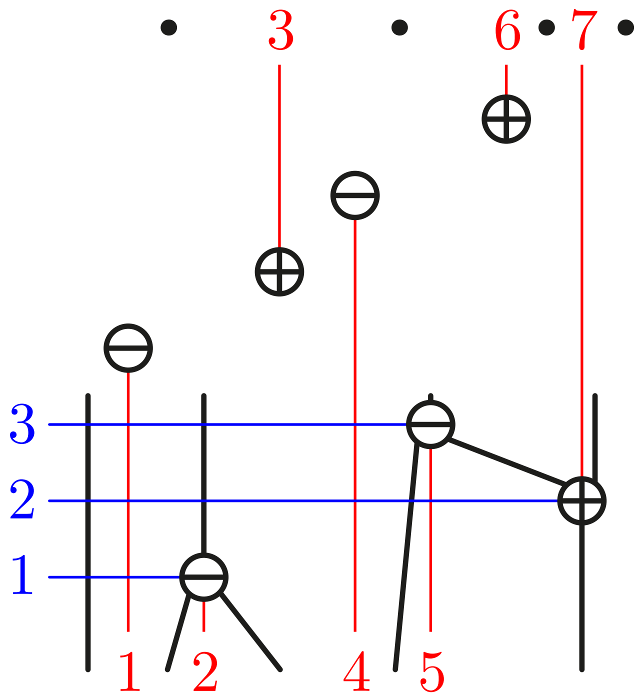

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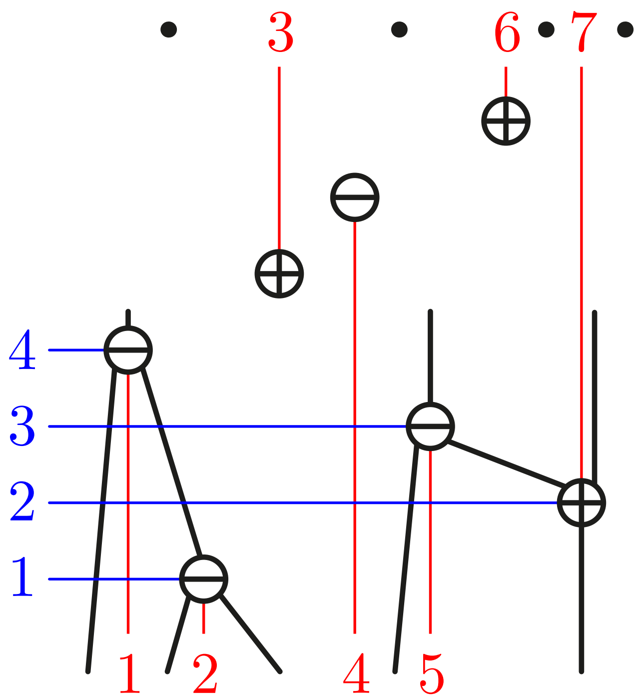

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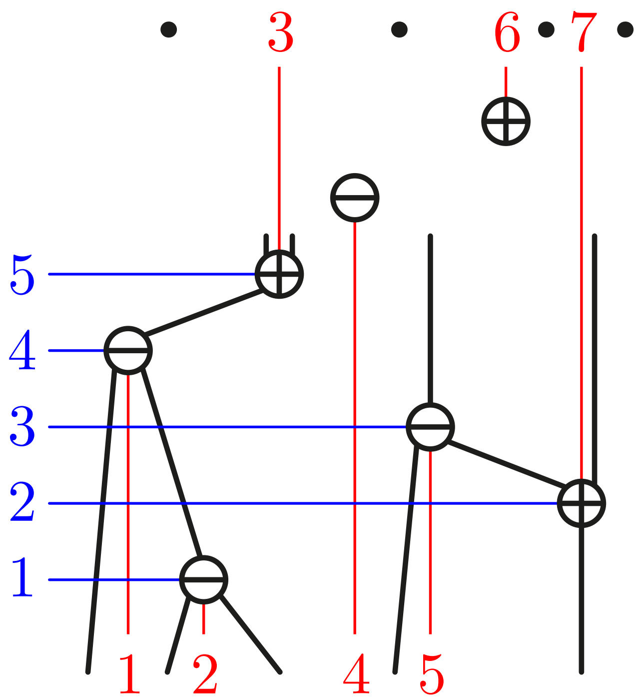

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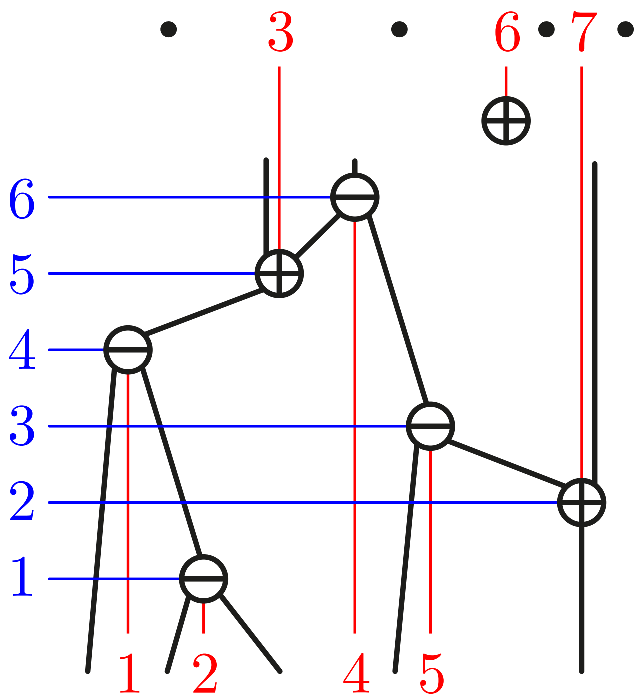

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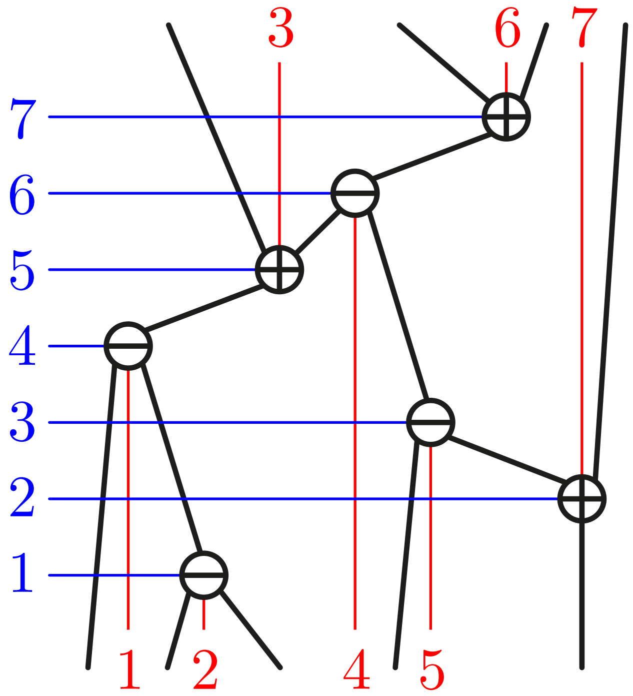

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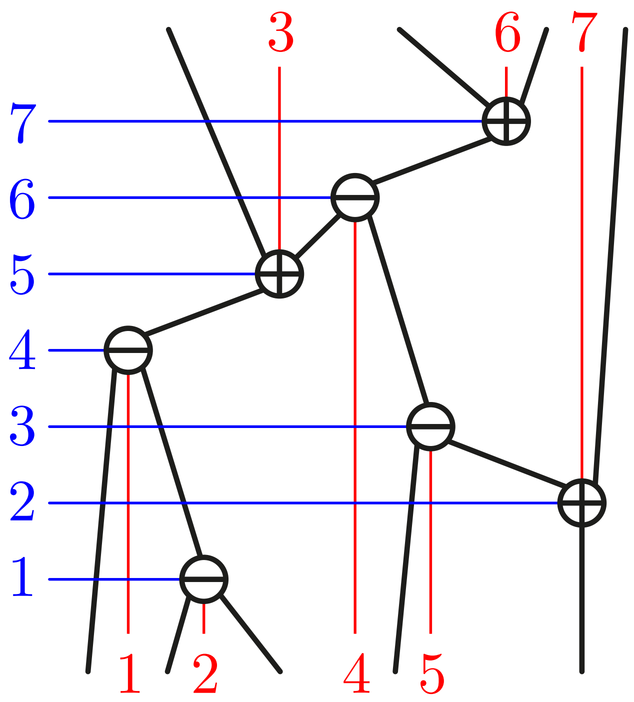


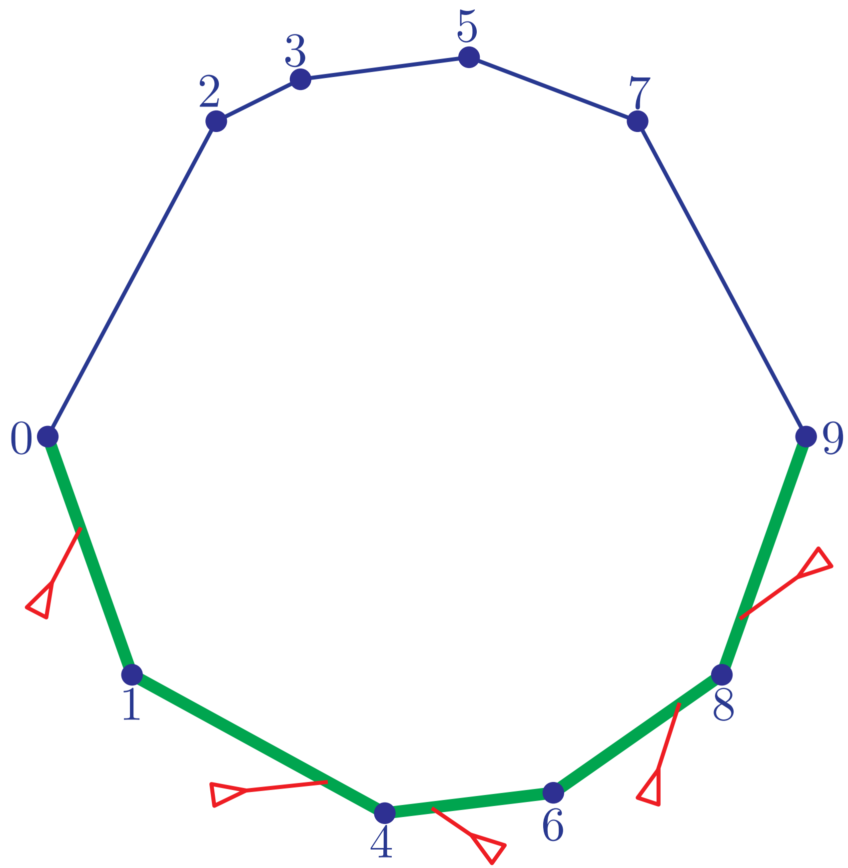
$\mathbf{P}(\tau)$ = $\mathbf{P}$-symbol of τ = Cambrian tree produced by Cambrian correspondence
$\mathbf{Q}(\tau)$ = $\mathbf{Q}$-symbol of τ = increasing tree produced by Cambrian correspondence

(analogy to Robinson-Schensted algorithm)

CAMBRIAN CORRESPONDENCE AND TRIANGULATIONS

Cambrian map $=$ signed permutation $\longmapsto$ triangulation

Exm: signed permutation $\underline{4}\overline{3}\underline{816}\overline{257}$



Reading. Cambrian lattices 2006

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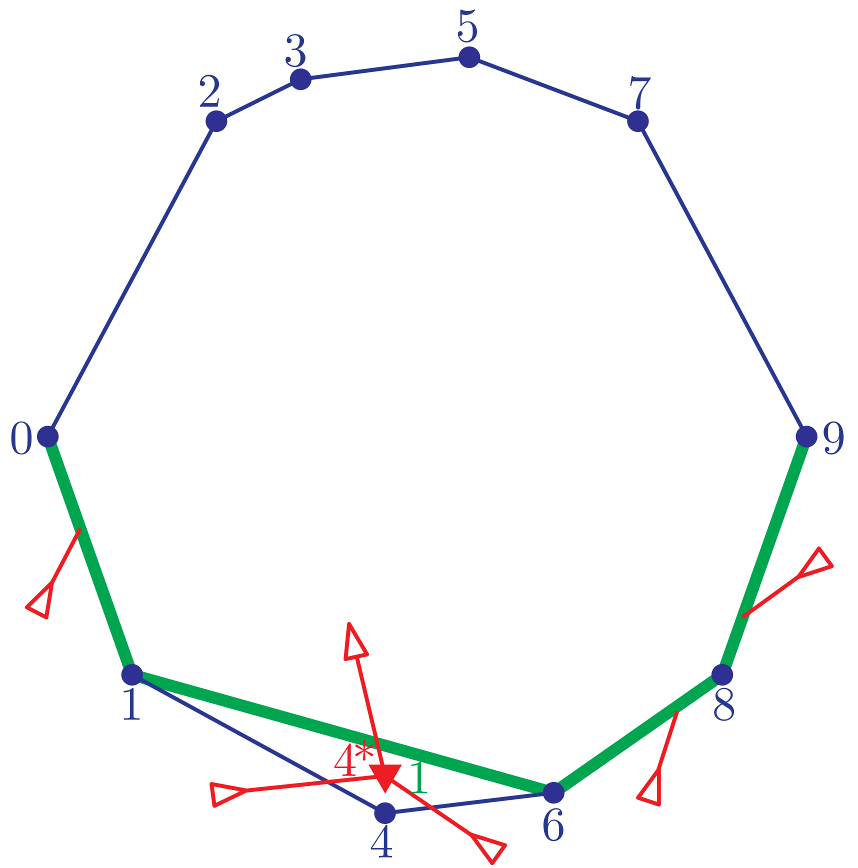

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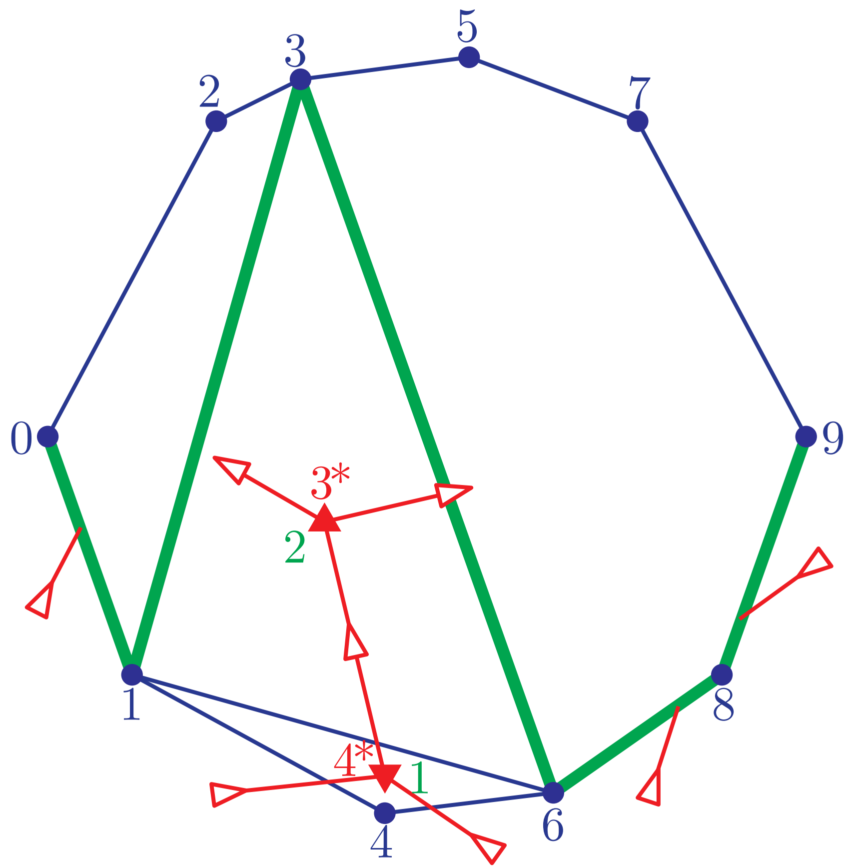

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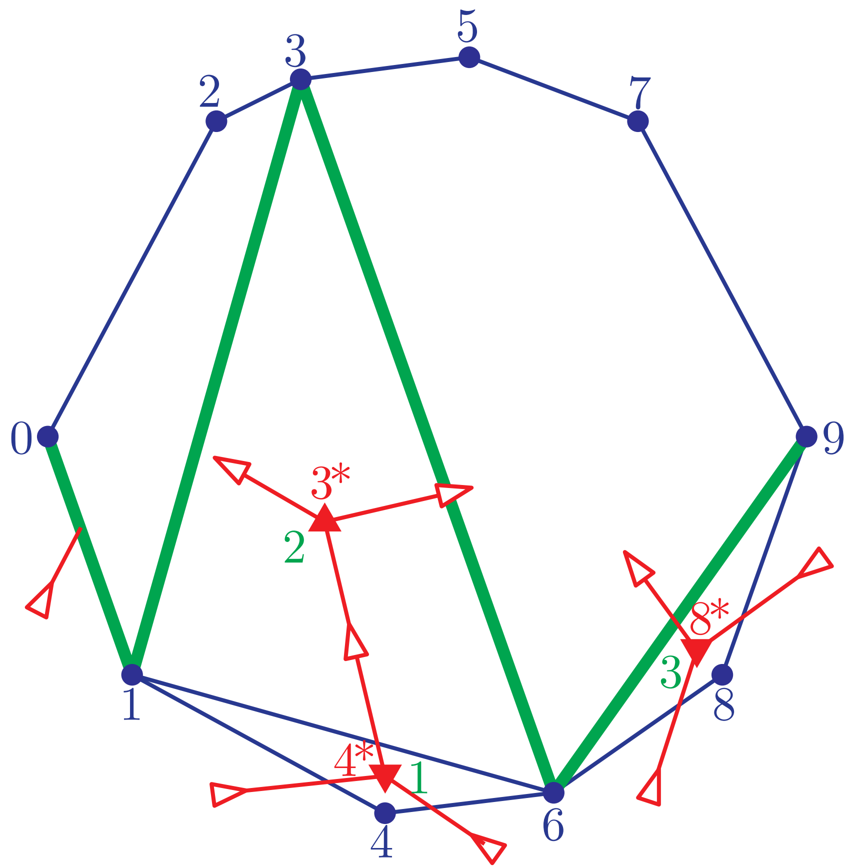

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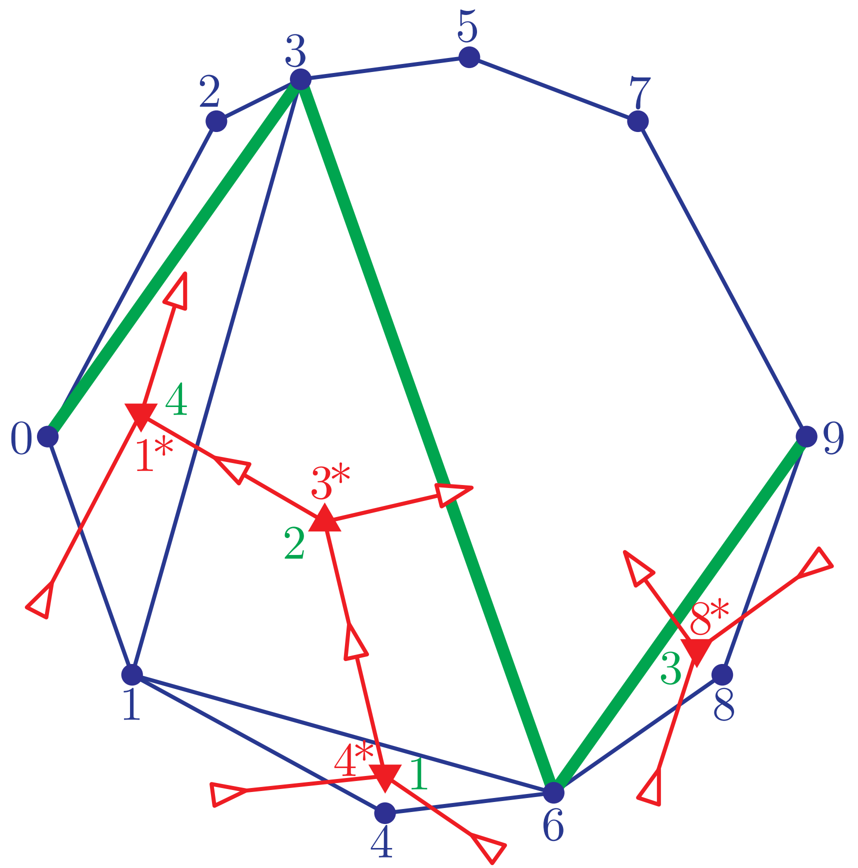

