Combinatorial description of the cohomology of the affine flag variety

Seung Jin Lee†

School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro Dongdaemun-gu, Seoul 130-722, Republic of Korea.

Abstract. We construct the affine version of the Fomin-Kirillov algebra, called the affine FK algebra, to investigate the combinatorics of affine Schubert calculus for type A. We introduce Murnaghan-Nakayama elements and Dunkl elements in the affine FK algebra. We show that they are commutative as Bruhat operators, and the commutative algebra generated by these operators is isomorphic to the cohomology of the affine flag variety. As a byproduct, we obtain Murnaghan-Nakayama rules both for the affine Schubert polynomials and affine Stanley symmetric functions. This enable us to express k-Schur functions in terms of power sum symmetric functions. We also provide the definition of the affine Schubert polynomials, polynomial representatives of the Schubert basis in the cohomology of the affine flag variety.

Résumé. Nous construisons la version affine de l'algèbre Fomin-Kirillov, appelé l'algèbre FK affine, pour enquêter sur la combinatoire du calcul de Schubert affine pour le type A. Nous introduisons des éléments Murnaghan-Nakayama et éléments de Dunkl dans l'algèbre FK affine. Nous montrons qu'ils sont commutative comme opérateurs Bruhat, et l'algèbre commutative généré par ces opérateurs est isomorphe à la cohomologie de la variété affine du pavillon. En tant que sous-produit, on obtient règles de Murnaghan-Nakayama tant pour les polynômes de Schubert affines et les fonctions symétriques de Stanley affines. Cela nous permet d'exprimer des fonctions k -Schur en termes de fonctions symétriques puissance de somme. Nous fournissons également la définition de les polynômes de Schubert affines, des représentants polynômes de la base Schubert dans la cohomologie de la variété affine du pavillon.

Keywords. affine flag variety, affine Fomin-Kirillov algebra, affine nilCoxeter algebra, affine Schubert polynomials, k-Schur function, Murnaghan-Nakayama rule

1 Introduction

Fomin and Kirillov defined a certain quadratic algebra, also called the Fomin-Kirillov algebra, to better understand the combinatorics of the cohomology ring of the flag variety. They showed that the commutative subalgebra generated by Dunkl elements of degree 1 is isomorphic to the cohomology of the flag variety. Since then, a lot of variations for the quadratic algebra have been studied [Kir15, KM04, KM05a, KM05b, KM10, KM12]. For example, there are generalizations of the Fomin-Kirillov algebra

†Email: `lsjin@kias.re.kr`.

for K-theory, quantum, equivariant cohomology and for other finite types.

In this extended abstract, we introduce the affine Fomin-Kirillov algebra (affine FK algebra in short), generalizing the Fomin-Kirillov algebra to affine type A, to describe the cohomology of the affine flag variety. We introduce higher degree generators called Murnaghan-Nakayama elements in the affine FK algebra. Elements in the affine FK algebra induce the Bruhat actions on the affine nilCoxeter algebra $\mathbb{A}$, which is isomorphic to the homology of the affine flag variety as a $\mathbb{Q}$-module. We show that Murnaghan-Nakayama elements and Dunkl elements commute with each other as Bruhat operators and show that the algebra generated by these elements as Bruhat operators is isomorphic to the cohomology of the affine flag variety as subalgebras in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$. Our proof identifies three different operators on $\mathbb{A}$, namely: Bruhat operators, cap operators defined by the author [Lee14], and the operators defined by Berg, Saliola and Serrano [BSS14] on the affine nilCoxeter algebra $\mathbb{A}$. The identification combines algebraic, geometric and combinatorial components of affine Schubert calculus. Those three operators will be considered as elements in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$. In this extended abstract, we mainly discuss Bruhat operators and cap operators.

Positive integers $n \geq 2$ and $k = n - 1$ will be fixed throughout the paper. The coefficient ring of the cohomology is $\mathbb{Q}$, and related combinatorics will be adjusted accordingly although the three operators are well-defined over $\mathbb{Z}$.

1.1 Cap operators

The set $\{A_w : w \in \widetilde{S}_n\}$ forms a basis of $\mathbb{A}$ (see Section 2.3), where $\widetilde{S}_n$ is the affine symmetric group. There is a coproduct structure on $\mathbb{A}$ defined by

$$\Delta(A_w) = \sum p^w_{u,v} A_u \otimes A_v$$

where the sum is over all $u, v \in \widetilde{S}_n$ satisfying $\ell(w) = \ell(u) + \ell(v)$. Kostant and Kumar [KK86] showed that $p^w_{u,v}$ is the same as the structure coefficient for the cohomology of the affine flag variety. Note that $p^w_{u,v}$'s are nonnegative integers [Gra01].

For $u \in \widetilde{S}_n$, a *cap operator* D_u is defined by

$$D_u(A_w) = \sum p^w_{u,v} A_v.$$

where the sum is over all $v \in \widetilde{S}_n$ satisfying $\ell(w) = \ell(u) + \ell(v)$. Let ϕ_{id} be the map from $\mathbb{A}$ to $\mathbb{Q}$ by taking the coefficient of A_{id}. Then ϕ_{id} induces a $\mathbb{Q}$-module homomorphism $\phi_{\mathrm{id},*}$ from $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$ to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$. Note that the cohomology of the affine flag variety is isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$ over $\mathbb{Q}$ [KK86], and the Schubert basis ξ^w can be considered as an element in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$ defined by $\xi^w(A_v) = \delta_{w,v}$ for all $v \in \widetilde{S}_n$. However, the problem with this description is that there is no natural product structure on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{Q})$.

