Toric matrix Schubert varieties and root polytopes (extended abstract)

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Abstract. Start with a permutation matrix π and consider all matrices that can be obtained from π by taking downward row operations and rightward column operations; the closure of this set gives the matrix Schubert variety $\overline{X_\pi}$. We characterize when the ideal defining $\overline{X_\pi}$ is toric (with respect to a $2n-1$-dimensional torus) and study the associated polytope of its projectivization. We construct regular triangulations of these polytopes which we show are geometric realizations of a family of subword complexes. We also show that these complexes can be realized geometrically via regular triangulations of root polytopes. This implies that a family of β-Grothendieck polynomials are special cases of reduced forms in the subdivision algebra of root polytopes. We also write the volume and Ehrhart series of root polytopes in terms of β-Grothendieck polynomials. Subword complexes were introduced by Knutson and Miller in 2004, who showed that they are homeomorphic to balls or spheres and raised the question of their polytopal realizations.

Résumé. En partant d'une matrice de permutation π, considérons toutes les matrices qui peuvent être obtenues à partir de π en effectuant des opérations de ligne vers le bas et des opérations de colonne vers la droite ; l'adhérence de cet ensemble donne la variété Schubert de matrices $\overline{X_\pi}$. Nous caractérisons la situation où l'idéal définissant $\overline{X_\pi}$ est torique et étudions le polytope associé de sa projectivisation. Nous construisons des triangulations régulières de ces polytopes et nous montrons qu'elles sont des réalisations géométriques d'une famille de complexes de sous-mots. Nous montrons également que ces complexes peuvent être réalisés géométriquement par des triangulations régulières de polytopes de racines. Cela implique qu'une famille de polynômes β-Grothendieck sont des cas particuliers de formes réduites dans l'algèbre de subdivision de polytopes des racines. On peut aussi écrire le volume et la série d'Ehrhart des polytopes des racines en termes de β-Grothendieck polynômes. Les complexes de sous-mots ont été introduits par Knutson et Miller en 2004, qui ont soulevé la question de leurs réalisations polytopales.

Keywords. subword complex, root polytope, matrix Schubert variety, toric variety

1 Introduction

This is an extended abstract based on Escobar and Mészáros (2015a) and Escobar and Mészáros (2015b). We study the geometry of matrix Schubert varieties and give geometric realizations of a family of subword complexes. Matrix Schubert varieties were introduced by Fulton (1992) to study the degeneraci loci of

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flagged vector bundles. Knutson and Miller (2005) showed that Schubert polynomials are multidegrees of matrix Schubert varieties. Knutson and Miller (2004, 2005) introduced subword complexes to illustrate the combinatorics of Schubert polynomials and determinantal ideals, building up on the work of Fomin and Kirillov (1994); Bergeron and Billey (1993). Knutson and Miller proved that any subword complex is homeomorphic to a ball or a sphere and asked about their geometric realizations.

Given a matrix Schubert variety $\overline{X_\pi}$, it can be written as $\overline{X_\pi} = Y_\pi \times \mathbb{C}^q$ (where q is maximal possible). Our main results are as follows. We characterize when Y_π is toric (with respect to a $(\mathbb{C}^*)^{2n-1}$-action) and study the polytope $\Phi(\mathbb{P}(Y_\pi))$ corresponding to its projectivization. We construct a regular triangulation of $\Phi(\mathbb{P}(Y_\pi))$, induced from a degeneration to a root polytope, which we show are geometric realizations of a family of subword complexes. The following papers have partially answered the question about the geometric realization of spherical subword complexes: Stump (2011); Ceballos (2012); Pilaud and Pocchiola (2012); Pilaud and Santos (2012); Serrano and Stump (2012); Ceballos et al. (2014); Bergeron et al. (2015). This submission is based on Escobar and Mészáros (2015a,b), where we give the first realizations of a family of subword complexes which are homeomorphic to balls.

The roadmap of this paper is as follows. In Section 2 we define matrix Schubert varieties $\overline{X_\pi}$ and calculate the moment polytope $\Phi(\mathbb{P}(Y_\pi))$ of the projectivization of Y_π. In Section 3 we characterize when Y_π is toric and construct a regular triangulation of $\Phi(\mathbb{P}(Y_\pi))$. In Section 4 we define subword complexes, give geometric realizations of subword complexes homeomorphic to balls, and show how to express the volume and Ehrhart series of root polytopes in terms of Grothendieck polynomials. Finally, in Section 5 we give canonical triangulations of $\Phi(\mathbb{P}(Y_\pi))$ and show they are geometric realizations of pipe dream complexes for all π such that Y_π is toric.