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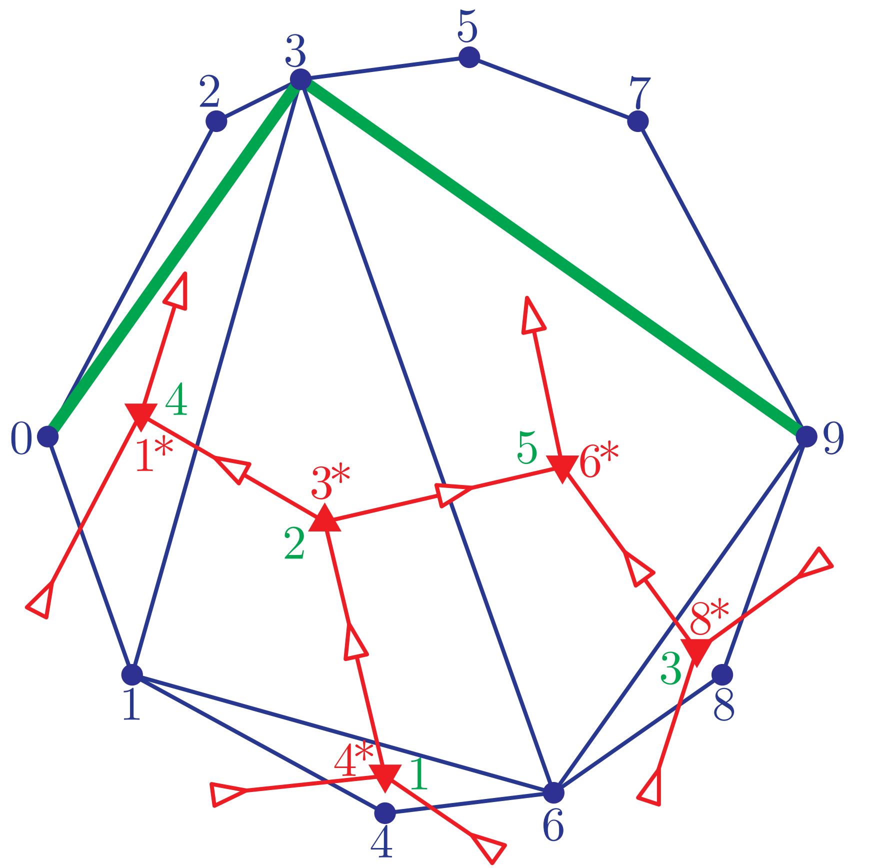

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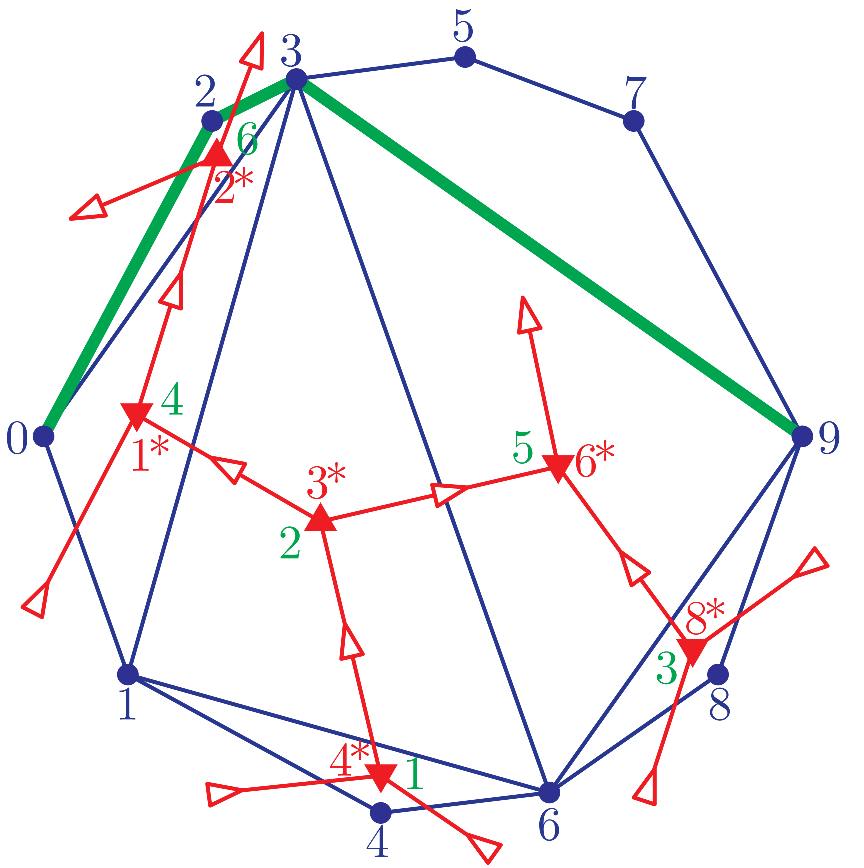

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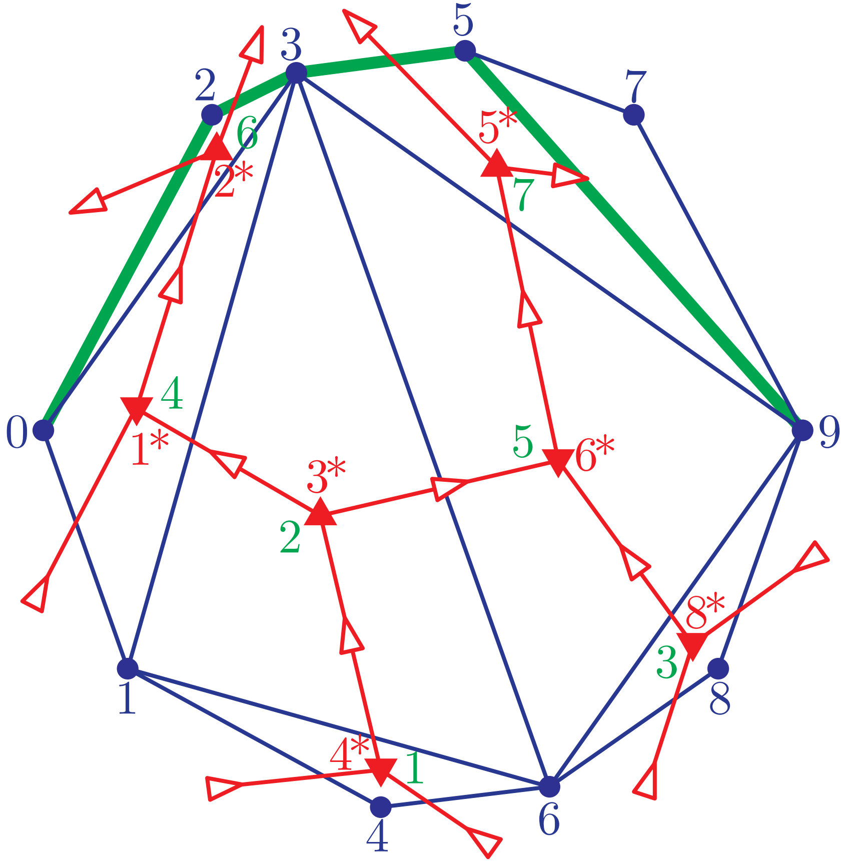

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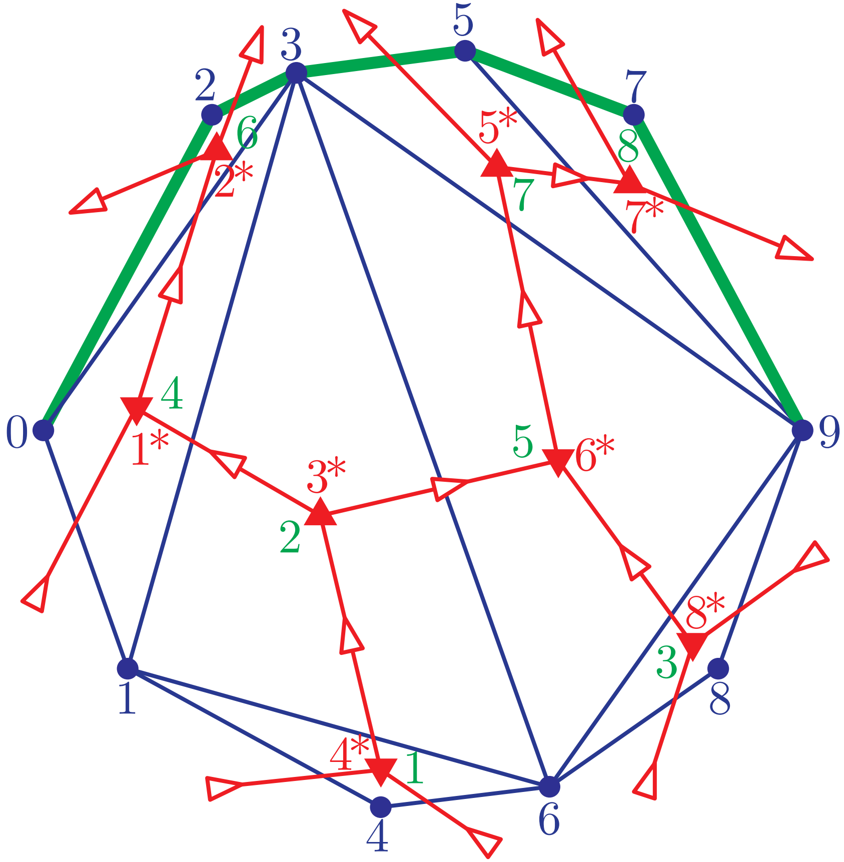


Reading. Cambrian lattices 2006

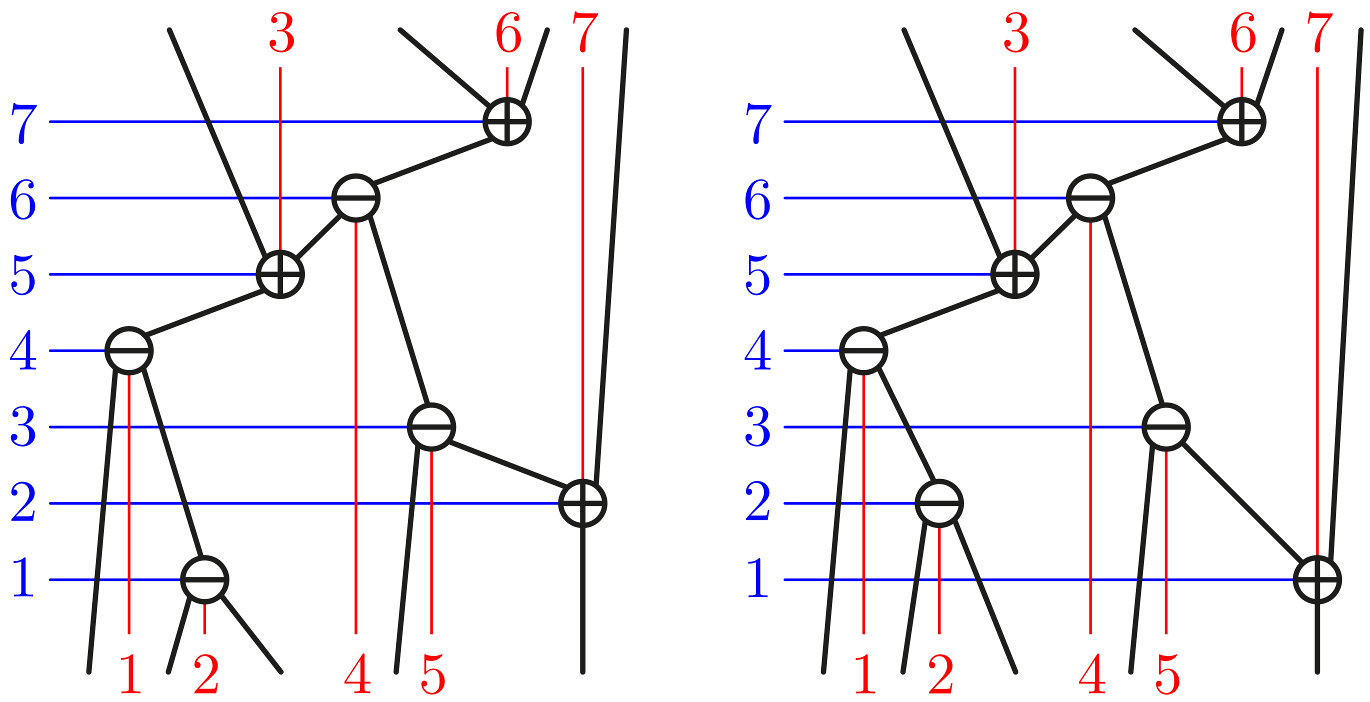
CAMBRIAN CONGRUENCE

ε-Cambrian congruence = transitive closure of the rewriting rules

$$UacVbW \equiv_\varepsilon UcaVbW \quad \text{if } a < b < c \text{ and } \varepsilon_b = -$$
$$UbVacW \equiv_\varepsilon UbVcaW \quad \text{if } a < b < c \text{ and } \varepsilon_b = +$$

where a, b, c are elements of $[n]$ while U, V, W are words on $[n]$

PROP. $\tau \equiv_\varepsilon \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$



$$2\bar{7}51\bar{3}4\bar{6} \equiv \bar{7}251\bar{3}4\bar{6}$$

CAMBRIAN CONGRUENCE

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PROP. $\tau \equiv_\varepsilon \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$

PROP. Cambrian congruence class labeled by Cambrian tree T

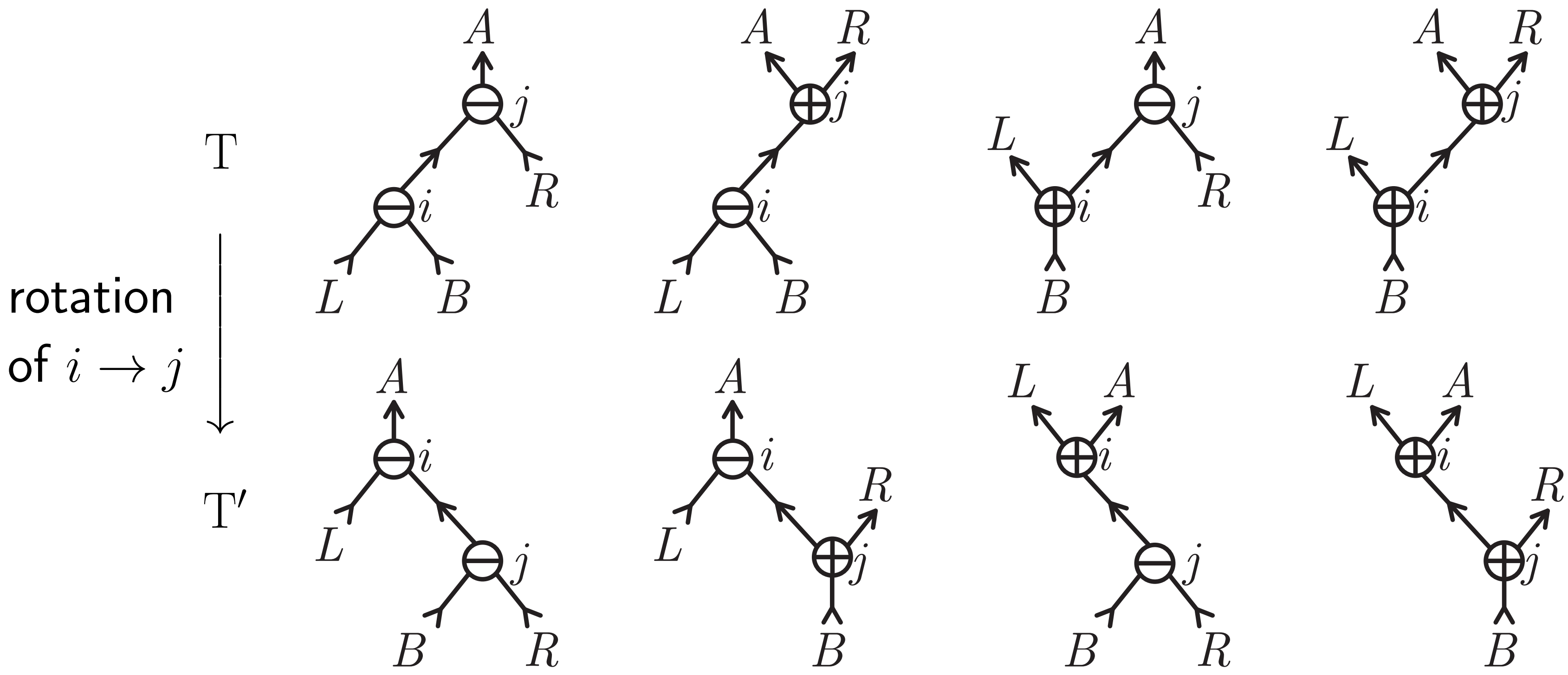
$$\{\tau \in \mathfrak{S}^\varepsilon \mid \mathbf{P}(\tau) = \mathrm{T}\} = \{\text{linear extensions of } \mathrm{T}\}$$

PROP. Cambrian classes are intervals of the weak order
minimums avoid $\overline{2}31$ and $31\underline{2}$ while maximums avoid $\overline{2}13$ and $13\underline{2}$

Reading. Cambrian lattices. 2006

ROTATIONS AND CAMBRIAN LATTICES

Rotation operation preserves Cambrian trees:



increasing rotation = rotation of edge $i \to j$ where $i < j$

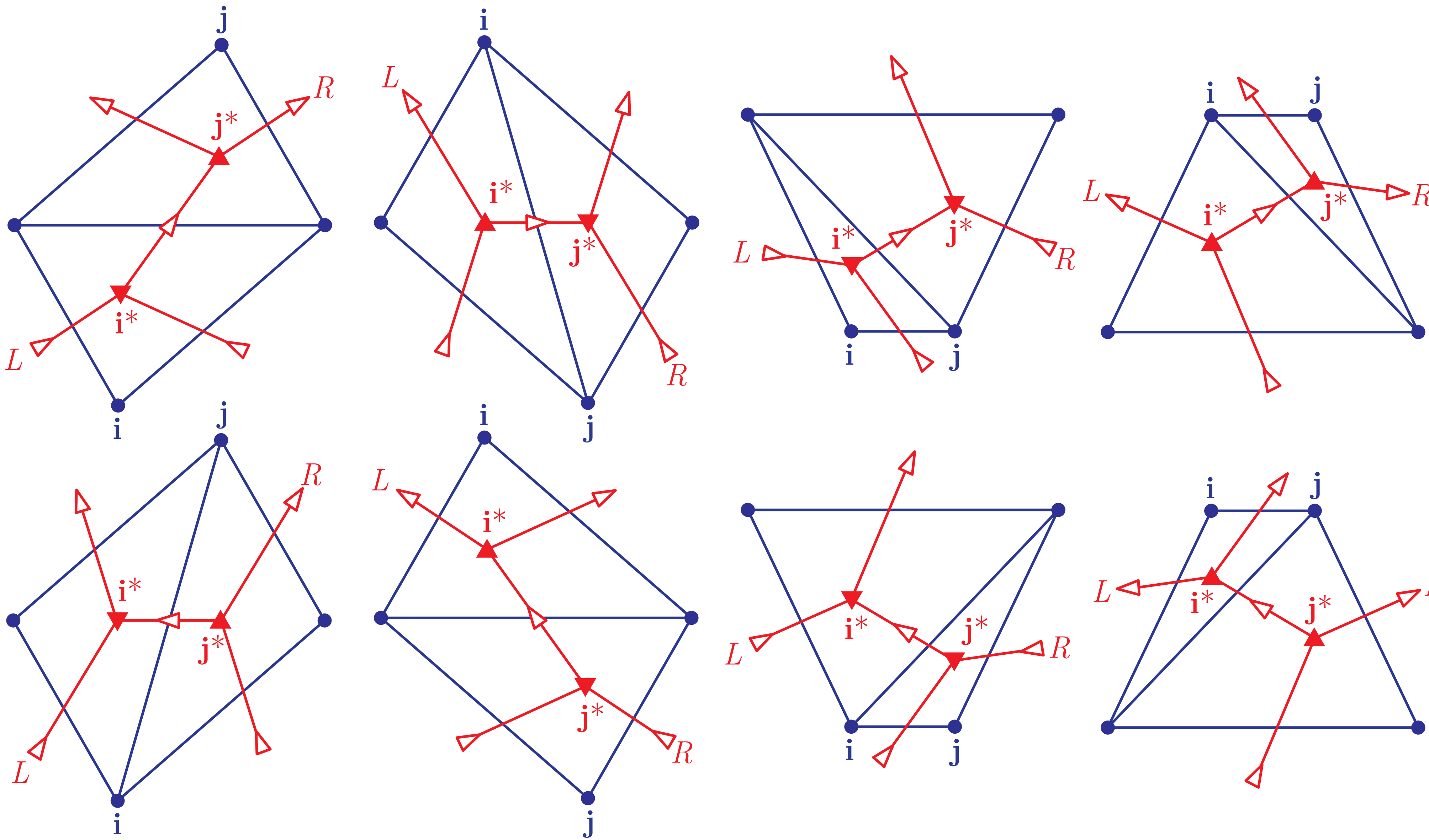
PROP. The transitive closure of the increasing rotation graph is the Cambrian lattice
$\mathbf{P}$ defines a lattice homomorphism from weak order to Cambrian lattice