One can avoid this problem by considering cap operators. It is obvious that the image of D_w via $\phi_{\mathrm{id},*}$ is ξ^w so that the cohomology of the affine flag variety can be identified with the subalgebra generated by D_w in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$, which naturally has the product structure by composition. Geometrically, $D_u(A_w)$ can be identified with $\xi^u \cap \xi_w$ where ξ^u is the Schubert class for u in the cohomology, ξ_w is the Schubert class for w in the homology of the affine flag variety, and $\cap$ is the cap product.

1.2 Bruhat operators

The affine Fomin-Kirillov algebra is generated by $[ij]$ for $i < j$ with distinct residues modulo n with certain quadratic relations among $[ij]$. An element $[ij]$ can be considered as a (right) *Bruhat action* on the affine nilCoxeter algebra defined by

$$A_w \cdot [ij] = \begin{cases} A_{wt_{ij}} & \text{if} \quad \ell(wt_{ij}) = \ell(w) - 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $w \in \widetilde{S}_n$, where t_{ij} is a transposition in $\widetilde{S}_n$. For an element $\mathbf{x}$ in the affine FK algebra, define $D_{\mathbf{x}}(A_w) = A_w \cdot \mathbf{x}$ when it is well-defined. In this paper, we only consider elements $\mathbf{x}$ such that $D_{\mathbf{x}}(A_w)$ are well-defined for all $w \in \widetilde{S}_n$. See Section 3 for details.

Now we are ready to state the main theorems in this paper.
For $0 \leq i < m < n$, let $\rho_{i,m}$ be the element $s_{-i}s_{-i+1} \dots s_{-1}s_{m-1-i}s_{m-2-i} \dots s_1 s_0$ in $\widetilde{S}_n$. Let $\tilde{\theta}_i$ (resp. $\mathbf{p}_m$) be the Dunkl elements (resp. MN elements) in the affine FK algebra defined in Section 4.

Theorem 1.1 *For $0 \leq i < m < n$, we have*

$$D_{\tilde{\theta}_i} = D_{s_{i+1}} - D_{s_i},$$
$$D_{\mathbf{p}_m} = \sum_{i=0}^{m-1} (-1)^i D_{\rho_{i,m}}.$$

Theorem 1.1 allows us to identify $D_{\tilde{\theta}_i}$ with $\xi^{i+1} - \xi^i$, and $D_{\mathbf{p}_m}$ with $\xi(m) := \sum_{i=0}^{m-1}(-1)^i \xi^{\rho_{i,m}}$. Since the first equation is immediate from the Chevalley rule, we mainly focus on properties of $\mathbf{p}_m$. Note that the second equation together with the definition of $\mathbf{p}_m$ provides MN rule for the cohomology of the affine flag variety, which also provides MN rule for the affine Stanley symmetric functions. It also implies a description of k-Schur functions in terms of power sum symmetric functions which gives the character table of the representation of the symmetric group whose Frobenius characteristic image is the k-Schur function.

It turns out that the cohomology of the affine flag variety $\hat{Fl}$ is generated by all $\xi^{i+1} - \xi^i$ and $\xi(m)$, and the subalgebra generated by $\xi^{i+1} - \xi^i$ (resp. $\xi(m)$) is isomorphic to the cohomology of the finite flag variety Fl_n (resp. the affine Grassmannian $\hat{Gr}$). It provides the following isomorphism

$$H^*(\hat{Fl}) \cong H^*(\hat{Gr}) \otimes H^*(Fl_n) \cong \mathbb{Q}[p_1, \dots, p_{n-1}, x_1, \dots, x_n]/\langle h_i(x) = 0 \quad \forall i \rangle =: R_n$$

where all p_m and x_i commute with each other and p_m (resp. x_i) corresponds to $\xi(m)$ (resp. $\xi^{i+1} - \xi^i$), and $h_i(x)$ is a homogeneous polynomial of degree i in $x_1, \dots, x_n$. The affine Schubert polynomial $\tilde{\mathfrak{S}}_w$ for $w \in \widetilde{S}_n$ is defined by the image of ξ^w in R_n via the above isomorphism. We provide an explicit definition of the affine Schubert polynomials in terms of divided difference operators without proof. The affine Schubert polynomials simultaneously generalize the affine Schubert polynomials and Schubert polynomials. See [Lee15] for details.

The paper is structured as follows: In Section 2, we review affine symmetric groups, symmetric functions, and the affine nilCoxeter algebra. In Section 3, we define the affine FK algebra and study the Bruhat operator acting on the affine nilCoxeter algebra. In Section 4, we define Dunkl elements and Murnaghan-Nakayama elements, and investigate relations among those elements. In particular, we list identities that uniquely determine Bruhat operators for MN elements. In Section 5, we prove Theorem 1.1. In section 6, we provide the Murnaghan-Nakayama rules both for the affine Schubert polynomial and the affine Stanley symmetric functions. we also discuss a new formula for k-Schur functions in terms of power sum symmetric functions as well as its relation with representation theory. In section 7, we define the affine Schubert polynomials in terms of divided difference operators.

