The twist for positroids

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Abstract. There are two reasonable ways to put a cluster structure on a positroid variety. In one, the initial seed is a set of Plücker coordinates. In the other, the initial seed consists of certain monomials in the edge weights of a plabic graph. We will describe an automorphism of the positroid variety, the twist, which takes one to the other. For the big positroid cell, this was already done by Marsh and Scott; we generalize their results to all positroid varieties. This also provides an inversion of the boundary measurement map which is more general than Talaska's, in that it works for all reduced plabic graphs rather than just Le-diagrams. This is the analogue for positroid varieties of the twist map of Berenstein, Fomin and Zelevinsky for double Bruhat cells. Our construction involved the combinatorics of dimer configurations on bipartite planar graphs.

Résumé. Deux méthodes sont bien connues pour munir d'une structure amassée une variété positroïde. La première utilise comme graine initiale un ensemble de coordonnées plückeriennes. Dans la seconde, la graine initiale est composée de certains monômes dans les poids des arêtes d'un graphe planaire bicolore. Nous décrivons un automorphisme de la variété positroïde, le torsion, qui prend l'une à l'autre. Pour la grosse cellule positroïde, cela a déjà été fait par Marsh et Scott; nous généralisons leurs résultats à toute variété positroïde. Notre automorphisme fournit aussi une application inverse de la fonction de mesure des bords, plus générale que celle de Talaska : la nôtre s'applique a tous les graphes planaires bicolores réduits, plutôt que seulement les Le-diagrammes. Notre construction est l'analogue pour les variétés positroïdes de la fonction de torsion de Berenstein-Fomin-Zelevinsky pour les doubles cellules de Bruhat. Notre approche utilise la combinatoire des configurations de dimères sur les graphes planaires bicolores.

Keywords. Positroid, planar bipartite graph, boundary measurement map, cluster algebra

1 Informal summary

The Grassmannian of k-planes in $\mathbb{C}^n$ admits a decomposition into *open positroid varieties* $\Pi^\circ(\mathcal{M})$, analogous to the decomposition of a semisimple Lie group into double Bruhat cells [FZ99]. Postnikov [Pos06] showed that an appropriate choice of *reduced graph G* defines a *boundary measurement map*

$$(\mathbb{C}^\times)^{\mathrm{Edges}(G)}/\mathrm{Gauge} \longrightarrow \Pi^\circ(\mathcal{M})$$

Among other properties, this map can be used to parametrize the 'totally positive part' of $\Pi^\circ(\mathcal{M})$.

Scott [Sco06] demonstrated that each face in the reduced graph G determines a homogeneous coordinate on $\Pi^\circ(\mathcal{M})$. Taking all of the faces of G, we obtain a collection of coordinates which collectively

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form a 'cluster' in a conjectural *cluster structure* on $\Pi^\circ(\mathcal{M})$. These homogeneous coordinates collectively define a rational coordinate chart, the *face Plücker map*:

$$\Pi^\circ(\mathcal{M}) \dashrightarrow \mathbb{C}^{\text{Faces(G)}}/\text{Scaling}$$

Despite the fact that these two maps are both defined by the same combinatorial input (a choice of reduced graph), the relation between them has been elusive.

Moreover, the results of Postnikov and Scott are weaker than we have stated in the two proceeding paragraphs. Postnikov only shows that the boundary measurement map exists as a rational map, which is well defined on $(\mathbb{R}_{>0})^{\text{Edges}(G)}/\text{Gauge}$. Scott does not show that coordinates of the face Plücker map generate the function field of $\Pi^\circ(\mathcal{M})$, or even that they are nonzero, as one would hope for rational coordinates![i] In fairness, at the time Postnikov and Scott were working, the algebraic structure on $\Pi^\circ(\mathcal{M})$ had not been defined, so these questions would have been difficult to formulate. However, with our current knowledge, these omissions are a major gap in our understanding.

In this paper, we relate the two maps by introducing a *twist automorphism* $\vec{\tau}$ of each open positroid variety. The main theorem of this paper then states that the composition

$$(\mathbb{C}^\times)^{\text{Edges}(G)}/\text{Gauge} \longrightarrow \Pi^\circ(\mathcal{M}) \xrightarrow{\vec{\tau}} \Pi^\circ(\mathcal{M}) \dashrightarrow \mathbb{C}^{\text{Faces}(G)}/\text{Scaling}$$

is given by a monomial in each coordinate, which is defined by a distinguished *matching* on G. Furthermore, the composite map of tori is an isomorphism.

As a consequence, we deduce that the boundary measurement map is a well defined map from the torus $(\mathbb{C}^\times)^{\text{Edges}(G)}/\text{Gauge}$ to $\Pi^\circ(\mathcal{M})$. We also learn that the face Plücker map is well defined on an open torus, and gives rational coordinates on $\Pi^\circ(\mathcal{M})$. Thus, we show that the statements of the first two paragraphs are correct after all. Furthermore, we obtain explicit birational inverses to these maps.