Reading. Cambrian lattices. 2006

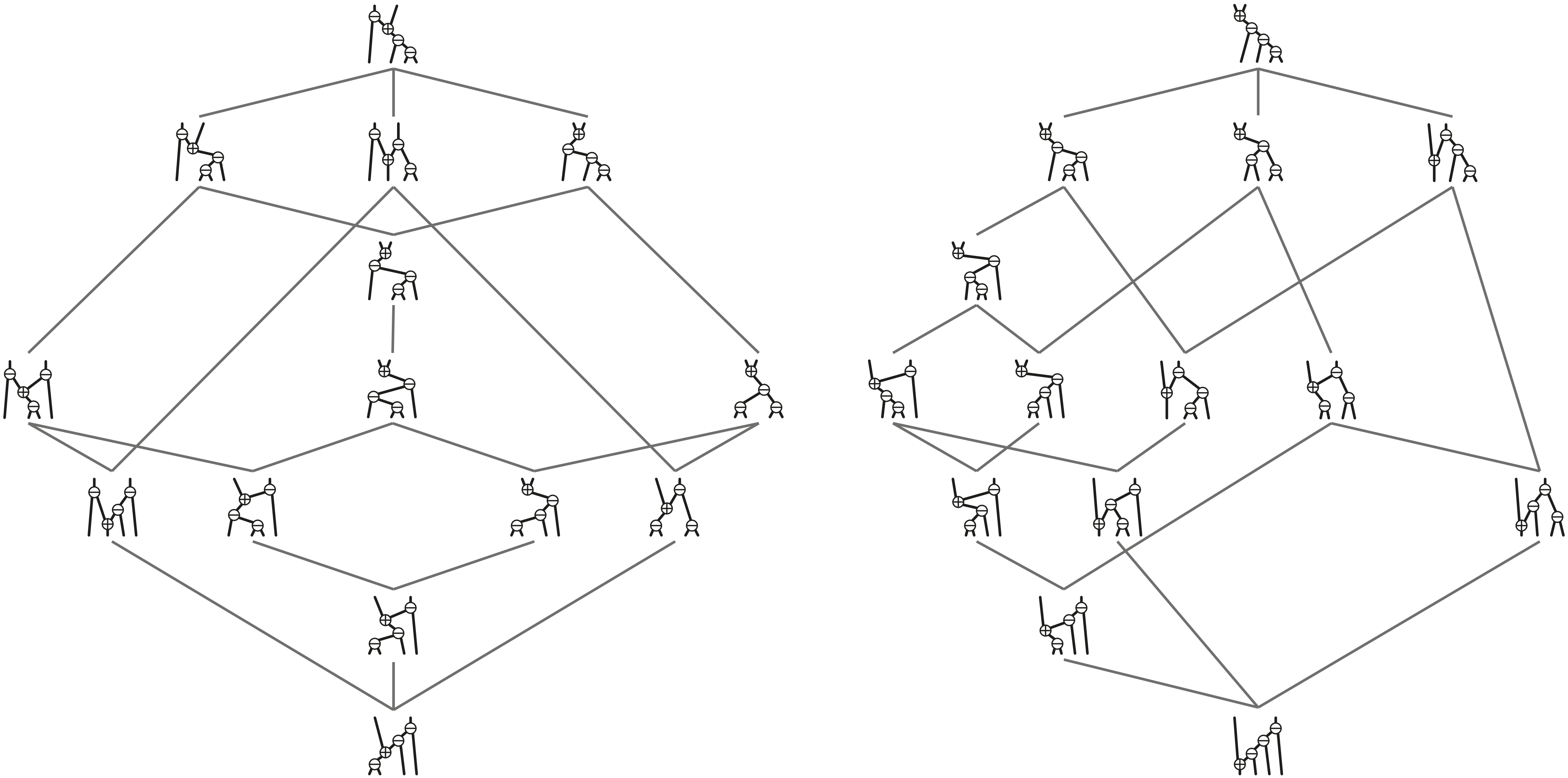
(rotation on Cambrian trees correspond to flips in triangulations)

ROTATIONS AND FLIPS

Rotation on Cambrian trees $\longleftrightarrow$ flips on triangulations



ROTATIONS AND CAMBRIAN LATTICES



Reading. Cambrian lattices. 2006

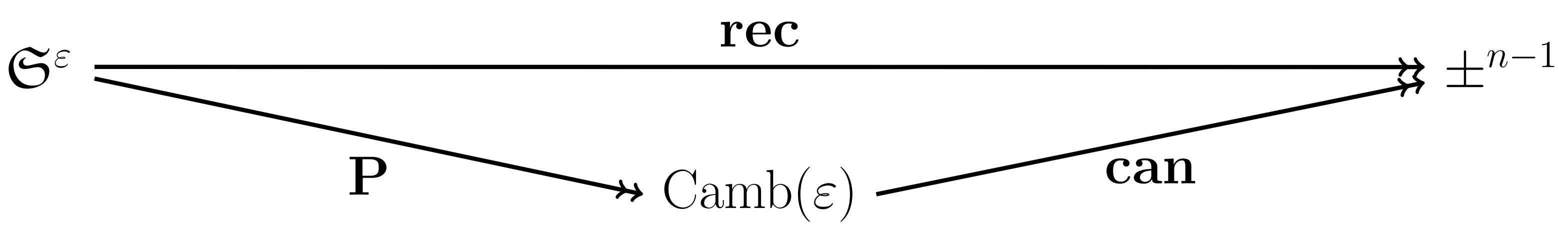
CANOPY

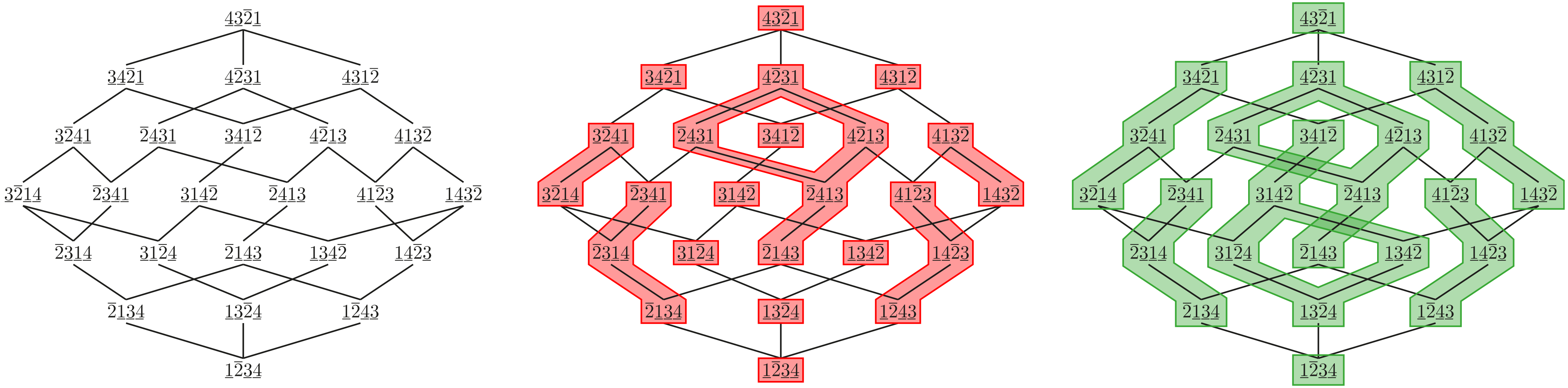
vertices i and $i+1$ are always comparable in a Cambrian tree

Canopy of a Cambrian tree T = sequence $\mathbf{can}(\mathrm{T}) \in \pm^{n-1}$ defined by

$$\mathbf{can}(\mathrm{T})_i = \begin{cases} - & \text{if } i \text{ above } i+1 \text{ in } \mathrm{T} \\ + & \text{if } i \text{ below } i+1 \text{ in } \mathrm{T} \end{cases}$$

PROP. $\mathbf{P}$, $\mathbf{can}$, and $\mathbf{rec}$ define lattice homomorphisms:





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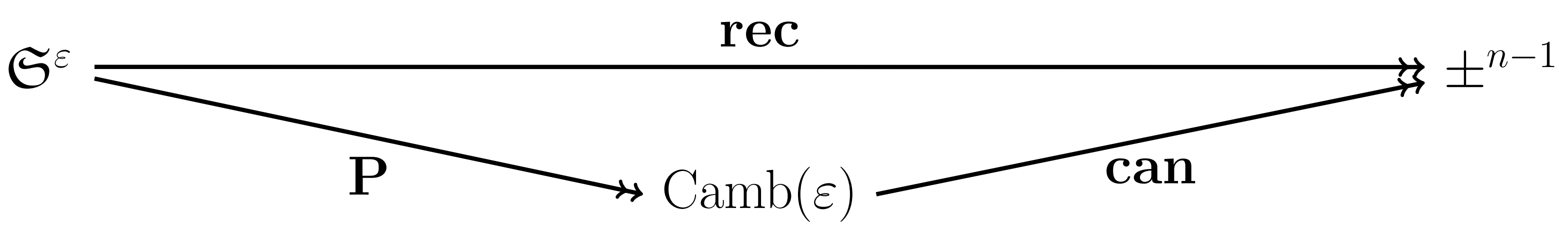
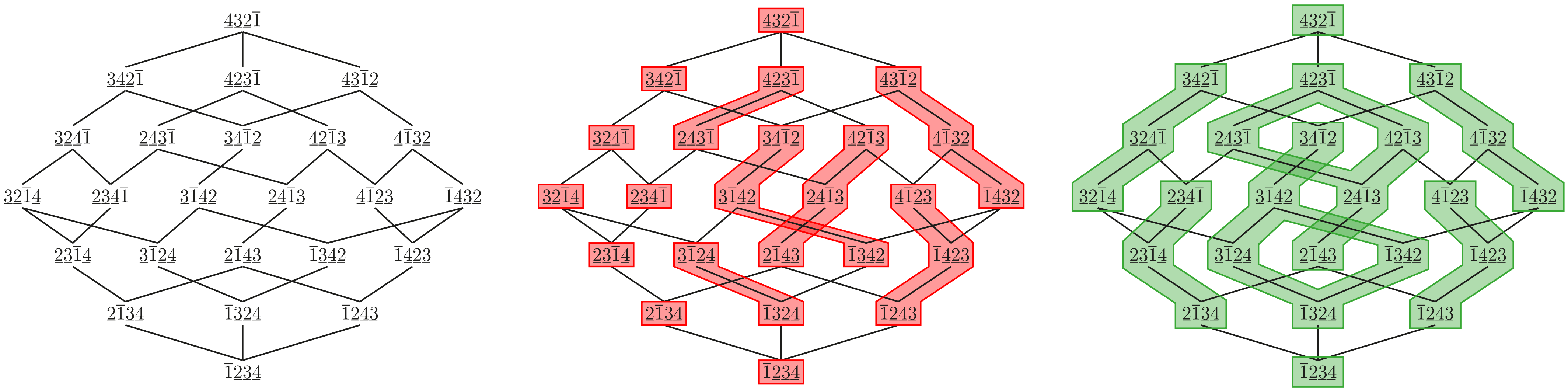


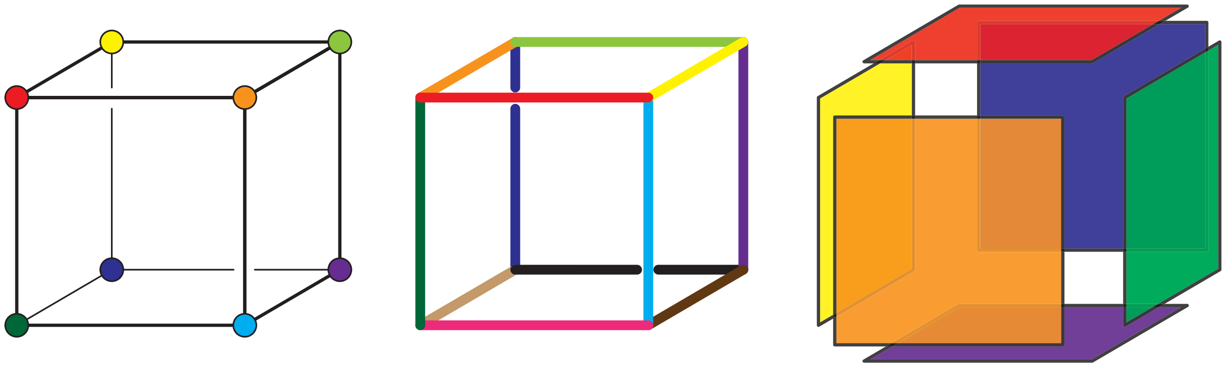


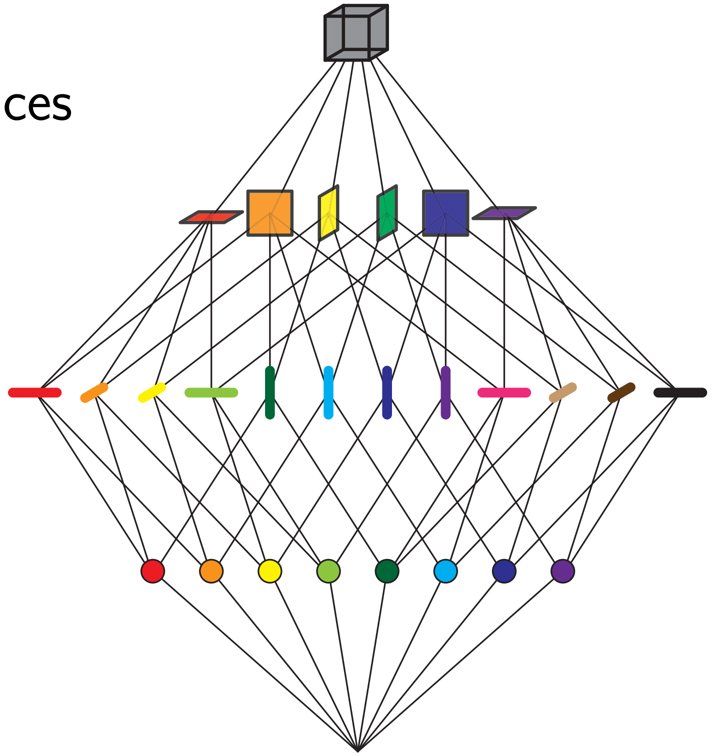
GEOMETRY

POLYTOPES & COMBINATORICS

polytope $=$ convex hull of a finite set of $\mathbb{R}^d$
$=$ bounded intersection of finitely many half-spaces

face $=$ intersection with a supporting hyperplane
face lattice $=$ all the faces with their inclusion relations

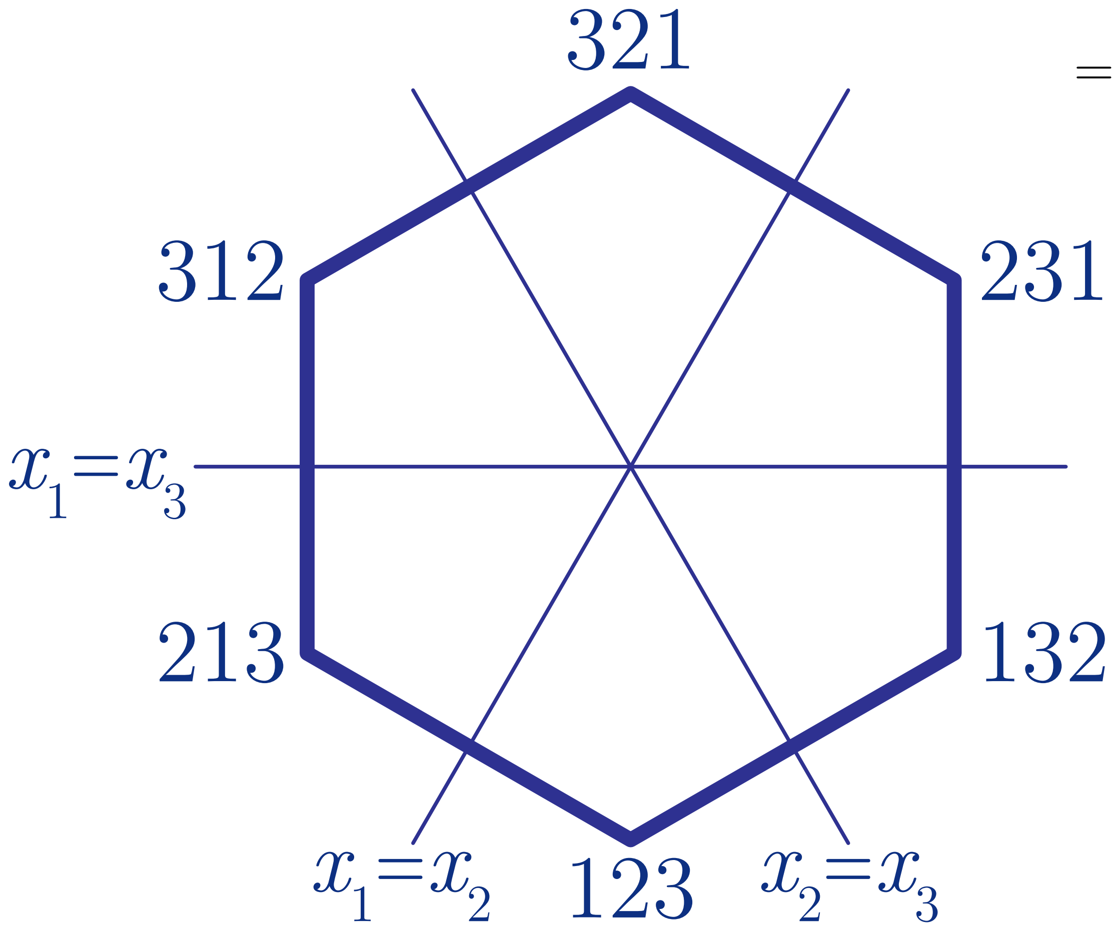




Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?

PERMUTAHEDRON

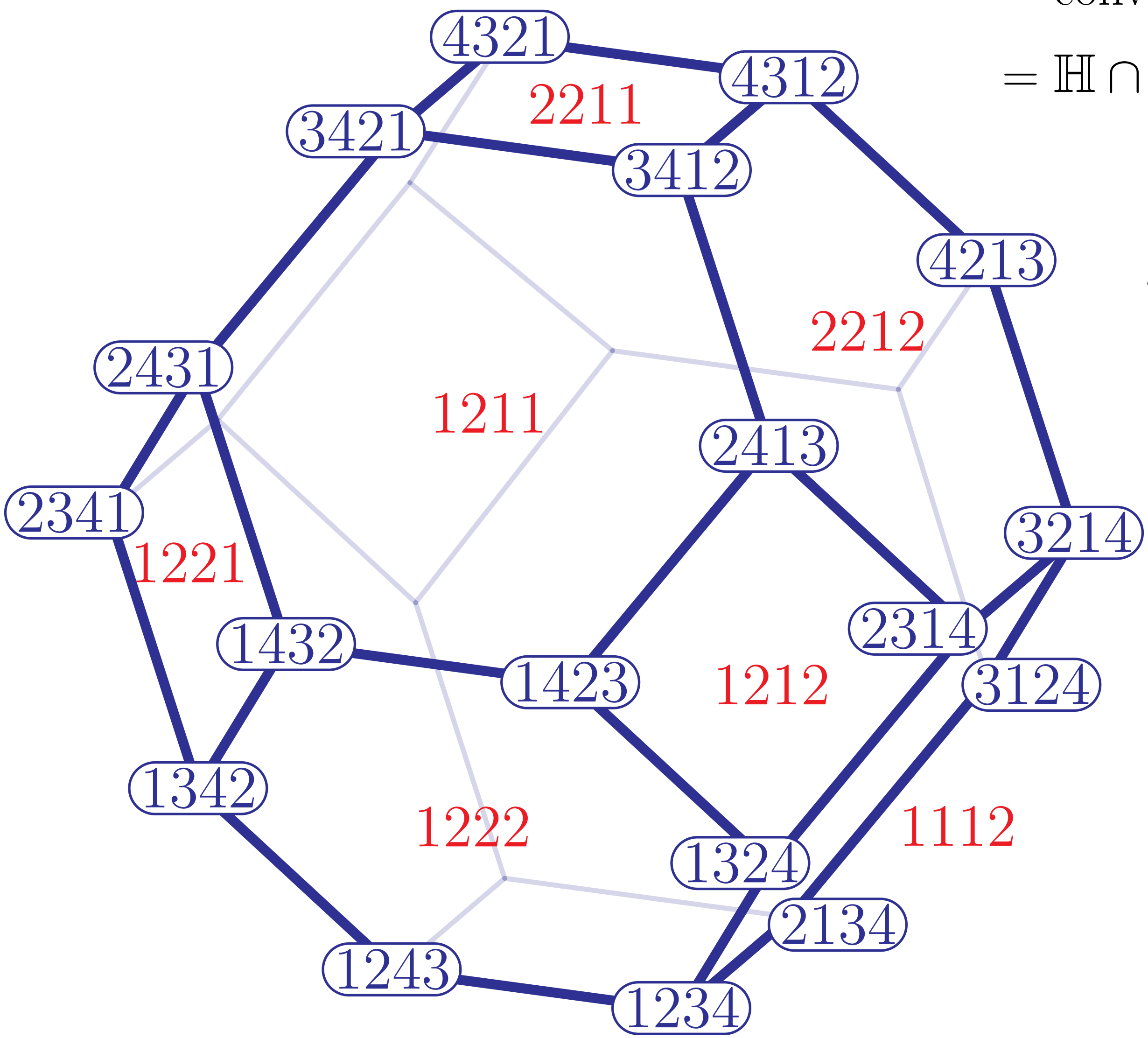


Permutohedron $\mathsf{Perm}(n)$

$$= \operatorname{conv}\left\{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \mathbf{H}^{\geq}(J)$$