2 Preliminaries

2.1 Affine symmetric group

Let I be the set $\{0, 1, \dots, n-1\} = \mathbb{Z}/n\mathbb{Z}$. Let $\widetilde{S}_n$ denote the affine symmetric group with simple generators $s_0, s_1, \dots, s_{n-1}$ satisfying the relations

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \\ s_i s_j &= s_j s_i \qquad\qquad \text{if } i-j \neq 1, -1. \end{aligned}$$

where indices are taken modulo n. An element of the affine symmetric group may be written as a word in the generators s_i. A *reduced word* of the element is a word of minimal length. The *length* of w, denoted $\ell(w)$, is the number of generators in any reduced word of w. The *Bruhat order*, also called *strong order*, on affine symmetric group elements is a partial order where $u < w$ if there is a reduced word for u that is a subword of a reduced word for w. If $u < w$ and $\ell(u) = \ell(w) - 1$, we write $u \lessdot w$. The subgroup of $\widetilde{S}_n$ generated by $\{s_1, \cdots, s_{n-1}\}$ is naturally isomorphic to the symmetric group S_n. The 0-Grassmannian elements are minimal length coset representatives of $\widetilde{S}_n/S_n$. In other words, w is 0-Grassmannian if and only if all reduced words of w end with s_0.

2.2 Symmetric functions

Let Λ denote the ring of symmetric functions over $\mathbb{Q}$. For a partition λ, we let $m_\lambda, h_\lambda, p_\lambda, s_\lambda$ denote the monomial, homogeneous, power sum and Schur symmetric functions, respectively, indexed by λ. Each of these families forms a basis of Λ. Let $\langle \cdot, \cdot \rangle$ be the Hall inner product on Λ satisfying $\langle m_\lambda, h_\mu \rangle = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$ for partitions λ, μ.

Let $\Lambda_{(k)}$ denote the subalgebra generated by $h_1, h_2, \dots, h_k$. The elements h_λ with $\lambda_1 \le k$ form a basis of $\Lambda_{(k)}$. We call a partition λ with $\lambda_1 \le k$ a *k-bounded partition*. Note that there is a bijection between the set of k-bounded partitions and the set of 0-Grassmannian elements in $\widetilde{S}_n$ [LM05]. Let $\Lambda^{(k)} = \Lambda/I_k$ denote the quotient of Λ by the ideal I_k generated by m_λ with $\lambda_1 > k$. The image of the elements m_λ with $\lambda_1 \le k$ form a basis of $\Lambda^{(k)}$. Note that I_k is isomorphic to the ideal generated by p_λ for $\lambda_1 > k$, so that p_λ for k-bounded partitions λ form a basis of $\Lambda^{(k)}$. There is another remarkable basis for $\Lambda_{(k)}$ and $\Lambda^{(k)}$. For a k-bounded partition λ, a k-Schur function $s_\lambda^{(k)}$ and an affine Schur function $\tilde{F}_\lambda$ are defined in [LM07a, LM07b, Lam06]. Lam [Lam08] showed that the k-Schur functions (resp. the affine Schur functions) are representatives of the Schubert basis of the homology (resp. the cohomology) of the affine Grassmannian $\hat{Gr}$ for $SL(n)$ via the isomorphism of Hopf-algebras

$$\Lambda_{(k)} \cong H_*(\hat{Gr})$$
$$\Lambda^{(k)} \cong H^*(\hat{Gr}).$$

The restriction of the Hall inner product on $\Lambda^{(k)} \times \Lambda_{(k)}$ gives the identity $\langle \tilde{F}_\lambda, s_\mu^{(k)} \rangle = \delta_{\lambda,\mu}$. Since we do not use the definitions of k-Schur functions, affine Schur functions, and affine Stanley symmetric functions in this paper, definitions are omitted. See [Lam06, Lam08, LLMS10] for more details.

2.3 Affine nilCoxeter algebra

The *affine nilCoxeter algebra* $\mathbb{A}$ is the algebra generated by $A_0, A_1, \ldots, A_{n-1}$ over $\mathbb{Z}$, satisfying

$$\begin{aligned} A_i^2 &= 0 \\ A_i A_{i+1} A_i &= A_{i+1} A_i A_{i+1} \\ A_i A_j &= A_j A_i && \text{if } i - j \neq 1, -1. \end{aligned}$$

where the indices are taken modulo n. The subalgebra $\mathbb{A}_f$ of $\mathbb{A}$ generated by A_i for $i \neq 0$ is isomorphic to the nilCoxeter algebra studied by Fomin and Stanley [FS94]. The simple generators A_i are considered as the *divided difference operators*.

The A_i satisfy the same braid relations as the s_i in $\widetilde{S}_n$, i.e., $A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}$. Therefore it makes sense to define

$$\begin{aligned} A_w &= A_{i_1} \cdots A_{i_l} && \text{where} \\ w &= s_{i_1} \cdots s_{i_l} && \text{is a reduced decomposition.} \end{aligned}$$

Lam [Lam06] defined certain elements $\mathbf{h}_i$ in $\mathbb{A}$ for $i < n$ and showed that the elements $\{\mathbf{h}_i\}_{i<n}$ commute and freely generate a subalgebra $\mathbb{B}$ of $\mathbb{A}$ called the *affine Fomin-Stanley algebra*. It is well-known that $\mathbb{B}$ is isomorphic to $\Lambda_{(k)}$ via the map sending $\mathbf{h}_i$ to h_i. Therefore, the set $\{\mathbf{h}_\lambda = \mathbf{h}_{\lambda_1} \ldots \mathbf{h}_{\lambda_l} \mid \lambda_1 \leq k\}$ forms a basis of $\mathbb{B}$.