This abstract explains the definitions necessary to formulate all of our results in detail, and explains their consequences. The proofs will appear in a separate manuscript.

2 Notations

We use the following standard notations for combinatorial sets:

$$[n] := \{1, 2, ..., n\}$$

$$\binom{[n]}{k} := \{I \subset [n] \mid |I| = k\}, \text{ the set of } k\text{-element subsets of } [n].$$

We write $G(k,n)$ for the Grassmannian of k-planes in $\mathbb{C}^n$. For V a $\mathbb{C}$-vector space, $\mathbb{P}(V)$ is the projective space of lines in V. We will write $\mathbb{G}_m$ for the nonzero complex numbers, considered as an abelian group. For any finite set X, we write $\mathbb{C}^X$ for the $\mathbb{C}$-vector space with basis labeled by X, and write $\mathbb{R}^X$ and $\mathbb{G}_m^X$ similarly.

For a $k \times n$ matrix A and $a \in [n]$, let A_a denote the ath column of A. Given a k-element set $I \subset [n]$, write it as $I = \{i_1 < i_2 < \cdots < i_k\}$ and define the ***I*th maximal minor** of A by

$$\Delta_I(A) := \det(A_{i_1}, A_{i_2}, ..., A_{i_k})$$

that is, the determinant of the matrix with columns $A_{i_1}, A_{i_2}, ..., A_{i_k}$.

[i] Since $\Pi^\circ(\mathcal{M})$ and $\mathbb{C}^{\text{Faces(G)}}/\text{Scaling}$ have the same dimension, a coordinate of zero would imply that face Plücker map is not an inclusion almost everywhere.

3 Positroids and positroid varieties

The definitions in this section can all be found in Knutson, Lam and Speyer [KLS13], and are due either to those authors or to Postnikov [Pos06].

Given a k-dimensional subspace $V \subset \mathbb{C}^n$, the corresponding *matroid* is the collection of k-element subsets[(ii)]

$$\mathcal{M} = \{I \subset [n] \mid \text{the projection } \mathbb{C}^n \to \mathbb{C}^I \text{ restricts to an isomorphism } V \xrightarrow{\sim} \mathbb{C}^I\}$$

The Grassmannian $G(k,n)$ can then be decomposed into pieces, each parametrizing those subspaces with a fixed matroid. Unfortunately, this decomposition is incredibly poorly-behaved; its many transgressions are explored elsewhere [Mnë88], [Stu87], [GGMS87]. We focus on a related decomposition of $G(k,n)$ which is much nicer.

Positroids are a special class of matroid with many equivalent characterizations. The shortest definition [Pos06] is that a positroid is a matroid $\mathcal{M}$ with a 'totally non-negative' representation. That is, there exists a (non-unique) subspace $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ with matroid $\mathcal{M}$, such that, for each k-element $I \subset [n]$, the determinant of the projection $\mathbb{R}^k \hookrightarrow \mathbb{R}^n \to \mathbb{R}^I$ is a non-negative real number. Every matroid $\mathcal{M}$ has a ***positroid envelope***; the unique smallest positroid containing $\mathcal{M}$ [KLS13, Section 3].

Given a positroid $\mathcal{M}$, the ***(open) positroid variety*** $\Pi^\circ(\mathcal{M})$ is the subvariety of $G(k,n)$ parametrizing subspaces whose matroid has positroid envelope $\mathcal{M}$. So we obtain a stratification

$$Gr(k,n) = \bigsqcup_{\substack{\text{positroids } \mathcal{M} \\ \text{of rank } k \text{ on } [n]}} \Pi^\circ(\mathcal{M})$$

which groups together matroid strata with the same positroid envelope. This decomposition of $G(k,n)$ arises naturally from several different perspectives and the positroid varieties avoid many of the pathologies exhibited by the matroid strata.

While the Grassmannian and its decomposition are the intrinsically interesting objects, the results of this paper will be most easily stated on the affine cone $\widetilde{G(k,n)}$ over the Plücker embedding of the Grassmannian. Denote by $\widetilde{\Pi}^\circ(\mathcal{M})$ the lift of a positroid variety $\Pi^\circ(\mathcal{M})$ to $\widetilde{G(k,n)} \setminus \{0\}$.

We write $\widetilde{\Pi}(\mathcal{M})$ for the closure of $\widetilde{\Pi}^\circ(\mathcal{M})$ in $\mathbb{C}^{\binom{[n]}{k}}$; we write $\Pi(\mathcal{M})$ for the closure of $\Pi^\circ(\mathcal{M})$ in $G(k,n)$. The origin of $\mathbb{C}^{\binom{[n]}{k}}$ is in every $\widetilde{\Pi}(\mathcal{M})$ and in no $\widetilde{\Pi}^\circ(\mathcal{M})$.