PERMUTAHEDRON



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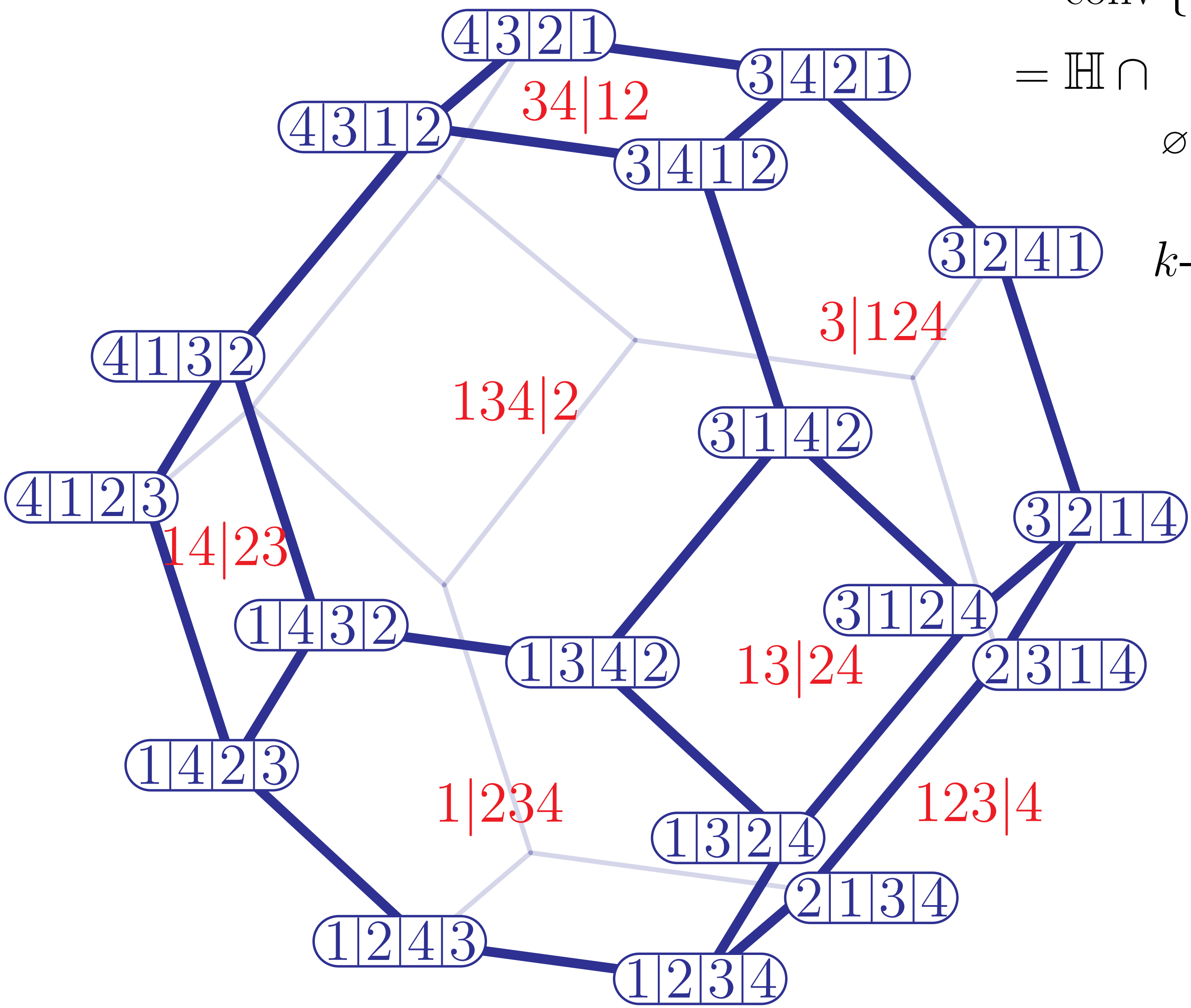
k-faces of $\mathsf{Perm}(n)$

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

PERMUTAHEDRON



Permutohedron Perm(n)

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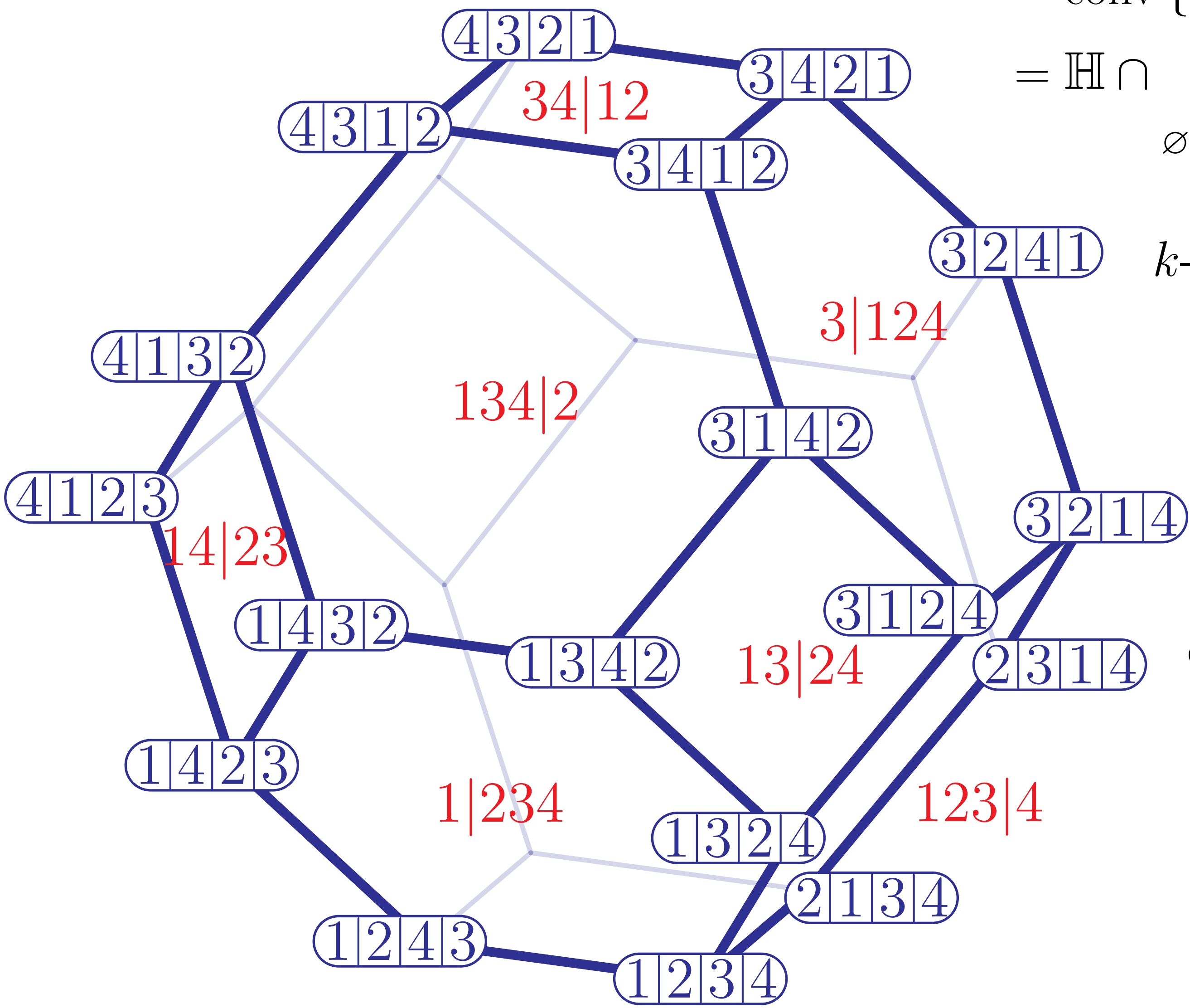
k-faces of Perm(n)

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

PERMUTAHEDRON



Permutohedron Perm(n)

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k-faces of Perm(n)

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

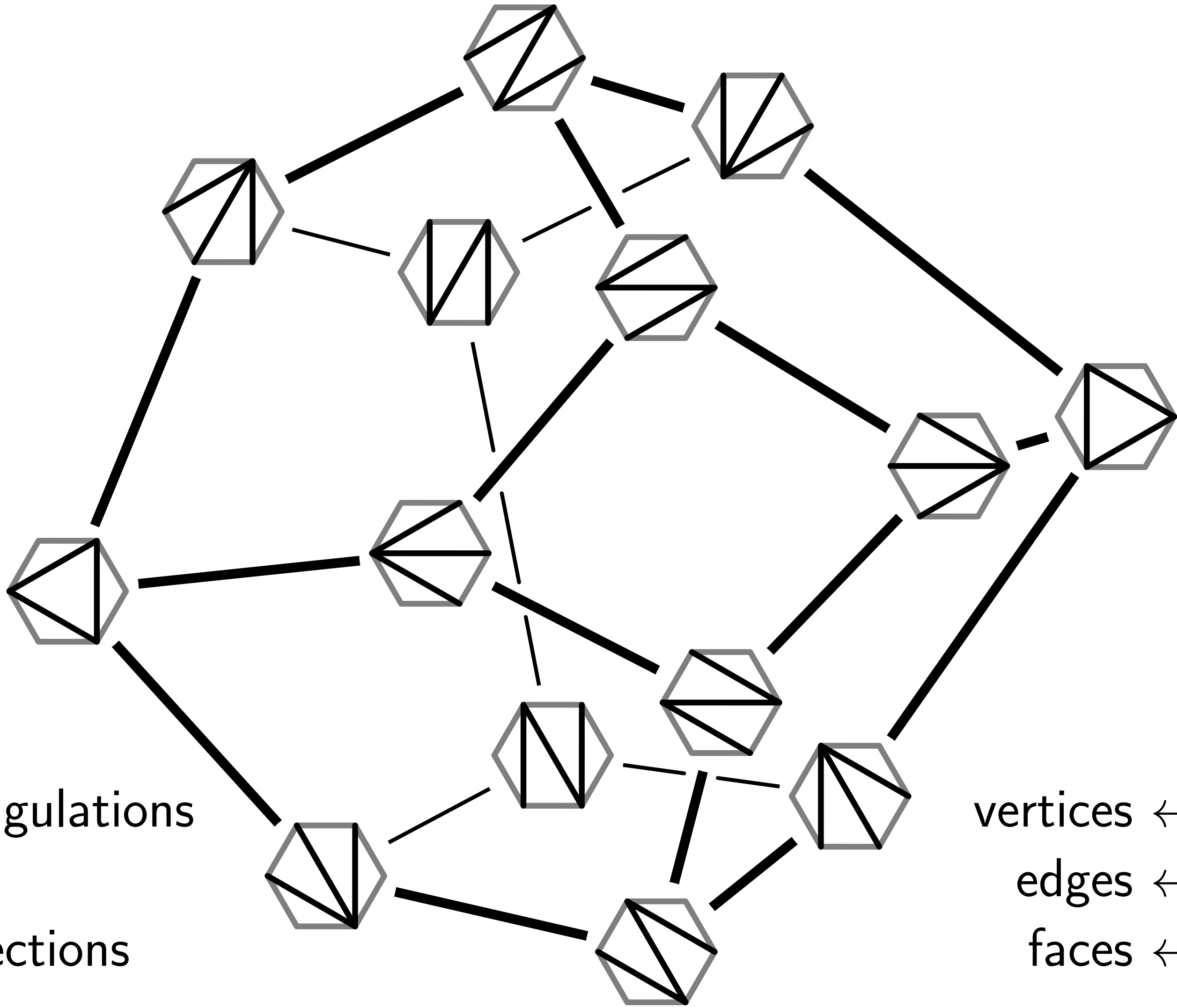
$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

connections to

- inversion sets
- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

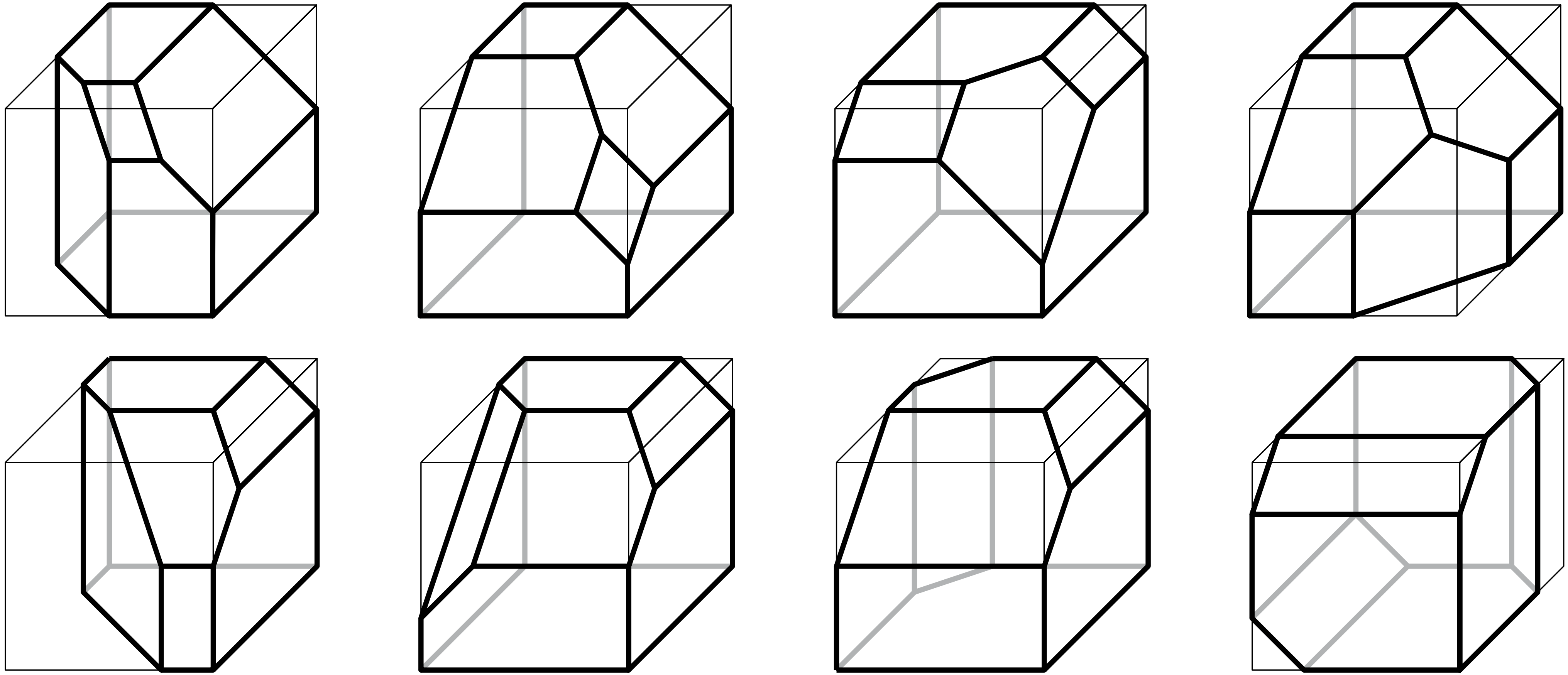


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

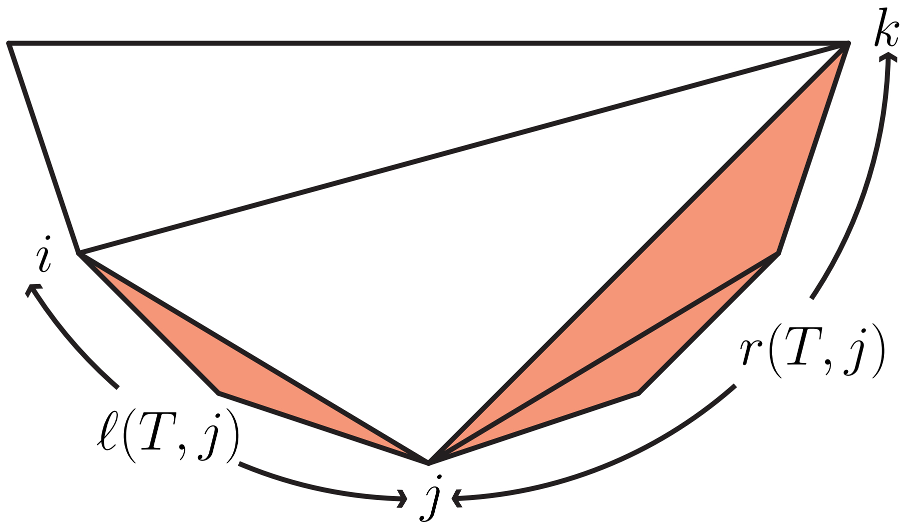


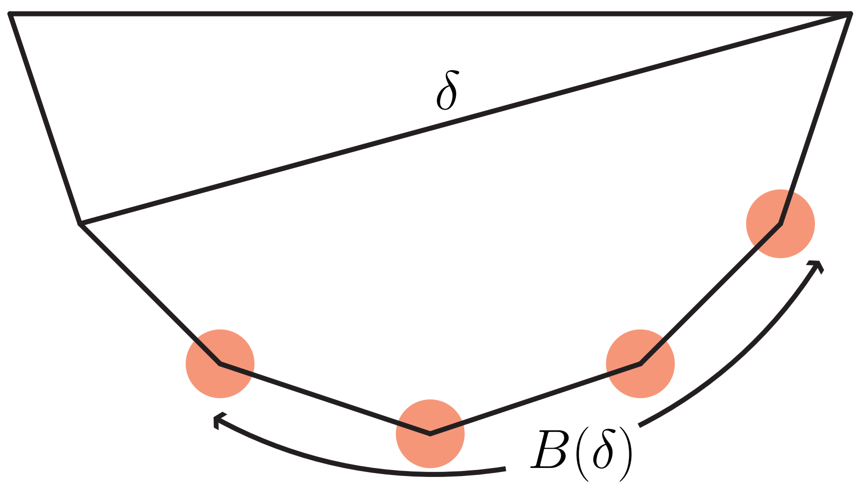
(Pictures by Ceballos-Santos-Ziegler)

Lee ('89), Gel'fand-Kapranov-Zelevinski ('94), Billera-Filliman-Sturmfels ('90), . . . , Ceballos-Santos-Ziegler ('11)
Loday ('04), Hohlweg-Lange ('07), Hohlweg-Lange-Thomas ('12), P.-Santos ('12), P.-Stump ('12$^+$), Lange-P. ('13$^+$)

LODAY'S ASSOCIAHEDRON

$$\text{Loday's associahedron} = \operatorname{conv}\{L(T) \mid T \text{ triangulation of the } (n+3)\text{-gon}\}$$
$$= \mathbb{H} \cap \bigcap_{\substack{\delta \text{ diagonal} \\ \text{of the } (n+3)\text{-gon}}} \mathbf{H}^{\geq}(\delta)$$