3 Affine Fomin-Kirillov algebra

For $i \in \mathbb{Z}$, let $\bar{i}$ be the residue of i modulo n.

Definition 3.1 *Let A be the free algebra generated by $\mathcal{S} = \{[ij] : i, j \in \mathbb{Z}, i < j, \bar{i} \neq \bar{j}\}$. Let $A(N)$ be the subalgebra of A generated by elements $[ij]$ with $|i|, |j| \geq N$. Then we have a filtration*

$$A = A(0) \supset A(1) \supset A(2) \supset \cdots .$$

Let $\mathcal{A}$ be the inverse limit $\varprojlim (A/A(i))$.

An element $\mathbf{x}$ in $\mathcal{A}$ can be written as a (possibly infinite) linear combination of *noncommutative monomials* $[j_{1,1}, j_{1,2}][j_{2,1}, j_{2,2}] \cdots [j_{m,1}, j_{m,2}]$. For $i > j$, we use the convention $[ij] = -[ji]$.

Define the *Bruhat action* of $[ij]$ on $\mathbb{A}$ by

$$A_w \cdot [ij] = \begin{cases} A_{wt_{ij}} & \text{if} \quad \ell(wt_{ij}) = \ell(w) - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

where t_{ij} is a transposition exchanging i and j in $\widetilde{S}_n$. For an element in A, one can define an action on $\mathbb{A}$ extended linearly. For an element $\mathbf{x}$ in $\mathcal{A}$, even if $\mathbf{x}$ is an infinite summation of product of $[ij]$'s, it is possible that all but finitely many terms in $\mathbf{x}$ vanish when acting with $\mathbf{x}$ on an element A_w. An element $\mathbf{x}$ in $\mathcal{A}$ gives a *valid action* on $\mathbb{A}$ if all but finitely many terms in $\mathbf{x}$ vanish when acting with $\mathbf{x}$ on any element A_w. Let $\mathcal{E}$ be the subalgebra of $\mathcal{A}$ consisting of elements which give a valid action on $\mathbb{A}$. All elements in $\mathcal{A}$ in this paper containing an infinite sum have a valid action on $\mathbb{A}$. Define the map $D : \mathcal{E} \to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$ by sending $\mathbf{x}$ to $D_{\mathbf{x}}$, where $D_{\mathbf{x}}(A_v) := A_v \cdot \mathbf{x}$. We call $D_{\mathbf{x}}$ a *Bruhat operator* for $\mathbf{x}$. We often say "$\mathbf{x}$ as a Bruhat operator" instead of $D_{\mathbf{x}}$ since we are mainly interested in describing the cohomology of the affine flag variety as a subalgebra in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{A})$.

As Bruhat operators, we have following relations between the operators $[ij]$.

(a) $[ij]^2 = 0$.

(b) $[ij][kl] = [kl][ij]$ if $\bar{i}, \bar{j}, \bar{k}, \bar{l}$ are all distinct.

(c) For i, j, k with distinct residues, $[ij][jk] = [jk][ik] + [ik][ij]$ and $[jk][ij] = [ik][jk] + [ik][ij]$.

(d) For distinct i, j with $\bar{i} \neq \bar{j}$, $\sum_{\overline{i'}=\bar{i}, \overline{j'}=\bar{j}} [ij'][ji'] = 0$.

(e) $[i, j] = [i+n, j+n]$.

Note that the relations (a)-(c) are analogous to those in the definition of the Fomin-Kirillov algebra, and proofs for these relations are similar. The relation (d) is an affine type A analogue of the quadratic relation in the bracket algebra which is a generalization of the Fomin-Kirillov algebra to (classical) Coxeter groups [KM04]. The relation (e) is obvious since we have $t_{i,j} = t_{i+n,j+n}$ as elements in the affine symmetric group. The quotient algebra of $\mathcal{E}$ modulo relations (a)-(e) is called the *affine Fomin-Kirillov algebra* $\widetilde{FK}_n$.

Note that there is a (left) $\widetilde{S}_n$- action on $\widetilde{FK}_n$ (and $\mathcal{E}$) defined by $w[ij] = [w(i), w(j)]$ for $w \in \widetilde{S}_n$ and $[ij] \in \mathcal{S}$. Indeed, one can check that the two-sided ideal generated by relations (a)-(e) is invariant under the $\widetilde{S}_n$-action.

4 Dunkl elements and Murnaghan-Nakayama elements

In this section, we define Dunkl elements and MN elements and investigate identities among these elements. For $i \in \mathbb{Z}$, a Dunkl element $\tilde{\theta}_i$ can be defined in an analogous way to the definition of the Dunkl element θ_i in the Fomin-Kirillov algebra defined in [FK99].