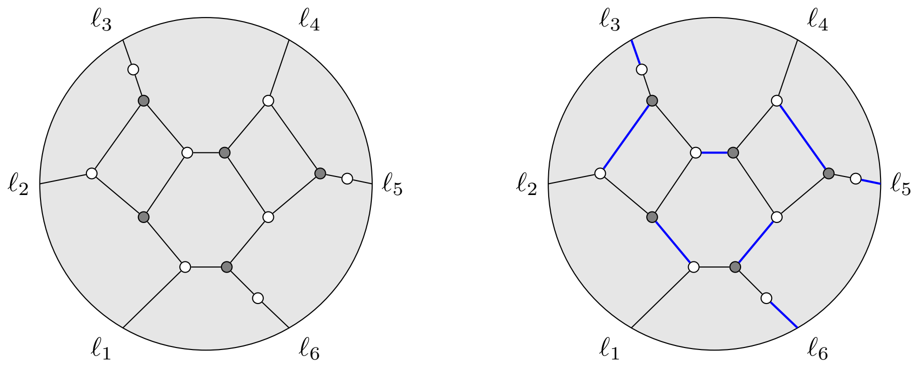
4 The boundary measurement map

Let G be a graph embedded in a disc, with a 2-coloring of its internal vertices as either black or white (Figure 1a). We require that each boundary vertex is adjacent to one white vertex and no other vertices. Let n denote the number of boundary vertices, and index the boundary vertices as $\ell_1, \ell_2, ..., \ell_n$ in a clockwise order.

A ***matching*** of G is a collection of edges in G which cover each internal vertex exactly once. For a matching M, we let ∂M denote the subset of the boundary vertices covered by M, which we identify with a subset of $[n] := \{1,2,...,n\}$ (Figure 1b). That is,

$$\partial M := \{i : \ell_i \text{ is covered by } M\} \subset [n]$$

(ii) Throughout, a matroid is a collection of 'bases', rather than 'independent sets' or other conventions.



(a) A 2-colored graph embedded in the disc. (b) A matching with boundary 356.

Fig. 1: A graph and a matching.

The cardinality k of ∂M is constant for any matching of G, and given by

$$k := (\#\text{ of white vertices}) - (\#\text{ of black vertices})$$

As long as G admits a matching, the graph G determines a positroid by

$$\mathcal{M} := \{I \subset [n] \mid \text{there exists a matching } M \text{ with } \partial M = I\}$$

A graph G as above is ***reduced*** if the number of *faces* of G (that is, components of the complement in the disc) is minimal among all graphs with the same positroid as G.

The matchings of G with a fixed boundary may be collected into a ***partition function*** as follows. Let $\{z_e\}$ be a set of formal variables indexed by edges e of G. For a matching M of G, define $z^M := \prod_{e \in M} z_e$, and for a k-element subset I of $[n]$, define

$$D_I := \sum_{\substack{\text{matchings } M \\ \text{with } \partial M = I}} z^M$$

Plugging complex numbers into the formal variables realizes D_I as a regular function $\mathbb{C}^E \to \mathbb{C}$, where E denotes the set of edges of G. Running over all k-elements subsets of $[n]$, the partition functions define a regular map

$$\mathbb{C}^E \longrightarrow \mathbb{C}^{\binom{[n]}{k}}$$

which we refer to as the ***boundary measurement map***. The partition functions are not algebraically independent, so this map lands in a subvariety.

Theorem 4.1 *For any graph G as above, the partition functions satisfy the Plücker relations. Therefore, the map $\mathbb{C}^E \to \mathbb{C}^{\binom{[n]}{k}}$ with coordinates $\{D_I\}$ has image contained $\widetilde{G(k,n)} \subset \mathbb{C}^{\binom{[n]}{k}}$.*

The correct attribution for this result is difficult; it can be found explicitly in Thomas Lam's lecture notes [Lam15] but is already implicit, to various degrees, in [PSW09], [Tal08], [Kuo06] ,[Kuo04], [Pos06] and [Kas67]. The second author's note [Spe15] gives a quick proof of this result.

The boundary measurement map is almost never injective because of the following ***gauge transformations***: if v is an internal vertex of G, (z_e) is a point of $\mathbb{C}^E$, and t is a nonzero complex number, then define a new point (z'_e) of $\mathbb{C}^E$ by

$$z'_e = \begin{cases} tz_e & v \in e \\ z_E & \text{otherwise} \end{cases}.$$

Since each matching of G contains exactly one edge covering v, we know that $(z')^M = t(z^M)$ and that $D_I(z') = tD_I(z)$.

The gauge transformations can be encoded more elegantly as follows. The group $\mathbb{G}_m^E$ acts on $\mathbb{C}^E$ by scaling the individual coordinates; in this way, $\mathbb{G}_m^E$ may be identified with ways to assign a nonzero 'weight' to each edge. Letting V denote the set of internal vertices of G, the action of $\mathbb{G}_m^V$ by gauge transformations is equivalent to a regular map of groups

$$\mathbb{G}_m^V \longrightarrow \mathbb{G}_m^E$$

where the coordinate at each edge is the product of the coordinates at its endpoints.