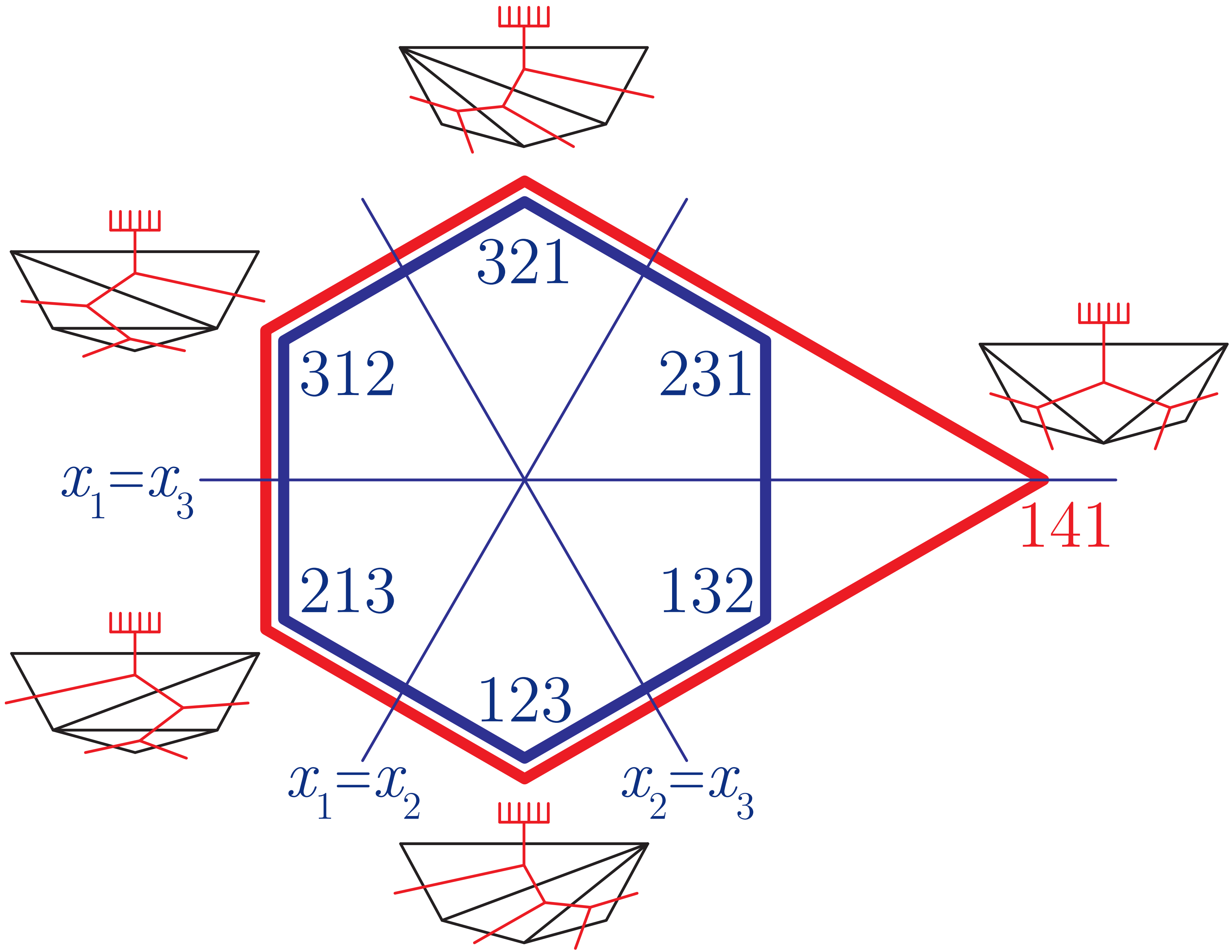




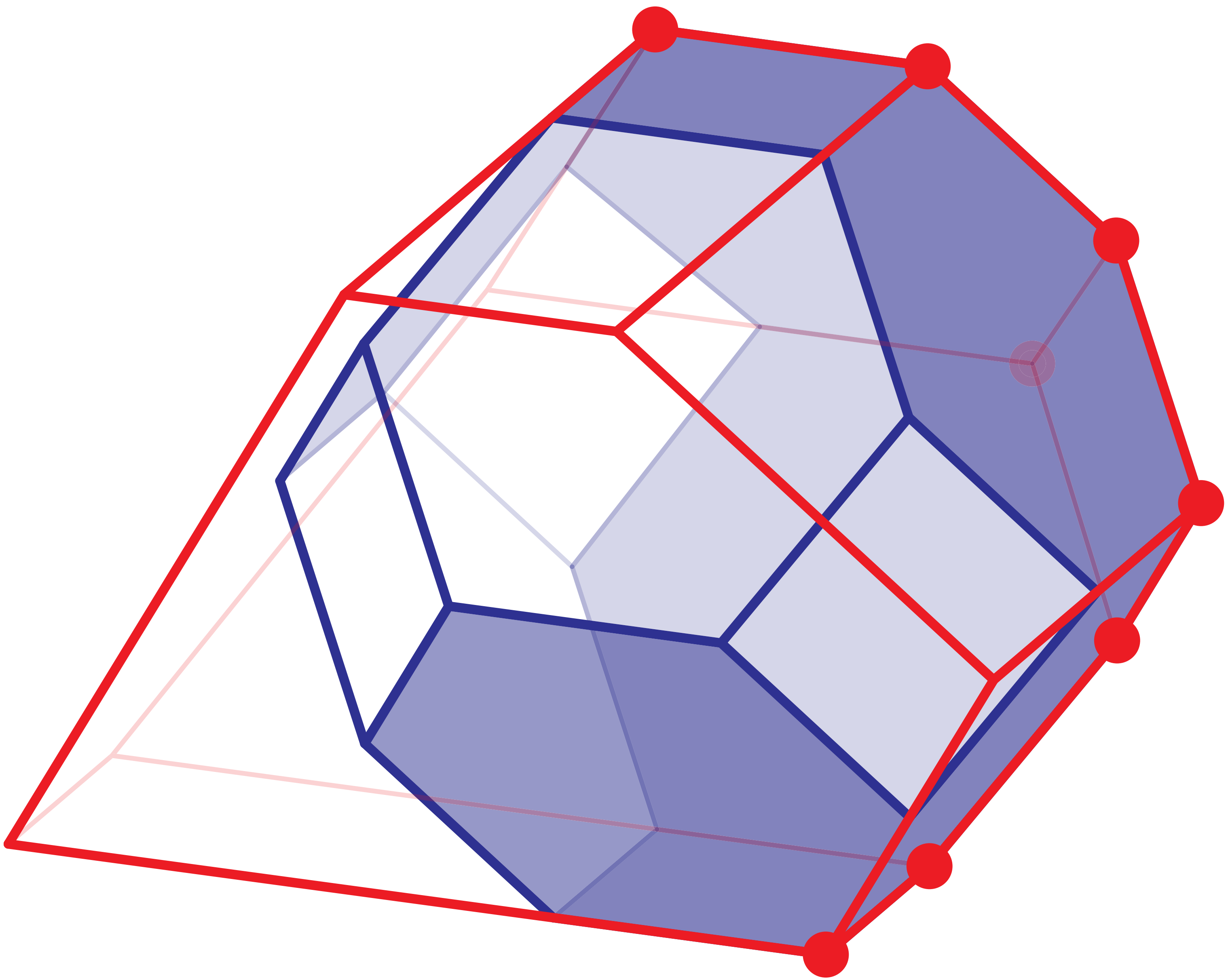
$$L(T) = \big(\ell(T,j)\cdot r(T,j)\big)_{j\in[n+1]} \qquad \mathbf{H}^{\geq}(\delta) = \left\{\mathbf{x}\in\mathbb{R}^{n+1} \;\middle|\; \sum_{j\in B(\delta)} x_j \geq \binom{|B(\delta)|+1}{2}\right\}$$

Loday, *Realization of the Stasheff polytope* ('04)

LODAY'S ASSOCIAHEDRON AND PERMUTAHEDRON



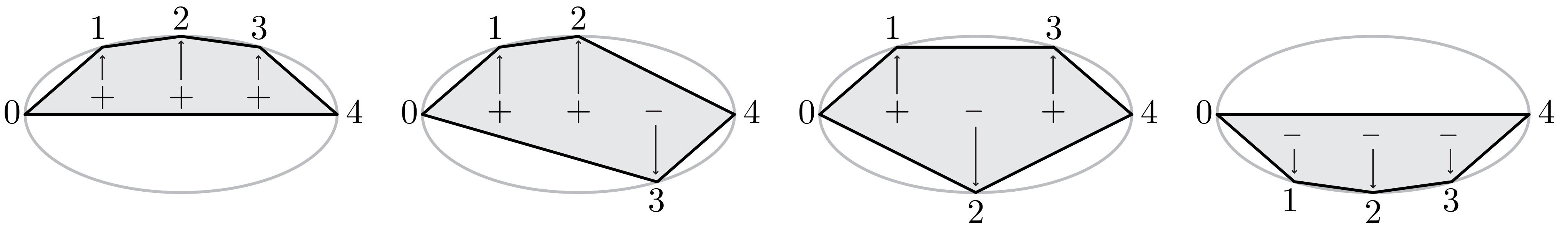
ASSOCIAHEDRON AND PERMUTAHEDRON



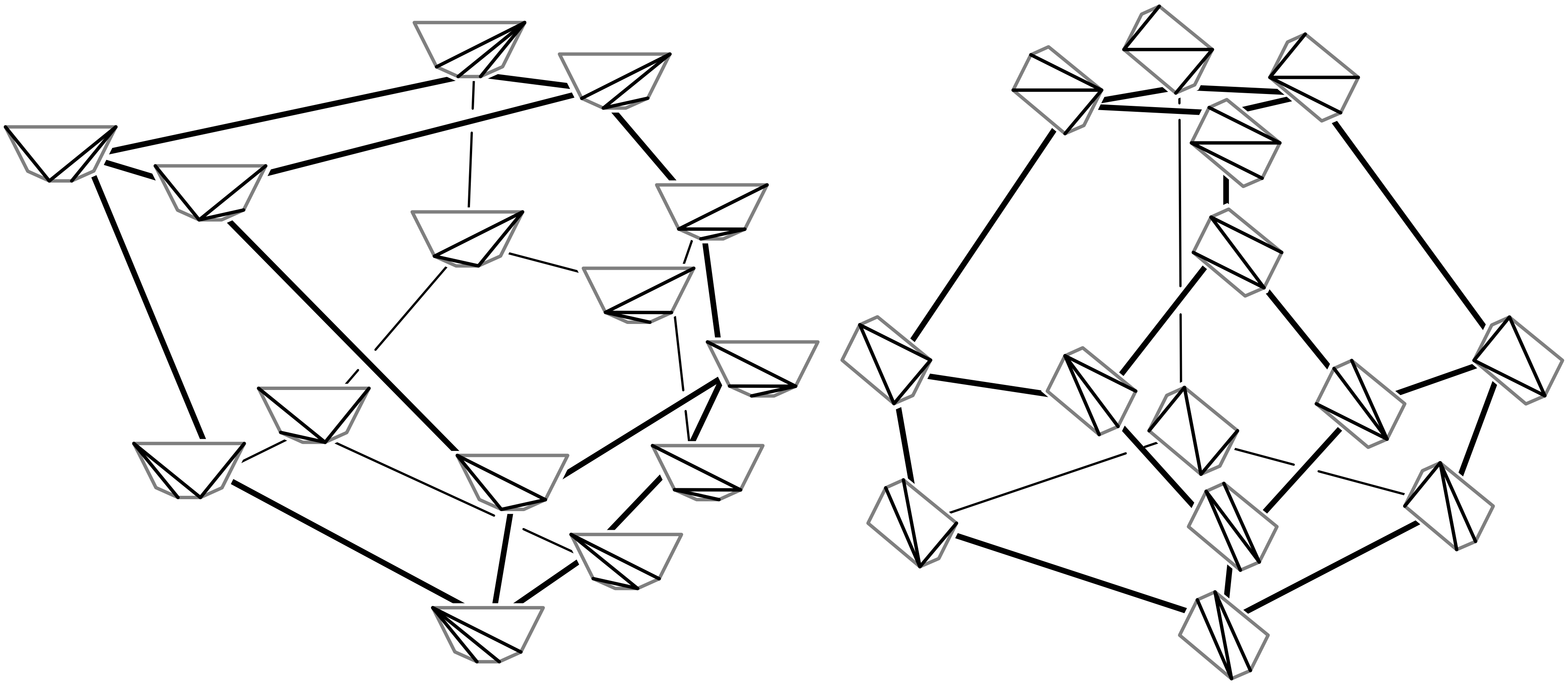
The associahedron is obtained from the permutahedron by removing facets

HOHLWEG & LANGE'S ASSOCIAHEDRA

Can also replace Loday's $(n+3)$-gon by others...



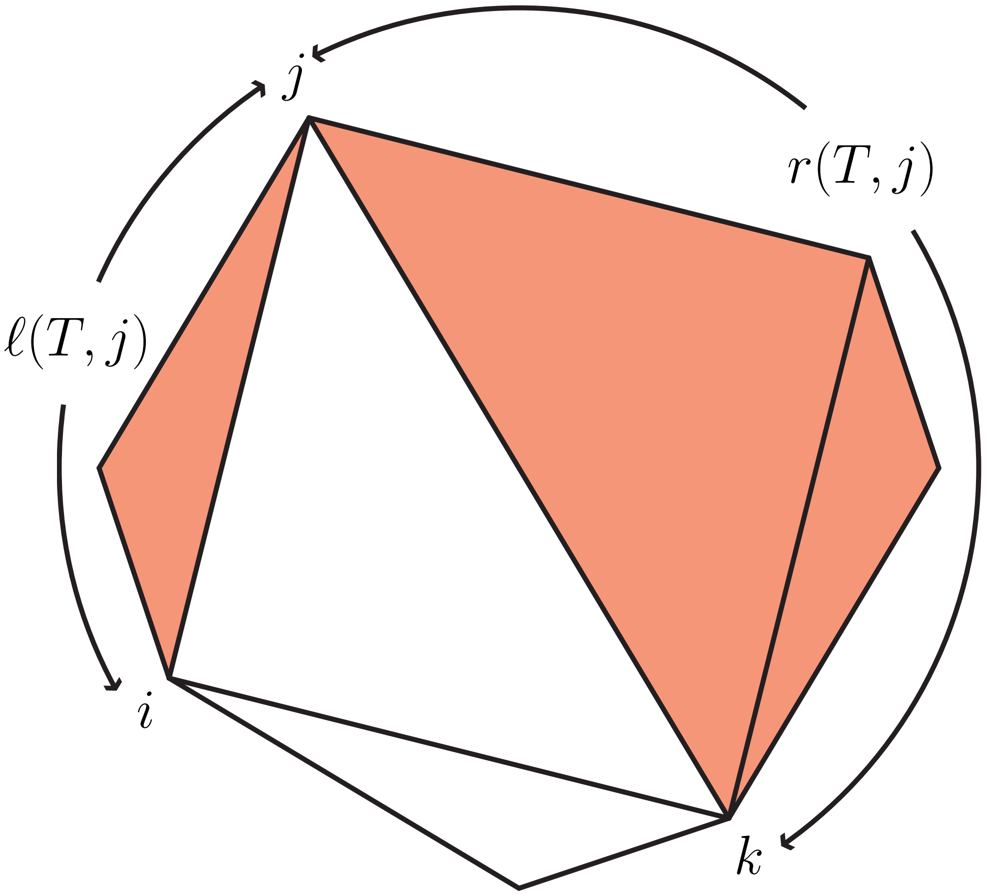
... to obtain different realizations of the associahedron

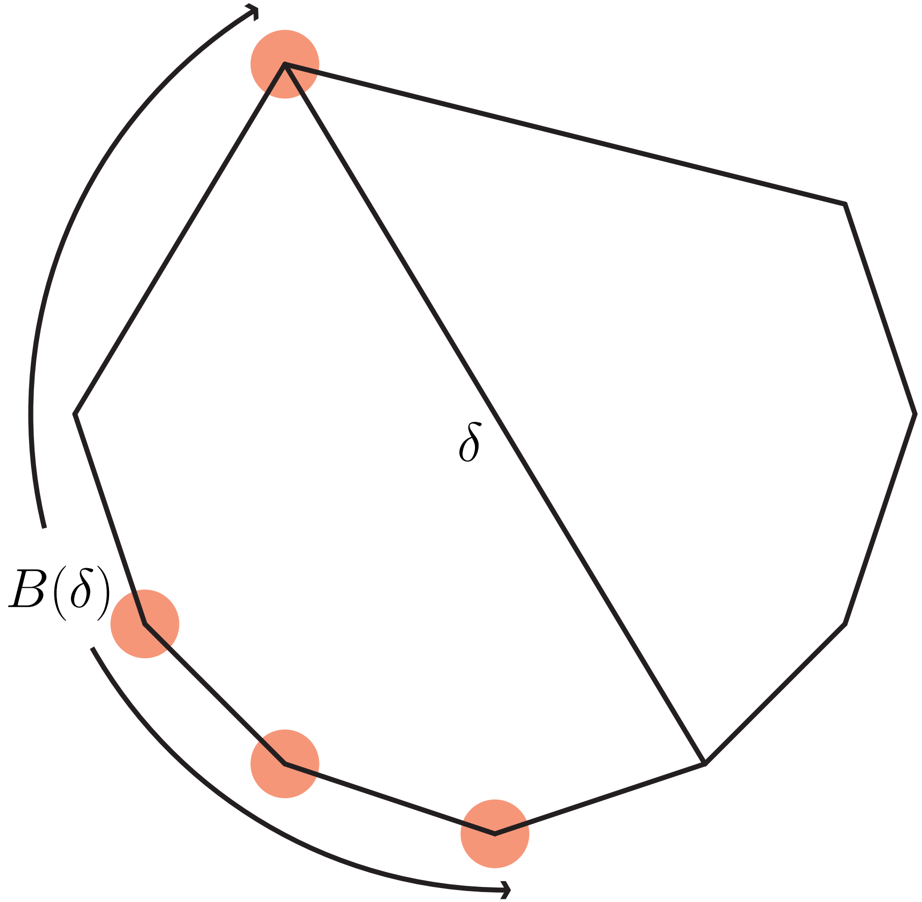


Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

HOHLWEG & LANGE'S ASSOCIAHEDRA

$$\mathsf{Asso}(P) \;=\; \operatorname{conv}\{HL(T) \mid T \text{ triangulation of } P\} \;=\; \mathbb{H} \cap \bigcap_{\delta \text{ diagonal of } P} \mathbf{H}^{\geq}(\delta)$$





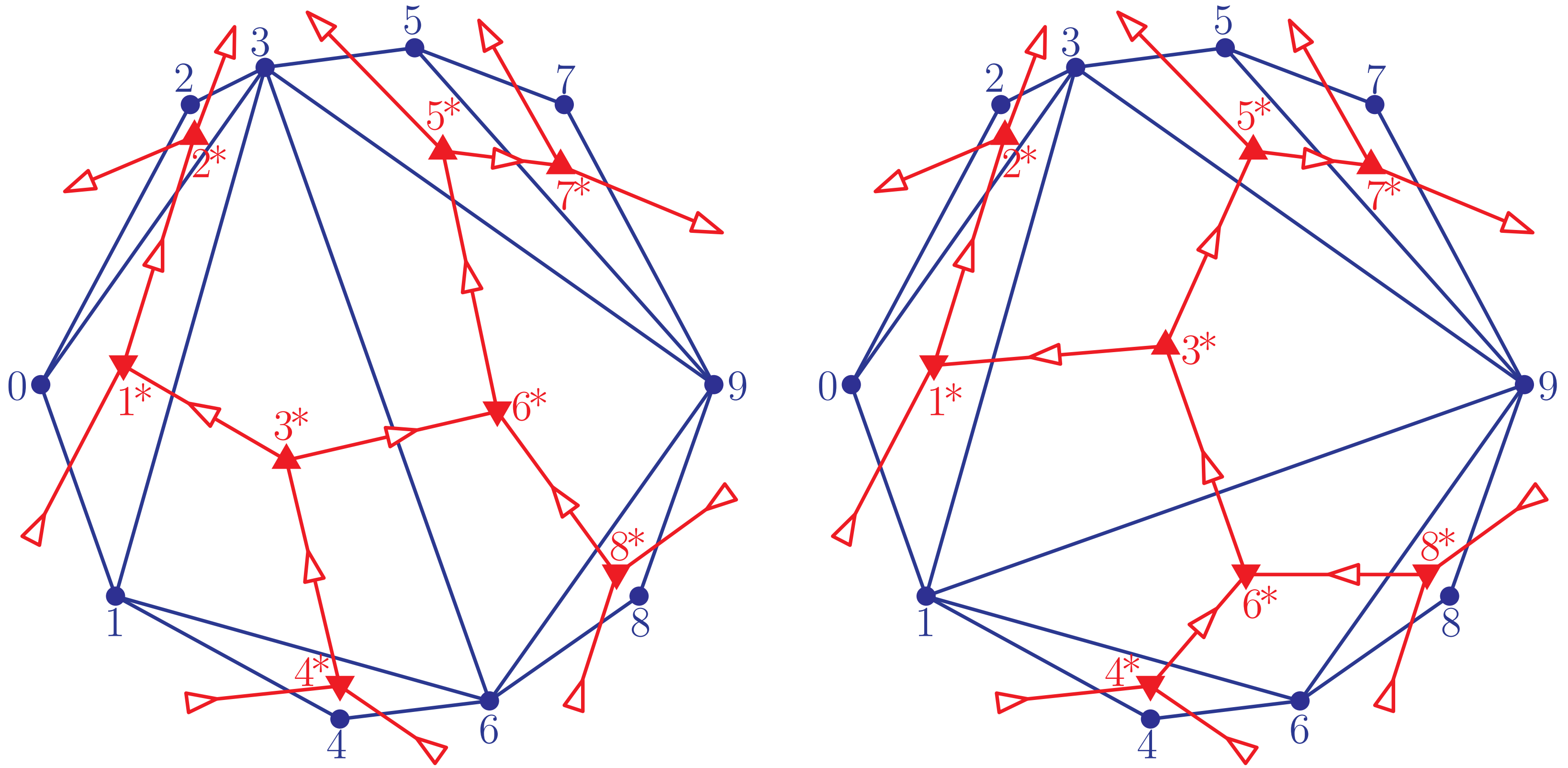
$$HL(T)_j = \begin{cases} \ell(T,j)\cdot r(T,j) & \text{if } j \text{ down} \\ n+2-\ell(T,j)\cdot r(T,j) & \text{if } j \text{ up} \end{cases}$$

$$\mathbf{H}^{\geq}(\delta) = \left\{ \mathbf{x} \;\middle|\; \sum_{j \in B(\delta)} x_j \geq \binom{|B(\delta)|+1}{2} \right\}$$

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

ASSOCIAHEDRA AND CAMBRIAN TREES

Lange-P., *Using spines to revisit a construction of the associahedron* ('13$^+$)



Cambrian trees = labeled and oriented dual binary trees

Alternative vertex description of Hohlweg-Lange's associahedra:

$$HL(T)_j = \begin{cases} |\{\pi \text{ maximal path in } T \text{ with 2 incoming arcs at } j\}| & \text{if } j \text{ down} \\ n+2-|\{\pi \text{ maximal path in } T \text{ with 2 outgoing arcs at } j\}| & \text{if } j \text{ up} \end{cases}$$

CAMBRIAN TREES AND NORMAL CONES

Incidence cone $\mathrm{C}(\mathrm{T}) = \mathrm{cone}\,\{e_i - e_j \mid \text{for all } i \to j \text{ in } \mathrm{T}\}$