2 Matrix Schubert varieties

Given a matrix Schubert variety $\overline{X_\pi}$ we define a variety $Y_\pi \hookrightarrow \overline{X_\pi}$ and characterize for which π, the variety Y_π is toric using the diagram of π. For such π, we construct a regular triangulation of its corresponding polytope, which we show is a geometric realization of a family of subword complexes, see Proposition 3.1 and Theorem 5.3.

Let M_n denote $n \times n$ matrices over $\mathbb{C}$, B_+ denote upper triangular invertible $n \times n$ matrices and B_- denote lower triangular invertible $n \times n$ matrices. We let $\pi \in S_n$ denote both a permutation and its corresponding permutation matrix, where its (i, j)-th entry is

$$(\pi)_{(i,j)} = \begin{cases} 1, & \text{if } \pi(j) = i, \\ 0, & \text{else.} \end{cases}$$

The multiplication on the left by matrices in B_- corresponds to downward row operations and multiplication on the right by matrices in B_+corresponds to rightward column operations. This multiplication gives a left action of $B_- \times B_+$ on M_n defined by

$$(X, Y) \cdot M := XMY^{-1}. \tag{1}$$

Given $1 \leq a \leq m$ and $1 \leq b \leq m$, let $M_{(a,b)}$ denote the upper left $a \times b$ submatrix of the matrix M. Define a **rank function** of a matrix M to be $r_M(a, b) := \text{rank}(M_{(a,b)})$. We then have that $M \in B_-\pi B_+$ if and only if $r_M(a, b) = r_\pi(a, b)$ for all $(a, b) \in [m] \times [m]$.

Definition 2.1 *The* **matrix Schubert variety** *of π is $\overline{X_\pi} := \overline{B_-\pi B_+}$, i.e. the Zariksi closure of its $(B_- \times B_+)$-orbit inside $M_n = \mathbb{C}^{n^2}$.*

Fulton studied this affine variety in Fulton (1992). We summarize some of his results here.

Theorem 2.2 *(Fulton, 1992, Proposition 3.3) The matrix Schubert variety $\overline{X_\pi}$ is an irreducible variety of dimension $n^2 - \ell(\pi)$ defined as a scheme by the equations $r_M(a,b) \leq r_\pi(a,b)$ for all $(a,b) \in [n] \times [n]$.*

Some of these inequalities are implied by others, and Fulton described the minimal set of rank conditions.

Definition 2.3 *The* **(Rothe) diagram** *of a permutation π is the collection of boxes $D(\pi) = \{(\pi_j, i) : i < j, \pi_i > \pi_j\}$. It can be visualized by considering the boxes left in the $n \times n$ grid after we cross out the boxes appearing south and east of each 1 in the permutation matrix for π.*

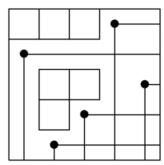

Fig. 1: The diagram for $\pi = [25413]$.

Definition 2.4 **Fulton's essential set** *$Ess(\pi)$ is the set consisting of the south-east corners of $D(\pi)$.*

Theorem 2.5 *(Fulton, 1992, Lemma 3.10) The ideal defining the variety $\overline{X_\pi}$ is generated by the equations $r_M(a,b) \leq r_\pi(a,b)$ for all $(a,b) \in Ess(\pi)$.*

We now define some regions inside the $(n \times n)$-grid and some varieties corresponding to these regions, including Y_π.

Definition 2.6 *The* **dominant piece***, denoted $dom(\pi)$, of a permutation π is the connected component of the diagram of π containing the box $(1,1)$, or empty if $\pi(1) = 1$.*

Definition 2.7 *Let $\boldsymbol{NW(\pi)}$ denote the union over the entries north-west of some box in $D(\pi)$. Let $\boldsymbol{L(\pi)} := NW(\pi) - dom(\pi)$ and let $\boldsymbol{L'(\pi)} := L(\pi) - D(\pi)$.*

See Figure 2 for an example.

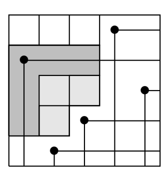

Fig. 2: Given $\pi = [25413]$, $L(\pi)$ consists of all the gray boxes and $L'(\pi)$ consists of only the darker gray boxes.