Let $\mathbb{G}_m^{V-1}$ denote the subgroup of $\mathbb{G}_m^V$ such that the product of the coordinates is 1; equivalently, this is the subgroup of the gauge group which leaves the partition functions invariant.

Before this paper, the following was known but not written explicitly.

Proposition 4.2 *For a graph G with positroid $\mathcal{M}$, the map $\mathbb{G}_m^E \to \mathbb{C}^{\binom{[n]}{k}}$ given in Plücker coordinates by the partition functions D_I factors through $\mathbb{G}_m^E/\mathbb{G}_m^{V-1}$ and lands in $\widetilde{\Pi}(\mathcal{M})$.*

We will improve this to:

Proposition 4.3 *For a reduced graph G with positroid $\mathcal{M}$, the map $\mathbb{G}_m^E \to \mathbb{C}^{\binom{[n]}{k}}$ given in Plücker coordinates by the partition functions D_I factors through $\mathbb{G}_m^E/\mathbb{G}_m^{V-1}$ and lands in $\widetilde{\Pi}^\circ(\mathcal{M})$, giving a map.*

$$\widetilde{\mathbb{D}} : \mathbb{G}_m^E/\mathbb{G}_m^{V-1} \longrightarrow \widetilde{\Pi}^\circ(\mathcal{M})$$

The map $\widetilde{\mathbb{D}}$ descend to a well-defined quotient map

$$\mathbb{D} : \mathbb{G}_m^E/\mathbb{G}_m^{V} \longrightarrow \Pi^\circ(\mathcal{M})$$

Moreover, the maps $\mathbb{D}$ and $\widetilde{\mathbb{D}}$ are open inclusions.

We will refer to the maps $\mathbb{D}$ and $\widetilde{\mathbb{D}}$ as ***boundary measurement maps***. These are equivalent to the boundary measurement map of Postnikov [Pos06].

Example 4.4 *Consider the graph G in Figure 1a. Of all the subsets of $[6]$ in $\binom{[6]}{3}$, only $\{1,2,3\}$ is not the boundary of a matching. The open positroid variety $\Pi^\circ(\mathcal{M})$ is defined inside $G(3,6)$ by the vanishing of the Plücker coordinate Δ_{123} and the non-vanishing of $\Delta_{124}, \Delta_{234}, \Delta_{345}, \Delta_{456}, \Delta_{156}$, and Δ_{126}.*[iii] *As a consequence, the closure $\Pi(\mathcal{M})$ is the Schubert divisor in $G(3,6)$.*

[iii] The non-vanishing of these Plücker coordinates removes subspaces that have a smaller positroid envelope than $\mathcal{M}$.

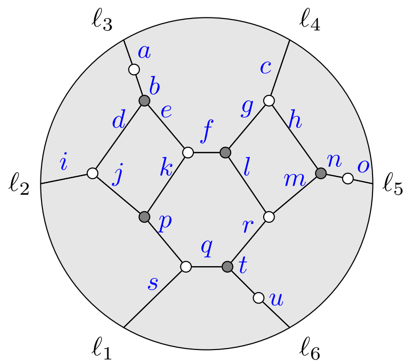


Fig. 2: A general set of edge weights.

Let us describe a general point in $\mathbb{G}_m^E$ by assigning an unknown weight in $\mathbb{G}_m$ to each edge in G, as in Figure 2. As a consequence of Theorem 4.1, there exists a 3×6 matrix such that, for any $I \in \binom{[6]}{3}$, the minor with columns in I is equal to D_I. One such matrix is given below in (1).[iv]

$$\begin{bmatrix} 1 & 0 & -\frac{aep}{bks} & 0 & \frac{fmop}{klns} & \frac{klqu+fpru}{klst} \\ 0 & 1 & \frac{adk+aej}{bik} & 0 & -\frac{fjmo}{ikln} & -\frac{fjru}{iklt} \\ 0 & 0 & 0 & bciklnst & bikost(hl+gm) & bgiknrsu \end{bmatrix} \tag{1}$$

The boundary measurement map $\mathbb{D}$ for G is the map which sends the edge weights given in Figure 2 to the row-span of the matrix in (1). *The content of Proposition 4.3 is that the map $\mathbb{D}$ is unaffected by the gauge action (for example, replacing b, d, and e by $\lambda b, \lambda d$, and λe), and that the image lands in $\Pi^\circ(\mathcal{M})$.*

5 Plücker coordinates associated to faces

In [Pos06], Postnikov showed how a reduced graph determines a collection of ***strands***: oriented curves in the disc beginning and ending at boundary vertices of G (Figure 3a).