Definition 4.1 *For $i \in \mathbb{Z}$, define a Dunkl element $\tilde{\theta}_i$ by $\sum_{j \in \mathbb{Z}, \bar{j} \neq \bar{i}} [ij]$.*

We define Murnaghan-Nakayama elements $\mathbf{p}_m(i)$ (MN elements in short) in $\widetilde{FK}_n$ as a generalization of $\theta_1^m + \cdots + \theta_i^m$ in the Fomin-Kirillov algebra. Unlike the finite case, MN elements are not generated by Dunkl elements $\tilde{\theta}_i$. We define MN elements by investigating the combinatorics of the Fomin-Kirillov algebra studied by Mészáros, Panova, Postnikov [MPP14] and generalizing them to affine case.

Let $\mathcal{D}$ be the 2-dimensional infinite grid. A *box* is specified by its position (i, j) when the vertices of the box are $(i, j), (i, j+1), (i+1, j), (i+1, j+1)$. Let $\mathcal{D}_a$ be the set of all boxes at (i, j) with $i \leq a < j$. A *diagram* D on $\mathcal{D}_a$ is a finite collection of boxes in $\mathcal{D}_a$. For a diagram D on $\mathcal{D}_a$, we associate a graph with the vertex set $\mathbb{Z}$ obtained by adding an edge between i and j for each box at (i, j) in D. We say that a diagram D is a *connected tree* if the associated graph consists of all but finitely many isolated points and a single tree, and all vertices in the tree have distinct residues modulo n. Let Supp(D) be the set consisting of indices of all vertices in the single tree in the associated graph for D and $c(D)$ the number of vertices in the tree with index $\leq a$. Note that the box at $(i, i+np)$ does not appear in a connected tree for any $i, p \in \mathbb{Z}$.

A *labeling* D_L on a diagram D is a bijection from the set $\{1, 2, \ldots, |D|\}$ to the set of boxes in D. For a labeling L of a connected tree D, one can associate an element in the affine FK algebra defined by $x_{D_L} = [D_L(1)][D_L(2)] \ldots [D_L(|D|)]$ where $[D_L(i)]$ is $[a_i b_i]$ for the i-th box placed at (a_i, b_i). We call two labelings L and L' equivalent if we have $x_{D_L} = x_{D_{L'}}$ by only using commutation relation.

The following lemma is an obvious generalization of [MPP14, Lemma7].

Lemma 4.2 *Let v, l be positive integers and D be a connected tree in $\mathcal{D}_a$ with $l + v$ boxes contained in l rows and $v + 1$ columns. Then the following two sets are equal:*

1. *The equivalent classes of labelings of D such that the class contains a labeling with: $i_1, \ldots, i_l$ are distinct, $j_1 \leq \cdots, \leq j_l$, $j_{l+1}, \ldots, j_{l+v}$ are distinct, $i_{l+1} \leq \cdots \leq i_{l+v}$.*
2. *The equivalent classes of labelings of D such that the class contains a labeling with: $i_1, \ldots, i_{l-1}$ are distinct, $j_1 \leq \cdots, \leq j_{l-1}$, $j_l, \ldots, j_{l+v}$ are distinct, $i_l \leq \cdots \leq i_{l+v}$.*

Let $M(D) = \{D_{L_1}, \ldots, D_{L_h}\}$ be the set of representative labelings of equivalent classes in Lemma 4.2.

Definition 4.3 *Let m and a be positive integers. Define $\mathbf{p}_m(a)$ in $\widetilde{FK}_n$ by*

$$\mathbf{p}_m(a) = \sum_{D \in \mathcal{D}_a} \sum_{D_L \in M(D)} (-1)^{c(D)-1} x_{D_L}$$

where the first sum runs over connected trees in $\mathcal{D}_a$. We denote $\mathbf{p}_m(0)$ by $\mathbf{p}_m$.

We list few identities among Dunkl elements and MN elements.

Theorem 4.4 *For $m < n, i \in \mathbb{Z}$, we have*

$$\mathbf{p}_m(i) + \tilde{\theta}_{i+1}^m = \mathbf{p}_m(i+1)$$

in $\widetilde{FK}_n$.

Theorem 4.5 *For $m > 0$, we have*

$$\sum_{i=1}^{n} \tilde{\theta}_i^m = 0.$$

Proof. It is obvious from Theorem 4.4 and the fact $\mathbf{p}_m(0) = \mathbf{p}_m(n)$ by the relation (e). □

Theorem 4.6 *For $i, a \in I$,we have*

$$s_i \mathbf{p}_m(a) = \begin{cases} \mathbf{p}_m(a) & \text{if } i \neq a \\ \mathbf{p}_m(i) + \tilde{\theta}_{i+1}^m - \tilde{\theta}_i^m. & i{=}a \end{cases} \tag{2}$$

For $\mathbf{x} = \mathbf{p}_m$, we call $D_{\mathbf{p}_m}$ a *Murnaghan-Nakayama operator* of degree m (a MN operator in short). The following theorems uniquely determine the MN operators.