Definition 2.8 *Given a permutation π, let $\boldsymbol{Y_\pi}$ be the projection of $\overline{X_\pi}$ onto the entries inside $L(\pi)$ and let $\boldsymbol{V_\pi}$ be the projection onto the entries not north-west of any box of $D(\pi)$.*

Theorem 2.5 implies that the entries in V_π are free in $\overline{X_\pi}$ and thus $V_\pi \cong \mathbb{C}^q$, where q is the number of boxes in the region defining V_π, and that $\overline{X_\pi} = Y_\pi \times V_\pi$. This, together with Theorem 2.2, imply that $\dim(Y_\pi) = |L'(\pi)|$ and that Y_π is irreducible.

Let T^n consist of $n \times n$ diagonal invertible matrices. The action defined in Equation (1) restricts to a $(T^n \times T^n)$-action on M_n. This yields a $(T^n \times T^n)$-action on $\overline{X_\pi}$ with $\text{Stab}(T^{2n}) = \{(a \cdot I, a \cdot I) : a \in \mathbb{C}^*\}$, as well as an action on Y_π and V_π. In Theorem 3.3 we characterize the π for which Y_π is a toric variety with respect to $T^{2n}/\text{Stab}(T^{2n})$ in terms of the shape of $L'(\pi)$. In other words, we characterize the π such that Y_π has a dense T^{2n}-orbit. We denote the quotient $T^{2n}/\text{Stab}(T^{2n})$ by T^{2n-1}. Note that $\overline{X_\pi}$ and Y_π are normal varieties by Fulton's realizations in Fulton (1992) as subvarieties of Schubert varieties, which are normal by De Concini and Lakshmibai (1981); Ramanan and Ramanathan (1985).

Since Y_π is an irreducible variety and toric varieties are also irreducible, in order to show that Y_π is a toric variety with respect to T^{2n-1}, it suffices to show that it has the same dimension as some T^{2n-1}-orbit. When p is a general point of Y_π, then $\overline{T^{2n} \cdot p} \subset Y_\pi$ is the affine toric variety associated to the T^{2n}-moment[(i)] cone of Y_π, which we denote by $\boldsymbol{\Phi(Y_\pi)}$, and $\dim(\overline{T^{2n} \cdot p}) = \dim(\Phi(Y_\pi))$. In Theorem 3.3 we classify when Y_π is a toric variety by classifying the π for which $\dim(\Phi(Y_\pi)) = \dim(Y_\pi)$.

To compute the dimension of the cone $\Phi(Y_\pi)$, we start by describing the cone $\Phi(\overline{X_\pi})$ corresponding to a T^{2n}-orbit of a general point q in $\overline{X_\pi}$; without loss of generality $q = (1, \ldots, 1)$. The orbit $\overline{T^{2n} \cdot q}$ is the Zariski closure of the image of a map $\varphi : T^{2n} \to \mathbb{C}^{n^2}$ where $\varphi(t) = (t^{a_{(1,1)}} q_{(1,1)}, \ldots, t^{a_{(n,n)}} q_{(n,n)})$ and $\Phi(\overline{X_\pi})$ is the cone spanned by the exponents $a_{(i,j)}$ of the monomials. Notice that the exponents are $x_i - y_j$, where the x_i are the standard basis for $\mathbb{R}^n \times 0$, and the y_j are the standard basis for $0 \times \mathbb{R}^n$, because if A and B are the diagonal matrices with diagonal entries $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, respectively, then for any matrix M the (i,j)-th entry of AMB^{-1} is $a_i b_j^{-1} M_{(i,j)}$. It follows that the moment cone $\Phi(\overline{X_\pi})$ is the cone spanned by the vectors in the set $\{x_i - y_j \mid (i,j) \in [n] \times [n]\}$. Now $Y_\pi \hookrightarrow \overline{X_\pi}$ by restricting $\overline{X_\pi}$ to the entries inside $L(\pi)$, and so $\Phi(Y_\pi)$ is the cone spanned by the set $\{x_i - y_j \mid (i,j) \in L(\pi)\}$.