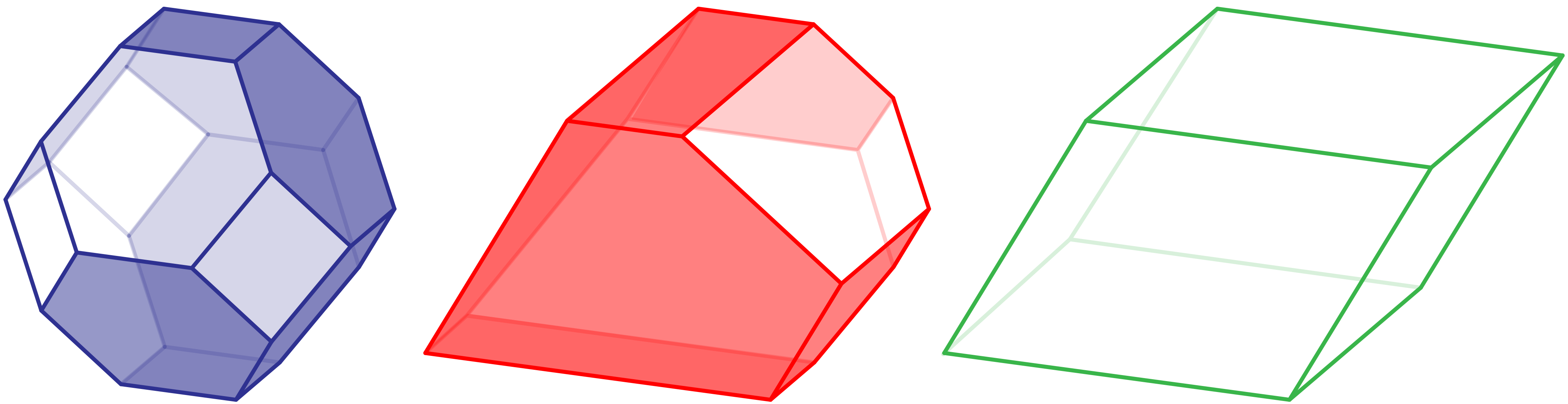
Braid cone $\mathrm{C}^\diamond(\mathrm{T}) = \{\mathbf{x} \in \mathbb{H} \mid x_i \le x_j \text{ for all } i \to j \text{ in } \mathrm{T}\}$

THEO. The cones form complete simplicial fans:

(i) $\{\mathrm{C}^\diamond(\tau) \mid \tau \in \mathfrak{S}_n\}$ = braid fan = normal fan of the permutahedron

(ii) $\{\mathrm{C}^\diamond(\mathrm{T}) \mid \mathrm{T} \in \mathrm{Camb}(\varepsilon)\}$ = ε-Cambrian fan = normal fan of the ε-associahedron

(iii) $\left\{\mathrm{C}^\diamond(\chi) \;\middle|\; \chi \in \pm^{n-1}\right\}$ = boolean fan = normal fan of the parallelepiped



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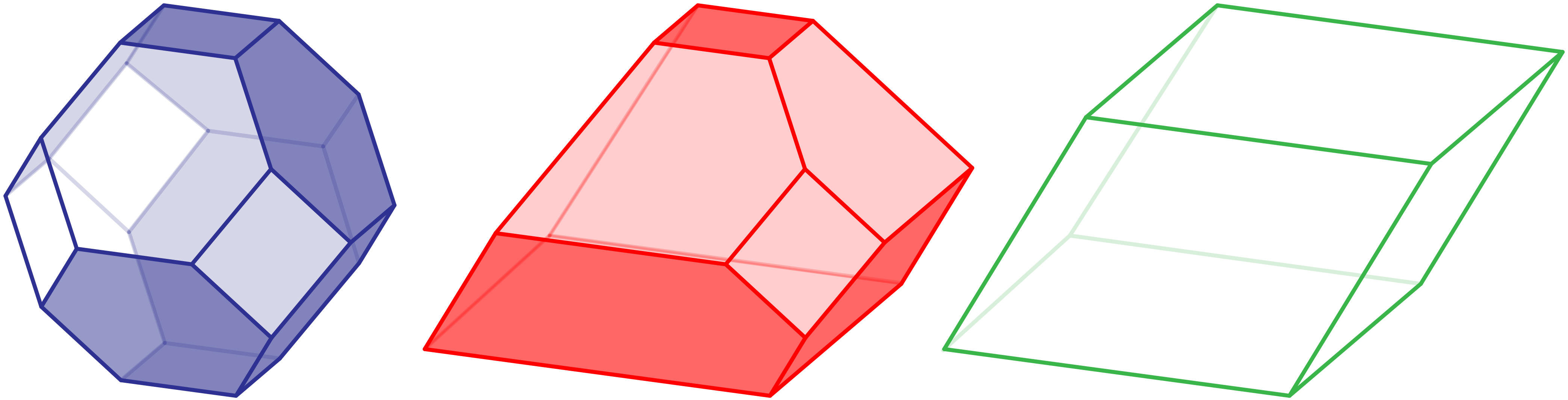

CAMBRIAN TREES AND NORMAL CONES

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(iii) $\left\{\mathrm{C}^\diamond(\chi) \;\middle|\; \chi \in \pm^{n-1}\right\}$ = boolean fan = normal fan of the parallelepiped

Characterization of fibers in terms of cones:

$$\begin{aligned}
\mathrm{T} = \mathbf{P}(\tau) \iff \mathrm{C}(\mathrm{T}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\mathrm{T}) \supseteq \mathrm{C}^\diamond(\tau),\\
\chi = \mathbf{can}(\mathrm{T}) \iff \mathrm{C}(\chi) \subseteq \mathrm{C}(\mathrm{T}) \iff \mathrm{C}^\diamond(\chi) \supseteq \mathrm{C}^\diamond(\mathrm{T}),\\
\chi = \mathbf{rec}(\tau) \iff \mathrm{C}(\chi) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\chi) \supseteq \mathrm{C}^\diamond(\tau).
\end{aligned}$$

ALGEBRA

SHUFFLE AND CONVOLUTION

For $n, n' \in \mathbb{N}$, consider the set of perms of $\mathfrak{S}_{n+n'}$ with at most one descent, at position n:

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n')\}$$

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

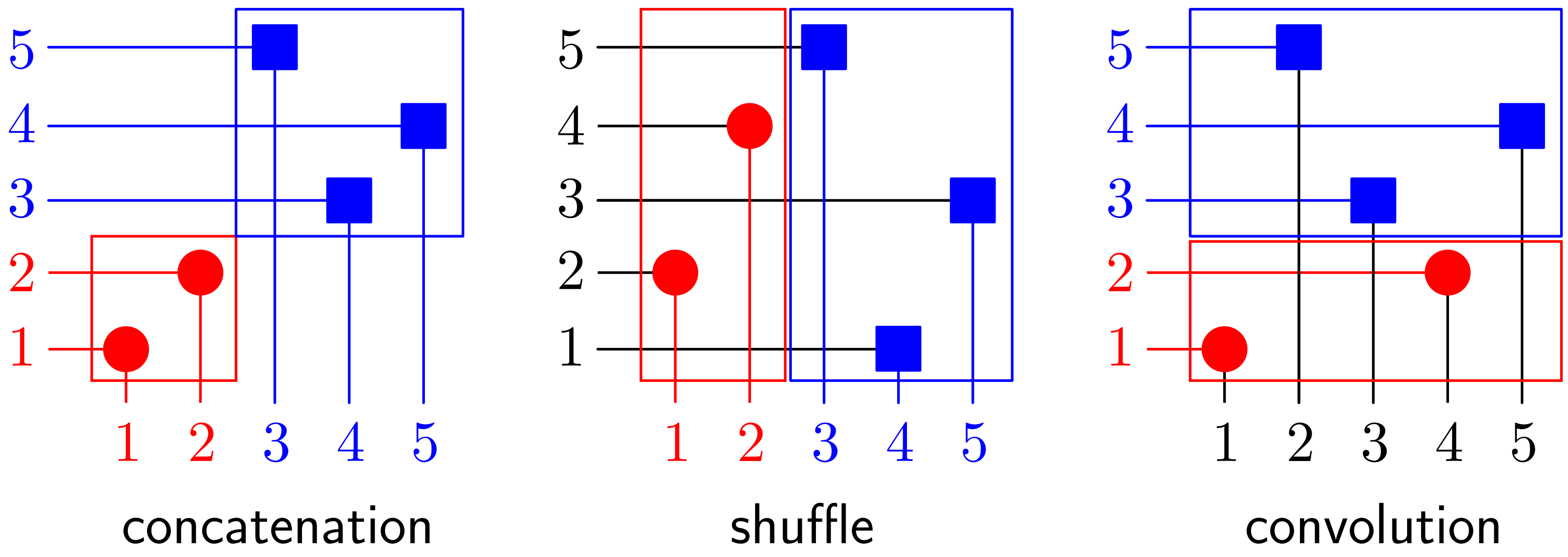
shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1)+n, \dots, \tau'(n')+n] \in \mathfrak{S}_{n+n'}$

shifted shuffle product $\tau \,\bar{\sqcup\!\sqcup}\, \tau' = \left\{(\tau\bar{\tau}') \circ \pi^{-1} \;\middle|\; \pi \in \mathfrak{S}^{(n,n')}\right\} \subset \mathfrak{S}_{n+n'}$

convolution product $\tau \star \tau' = \left\{\pi \circ (\tau\bar{\tau}') \;\middle|\; \pi \in \mathfrak{S}^{(n,n')}\right\} \subset \mathfrak{S}_{n+n'}$

Exm: $12 \,\bar{\sqcup\!\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



MALVENUTO-REUTENAUER ALGEBRA

DEF. Combinatorial Hopf Algebra $=$ combinatorial vector space $\mathcal{B}$ endowed with

$$\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$$

$$\text{coproduct } \triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

which are "compatible", ie.

$$\begin{array}{ccccc}
\mathcal{B} \otimes \mathcal{B} & \xrightarrow{\quad \cdot \quad} & \mathcal{B} & \xrightarrow{\quad \triangle \quad} & \mathcal{B} \otimes \mathcal{B} \\
\downarrow {\scriptstyle \triangle \otimes \triangle} & & & & \uparrow {\scriptstyle \cdot \otimes \cdot} \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & & \xrightarrow[I \otimes \mathrm{swap} \otimes I]{} & & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}
\end{array}$$

Malvenuto-Reteunauer algebra $=$ Hopf algebra FQSym with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ and where

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

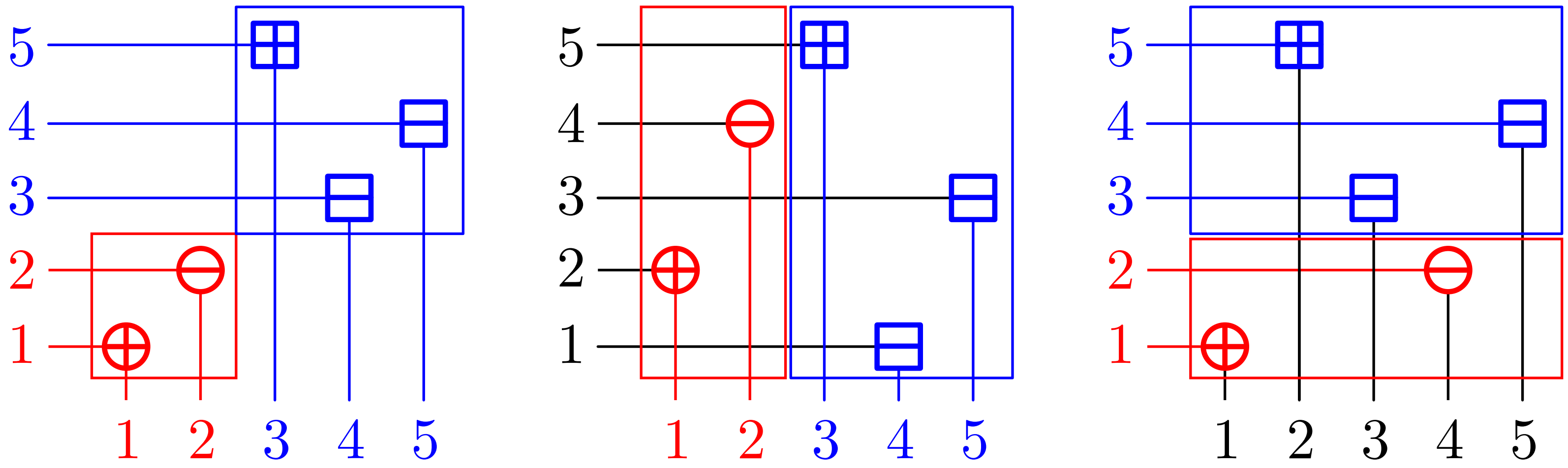
SIGNED VERSION

For signed permutations:

- signs are attached to values in the shuffle product
- signs are attached to positions in the convolution product

Exm: $\bar{1}2 \sqcup\!\sqcup 23\bar{1} = \{\bar{1}2453\bar{3}, \bar{1}4253\bar{3}, \bar{1}452\bar{3}, \bar{1}45\bar{3}2, 4\bar{1}25\bar{3}, 4\bar{1}52\bar{3}, 4\bar{1}5\bar{3}2, 45\bar{1}2\bar{3}, 45\bar{1}\bar{3}2, 45\bar{3}\bar{1}2\},$

$\bar{1}2 \star 23\bar{1} = \{\bar{1}245\bar{3}, \bar{1}345\bar{2}, \bar{1}435\bar{2}, \bar{1}534\bar{2}, \bar{2}345\bar{1}, \bar{2}435\bar{1}, \bar{2}534\bar{1}, \bar{3}425\bar{1}, \bar{3}524\bar{1}, \bar{4}523\bar{1}\}.$



concatenation shuffle convolution

$\mathrm{FQSym}_{\pm}$ = Hopf algebra with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}_\pm}$ and where

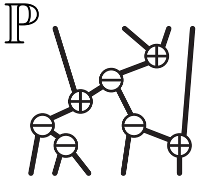
$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \sqcup\!\sqcup \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

CAMBRIAN ALGEBRA AS SUBALGEBRA OF FQSym$_\pm$

Cambrian algebra = vector subspace Camb of FQSym$_\pm$ generated by

$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau \in \mathfrak{S}_\pm \\ \mathbf{P}(\tau) = \mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T})} \mathbb{F}_\tau,$$

for all Cambrian trees T.

Exm: $\mathbb{P}$ 

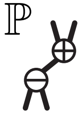
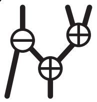
$$= \mathbb{F}_{\underline{21}\overline{37}\underline{54}\overline{6}} + \mathbb{F}_{\underline{21}\overline{73}\underline{54}\overline{6}} + \mathbb{F}_{\underline{21}\overline{7}\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{1}\overline{3}\underline{54}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{15}\overline{3}\underline{4}\overline{6}}$$
$$+ \mathbb{F}_{\underline{2}\overline{7}\underline{51}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{21}\overline{3}\underline{54}\overline{6}} + \mathbb{F}_{\overline{7}\underline{215}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{251}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{521}\overline{3}\underline{4}\overline{6}}$$

THEO. Camb is a subalgebra of FQSym$_\pm$

(ie. the Cambrian congruence is "compatible" with the product and coproduct in FQSym$_\pm$)

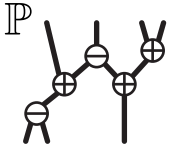
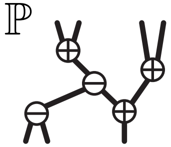
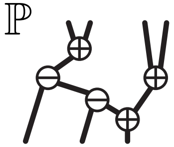
GAME: Explain the product and coproduct directly on the Cambrian trees...

PRODUCT IN CAMBRIAN ALGEBRA

$\mathbb{P}$  $\cdot\ \mathbb{P}$ 

$$= \mathbb{F}_{\underline{1}\overline{2}} \cdot \left(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{2}\overline{3}\underline{1}}\right)$$

$$= \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{F}_{\underline{1}\overline{2}\overline{4}\overline{5}\underline{3}} + \mathbb{F}_{\underline{1}\overline{4}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\underline{1}\overline{4}\overline{2}\overline{5}\underline{3}} + \mathbb{F}_{\underline{1}\overline{4}\overline{5}\overline{2}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\overline{5}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\overline{2}\underline{3}} + \mathbb{F}_{\overline{4}\overline{5}\underline{1}\overline{2}\underline{3}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{2}\overline{5}} + \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{5}\overline{2}} \\ + \mathbb{F}_{\underline{1}\overline{4}\overline{5}\underline{3}\overline{2}} + \mathbb{F}_{\overline{4}\underline{1}\underline{3}\overline{2}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\underline{3}\overline{5}\overline{2}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\underline{3}\overline{2}} \\ + \mathbb{F}_{\overline{4}\overline{5}\underline{1}\underline{3}\overline{2}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\overline{4}\underline{3}\underline{1}\overline{2}\overline{5}} + \mathbb{F}_{\overline{4}\underline{3}\underline{1}\overline{5}\overline{2}} \\ + \mathbb{F}_{\overline{4}\underline{3}\overline{5}\underline{1}\overline{2}} + \mathbb{F}_{\overline{4}\overline{5}\underline{3}\underline{1}\overline{2}} \end{pmatrix}$$

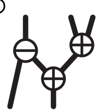
$= \mathbb{P}$  $+\ \mathbb{P}$  $+\ \mathbb{P}$ 

PROP. For any Cambrian trees T and T',

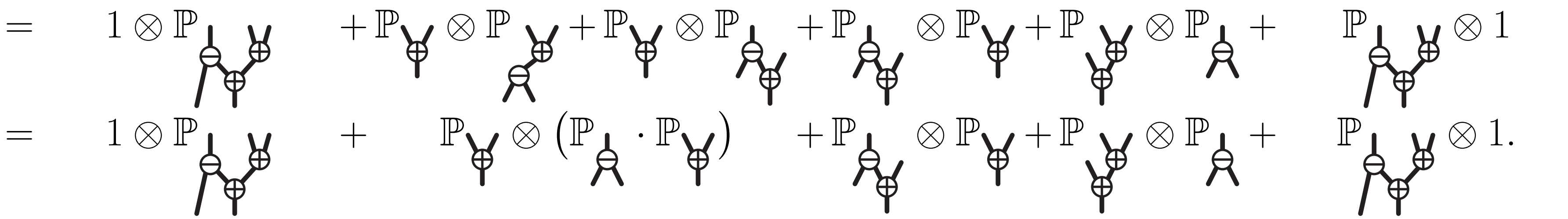
$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}}$$

where S runs over the interval $\left[\mathrm{T} \nearrow \bar{\mathrm{T}}',\ \mathrm{T} \nwarrow \bar{\mathrm{T}}'\right]$ in the $\varepsilon(\mathrm{T})\varepsilon(\mathrm{T}')$-Cambrian lattice

COPRODUCT IN CAMBRIAN ALGEBRA

$\triangle\mathbb{P}$  $= \triangle\left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right)$

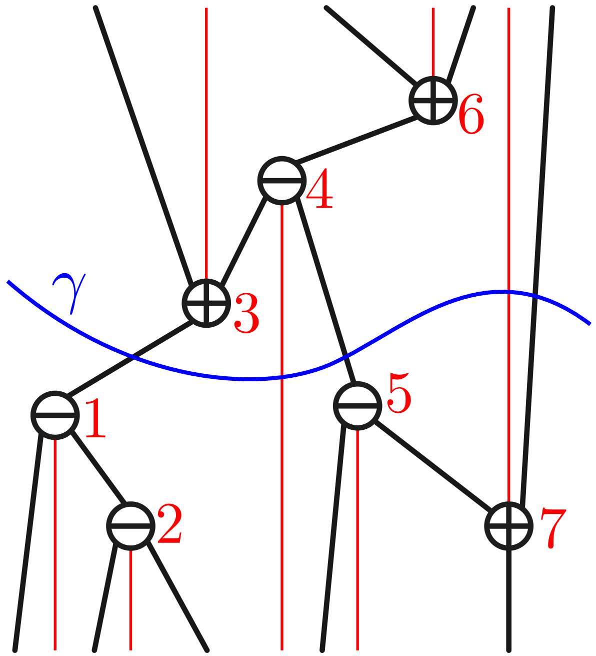
$$= 1 \otimes \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\underline{1}\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\bar{2}\underline{1}} + \mathbb{F}_{\bar{2}\underline{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{\bar{1}\bar{2}} \otimes \mathbb{F}_{\underline{1}} + \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) \otimes 1$$