The strands do not self-intersect (except possibly at the boundary), so each one subdivides the disc into two components. The orientation of a strand distinguishes these components as the 'left side' and the 'right side'. Each face of G is on the left side of exactly k-many strands, where k again denotes the number of white vertices minus the number of black vertices.

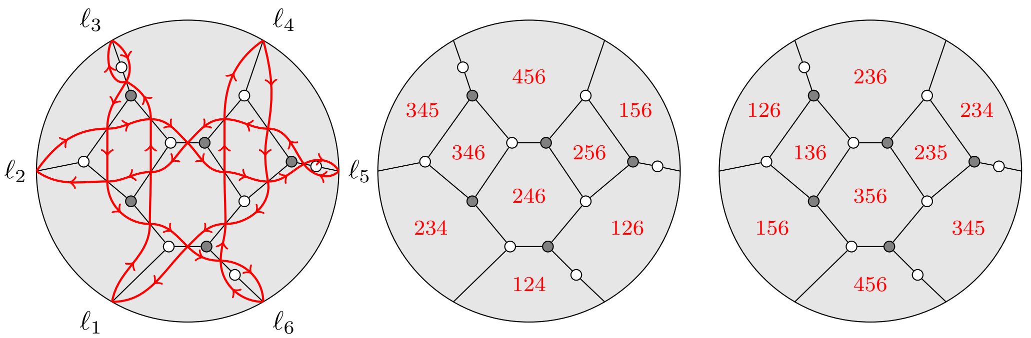
There are two natural ways to use a collection of k-many strands to determine a k-element subset of $[n]$: identify each strand either with the index of its *source* vertex, or with the index of its *target* vertex. In this paper we will be forced to work with both conventions.

Given a face f of G, define the following two k-element subsets of $[n]$ (see Figures 3b and 3c).

$$\overset{\bullet\leftarrow}{I}(f) := \{ i \in [n] \mid f \text{ is to the left of the strand ending at } \ell_i \}$$

$$\overset{\bullet\rightarrow}{I}(f) := \{ i \in [n] \mid f \text{ is to the left of the strand starting at } \ell_i \}$$

[iv] Note that such a matrix is only determined up to the left action of $SL_3(\mathbb{C})$; however, its row-span is uniquely determined.



(a) The strands of the graph. (b) Target-labeling of the faces. (c) Source-labeling of the faces.

Fig. 3: Two ways to associate a k-element subset of $[n]$ to a face.

For any k-element subset I of $[n]$, let Δ_I denote the Plücker coordinate on $\widetilde{G(k,n)}$ indexed by I. Hence, each face f in G determines two Plücker coordinates, given by $\Delta_{\overset{\bullet\rightarrow}{I}(f)}$ and $\Delta_{\overset{\bullet\leftarrow}{I}(f)}$.

Letting F denote the set of faces of G, this determines a pair of regular maps

$$\overset{\bullet\leftarrow}{\mathbb{F}} : \widetilde{\Pi}^\circ(\mathcal{M}) \longrightarrow \mathbb{C}^F$$

$$\overset{\bullet\rightarrow}{\mathbb{F}} : \widetilde{\Pi}^\circ(\mathcal{M}) \longrightarrow \mathbb{C}^F$$

where the coordinate corresponding to a face is the appropriate Plücker coordinate.

It is anticipated that the Plücker coordinates $(\Delta_{\overset{\bullet\rightarrow}{I}(f)})_{f\in F}$ form a cluster for a cluster structure on $\widetilde{\Pi}^\circ(\mathcal{M})$. In particular, this would imply:

Proposition 5.1 *Let T be the open subset of $\widetilde{\Pi}^\circ(\mathcal{M})$ where the Plücker coordinates $(\Delta_{\overset{\bullet\rightarrow}{I}(f)})_{f\in F}$ are nonzero. Then $\overset{\bullet\rightarrow}{\mathbb{F}}$ provides an isomorphism $T \to \mathbb{G}_m^F$. The analogous result holds for $\overset{\bullet\leftarrow}{I}(f)$ and $\overset{\bullet\leftarrow}{\mathbb{F}}$.*

While we do not construct a cluster structure on $\widetilde{\Pi}^\circ(\mathcal{M})$ (see [Lec14] for the best current results), we do establish Proposition 5.1 as a consequence of our combinatorial results.

6 Extremal matchings and a basis of gauge invariant characters

We explained above the importance of thinking of the boundary measurement map as a function on $\mathbb{G}_m^E/\mathbb{G}_m^{V-1}$. We will now give a basis of characters for this quotient torus. Since G is a planar graph embedded in a disc, $\#E - (\#V - 1) = \#F$, so we should expect one character for each face of G. Note that, if M is any matching of G, and $z \in \mathbb{G}_m^E$, then $z^M = \prod_{e\in M} z_e$ descends to the quotient $\mathbb{G}_m^E/\mathbb{G}_m^{V-1}$.

