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# *Hook formulas for skew shapes (extended abstract)*

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**Abstract.** The celebrated *hook-length formula* gives a product formula for the number of standard Young tableaux of a straight shape. In 2014, Naruse announced a more general formula for the number of standard Young tableaux of skew shapes as a positive sum over *excited diagrams* of products of hook-lengths. We give two $q$-analogues of Naruse's formula for the skew Schur functions and for counting reverse plane partitions of skew shapes. We also apply our results to border strip shapes and their generalizations.

**Résumé.** La formule des équerres donne une formule de produit pour le nombre de tableaux standard de forme droite. En 2014, Naruse a annoncé une formule pour le nombre de tableaux standard de forme gauche comme une somme positive sur les *diagrammes excitées* de produits des équerres. Nous donnons deux $q$-analogues de la formule pour les tableaux semistandard et les partitions planes inversées. Nous appliquons aussi nos résultats aux diagrammes rubans et ses généralisations.

**Keywords.** hook-length formula, skew shapes, Hillman-Grassl correspondence, RSK algorithm, excited diagrams, pleasant diagrams, Catalan numbers, Dyck paths, Euler numbers, Schröder numbers

## 1 Introduction

### *1.1 Foreword*

The classical *hook-length formula* is a beautiful result that is both mysterious and extremely well studied. Discovered by Frame, Robinson and Thrall [FRT] in 1954, it has numerous proofs, such as probabilistic, bijective, inductive, etc. In a way it is a perfect result in enumerative combinatorics – clean, concise and generalizing several others (binomial coefficients, Catalan numbers, etc.) Still, the real nature of the formula remains unexplained, making potential generalizations an interesting challenge.

For skew shapes, there is no product formula for the number $f^{\lambda/\mu}$ of standard Young tableaux. However, there is a determinantal formula using the Jacobi-Trudi identity [S2, §7.16], allowing an efficient computation of these numbers. Alternatively, $f^{\lambda/\mu}$ can be computed by a standard formula in terms of the *Littlewood-Richardson coefficients* $c^{\lambda}_{\mu,\nu}$, and a curious Okounkov-Olshanski formula [OO]; both formulas are positive but inefficient. Naruse's formula [Naru] is a remarkable formula generalizing the hook-length formula as a sum of "hook products" over certain *excited diagrams*.

The goals of this extended abstract are threefold. First, we give Naruse-style hook formulas for the Schur functions $s_{\lambda/\mu}(1, q, q^2, \ldots)$ and for the *reverse plane partitions*. In contrast with the straight shape these two formulas are quite different. Their proofs employ a combination of algebraic and bijective arguments, using factorial Schur functions and Hillman-Grassl bijection [HiG], respectively. The full details of the proofs are in [MPP]. Second, as a biproduct of our proofs we give a combinatorial proof of the Naruse's formula. Finally, we illustrate how in the special case of *alternating ribbons* $\delta_{m+2}/\delta_m$ we obtain two new summation formulas for the $q$-Euler numbers.

## 1.2 Hook-length formulas for straight and skew shapes

Recall the *(Semi) Standard Young tableaux* and *reverse plane partitions* of straight and skew shape. The number $f^\lambda$ of standard Young tableaux of shape $\lambda$ is given by the celebrated *hook-length formula*:

**Theorem 1.1 (Frame-Robinson-Thrall [FRT])** *For $\lambda$ a partition of $n$ we have*

$$f^\lambda = \frac{n!}{\prod_{u\in[\lambda]} h(u)}, \tag{1.1}$$

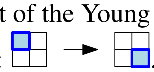
*where $[\lambda]$ is the Young diagram of $\lambda$ and $h(u) = \lambda_i - i + \lambda'_j - j + 1$ is the* hook-length *of the square $u = (i, j)$: the number of cells directly to the right and directly below $u$.*

There is no product formula for the number $f^{\lambda/\mu}$ of standard Young tableaux of skew shape $\lambda/\mu$. The following formula by Naruse involves the summation over *excited diagrams*. These diagrams were introduced independently in [IN] by Ikeda-Naruse and by Kreiman [Kre1] and Knutson-Miller-Yong [KMY, §5] in the context of Schubert calculus. These diagrams are a combinatorial model for reduced subwords of a given reduced word appearing in Billey's formula [Bil] for *Kostant polynomials*.