Theorem 4.7 *For $w \in \widetilde{S}_n$ and $w' \in S_n$, we have*

$$D_{\mathbf{p}_m}(A_w A_{w'}) = D_{\mathbf{p}_m}(A_w)A_{w'}.$$

Theorem 4.8 *For $\mathbf{h} \in \mathbb{B}$ and $w \in \widetilde{S}_n$,*

$$D_{\mathbf{p}_m}(\mathbf{h}A_w) = D_{\mathbf{p}_m}(\mathbf{h})A_w + \mathbf{h}D_{\mathbf{p}_m}(A_w).$$

Theorem 4.9 *For $1 \leq m \leq i < n$ and $a \in \mathbb{Z}$, we have*

$$D_{\mathbf{p}_m(a)}\mathbf{h}_i = \mathbf{h}_{i-m}.$$

5 Relations between operators

To show Theorem 1.1, it is enough to show that $\sum_{i=0}^{m-1}(-1)^i D_{\rho_{i,m}}$ also satisfies Theorem 4.7, 4.8, 4.9. First of all, Theorem 4.7 follows from the fact that all $\rho_{i,m}$ are 0-Grassmannian elements (See [Lee14]). Therefore, it is enough to show that $D_{\mathbf{p}_m}$ and $\sum_{i=0}^{m-1}(-1)^i D_{\rho_{i,m}}$ are the same operators as actions on $\mathbb{B} \cong \Lambda_{(k)}$. One can show that the restriction of $D_{\rho_{i,m}}$ is $\overline{s_{[m-i,1^i]}}^{\perp}$, where $s_{[m-i,1^i]}$ is the Schur function for the hook shape $[m-i, 1^i]$, $\overline{s_{[m-i,1^i]}}$ is the image of $s_{[m-i,1^i]}$ in $\Lambda_{(k)}$, and $f^{\perp}$ is an operator acting on $\Lambda^{(k)}$ adjoint to the multiplication by f for $f \in \Lambda_{(k)}$,. This follows from the comparison between cap operators and BSS operators denoted by D_J in [BSS14]. See [Lee14] for details.

Recall the following theorems about the power sum symmetric functions p_m: (see [Sta99] for instance):

$$p_m = \sum_{i=0}^{m-1}(-1)^i s_{[m-i,1^i]},$$

$$p_m^{\perp}(fg) = p_m^{\perp}(f)g + fp_m^{\perp}(g),$$
$$p_m^{\perp}(h_i) = h_{i-m}$$

for any symmetric functions f, g. Therefore, we proved that $D = \sum_{i=0}^{m-1}(-1)^i D_{\rho_{i,m}}$ satisfies the following identities.

1. $D(fg) = D(f)g + fD(g)$ for $f, g \in \mathbb{B}$.

2. $D(hA_v) = D(h)A_v$ for $h \in \mathbb{B}$ and a 0-Grassmannian element v.

3. $D(\mathbf{h}_i) = \mathbf{h}_{i-m}$

Note that the above identities uniquely determine D as an action on $\mathbb{A}$. Since $\mathbf{p}_m$ also satisfies the above identities by Theorem 4.7, 4.8, 4.9, the main theorem follows. □

6 Murnaghan-Nakayama rule for the affine flag variety

Recall that ξ^w is the Schubert class for w in the cohomology of the affine flag variety and $\xi(m) = \sum_{i=0}^{m}(-1)^i\xi^{\rho_{i,m}}$. Note that $\xi(m)$ maps to p_m via the map $p_1^* : H^*(\hat{Fl}) \to H^*(Gr) \cong \Lambda^{(k)}$.

Theorem 6.1 *For $w, u \in \widetilde{S}_n, m < n$, let $c^w_{m,u}$ be integers satisfying $D_{\mathbf{p}_m}(A_w) = \sum_u c^w_{m,u}A_u$. Then for $v \in \widetilde{S}_n$, we have*

$$\xi(m) \cup \xi^v = \sum_{w\in\widetilde{S}_n} c^w_{m,v}\xi^w.$$

Proof. For $u \in \widetilde{S}_n$, let ξ_u be the Schubert class for u in the homology of the affine flag variety and let $\langle\cdot,\cdot\rangle$ be the pairing between the cohomology and homology of the affine flag variety. Then we have

$$\langle \xi(m) \cup \xi^v, \xi_w\rangle = \langle \xi^v, D_{\mathbf{p}_m}(\xi_w)\rangle = c^w_{m,v}.$$

□

Therefore, the definition of $\mathbf{p}_m$ provide MN rule for the cohomology of the affine flag variety. One can also obtain the MN rule for the affine Stanley symmetric functions from the fact that the Stanley symmetric function $\tilde{F}_w$ is the pullback $p_1^*(\xi^w)$ where $p_1^* : H^*(\hat{Fl}) \to H^*(\hat{Gr})$ (See [Lam08] for details). By applying the pullback p_1^* to both sides of Theorem 6.1, we have the following MN rule.

Corollary 6.2 *For $m < n, v \in \widetilde{S}_n$, we have*

$$p_m\tilde{F}_v = \sum_{w\in\widetilde{S}_n} c^w_{m,v}\tilde{F}_w.$$

Example 6.3 *Consider the identity $\tilde{F}_{10}p_3 = \tilde{F}_{12310} - \tilde{F}_{20310} + \tilde{F}_{03210}$. Each term can be computed from the Bruhat actions of the following terms in $\mathbf{p}_m(0)$.*

$$\begin{aligned} s_1s_2s_3s_1s_0 \cdot [-2,1][-4,1][-1,1] &= s_1s_0 \\ s_2s_0s_3s_1s_0 \cdot [-4,1][-1,2][-1,1] &= s_1s_0 \\ s_0s_3s_2s_1s_0 \cdot [0,6][0,5][0,3] &= s_1s_0. \end{aligned}$$