For every face f of G, we define a matching $\overrightarrow{M}(f)$ as follows: An edge e of G appears in $\overrightarrow{M}(f)$ if and only if the face f is contained in the "downstream wedge" bounded by the two half strands flowing out of e and the edge of the disc; see Figure 4.

Proposition 6.1 *The set of edges $\overrightarrow{M}(f)$ is a matching of G, with boundary $\overset{\bullet\rightarrow}{I}(f)$. The list of characters $(z^{\overrightarrow{M}(f)})_{f\in F}$ is a basis of the character lattice of $\mathbb{G}_m^E/\mathbb{G}_m^{V-1}$.*

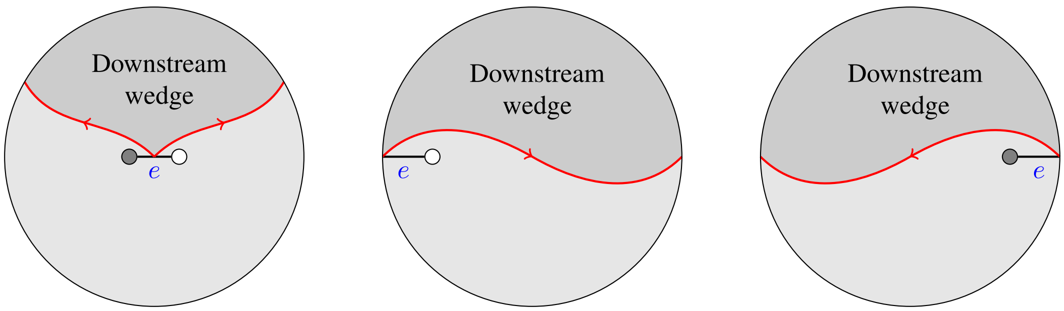


Fig. 4: The downstream wedge of an edge e

We similarly define matchings $\overleftarrow{M}(f)$ using upstream edges, with boundaries $\overset{\bullet\leftarrow}{I}(f)$, for which the analogous result holds.

Example 6.2 *The matching given in Figure 1b is the matching $\overrightarrow{M}(f)$, where f is the interior hexagonal face. The boundary 356 of $\overrightarrow{M}(f)$ coincides with the source-labeling of f, as shown in Figure 3c.*

Let $\overrightarrow{\mathbb{M}}$ and $\overleftarrow{\mathbb{M}}$ be the monomial maps $\mathbb{G}_m^E \longrightarrow \mathbb{G}_m^F$ where the f-coordinate of $\overleftarrow{\mathbb{M}}(z)$ is $z^{-\overleftarrow{M}(f)}$ and the f-coordinate of $\overrightarrow{\mathbb{M}}(z)$ is $z^{-\overrightarrow{M}(f)}$.

Corollary 6.3 *For a reduced graph G, the maps $\overleftarrow{\mathbb{M}}$ and $\overrightarrow{\mathbb{M}}$ descend to well-defined isomorphisms*

$$\mathbb{G}_m^E/\mathbb{G}_m^{V-1} \xrightarrow{\sim} \mathbb{G}_m^F$$

We will introduce some notations to discuss the inverse maps to $\overrightarrow{\mathbb{M}}$ and $\overleftarrow{\mathbb{M}}$. Let f be a face and e an edge of G. We say that e is an ***internal edge*** if both the endpoints of e are interior to G, and a boundary edge if one of those endpoints is on the boundary. Define

$$\partial_{fe} := \left\{ \begin{array}{cc} 1 & e \text{ is an internal edge in the boundary of } f \\ 1 & e \text{ is an external edge, and } f \text{ is directly downstream from } e \\ 0 & \text{otherwise} \end{array} \right\}.$$

Let

$$B_f := \#\text{ of edges } e \text{ in the boundary of } f \text{ such that } f \text{ is downstream from } e$$

When f is an internal face, B_f is always half the total number of edges in the boundary of f; if f is a boundary face then B_f is this quantity either rounded up or down.

Let $x \in \mathbb{G}_m^F$, we write the coordinates of x as x_f for $f \in F$. Then $(\overrightarrow{\mathbb{M}})^{-1}(x)$ is a point of the quotient torus $\mathbb{G}_m^E/\mathbb{G}_m^{V-1}$. It does not make sense to ask for the coordinates of a point in a quotient torus, but it does make sense to ask for the evaluation of any gauge invariant character at $(\overrightarrow{\mathbb{M}})^{-1}(x)$. Particularly, if M is any matching of G, it makes sense to compute $\left((\overrightarrow{\mathbb{M}})^{-1}(x)\right)^M$.

Proposition 6.4 *Using the above notations,*

$$\left((\overrightarrow{\mathbb{M}})^{-1}(x)\right)^M = \prod_{f\in F} x_f^{\#\{e\in M\,:\,\partial_{fe}=1\}-B_f+1}.$$

If we want to consider $\overleftarrow{\mathbb{M}}^{-1}$ instead, we simply need to change the word "downstream" to "upstream" in the definitions of ∂_{fe} and B_f.