**Theorem 1.2 (Naruse [Naru])** *For partitions $\lambda, \mu$ we have*

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in[\lambda]\setminus D} \frac{1}{h(u)}, \tag{1.2}$$

*where $\mathcal{E}(\lambda/\mu)$ is the set of* excited diagrams *of $\lambda/\mu$.*

An excited diagram of $\lambda/\mu$ is a subset of the Young diagram of $\lambda$ obtained from the Young diagram of $\mu$ by doing a sequence of *excited moves*: .

**Example 1.3** *There are five excited diagrams for the shape $(4321/21)$, see Figure 1. By Theorem 1.2 we have* $f^{(4321/21)} = 7!\left(\frac{1}{1^4\cdot 3^3} + \frac{1}{1^3\cdot 3^3\cdot 5} + \frac{1}{1^3\cdot 3^3\cdot 5} + \frac{1}{1^2\cdot 3^3\cdot 5^2} + \frac{1}{1^2\cdot 3^2\cdot 5^2\cdot 7}\right) = 61.$

## 1.3 $q$-analogues of hook-length formulas

There is a $q$-analogue by Stanley [S1, Thm. 15.3 and Eq. 57] for the generating series of semistandard Young tableaux of shape $\lambda$. Let $b(\lambda) := \sum_i (i-1)\lambda_i$.

**Theorem 1.4 (Stanley [S1])**

$$\sum_{T\in\mathrm{SSYT}(\lambda)} q^{|T|} = \frac{q^{b(\lambda)}}{\prod_{u\in[\lambda]}(1-q^{h(u)})}, \tag{1.3}$$

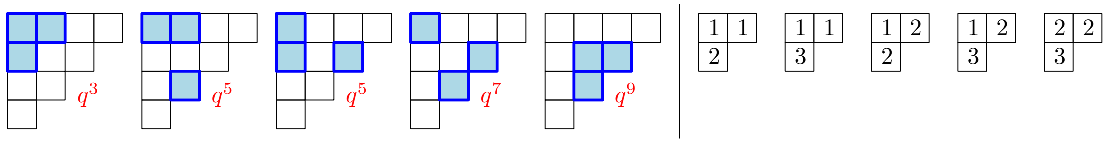



**Fig. 1:** Left: the $C_3 = 5$ excited diagrams for $(4321/21)$, their associated statistic $a(D)$, right: the five corresponding flagged tableaux in $\mathcal{F}(\mu,(2,3))$.

By Stanley's theory of *P-partitions* [S2, Prop. 7.19.11] or using a geometric argument [Pak, Lemma 1], one can show that the hook-length formula (1.1) follows from (1.3). We give a $q$-analogue of Naruse's as a specialization of a skew Schur functions.

**Theorem 1.5 (Morales-Pak-Panova [MPP])**

$$\sum_{T \in \mathrm{SSYT}(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in[\lambda]\setminus D} \left[\frac{q^{\lambda'_j - i}}{1-q^{h(i,j)}}\right]. \tag{1.4}$$

Unlike the case of straight shapes, the enumeration of semistandard tableaux and of reverse plane partitions (RPP) of skew shape are not equivalent. So we also give an expression for the generating series of reverse plane partitions of skew shape in terms of a richer class of diagrams called *pleasant diagrams*. These diagrams are defined as the supports of arrays that are mapped to RPP of shape $\lambda/\mu$ by the inverse Hillman-Grassl correspondence. An alternative description (see [MPP, §6]) is subsets of complements of excited diagrams in $\mathcal{E}(\lambda/\mu)$.

**Theorem 1.6 (Morales-Pak-Panova [MPP])**

$$\sum_{\pi \in \mathrm{RPP}(\lambda/\mu)} q^{|\pi|} = \sum_{S \in \mathcal{P}(\lambda/\mu)} \prod_{u\in S} \left[\frac{q^{h(u)}}{1-q^{h(u)}}\right]. \tag{1.5}$$

*where $\mathcal{P}(\lambda/\mu)$ is the set of pleasant diagrams.*

This result is proved using the Hillman-Grassl correspondence. As a corollary of this result we derive combinatorially Naruse's formula (1.2) (see [MPP, §6]).

## 1.4 Enumerative corollaries

We give enumerative formulas derived from Naruse's hook-length formula and our $q$-analogues involving Catalan, Euler and Schröder numbers. We highlight three of these formulas.