By applying Theorem 6.2 repeatedly from $v = id$, one can write p_λ in terms of linear combination of the affine Schur functions $\tilde{F}_v$. By taking dual, one can write a k-Schur function $s_u^{(k)}$ in terms of power sum symmetric functions p_μ. Note that the k-Schur function $s_u^{(k)}$ is known to be Schur-positive [LLMS13], so there exists a S_n-representation whose Frobenius image is $s_u^{(k)}$. Therefore the description of k-Schur function in terms of p_μ provides the character table of the represntation. Note that Chen and Haiman [CH08] conjectured t-graded S_n-representation whose Frobenius image is the involution of the t-dependent k-Schur function $\omega s_u^{(k)}[X;t]$. It would be interesting if the combinatorics studied by Chen and Haiman is related to the combinatorics studied in this paper.

7 Affine Schubert polynomials

In this section, we provide the definition of the affine Schubert polynomials, polynomial representatives of the Schubert class in the cohomology of the affine flag variety $\hat{Fl}$. We start with the following theorem proved in [Lee14].

Theorem 7.1 *The cohomology of the affine flag variety $\hat{Fl}$ is generated by $\xi(m)$ for $m < n$ and $\xi^{s_{i+1}} - \xi^{s_i}$ for $i \in \mathbb{Z}/n\mathbb{Z}$, and the subalgebra generated by $\xi(m)$ (resp. $\xi^{s_{i+1}} - \xi^{s_i}$) is isomorphic to the cohomology of the affine Grassmannian (resp. finite flag variety). Moreover, under the isomorphisms we have $H^*(\hat{Fl}) \cong H^*(\hat{Gr}) \otimes H^*(Fl_n)$.*

Note that $H^*(\hat{Gr})$ is isomorphic to $\Lambda^{(k)} \cong \mathbb{Q}[p_1, \dots, p_k]$ and $H^*(Fl_n)$ is isomorphic to $\mathbb{Q}[x_1, \dots, x_n]$ modulo an ideal $J = \langle h_i(x) = 0 \quad \forall i \rangle$ where h_i is the homogeneous symmetric function of degree i. For $w \in \widetilde{S}_n$, the affine Schubert polynomial $\tilde{\mathfrak{S}}_w$ for w is defined as an element in $\mathbb{Q}[p_1, \dots, p_k, x_1, \dots, x_n]/J$ corresponding to the Schubert basis ξ^w in $H^*(\hat{Fl})$. One can explicitly define the affine Schubert polynomial in terms of divided difference operators in the following way.

Definition 7.2 *For $i \in \mathbb{Z}/n\mathbb{Z}$, the Weyl group action s_i and the divided difference operator $\partial_i := \frac{1-s_i}{x_i - x_{i+1}}$ on R_n can be uniquely defined by the following rules.*

1. *For $f, g \in H^*(\hat{Fl})$, we have $s_i(fg) = s_i(f)s_i(g)$. Therefore ∂_i satisfies the Leibniz's rule: for $f, g \in H^*(\hat{Fl})$, we have*
$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g).$$
2. *For nonzero i and for all m, we have $s_i(p_m) = p_m$ and $\partial_i(p_m) = 0$.*
3. *For $i = 0$, we have $s_0(p_m) = p_m + x_1^m - x_0^m$ and $\partial_0(p_m) = \sum_{j=0}^{m-1} x_1^{m-1-j} x_0^j$.*
4. *For all $i, j \in \mathbb{Z}/n\mathbb{Z}$, we have $s_i(x_j) = x_{s_i(j)}$ and $\partial_i(x_j) = \delta_{ij} - \delta_{i,j+1}$.*

Definition 7.3 *For $w \in \widetilde{S}_n$, the affine Schubert polynomial $\tilde{\mathfrak{S}}_w$ is the unique homogeneous element of degree $\ell(w)$ in R_n satisfying*

$$\partial_i \tilde{\mathfrak{S}}_w = \begin{cases} \tilde{\mathfrak{S}}_{ws_i} & \text{if } \ell(ws_i) = \ell(w) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

for $i \in \mathbb{Z}/n\mathbb{Z}$, with the initial condition $\tilde{\mathfrak{S}}_{id} = 1$.

The affine Schubert polynomials behave surprisingly well with the affine Stanley symmetric functions $\tilde{F}_w$ [Lam08] for $w \in \widetilde{S}_n$ and the Schubert polynomials $\mathfrak{S}_v$ for $v \in S_n$. First note that the divided difference operators ∂_i for nonzero i on the subalgebra of R_n generated by x_i's are the same as the divided difference operators defined by Lascoux and Schützenberger [LS82], so that $\tilde{\mathfrak{S}}_w$ for $w \in S_n$ is the Schubert polynomial $\mathfrak{S}_w$. Moreover the affine Schubert polynomial $\tilde{\mathfrak{S}}_w$ for 0-Grassmannian element w is the same as the affine Schur functions, and for $w \in \widetilde{S}_n$ the projection from R_n to $\mathbb{Q}[p_1, \dots, p_{n-1}]$ sends $\tilde{\mathfrak{S}}_w$ to the affine Stanley symmetric functions $\tilde{F}_w$ [Lam08].