7 The twists of a positroid variety

We now define a pair of mutually inverse automorphisms $\vec{\tau}$ and $\overleftarrow{\tau}$ of $\widetilde{\Pi}^\circ(\mathcal{M})$, called the *right twist* and *left twist*, respectively. The definitions of the twists are elementary, and use none of the combinatorics or geometry we have built up so far.

Let A denote a $k\times n$ matrix of rank k. In this introduction, assume for simplicity that A has no zero columns. Let A_i denote the ith column of A, with indices taken cyclically; that is, $A_{i+n}=A_i$. The ***right twist*** $\vec{\tau}(A)$ of A is the $k\times n$ matrix such that, for all i, the ith column $\vec{\tau}(A)_i$ satisfies the relations

$$\langle \vec{\tau}(A)_i \mid A_i\rangle = 1, \text{ and}$$

$$\langle \vec{\tau}(A)_i \mid A_j\rangle = 0 \text{ if } A_j \text{ is not in the span of } \{A_i, A_{i+1}, \ldots, A_{j-1}\}$$

Similarly, the ***left twist*** of A is the $k\times n$ matrix $\overleftarrow{\tau}(A)$ defined on columns by the relations

$$\langle \overleftarrow{\tau}(A)_i \mid A_i\rangle = 1, \text{ and}$$

$$\langle \overleftarrow{\tau}(A)_i \mid A_j\rangle = 0 \text{ if } A_j \text{ is not in the span of } \{A_i, A_{i-1}, \ldots, A_{j+1}\}$$

The reader is cautioned that these operations are only piecewise continuous on the space of matrices.

Example 7.1 *Each of the following matrices is the right twist of the matrix to its left, and the left twist of the matrix to its right.*

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1\\ -1 & 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 1 & 1\end{pmatrix} \underset{\overleftarrow{\tau}}{\overset{\vec{\tau}}{\rightleftarrows}} \begin{pmatrix} 1 & 0 & 1 & -1 & 0\\ 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\end{pmatrix} \underset{\overleftarrow{\tau}}{\overset{\vec{\tau}}{\rightleftarrows}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0\\ 0 & 1 & 1 & -1 & 0\\ 1 & -1 & 0 & 1 & 1\end{pmatrix}$$

As the example suggests, the two twists are inverse to each other.

Theorem 7.2 *If A is a $k\times n$ matrix of rank k, then $\vec{\tau}(\overleftarrow{\tau}(A)) = \overleftarrow{\tau}(\vec{\tau}(A)) = A$.*

The set of $k\times n$ matrices of rank k naturally projects onto $\widetilde{G(k,n)}$ and $G(k,n)$, in the latter case sending a matrix to the span of its rows. The twists descend to well-defined maps on these spaces as well. The twists become continuous when restricted to an individual positroid variety. More specifically:

Theorem 7.3 *For each positroid $\mathcal{M}$, the twists $\vec{\tau}$ and $\overleftarrow{\tau}$ restrict to mutually inverse, regular automorphisms of $\widetilde{\Pi}^\circ(\mathcal{M})$ and $\Pi^\circ(\mathcal{M})$.*

In the example, the reader can check that columns $\{1,2,3\}$ are linearly dependent in all three matrices, as are columns $\{3,4,5\}$; this reflects that the positroid corresponding to to the given matrices is, in each case, all 3-element subsets of $[5]$ except for those two.

8 The main theorem

We are now in a position to state the main theorem.

Theorem 8.1 *Let G be a reduced graph with positroid $\mathcal{M}$. The following diagram commutes, where dashed arrows denote rational maps.*

$$\begin{tikzcd}
\mathbb{G}_m^F \arrow[r, bend left, "\overrightarrow{\partial}"] & \mathbb{G}_m^E/\mathbb{G}_m^{V-1} \arrow[l, bend left, "\overleftarrow{\mathbb{M}}"] \arrow[r, bend left, "\overrightarrow{\mathbb{M}}"] \arrow[d, "\widetilde{\mathbb{D}}"'] & \mathbb{G}_m^F \arrow[l, bend left, "\overleftarrow{\partial}"] \\
\widetilde{\Pi}^\circ(\mathcal{M}) \arrow[u, dashed, "\overset{\bullet\leftarrow}{\mathbb{F}}"] \arrow[r, bend left, "\vec{\tau}"] & \widetilde{\Pi}^\circ(\mathcal{M}) \arrow[l, bend left, "\overleftarrow{\tau}"] \arrow[r, bend left, "\vec{\tau}"] & \widetilde{\Pi}^\circ(\mathcal{M}) \arrow[l, bend left, "\overleftarrow{\tau}"] \arrow[u, dashed, "\overset{\bullet\rightarrow}{\mathbb{F}}"']
\end{tikzcd}$$

More specifically, the diagram commutes as a diagram of rational maps, and any composition of maps beginning in the top row is regular.