Let $\mathsf{Alt}(n)$ be the the set of *alternating permutations* of size $n$. The size of this set is given by $E_n$, the $n$-th Euler number [OEIS, A000111]. Since $E_{2n+1} = f^{\delta_{n+2}/\delta_n}$ where $\delta_n = (n-1, n-2, \ldots, 1)$, Naruse's formula relates Euler numbers with excited diagrams of this shape. It turns out that these excited diagrams are in correspondence with Dyck paths (see Proposition 3.1 and Figure 1) and so

$$|\mathcal{E}(\delta_{n+2}/\delta_n)| = C_n$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n$-th Catalan number that counts the number of Dyck paths $\mathsf{Dyck}(n)$: paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that stay on or above the $x$-axis. Naruse's formula then gives the following identity of odd Euler numbers in terms of Dyck paths

**Corollary 1.7**

$$.\frac{E_{2n+1}}{(2n+1)!} = \sum_{\gamma\in\mathsf{Dyck}(n)} \Bigg[\prod_{(a,b)\in\gamma} \frac{1}{2b+1}\Bigg], \qquad (1.6)$$

*where $(a,b)\in\gamma$ denotes a point $(a,b)$ of the Dyck path $\gamma$.*

Consider the following two $q$-analogues of $E_n$[(i)]: $E_n(q) := \sum_{\sigma\in\mathsf{Alt}(n)} q^{\mathrm{maj}(\sigma^{-1})}$, and $E'_n(q) := \sum_{\sigma\in\mathsf{Alt}(n)} q^{\mathrm{maj}(\sigma^{-1}\kappa)}$, where $\mathrm{maj}(\sigma)$ is the *major index* of permutation $\sigma$ in $\mathfrak{S}_n$ and $\kappa$ is the permutation $\kappa = (13254\ldots)$. Then as a corollary of Theorem 1.5 for skew SSYT we have

**Corollary 1.8**

$$\frac{E_{2n+1}(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} = \sum_{\gamma\in\mathsf{Dyck}(n)} \Bigg[\prod_{(a,b)\in\gamma} \frac{q^b}{1-q^{2b+1}}\Bigg],$$

*where $(a,b)\in\gamma$ denotes a point $(a,b)$ of the path $\gamma$.*

Given a Dyck path $\gamma$, a *high-peak* $(c,d)$ of $\gamma$ is a peak of the path of height greater than one. As a corollary of Theorem 1.6 for skew RPP and Lemma 4.2 we have

**Corollary 1.9**

$$\frac{E'_{2n+1}(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} = \sum_{\gamma\in\mathsf{Dyck}(n)} \Bigg[\prod_{(c,d)\in\mathcal{HP}(\gamma)} q^{2d+1} \prod_{(a,b)\in\gamma} \frac{1}{1-q^{2b+1}}\Bigg],$$

*where $\mathcal{HP}(\gamma)$ denotes the set of high peaks of $\gamma$.*

### *1.5 Notation*

Let $\lambda = (\lambda_1,\ldots,\lambda_r), \mu = (\mu_1,\ldots,\mu_s)$ denote integer partitions of length $\ell(\lambda) = r$ and $\ell(\mu) = s$. The *size* of the partition is denoted by $|\lambda|$ and $\lambda'$ denotes the *conjugate partition* of $\lambda$. We use $[\lambda]$ to denote the Young diagram of the partition $\lambda$. A *skew shape* is denoted by $\lambda/\mu$.

We denote the set of reverse plane partitions and semistandard Young tableaux of shape $\lambda/\mu$ by $\mathrm{RPP}(\lambda/\mu)$ and $\mathrm{SSYT}(\lambda/\mu)$. We denote the generating function for $\mathrm{SSYT}(\lambda/\mu)$ as a specialization of the Schur function $s_{\lambda/\mu}(1,q,q^2,\ldots)$. We denote the staircase partition $(n-1,n-2,\ldots,1)$ by $\delta_n$.

Given the skew shape $\lambda/\mu$, let $P_{\lambda/\mu}$ be the poset of cells $(i,j)$ of $[\lambda/\mu]$ partially ordered by component. This poset is *naturally labelled*, unless otherwise stated.

## 2 Excited diagrams

In this section we define excited diagrams and relate them with *flagged tableaux*. The relation is based on a map by Kreiman [Kre1, §6] and also Knutson-Miller-Yong [KMY, §5]. Let $\lambda/\mu$ be a skew partition and $D$ be a subset of $[\lambda]$. A cell $u = (i,j) \in D$ is called *active* if $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$ are all in $[\lambda]\backslash D$. Let $u$ be an active cell of $D$, define $\alpha_u(D)$ to be the set obtained by replacing $(i,j)$ in $D$ by $(i+1,j+1)$. We call this replacement an *excited move*.

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(i) In the survey [S3, §2] our $E_n(q)$ is denoted by $E^\star_n(q)$, our $E'_n(q)$ does not appear to have been studied before.