The variety $\overline{X_\pi}$ is a cone, meaning that for any $z \in \overline{X_\pi}$ and $c \in \mathbb{C}$, we have that $cz \in \overline{X_\pi}$. We can therefore projectivize it, that is, we can take the projective variety

$$\mathbb{P}(\overline{X_\pi}) := \{[z_{(1,1)}, \ldots, z_{(n,n)}] : (z_{(1,1)}, \ldots, z_{(n,n)}) \in \overline{X_\pi}\} \subset \mathbb{CP}^{n^2-1},$$

and the same is true for Y_π. In this paper we study the **moment**[(ii)] **polytope** $\boldsymbol{\Phi(\mathbb{P}(Y_\pi))}$ of the projectivization of Y_π. This polytope is the convex hull of $(x_i - y_j)$ for (i,j) inside $L(\pi)$. The next section studies the properties of the moment polytopes $\Phi(\mathbb{P}(Y_\pi))$.

3 Understanding the polytope $\Phi(\mathbb{P}(Y_\pi))$

In this section we describe the polytope $\Phi(\mathbb{P}(Y_\pi)) = \text{ConvHull}(x_i - y_j \mid (i,j) \in L(\pi))$ for $\pi \in S_n$, the moment polytope of the projectivization of Y_π. This polytope is a root polytope, since its vertices are positive roots of type A_{n-1}. We will encounter slightly different root polytopes (acyclic root polytopes) in Section 4.2 when describing the realizations of a family of pipe dream complexes. In Section 5 we will give a map that transforms the root polytope $\Phi(\mathbb{P}(Y_\pi))$ into an acyclic root polytope for $\pi = 1\pi'$ with π' dominant. We set our notation for the first root polytopes now.

(i) The reason we use the word moment for these convex objects is because they arise in the context of symplectic and pre-symplectic geometry. For readers interested in the connection, we refer them to Cannas da Silva (2001, 2003); Berline and Vergne (2011).

(ii) See footnote (i).

3.1 Root polytopes and their triangulations

A **root polytope** (of type A_{n-1}) is the convex hull of some of the points $e_i - e_j$ for $1 \leq i < j \leq n$. Given a graph G on the vertex set $[n]$ we associate to it the root polytope

$$Q_G = \text{ConvHull}(e_i - e_j \mid (i,j) \in E(G), i < j). \tag{2}$$

Note that for every $\pi \in S_n$ we have that $L(\pi)$ *is a skew Ferrers diagram.* Given a skew Ferrers diagram D with r rows and c columns, label its rows by $1, 2, \ldots, r$ from top to bottom and its columns by $1, 2, \ldots, c$ from left to right. Define

$$G_D = (\{x_1, \ldots, x_r, y_1, \ldots, y_c\}, \{(x_i, y_j) \mid (i,j) \in D\}). \tag{3}$$

Then

$$\Phi(\mathbb{P}(Y_\pi)) = Q_{G_{L(\pi)}}. \tag{4}$$

Note that an edge $(x_i, y_j) \in G_D$ yields the vertex $e_i - e_{r+j}$ of the root polytope Q_{G_D}.

Given a drawing of a graph G so that its vertices $v_1, \ldots, v_n$ are arranged in this order on a horizontal line, and its edges are drawn above this line, we say that G is **noncrossing** if it has no edges (v_i, v_k) and (v_j, v_l) with $i < j < k < l$. A vertex v_i of G is said to be **nonalternating** if it has both an incoming and an outgoing edge; it is called **alternating** otherwise. The graph G is alternating if all its vertices are alternating.

Since being noncrossing depends on the drawing of the graph it is essential that we set a way to draw G_D. For the purposes of this paper the vertices of G_D are drawn from left to right in the following order: $x_r, \ldots, x_1, y_c, \ldots, y_1$.

Lemma 3.1 *Given a skew diagram D for which G_D has k components, the root polytope $Q_{G_D} = \bigcup_F Q_F$, where the union runs over all noncrossing alternating spanning forests of G_D with $|V(G_D)| - k$ edges and the simplices Q_F are interior disjoint and of the same dimension as Q_{G_D}.*

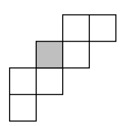

Fig. 3: For D the unshaded region, G_D is disconnected. In this case, D' equals D together with the shaded square.

We call the triangulation of Q_{G_D} given in Lemma 3.1 the **noncrossing alternating triangulation**, or **NAT** for short. These triangulations are closely related to the triangulations appearing in Gelfand et al. (1997) and Cellini and Marietti (2014). Recall that a triangulation of a polytope P is **regular** if there exists a concave piecewise linear function $f : P \to \mathbb{R}$ such that the regions of linearity of f are the maximal simplices in the triangulation.

Proposition 3.1 *For a skew diagram D, the NAT triangulation of Q_{G_D} described in Lemma 3.1 is a regular triangulation.*

3.2 Characterizing when Y_π is a toric variety

Now we are ready to use the above lemmas in order to characterize when Y_π is a toric variety.