The morphisms in this diagram either commute or anticommute with the $\mathbb{G}_m$ action on each variety, and so the diagram descends to a commutative diagram on the quotients $\mathbb{G}_m^F/\mathbb{G}_m$, $\mathbb{G}_m^E/\mathbb{G}_m^V$ and $\Pi^\circ(\mathcal{M})$.

As a corollary of Theorem 8.1, we obtain a combinatorial formula for the Plücker coordinates of a twisted point as a Laurent polynomial in the Plücker coordinates of the original point. Specifically, for any point $x \in \widetilde{\Pi}^\circ(\mathcal{M})$, and any k-element subset J of $[n]$, we obtain the following formula for the Plücker coordinate Δ_J of $\vec{\tau}(x)$:

$$\Delta_J(\vec{\tau}(x)) = \sum_{\substack{\text{matchings } M \\ \text{with } \partial M = J}} \prod_{f \in F} \Delta_{\overset{\bullet}{\vec{I}}(f)}(x)^{\#\{e \in M \,:\, \partial_{fe} = 1\} - B_f + 1}.$$

The notations in the above formula were introduced shortly before Proposition 6.4, and this formula is an immediate consequence of that result.

We obtain Proposition 4.3 as a corollary: $\overleftarrow{\partial} \circ \overset{\bullet\leftarrow}{\mathbb{F}} \circ \vec{\tau}$ gives a left inverse to $\mathbb{D}$, showing that $\mathbb{D}$ is an open immersion. Similarly, $\vec{\tau} \circ \mathbb{D} \circ \overleftarrow{\partial}$ is a right inverse to $\overset{\bullet\leftarrow}{\mathbb{F}}$ which can be used to establish Proposition 5.1.

Example 8.2 *Let us consider the theorem in the running example of Figure 1a. The boundary measurement map $\mathbb{D}$ sends the edge weights in Figure 2 to the row-span of the matrix in* (1)*. The right twist of this matrix is given below in* (2)*.*

$$\begin{bmatrix} 1 & \frac{dks+ejs}{eip} & \frac{bjs}{adp} & \frac{hrs}{cmq} & 0 & 0 \\ 0 & 1 & \frac{bi}{ad} & \frac{fhipr+ikq(hl+gm)}{cfjmq} & \frac{gikn}{fhjo} & 0 \\ 0 & 0 & 0 & \frac{1}{bciklnst} & \frac{1}{bhiklots} & \frac{1}{bgiknrsu} \end{bmatrix} \tag{2}$$

To determine the value of $\overset{\bullet\rightarrow}{\mathbb{F}}$ at the point in $\Pi^\circ(\mathcal{M})$ defined by this matrix, we compute the nine minors with columns given by the source labels of faces in G (cf. Figure 3c).

$$\begin{array}{ccc} \Delta_{156} = \frac{1}{bfhjorsu} & \Delta_{126} = \frac{1}{bgiknrsu} & \Delta_{236} = \frac{1}{aegipnru} \\ \Delta_{234} = \frac{1}{aceilpnt} & \Delta_{345} = \frac{1}{acdfmopt} & \Delta_{456} = \frac{1}{bcfjmpqu} \\ \Delta_{136} = \frac{1}{adgknrsu} & \Delta_{356} = \frac{1}{adfhporu} & \Delta_{235} = \frac{1}{aehilpot} \end{array}$$

We see that, for each face f in G, the value of $\Delta_{\overset{\bullet}{\vec{I}}(f)}$ on the matrix in (2) *is the reciprocal of the product of the edge weights in the extremal matching $\overrightarrow{M}(f)$. This is equivalent to the equality $\overrightarrow{\mathbb{M}} = \overset{\bullet\to}{\mathbb{F}} \circ \vec{\tau} \circ \mathbb{D}$, and thus the commutativity of the right square in Theorem 8.1.*

9 Earlier work

The twist map was constructed earlier by Marsh and Scott [MS13] for the largest positroid variety, although they only give explicit formulas for the composite map above when G is a certain standard reduced graph known as a Le diagram. Talaska [Tal11] provided a birational inverse to the boundary measurement map when G is a Le diagram; her inverse was not formulated in terms of a twist map and seems unlikely to generalize to other reduced plabic graphs. Any double wiring diagram for a type A double Bruhat cell can be converted to a reduced plabic graph for a corresponding positroid variety; in this setting, the twist map was defined by Berenstein, Fomin and Zelevinsky, and it was proved that an analogous composite map is an isomorphism of tori.

Our result build on the above results, but are stronger than any of them, because they work for all positroid varieties and all reduced plabic graphs, which none of the above results do. We also hope that the unified presentation in this paper clarifies the nature of the previous results.

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