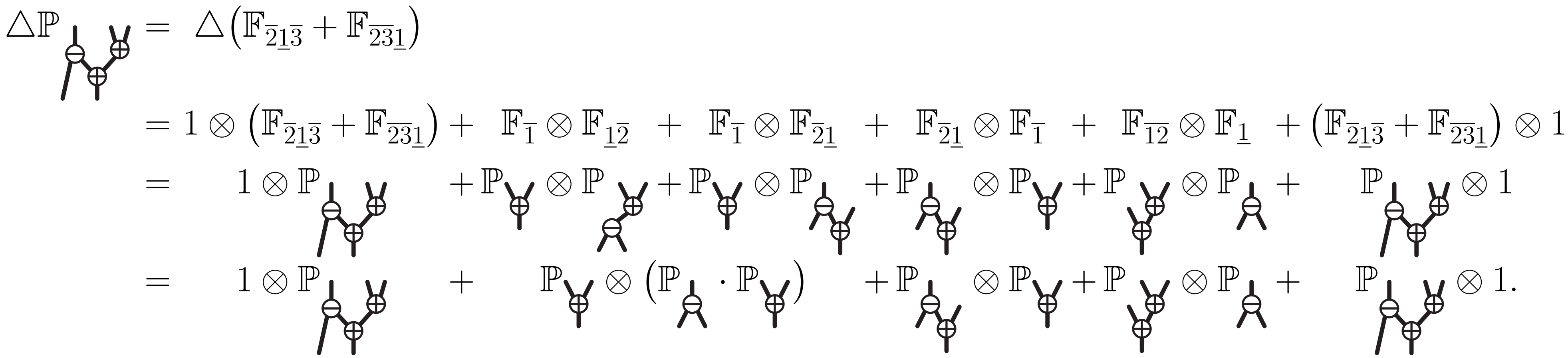
PROP. For any Cambrian tree S,

$$\triangle\mathbb{P}_{\mathrm{S}} = \sum_{\gamma} \Bigg(\prod_{\mathrm{T} \in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}} \Bigg) \otimes \Bigg(\prod_{\mathrm{T}' \in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'} \Bigg)$$

where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the Cambrian forests above and below γ respectively



COPRODUCT IN CAMBRIAN ALGEBRA

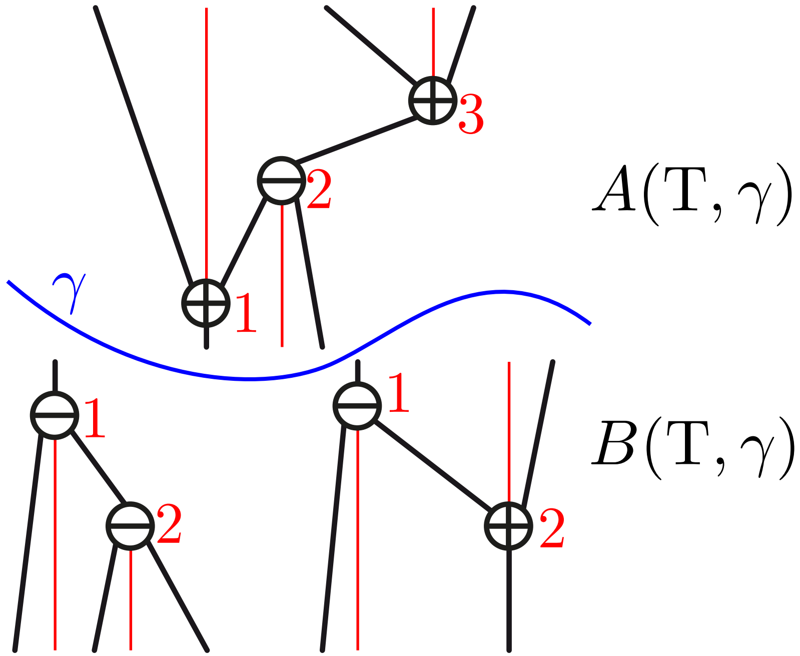


$$= 1\otimes\left(\mathbb{F}_{\bar2\underline1\bar3}+\mathbb{F}_{\bar2\bar3\underline1}\right) + \mathbb{F}_{\bar1}\otimes\mathbb{F}_{\underline1\bar2} + \mathbb{F}_{\bar1}\otimes\mathbb{F}_{\bar2\underline1} + \mathbb{F}_{\bar2\underline1}\otimes\mathbb{F}_{\bar1} + \mathbb{F}_{\bar1\bar2}\otimes\mathbb{F}_{\underline1} + \left(\mathbb{F}_{\bar2\underline1\bar3}+\mathbb{F}_{\bar2\bar3\underline1}\right)\otimes 1$$

PROP. For any Cambrian tree S,

$$\triangle\mathbb{P}_{\mathrm{S}} = \sum_{\gamma}\Bigg(\prod_{\mathrm{T}\in B(\mathrm{S},\gamma)}\mathbb{P}_{\mathrm{T}}\Bigg)\otimes\Bigg(\prod_{\mathrm{T}'\in A(\mathrm{S},\gamma)}\mathbb{P}_{\mathrm{T}'}\Bigg)$$

where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the Cambrian forests above and below γ respectively



DUAL CAMBRIAN ALGEBRA AS QUOTIENT OF $\mathsf{FQSym}_{\pm}^{*}$

$\mathsf{FQSym}_{\pm}^{*}$ = dual Hopf algebra with basis $(\mathbb{G}_{\tau})_{\tau \in \mathfrak{S}_{\pm}}$ and where

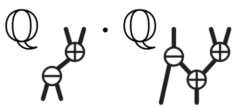
$$\mathbb{G}_{\tau} \cdot \mathbb{G}_{\tau'} = \sum_{\sigma \in \tau \star \tau'} \mathbb{G}_{\sigma} \qquad \text{and} \qquad \triangle \mathbb{G}_{\sigma} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{G}_{\tau} \otimes \mathbb{G}_{\tau'}$$

PROP. The graded dual Camb^{*} of the Cambrian algebra is isomorphic to the image of $\mathsf{FQSym}_{\pm}^{*}$ under the canonical projection

$$\pi : \mathbb{C}\langle A \rangle \longrightarrow \mathbb{C}\langle A \rangle / \equiv,$$

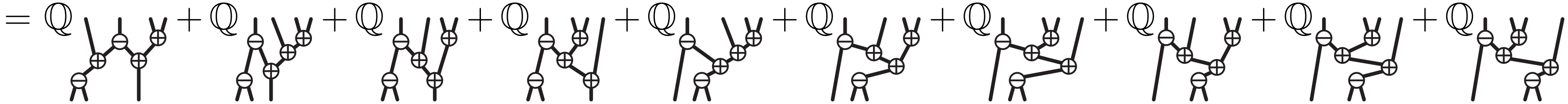
where $\equiv$ denotes the Cambrian congruence. The dual basis $\mathbb{Q}_{\mathrm{T}}$ of $\mathbb{P}_{\mathrm{T}}$ is expressed as $\mathbb{Q}_{\mathrm{T}} = \pi(\mathbb{G}_{\tau})$, where τ is any linear extension of T

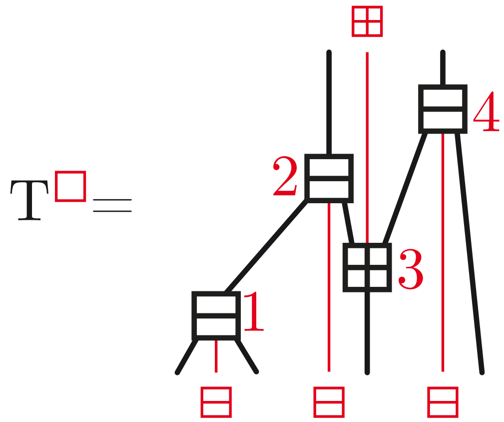
PRODUCT IN DUAL CAMBRIAN ALGEBRA

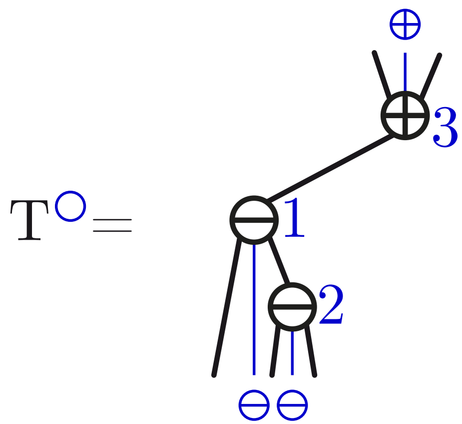


$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

$$= \mathbb{G}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\underline{3}\overline{4}\overline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{4}\underline{3}\overline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{5}\underline{3}\overline{2}\overline{4}} + \mathbb{G}_{\overline{2}\underline{3}\overline{4}\underline{1}\overline{5}} + \mathbb{G}_{\overline{2}\overline{4}\underline{3}\underline{1}\overline{5}} + \mathbb{G}_{\overline{2}\overline{5}\underline{3}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{4}\overline{2}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{5}\overline{2}\underline{1}\overline{4}} + \mathbb{G}_{\overline{4}\overline{5}\overline{2}\underline{1}\underline{3}}$$







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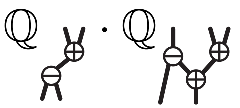
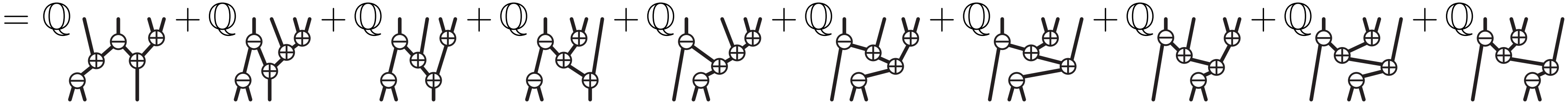
where s runs over all shuffles of $\varepsilon(\mathrm{T})$ and $\varepsilon(\mathrm{T}')$

PRODUCT IN DUAL CAMBRIAN ALGEBRA



$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

$$= \mathbb{G}_{\underline{1}\overline{24}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{34}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{43}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{53}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{34}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{43}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{53}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{42}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{52}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{52}\underline{1}\overline{3}}$$



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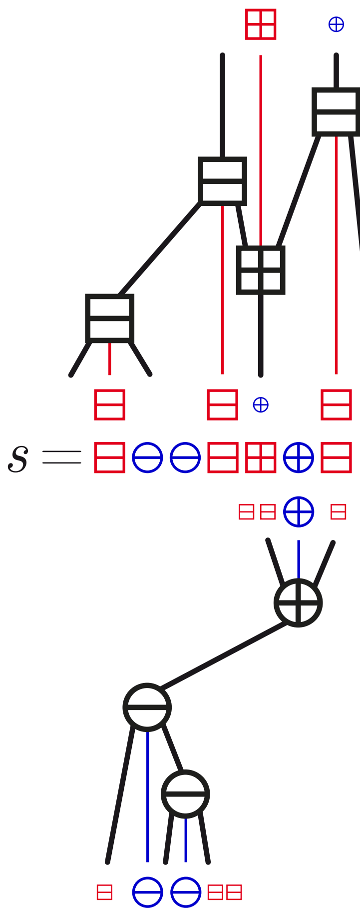

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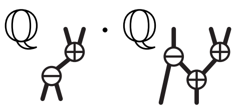
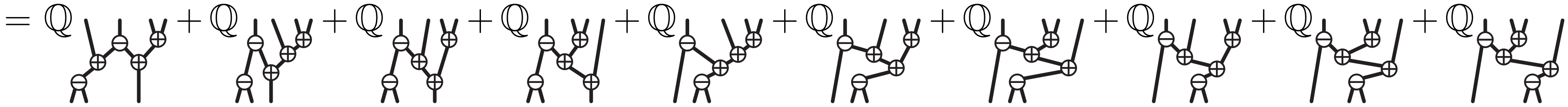


$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

$$= \mathbb{G}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{3}\overline{4}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{4}\overline{3}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{5}\overline{3}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{3}\overline{4}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{4}\overline{3}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{5}\overline{3}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{4}\overline{2}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{5}\overline{2}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{5}\overline{2}\underline{1}\overline{3}}$$



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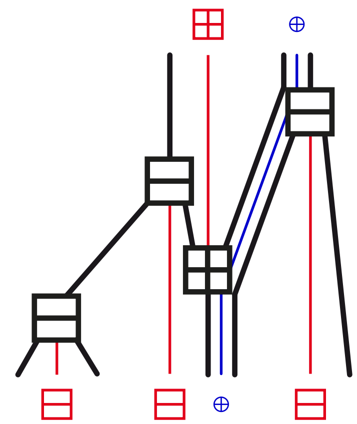
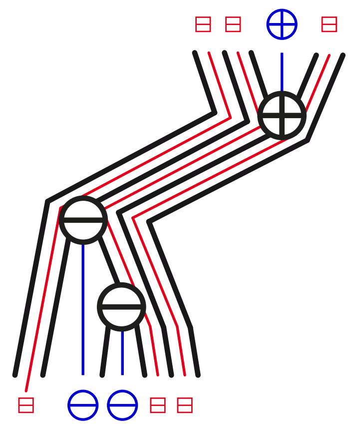
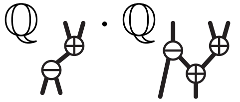




PRODUCT IN DUAL CAMBRIAN ALGEBRA



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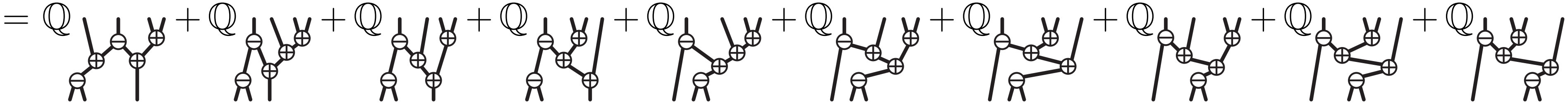
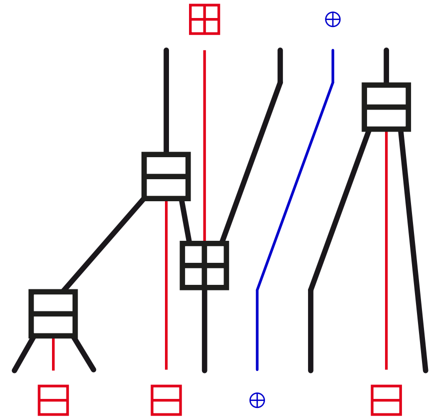
$$= \mathbb{G}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{3}\overline{4}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{4}\overline{3}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{5}\overline{3}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{3}\overline{4}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{4}\overline{3}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{5}\overline{3}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{4}\overline{2}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{5}\overline{2}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{5}\overline{2}\underline{1}\overline{3}}$$





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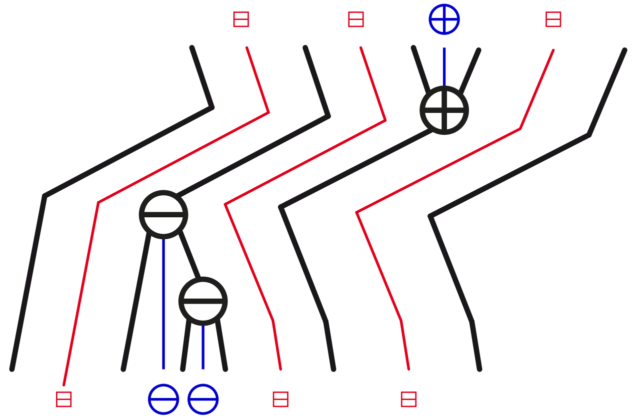

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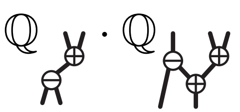
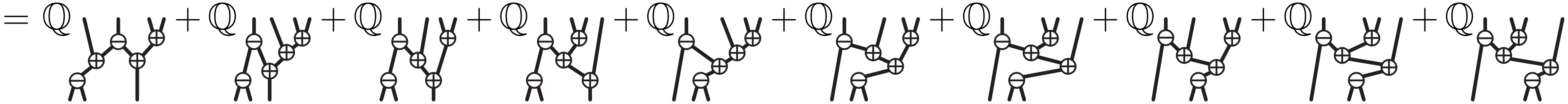


$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

$$= \mathbb{G}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{3}\overline{4}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{4}\overline{3}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{5}\overline{3}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{3}\overline{4}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{4}\overline{3}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{5}\overline{3}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{4}\overline{2}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{5}\overline{2}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{5}\overline{2}\underline{1}\overline{3}}$$



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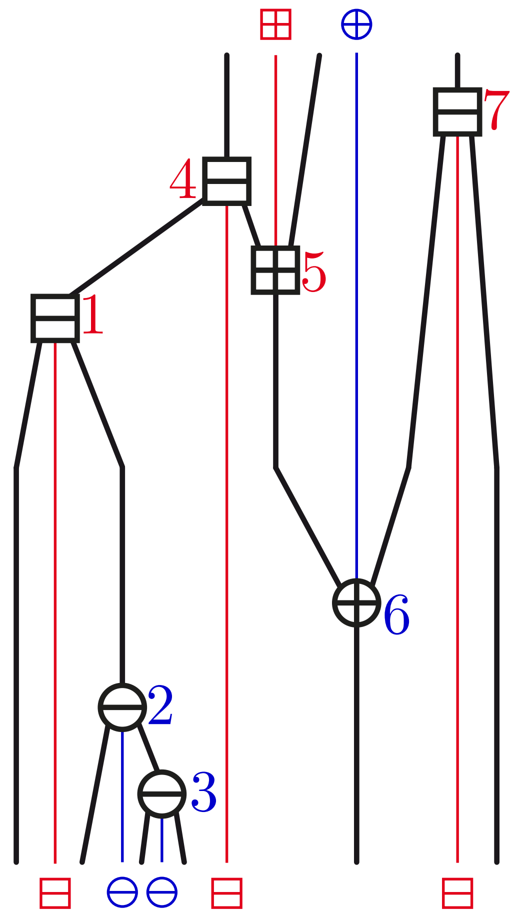
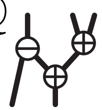
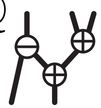
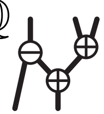


$= \mathrm{T}^{\bigcirc} s \mathrm{T}^{\square}$

$\triangle\mathbb{Q}$  $= \triangle\mathbb{G}_{\bar{2}\underline{1}\bar{3}}$

$$= 1 \otimes \mathbb{G}_{\bar{2}\underline{1}\bar{3}} + \mathbb{G}_{\underline{1}} \otimes \mathbb{G}_{\bar{1}\bar{2}} + \mathbb{G}_{\bar{2}\underline{1}} \otimes \mathbb{G}_{\bar{1}} + \mathbb{G}_{\bar{2}\underline{1}\bar{3}} \otimes 1$$

$= 1 \otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes 1.$

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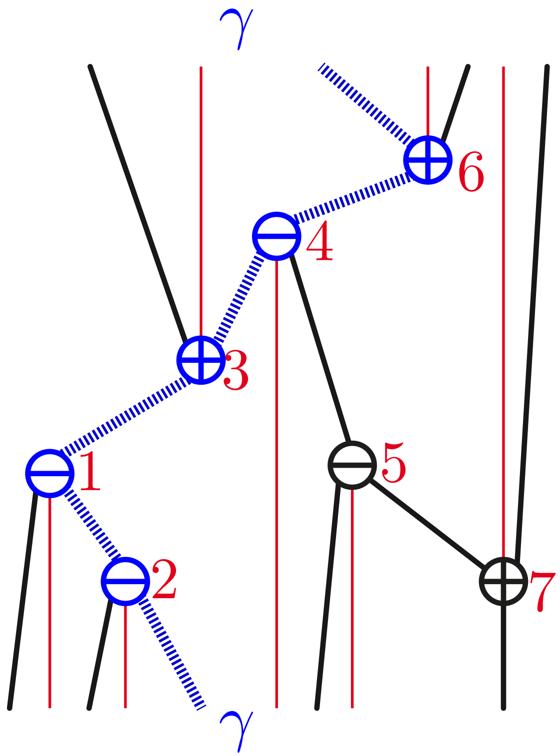
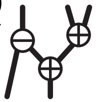
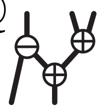
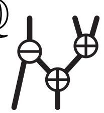
where γ runs over all gaps between vertices of S, and $L(\mathrm{S},\gamma)$ and $R(\mathrm{S},\gamma)$ denote the Cambrian trees left and right to γ respectively



COPRODUCT IN DUAL CAMBRIAN ALGEBRA

$\triangle\mathbb{Q}$  $= \triangle\mathbb{G}_{\overline{2}\underline{1}\overline{3}}$

$= 1 \otimes \mathbb{G}_{\overline{2}\underline{1}\overline{3}} + \mathbb{G}_{\underline{1}} \otimes \mathbb{G}_{\overline{12}} + \mathbb{G}_{\overline{2}\underline{1}} \otimes \mathbb{G}_{\overline{1}} + \mathbb{G}_{\overline{2}\underline{1}\overline{3}} \otimes 1$

$= 1 \otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes 1.$

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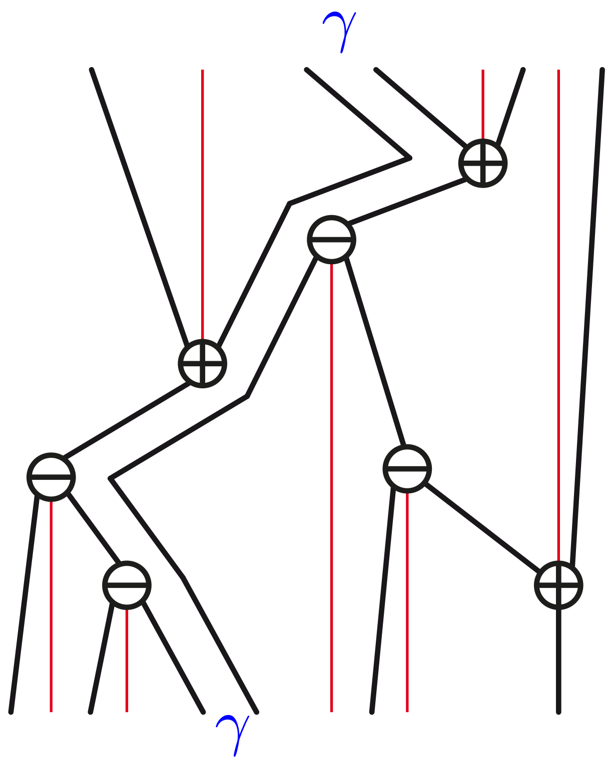