Lemma 3.2 *Given a skew diagram D with r rows and c columns, for which G_D has k components, the dimension of Q_{G_D} is $r + c - k - 1$.*

Theorem 3.3 *Y_π is a toric variety with respect to the T^{2n-1}-action if and only if $L'(\pi)$ consists of disjoint hooks that do not share a row or a column with each other.*

Proof: We have that $\dim(Y_\pi) = |L'(\pi)|$. Lemma 3.2 yields that the dimension of $\Phi(\mathbb{P}(Y_\pi))$ equals $|L'(\pi)| - 1$ if and only if $L'(\pi)$ consists of disjoint hooks that do not share a row or a column with each other. This suffices to prove the theorem. □

A **dominant permutation** is one for which its diagram has empty dominant piece and is in the shape of a partition. An immediate corollary of Theorem 3.3 is the following.

Corollary 3.4 *If π' is a dominant permutation on $2, 3, \ldots, n$ then $Y_{1\pi'}$ is a toric variety.*

4 On geometric realizations of subword complexes

In this section we give some geometric realizations of subword complexes homeomorphic to balls. In Section 4.1 we show that the NAT triangulations studied in the previous section geometrically realize certain subword complexes. In Section 4.2 we use acyclic root polytopes to give geometric realizations for pipe dream complexes of permutations $\pi = 1\pi'$ with π' dominant.

The symmetric group S_n is generated by the adjacent transpositions $s_1, \ldots, s_{n-1}$, where s_i transposes $i \leftrightarrow i+1$. Let $Q = (q_1, \ldots, q_m)$ be a word in $\{s_1, \ldots, s_{n-1}\}$. A **subword** $J = (r_1, \ldots, r_m)$ of Q is a word obtained from Q by replacing some of its letters by $-$. There are a total of $2^{|Q|}$ subwords of Q. Given a subword J, we denote by $Q \setminus J$ the subword with k-th entry equal to $-$ if $r_k \neq -$ and equal to q_k otherwise for, $k = 1, \ldots, m$. For example, $J = (s_1, -, s_3, -, s_2)$ is a subword of $Q = (s_1, s_2, s_3, s_1, s_2)$ and $Q \setminus J = (-, s_2, -, s_1, -)$. Given a subword J we denote by $\prod J$ the product of the letters in J, from left to right, with $-$ behaving as the identity.

Definition 4.1 *Knutson and Miller (2004, 2005) Let $Q = (q_1, \ldots, q_m)$ be a word in $\{s_1, \ldots, s_{n-1}\}$ and $\pi \in S_n$. The **subword complex** $\Delta(Q, \pi)$ is the simplicial complex on the vertex set Q whose facets are the subwords F of Q such that the product $\prod(Q \setminus F)$ is a reduced expression for π. The **pipe dream complex** $PD(\pi)$ is the subword complex $\Delta(Q, \pi)$ corresponding to the triangular word $Q = (s_{n-1}, s_{n-2}, s_{n-1}, \ldots, s_1, s_2, \ldots, s_{n-1})$ and π.*

4.1 Realization by NAT triangulations

Given a permutation $\pi \in S_n$, let $\overline{L(\pi)}$ be the mirror image of the skew shape $L(\pi)$. Fill in the boxes of $\overline{L(\pi)}$ with transpositions starting with $s_1, s_2, \ldots$ on the first column, $s_2, s_3, \ldots$ on the second column, and so on. Let $Q(\overline{L(\pi)})$ be the word given by reading the transpositions in the boxes of $\overline{L(\pi)}$ from left to right and from bottom to top. Let $P(\pi) = \overline{L(\pi)} - B(\pi)$ where $B(\pi)$ is as follows. In each connected part of $\overline{L(\pi)}$ draw the lowestmost path from its top left box to its bottommost rightmost box. These boxes constitute $B(\pi)$. Let $p(\pi)$ be the permutation obtained from reading the transpositions in the boxes of $P(\pi)$ from left to right and from bottom to top. See Figure 4.

Fig. 4: On the left we have $L(\pi)$ and on the right $\overline{L(\pi)}$ for $\pi = [14325]$. Note that $p(\pi) = s_4 s_2 s_3 = [13524]$.