**Definition 2.1 (Excited diagrams)** *An* excited diagram *of $\lambda/\mu$ is a subset of $[\lambda]$ obtained from $[\mu]$ after a sequence of excited moves on active cells. Let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of $\lambda/\mu$.*

For $\mathbf{f} = (f_1, f_2, \ldots, f_{\ell(\mu)})$ a list of nonnegative integers, let $\mathcal{F}(\mu, \mathbf{f})$ be the set of SSYT $T$ of shape $\mu$ where the entries in the $i$-th row of $T$ are $\leq f_i$. Such tableaux are called *flagged SSYT* and they first appeared in work of Lascoux and Schützenberger and of Wachs [Wac].

Given a skew shape $\lambda/\mu$, each row $i$ of $\mu$ is between the rows of two corners of $\mu$. Denote such rows by $k_{i-1}$ and $k_i$ respectively. When a corner of $\mu$ is in row $k$, let $f'_k$ be the last row of diagonal $\mathsf{d}_{\mu_k - k}$ in $\lambda$. Lastly, let $\mathbf{f}^{(\lambda/\mu)}$ be the vector[(ii)] $(f_1, f_2, \ldots, f_{\ell(\mu)})$ where $f_i = f'_{k_i}$ where $k_i$ is the row of the corner of $\mu$ at or immediately after row $i$. Let $\mathcal{F}(\lambda/\mu) := \mathcal{F}(\mu, \mathbf{f}^{(\lambda/\mu)})$. See Figure 2: Left.

Next we show that excited diagrams in $\mathcal{E}(\lambda/\mu)$ are in bijection with flagged tableaux in $\mathcal{F}(\lambda/\mu)$.

Given $D \in \mathcal{E}(\lambda/\mu)$, we define $\varphi(D) := T$ as follows: Each cell $(x, y)$ of $[\mu]$ corresponds to a cell $(i_x, j_y)$ of $D$. We let $T$ be the tableau of shape $\mu$ with $T_{x,y} = i_x$. See Figure 1 right panel for the image of this map on the excited diagrams of shape $\delta_5/\delta_3$.

**Proposition 2.2** *We have that $|\mathcal{E}(\lambda/\mu)| = |\mathcal{F}(\lambda/\mu)|$ and the map $\varphi$ is a bijection between these two sets.*

# 3 Excited diagrams and SSYT of border strips and thick strips

In the next two sections we focus on the case of the *thick strip* $\delta_{n+2k}/\delta_n$ where $\delta_n$ denotes the staircase shape $(n-1, n-2, \ldots, 2, 1)$. We first study the excited diagrams $\mathcal{E}(\delta_{n+2k}/\delta_n)$ and the number of SYT of this shape via the NHLF (Theorem 1.2) and our first $q$-analogue for SSYT of this shape (Theorem 1.5).

## 3.1 Excited diagrams and Catalan numbers

Let $\mathsf{FanDyck}(k, n)$ be the set of tuples $(\gamma_1, \ldots, \gamma_k)$ of $k$ noncrossing Dyck paths from $(0, 0)$ to $(2n, 0)$ (see Figure 2: Center). We call such tuples *$k$-fans of Dyck paths*. We show that excited diagrams in $\mathcal{E}(\delta_{n+2k}/\delta_n)$ are in correspondence with non-crossing Dyck paths.

**Proposition 3.1** *The number of excited diagrams in $\mathcal{E}(\delta_{n+2k}/\delta_n)$ is equal to the number of fans of Dyck paths in $\mathsf{FanDyck}(k, n)$: $\big|\mathcal{E}(\delta_{n+2k}/\delta_n)\big| = \big|\mathsf{FanDyck}(k, n)\big|$.*

**Proof:** We start with the case $k = 1$. By Proposition 2.2, excited diagrams in $\mathcal{E}(\delta_{n+2}/\delta_n)$ are in bijection with flagged tableaux of shape $\delta_n$ with flag $(2, 3, \ldots, n)$. It is well known and easy to see that these are in bijection with Dyck paths (row $i$ of these tableaux have entries $i$ and $i + 1$, the boundary between these values outlines the Dyck path $\gamma$), as illustrated in Figure 1. For general $k$, by the same argument, the excited diagrams in $\mathcal{E}(\delta_{n+2k}/\delta_n)$ are in bijection with flagged tableaux of shape $\delta_n$ with flag $(k+1, k+2, \ldots, k+n-1)$. These tableaux correspond to $k$-tuples $(\gamma_1, \ldots, \gamma_k)$ of $k$ noncrossing Dyck paths . □

**Corollary 3.2** *We have:* $|\mathcal{E}(\delta_{n+2}/\delta_n)| = C_n, \quad |\mathcal{E}(\delta_{n+4}/\delta_n)| = C_n C_{n+2} - C_{n+1}^2,$

$$|\mathcal{E}(\delta_{n+2k}/\delta_n)| \;=\; \det[C_{n-2+i+j}]_{i,j=1}^{k} = \prod_{1 \leq i < j \leq n} \frac{2k + i + j - 1}{i + j - 1}\,. \tag{3.1}$$

[(ii)] In [KMY], the vector $\mathbf{f}^{\lambda/\mu}$ is called a *flagging*.