The affine Schubert polynomials can be computed from the affine Stanley functions. For $w \in \widetilde{S}_n$, let v be an element in $\widetilde{S}_n$ such that wv is 0-Grassmannian with $\ell(wv) = \ell(w) + \ell(v)$. There is always such a v for any w. Let $\tilde{F}_{wv}$ be the affine Schur function for wv. Then we have

$$\tilde{\mathfrak{S}}_w = \partial_{v^{-1}} \tilde{\mathfrak{S}}_{wv} = \partial_{v^{-1}} \tilde{F}_{wv}.$$

Note that there is a formula for the expansion of the affine Schur functions in terms of power sum symmetric functions [BSZ11], so that one can compute the affine Schubert polynomials from Definition 7.2, 7.3.

Acknowledgments

I thank Thomas Lam, Leonardo Mihalcea, Mark Shimozono for helpful discussions.

References

[BSS14] C. Berg, F. Saliola, and L. Serrano. Pieri operators on the affine nilCoxeter algebra. *Trans. Amer. Math. Soc.*, 366:531–546, 2014.

[BSZ11] J. Bandlow, A. Schilling, and M. Zabrocki. The Murnaghan-Nakayama rule for k-Schur functions. *J. Comb. Theory. Ser. A*, 118(5):1588–1607, 2011.

[CH08] L. Chen and M. Haiman. A representation-theoretic model for k-atoms. *J. Comb. Theory. Ser. A*, pages Talk 1039–05–169 at the AMS meeting in Claremont, McKenna, 2008.

[FK99] S. Fomin and A. Kirillov. Quadratic algebras, Dunkl elements, and Schubert calculus. *Adv. Geom., Progress in Mathematics*, 172:147–182, 1999.

[FS94] S. Fomin and R. Stanley. Schubert polynomials and the nil-Coxeter algebra. *Adv. Math.*, 103(2):196–207, 1994.

[Gra01] W. Graham. Positivity in equivariant Schubert calculus. *Duke Math. J.*, 109(3):599–614, 2001.

[Kir15] A. Kirillov. On some quadratic algebras i$\frac{1}{2}$: Combinatorics of Dunkl and Gaudin elements, Schubert, Grothendieck, Fuss-Catalan, universal Tutte and reduced polynomials. *Duke Math. J.*, preprint RIMS-1817, 2015.

[KK86] B. Kostant and S. Kumar. The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. *Adv. Math.*, 62(3):187–237, 1986.

[KM04] A. Kirillov and T. Maeno. Noncommutative algebras related with Schubert calculus on Coxeter groups. *European Journal of Combinatorics*, 25(3):1301–1325, 2004.

[KM05a] A. Kirillov and T. Maeno. A note on quantization on Nichols algebra model for Schubert calculus on Weyl groups. *Lett. in Math. Phys.*, 72:233–241, 2005.

[KM05b] A. Kirillov and T. Maeno. On some noncommutative algebras related with k-theory of flag varieties. *IRMN*, 60:3753–3789, 2005.

[KM10] A. Kirillov and T. Maeno. Nichols-Woronowicz model of coinvariant algebra of complex reflection groups. *J. Pure Appl. Algebra*, 214:402–409, 2010.

[KM12] A. Kirillov and T. Maeno. Affine nil-Hecke algebras and braided differential structure on affine Weyl groups. *Publ. RIMS*, 48:215–228, 2012.

[Lam06] T. Lam. Affine Stanley symmetric functions. *Amer. J. Math.*, 128(6):1553–1586, 2006.

[Lam08] T. Lam. Schubert polynomials for the affine Grassmannian. *J. Amer. Math. Soc.*, 21(1):259–281, 2008.

[Lee14] S. Lee. Pieri rule for the affine flag variety. *arXiv:1406.4246v2*, 2014.

[Lee15] S. Lee. Combinatorial description of the cohomology of the affine flag variety. *arXiv:1506.02390*, 2015.

[LLMS10] T. Lam, L. Lapointe, J. Morse, and M. Shimozono. Affine insertion and Pieri rules for the affine Grassmannian. *Memoirs of the AMS*, 208(977), 2010.

[LLMS13] T. Lam, L. Lapointe, J. Morse, and M. Shimozono. k-shape poset and branching of k-schur functions. *Mem. Amer. Math. Soc.*, 223(1050), 2013.

[LM05] L. Lapointe and J. Morse. Tableaux on $k+1$-cores, reduced words for affine permutations, and k-schur expansions. *J. Combin. Theory Ser. A*, 112(1):44–81, 2005.

[LM07a] L. Lapointe and J. Morse. A k-tableaux characterization of k-Schur functions. *Adv. Math.*, 213(1):183–204, 2007.

[LM07b] L. Lapointe and J. Morse. Quantum cohomology and the k-Schur basis. *Trans. Amer. Math. Soc.*, pages PII S 0002–9947(07)04287–0 (to appear in print)., 2007.

[LS82] A. Lascoux and M. Schützenberger. Polynômes de Schubert. *C. R. Acad. Sci. Paris. Sér. I Math.*, 294(13):447–450, 1982.

[MPP14] K. Mészáros, G. Panova, and A. Postnikov. Schur times Schubert via the Fomin-Kirillov algebra. *Electr. J. Comb.*, 21(1), 2014.

[Sta99] R. Stanley. Enumerative combinatorics. *Cambridge University Press*, 2, 1999.