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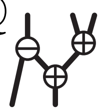
$\triangle\mathbb{Q}$ 

$$= \triangle\mathbb{G}_{\bar{2}\underline{1}\bar{3}}$$

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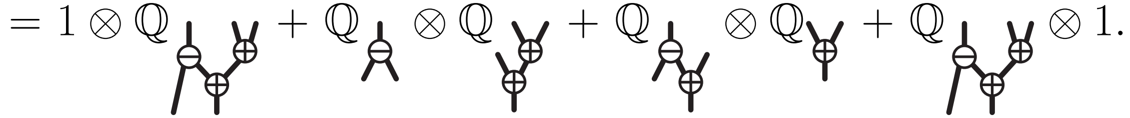

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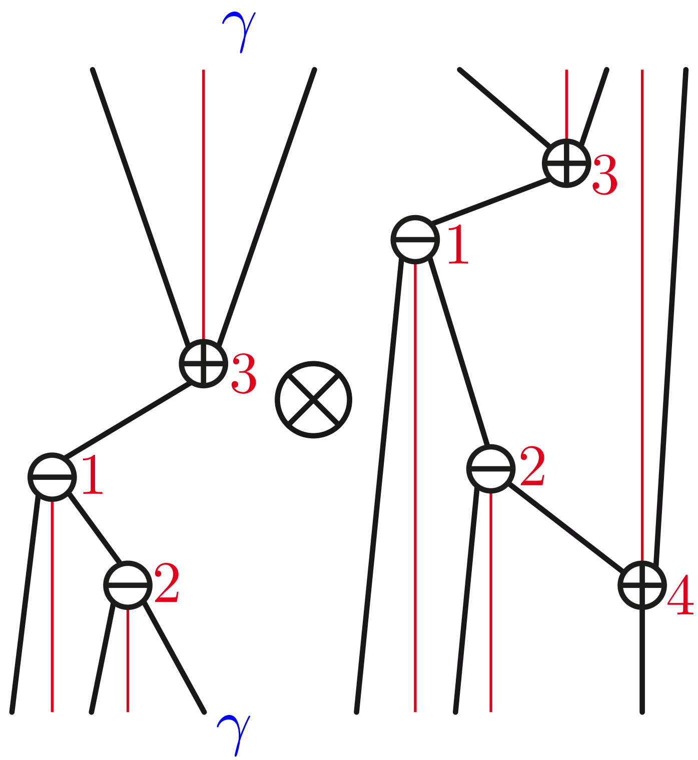
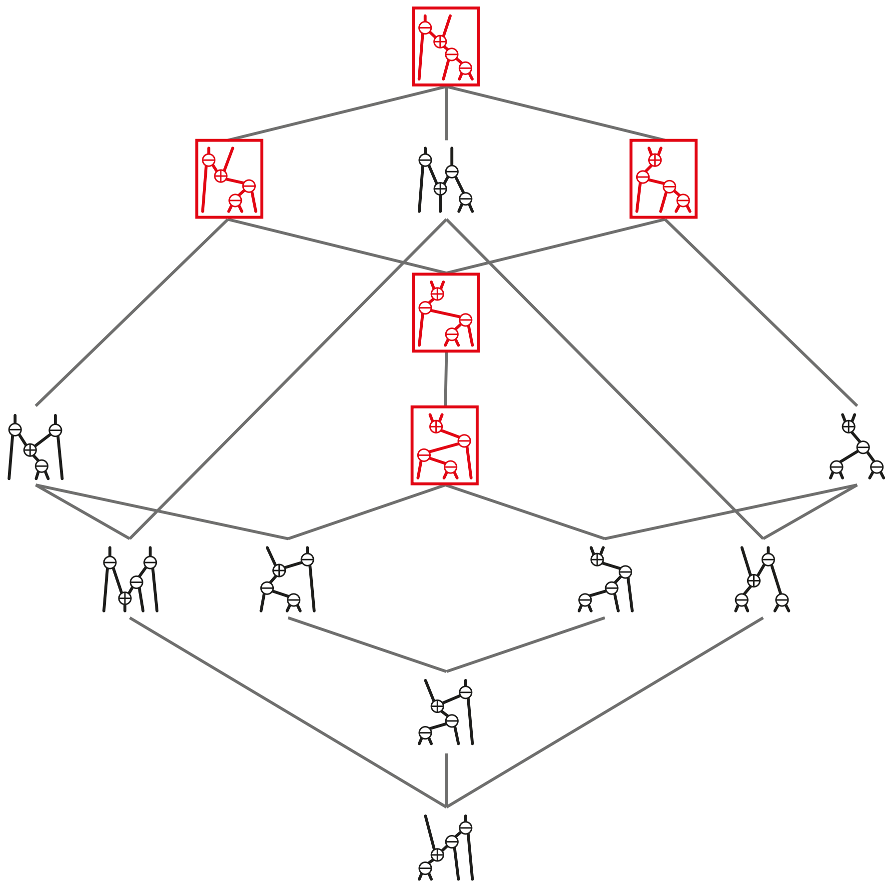
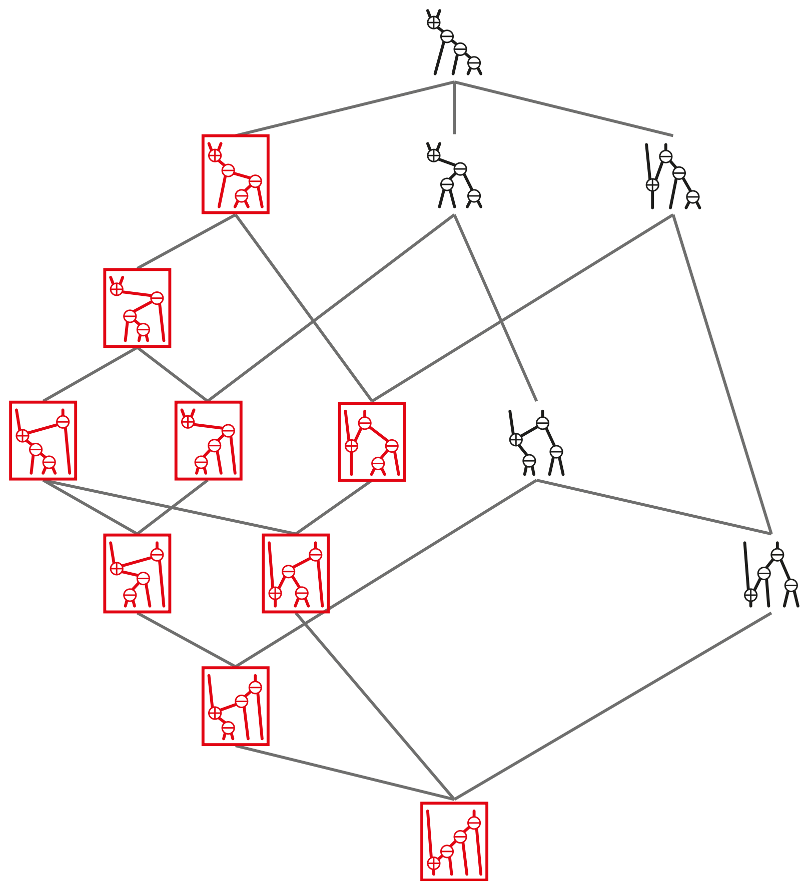


MULTIPLICATIVE BASES

Define
$$\mathbb{E}^{\mathrm{T}} := \sum_{\mathrm{T} \le \mathrm{T}'} \mathbb{P}_{\mathrm{T}'} \qquad \text{and} \qquad \mathbb{H}^{\mathrm{T}} := \sum_{\mathrm{T}' \le \mathrm{T}} \mathbb{P}_{\mathrm{T}'}.$$





PROP. $(\mathbb{E}^{\mathrm{T}})_{\mathrm{T} \in \mathrm{Camb}}$ and $(\mathbb{H}^{\mathrm{T}})_{\mathrm{T} \in \mathrm{Camb}}$ are multiplicative bases of Camb, ie.

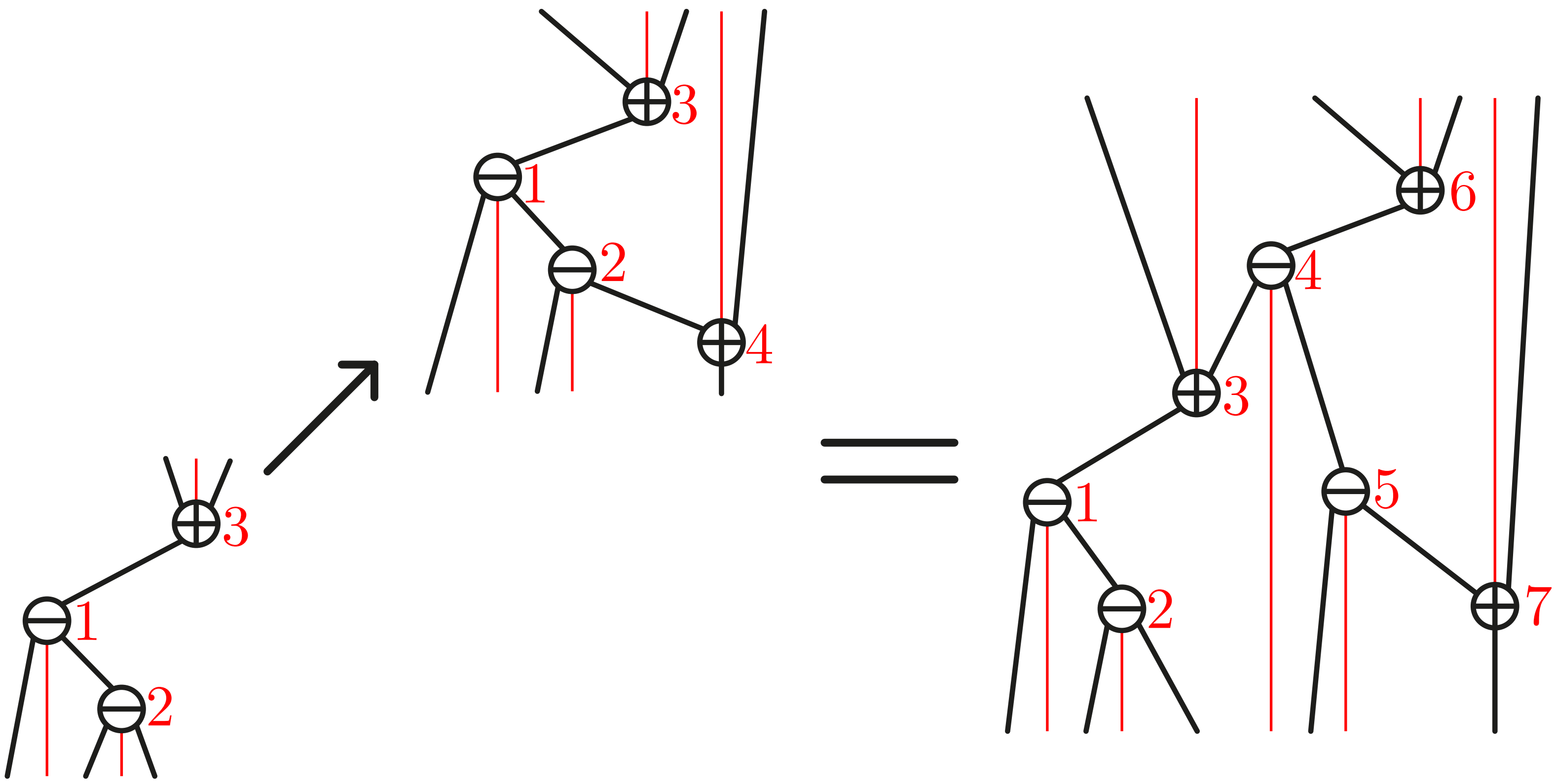
$$\mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'} = \mathbb{E}^{\mathrm{T} \nearrow \bar{\mathrm{T}}'} \qquad \text{and} \qquad \mathbb{H}^{\mathrm{T}} \cdot \mathbb{H}^{\mathrm{T}'} = \mathbb{H}^{\mathrm{T} \nwarrow \bar{\mathrm{T}}'}$$

INDECOMPOSABLE ELEMENTS

PROP. The following properties are equivalent for a Cambrian tree S:

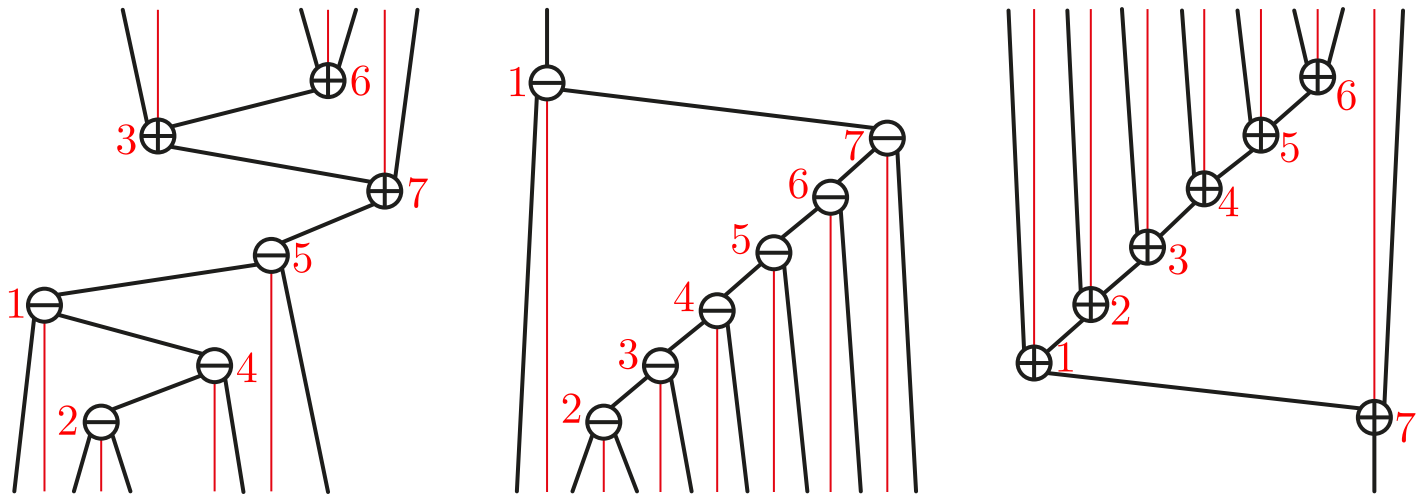
- $\mathbb{E}^{\mathrm{S}}$ can be decomposed into a product $\mathbb{E}^{\mathrm{S}} = \mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'}$ for non-empty T, T'
- $([k] \parallel [n] \smallsetminus [k])$ is an edge cut of S for some $k \in [n]$
- at least one linear extension τ of S is decomposable, ie. $\tau([k]) = [k]$ for some $k \in [n]$

The tree S is then called $\mathbb{E}$-decomposable

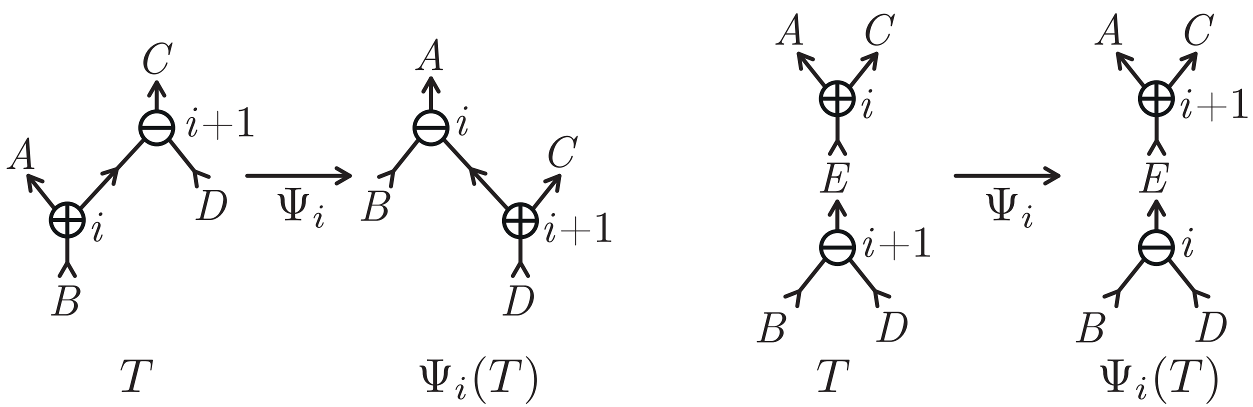


INDECOMPOSABLE ELEMENTS

PROP. For any signature $\varepsilon \in \pm^n$, the set of $\mathbb{E}$-indecomposable ε-Cambrian trees forms a principal upper ideal of the ε-Cambrian lattice



PROP. For any signature $\varepsilon \in \pm^n$, there are C_{n-1} $\mathbb{E}$-indecomposable ε-Cambrian trees. Therefore, there are $2^n C_{n-1}$ $\mathbb{E}$-indecomposable Cambrian trees on n vertices



PERSPECTIVES

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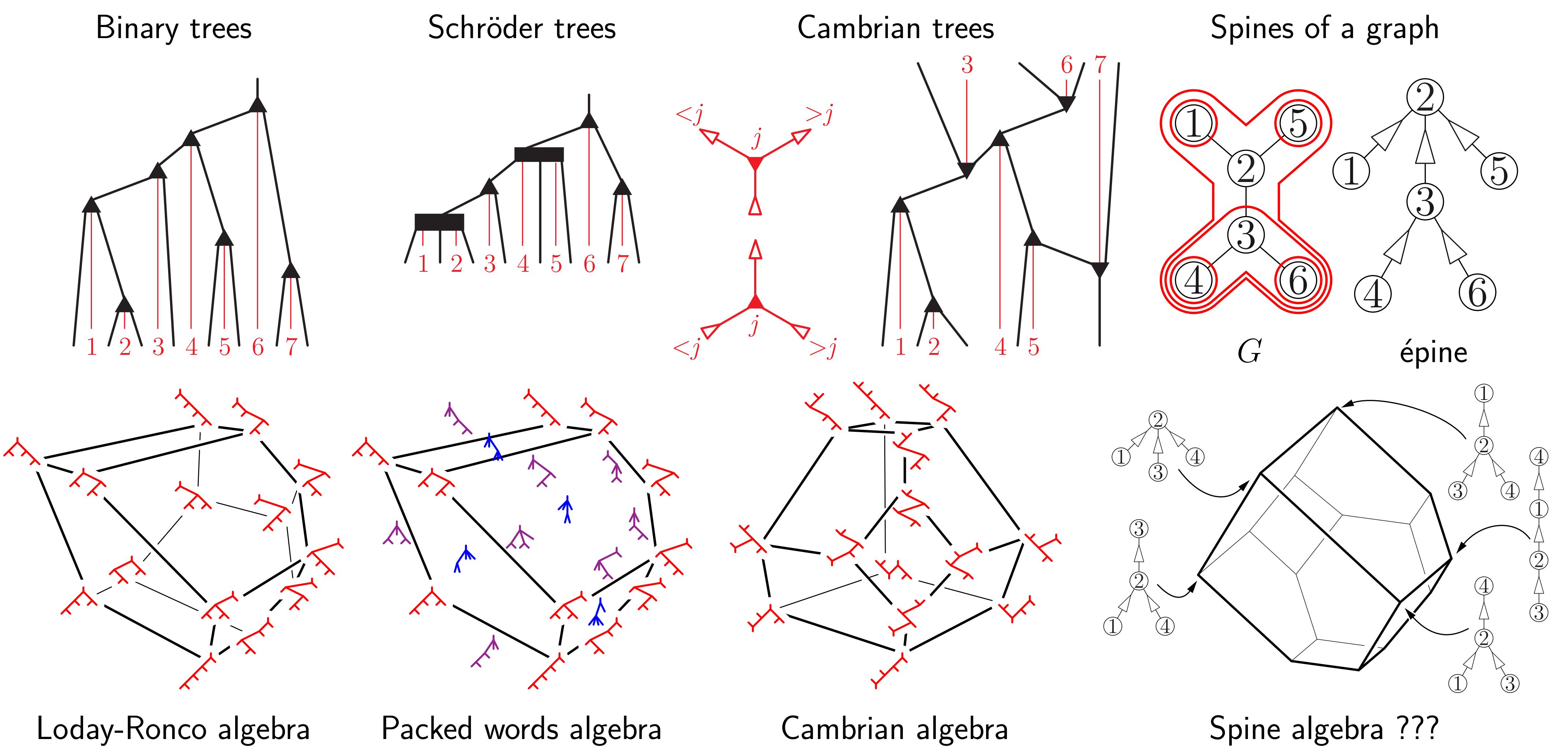
Extend combinatorial, geometric and algebraic properties of binary trees to further families of trees...



THANK YOU