Theorem 4.2 *For $\pi \in S_n$, the noncrossing alternating triangulation of the root polytope $Q_{G_{L(\pi)}}$ is a geometric realization of the subword complex $\Delta(Q(\overline{L(\pi)}), p(\pi))$.*

4.2 Realizations for pipe dream complexes of dominant permutations by acyclic root polytopes

In this section we give a geometric realization for a different family of subword complexes using acyclic root polytopes. We show that the pipe dream complex $PD(\pi)$ of a permutation $\pi = 1\pi'$, with π' dominant, can be geometrically realized as the canonical triangulation of an acyclic root polytope $\mathcal{P}(T(\pi))$. These polytopes are closely related to the root polytopes of Section 3.1.

We begin by defining acyclic root polytopes. Let G be an acyclic graph on the vertex set $[n+1]$. Define

$$\mathcal{V}_G = \{e_i - e_j \mid (i,j) \in E(G), i < j\}, \text{ a set of vectors associated to } G;$$

$$\text{cone}(G) = \langle \mathcal{V}_G \rangle := \{ \sum_{e_i - e_j \in \mathcal{V}_G} c_{ij}(e_i - e_j) \mid c_{ij} \geq 0\}, \text{ the cone associated to } G; \text{ and}$$

$$\overline{\mathcal{V}}_G = \Phi^+ \cap \text{cone}(G), \text{ all the positive roots of type } A_n \text{ contained in } \text{cone}(G),$$

where $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\}$ is the set of positive roots of type A_n. The **acyclic root polytope** $\mathcal{P}(G)$ associated to the acyclic graph G is

$$\mathcal{P}(G) = \text{ConvHull}(0, e_i - e_j \mid e_i - e_j \in \overline{\mathcal{V}}_G). \tag{5}$$

Theorem 4.3 *Mészáros (2011) Let $T_1, \ldots, T_k$ be the noncrossing alternating spanning trees of the directed transitive closure of the acyclic graph G. Then $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ are top dimensional simplices in a regular triangulation of $\mathcal{P}(G)$ called the **canonical triangulation**.*

The main tool developed in Mészáros (2011) which is used to construct the canonical triangulation of Theorem 4.3 is the subdivision algebra. Subdivision algebras have since been utilized in solving various problems in Mészáros (2014); Mészáros (2015); Mészáros (2015); Mészáros (2016); Mészáros (2015); Mészáros and Morales (2015).

When $PD(\pi)$ is not a ball, it is usually a cone over a list of its vertices, namely those that are in all its facets. Let **cone**(π) denote the set of vertices of $PD(\pi)$ that are in all its facets. We define the **core** of π to be the restriction of $PD(\pi)$ to the set of vertices not in cone(π). Then $PD(\pi)$ is obtained from its core by iteratively coning core(π) over the vertices in cone(π). Translating to pipe dream complexes, the core is the restriction to the entries in the $n \times n$ matrix that are a cross in some reduced pipe dream for π. We refer to the region itself as the **core region**, and denote it by cr(π). Let $\pi = 1\pi'$, where π' is dominant. Denote by $\boldsymbol{\mathcal{S}(\pi)}$ the subword complex which is the core(π) coned over the vertex of $PD(\pi)$ corresponding to the entry $(1,1)$. Denote the region which is the union of $(1,1)$ and cr(π) by $\boldsymbol{R(\pi)}$.

Proposition 4.1 *Let $\pi = 1\pi'$ with π' dominant. Then $R(\pi) = NW(\pi) - Ess(\pi)$.*

The SE boundary of the core region starting from the southwest (SW) corner of it to the northeast (NE) corner can be described as a series of east (E) and north (N) steps. We construct the graph $T(\pi)$ by looking at the E and N steps bounding the SE boundary of $NW(\pi) - Ess(\pi)$. Let A be the set consisting of all the N steps together with the E steps that do not bound a box in $Ess(\pi)$. Suppose $|A| = m$, as we travel the SE boundary from the SW corner, we label, in order, the E steps and N steps in A with the elements of the sequence $(\alpha_1, \dots, \alpha_m)$. For the E steps that we did not assign an α_i, we consider their label to be the α_i assigned to the N step directly preceding them. See Figure 6 for an example of the labelling.

Given a diagram of a permutation there are two reduced pipe dreams for π with special names: the **bottom reduced pipe dream of** π obtained by aligning the diagram to the left and replacing the boxes with crosses, and the **top reduced pipe dream of** π obtained in a similar fashion, but by aligning the diagram up. See Figure 5 for an example of the bottom reduced pipe dream.

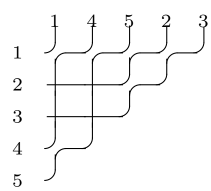

Fig. 5: The bottom reduced pipe dream for [14523] obtained by aligning the diagram to the left.

Consider the bottom reduced pipe dream drawn inside $R(\pi)$ and with elbows replaced by dots. Drop these dots south. Define $T(\pi)$ to be the tree with vertices $V = \{\alpha_1, \dots, \alpha_m\}$ such that there is an edge between vertices α_i and α_j with $i < j$ if there is a dot in the entry in the column of the E step labeled $\alpha_i i$ and in the row of the N step labeled $\alpha_i j$. See Figure 6 for an example.

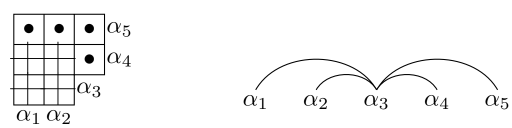

Fig. 6: On the left we have $NW(\pi) - Ess(\pi)$ for $\pi = [14523]$ with its SW boundary labelled by $(\alpha_1, \dots, \alpha_m)$ and the bottom reduced pipe dream drawn inside $NW(\pi) - Ess(\pi)$ with dots instead of elbows. We then drop the dots to the south to get the edges of $T(\pi)$, which is depicted on the right.

Theorem 4.4 *Let $\pi = 1\pi' \in S_n$, where π' is dominant. Let $\mathcal{C}^2(\pi)$ be the core of $PD(\pi)$ coned over twice. The canonical triangulation of the root polytope $\mathcal{P}(T(\pi))$ is a geometric realization of $\mathcal{C}^2(\pi)$.*

We now mention some of the corollaries to this Theorem.

Corollary 4.5 *The volume of the root polytope $\mathcal{P}(T(1\pi'))$, for π' dominant is equal to the number of reduced pipe dreams of $1\pi'$.*

Recall that for a polytope $\mathcal{P} \subset \mathbb{R}^N$, the t^{th} dilate of $\mathcal{P}$ is $t\mathcal{P} = \{(tx_1, \ldots, tx_N) \mid (x_1, \ldots, x_N) \in \mathcal{P}\}$. The number of lattice points of $t\mathcal{P}$, where t is a nonnegative integer and $\mathcal{P}$ is a convex polytope, is given by the **Ehrhart function** $i(\mathcal{P}, t)$. If $\mathcal{P}$ has integral vertices then $i(\mathcal{P}, t)$ is a polynomial.

In the spirit of Knutson and Miller (2004); Fomin and Kirillov (1994), the **double β-Grothendieck polynomial** $\mathfrak{G}_w^\beta(\mathbf{x}, \mathbf{y})$ for $w \in S_n$, where $\mathbf{x} = (x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_1, \ldots, y_{n-1})$ is

$$\mathfrak{G}_w^\beta(\mathbf{x}, \mathbf{y}) = \sum_{P \in \mathrm{Pipes}(w)} \beta^{codim_{PD(w)} F(P)} wt_{x,y}(P), \tag{6}$$

where $\mathrm{Pipes}(w)$ is the set of all pipe dreams of w (both reduced and nonreduced), $F(P)$ is the interior face in $PD(w)$ labeled by the pipe dream P, $codim_{PD(w)} F(P)$ denotes the codimension of $F(P)$ in $PD(w)$ and $wt_{x,y}(P) = \prod_{(i,j) \in \mathrm{cross}(P)} (x_i - y_j)$, with $\mathrm{cross}(P)$ being the set of positions where P has a cross. Note that in the product $\prod_{(i,j) \in \mathrm{cross}(P)} (x_i - y_j)$ we are assuming a certain labeling of rows and columns. Conventionally, rows are labeled increasingly from top to bottom and columns are labeled increasingly from left to right.

Corollary 4.6 *Let $\pi = 1\pi'$, where π' is a dominant permutation. Then*

$$\mathfrak{G}_\pi^{\beta-1}(\mathbf{1}, \mathbf{0}) = \sum_{m \geq 0} (i(\mathcal{P}(T(\pi)), m)\beta^m)(1-\beta)^{\dim(\mathcal{P}(T(\pi)))+1}. \tag{7}$$

5 Degeneration of moment polytopes into acyclic root polytopes

In this section we explain how to map the root polytope $\Phi(\mathbb{P}(Y_\pi))$ to the acyclic root polytope $\mathcal{P}(T(\pi))$. We then use this map to triangulate $\Phi(\mathbb{P}(Y_\pi))$ based on the triangulation of $\mathcal{P}(T(\pi))$.

Theorem 5.1 *Given $\pi = 1\pi'$, with π' dominant, the moment polytope $\Phi(\mathbb{P}(Y_\pi))$ can be degenerated into the root polytope $\mathcal{P}(T(\pi))$.*

Proof: Consider the linear map from $\Phi(\mathbb{P}(Y_\pi)) \to \mathcal{P}(T(\pi))$ that is the composition of the maps K and L, where L is the map

$$L(x_i) = -e_j, \text{ where } \alpha_j \text{ is the label of step } N \text{ on row } i, and$$

$$L(y_i) = \begin{cases} 0 & \text{if } (a,i) \in Ess(\pi) \text{ for some } a, \\ -e_j & \text{where } \alpha_j \text{ is the label of step } E \text{ on column } i, \end{cases}$$

and K is the map given by

$$K(y_j) = \begin{cases} x_i & \text{if } (i,j) \in Ess(\pi), \\ y_j & \text{if there is no } a \text{ such that } (a,j) \in Ess(\pi). \end{cases}$$

$$K(x_i) = x_i.$$

See Figure 8 for an example of these maps. Then this maps $\Phi(\mathbb{P}(Y_\pi))$ to $\mathcal{P}(T(\pi))$. □

The degeneration $L \circ K : \Phi(\mathbb{P}(Y_\pi)) \to \mathcal{P}(T(\pi))$ consists of contracting the face of $\Phi(\mathbb{P}(Y_\pi))$ corresponding to $Ess(\pi)$ to a point and moving this point to the origin while tweaking the vertices of $\Phi(\mathbb{P}(Y_\pi))$ that are of the form $\frac{1}{2}(x_i - y_j)$ where (i,j) is north of an entry of $Ess(\pi)$ and not in $\mathrm{dom}(\pi)$.

$\frac{1}{2}$

x_1-y_1	x_1-y_2	x_1-y_3
x_2-y_1	x_2-y_2	x_2-y_3
x_3-y_1	x_3-y_2	x_3-y_3

$\overset{K}{\longmapsto}$

x_1-y_1	x_1-y_2	x_1-x_3
x_2-y_1	x_2-y_2	x_2-x_3
x_3-y_1	x_3-y_2	0

Fig. 7: The map K for $Y_{[1243]}$.

x_1-y_1	x_1-y_2	x_1-x_3
x_2-y_1	x_2-y_2	x_2-x_3
x_3-y_1	x_3-y_2	0

$\overset{L}{\longmapsto}$

e_1-e_5	e_2-e_5	e_3-e_5	α_5
e_1-e_4	e_2-e_4	e_3-e_4	α_4
e_1-e_3	e_2-e_3	α_3	
α_1	α_2		

Fig. 8: The map L for $Y_{[1243]}$.

5.1 Triangulating $\Phi(\mathbb{P}(Y_\pi))$ and geometric realization of subword complexes

The preimage of the canonical triangulation of $\mathcal{P}(T(\pi))$ for $\pi = 1\pi'$, with π' dominant, under the linear map $L \circ K$ is a triangulation of $\Phi(\mathbb{P}(Y_\pi))$. This is yet another way to geometrically realize the pipe dream complex $PD(\pi)$ for these permutations.

Theorem 5.2 *Let $\Delta_1, \ldots, \Delta_k$ be the top dimensional simplices in the canonical triangulation of $\mathcal{P}(T(\pi))$ for $\pi = 1\pi'$, where π' is dominant. Then $P_i := (L \circ K)^{-1}(\Delta_i)$, $i \in [k]$, are the top dimensional simplices in a triangulation of $\Phi(\mathbb{P}(Y_\pi))$ which we call its **canonical triangulation**.*

Denote by $\boldsymbol{\mathcal{C}(\pi)}$ the core of the pipe dream complex $PD(\pi)$ and by $\mathcal{C}^i(\pi)$ the core $\mathcal{C}(\pi)$ coned over i times.

Theorem 5.3 *The canonical triangulation of $\Phi(\mathbb{P}(Y_\pi))$, for $\pi = 1\pi'$, with π' dominant, is a geometric realization of $\mathcal{C}^{|Ess(\pi)|+1}(\pi)$. Using the characterization of toric Y_π of Theorem 3.3, one can extend this geometric realization to realizations of pipe dream complexes for all π such that Y_π is toric.*

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