**Proof:** By Proposition 2.2, we have $|\mathcal{E}(\delta_{n+2k}/\delta_n)| = |\mathsf{FanDyck}(k,n)|$. On the other hand, the fans of paths in $\mathsf{FanDyck}(k,n)$ are counted by the given determinant of Catalan numbers, and also by the given product formula. □

**Remark 3.3** *Fans of Dyck paths in* $\mathsf{FanDyck}(k,n)$ *are equinumerous with* $k$-triangulations *of an* $(n+2k)$-*gon [Jon] (see also [S4, A12]).*

## 3.2 Determinantal identity of Schur functions of thick strips

Observe that SYT of shape $\delta_{n+2}/\delta_n$ are in bijection with *alternating permutations* of size $2n+1$. These permutations are counted by the odd Euler number $E_{2n+1}$. Thus, $f^{\delta_{n+2}/\delta_n} = E_{2n+1}$.

Let $E_n(q)$ be as in the introduction, the $q$-analogue of Euler numbers.(iii)

**Example 3.4** *We have:* $E_1(q) = E_2(q) = 1$, $E_3(q) = q^2+q$, $E_4(q) = q^4+q^3+2q^2+q$, *and* $E_5(q) = q^8+2q^7+3q^6+4q^5+3q^4+2q^3+q^2$.

By the theory of $(P,\omega)$-partitions, we have:

$$E_{2n+1}(q)/\left(\prod_{i=1}^{2n+1}(1-q^i)\right) = s_{\delta_{n+2}/\delta_n}(1,q,q^2,\ldots). \tag{3.2}$$

Denote the RHS above by $\widetilde{E}_{2n+1}(q)$. The following result is a special case of a general theorem in [LasP] which gives an expression for $s_{\lambda/\mu}(\mathbf{x})$ as a determinant of Schur functions of *rim ribbons*. We consider only the case $\lambda/\mu = \delta_{n+2k}/\delta_n$, the rim ribbons are the strips $\delta_{m+2}/\delta_m$.

**Theorem 3.5 (Lascoux–Pragacz [LasP])** *We have:* $s_{\delta_{n+2k}/\delta_n}(\mathbf{x}) = \det\left[s_{\delta_{n+i+j}/\delta_{n-2+i+j}}(\mathbf{x})\right]_{i,j=1}^{k}$.

Taking the specialization $1,q,q^2,\ldots$ in Theorem 3.5 we get the following identity.

**Corollary 3.6** *We have:* $s_{\delta_{n+2k}/\delta_n}(1,q,q^2,\ldots) = \det\left[\widetilde{E}_{2(n+i+j)-3}(q)\right]_{i,j=1}^{k}$.

Taking the limit $q\to 1$ in Corollary 3.6 we get corresponding identities for $f^{\delta_{n+2k}/\delta_n}$.

**Corollary 3.7** *We have:* $f^{\delta_{n+2k}/\delta_n}\,/\,|\delta_{n+2k}/\delta_n|! = \det\left[\widehat{E}_{2(n+i+j)-3}\right]_{i,j=1}^{k}$ *where* $\widehat{E}_n := E_n/n!$.

## 3.3 SYT and Euler numbers

We use the NHLF to obtain an expression for $f^{\delta_{n+2}/\delta_n} = E_{2n+1}$ in terms of Dyck paths.

**Proof of Corollary 1.7:** By the NHLF, we have

$$f^{\delta_{n+2}/\delta_n} = |\delta_{n+2}/\delta_n|! \sum_{D\in\mathcal{E}(\delta_{n+2}/\delta_n)} \prod_{u\in\overline{D}} \frac{1}{h(u)}, \tag{3.3}$$

(iii) In the survey [S3, §2], our $E_n(q)$ is denoted by $E_n^\star(q)$.

where $\overline{D} = [\delta_{n+2}/\delta_n] \setminus D$. Now $|\delta_{n+2}/\delta_n| = (2n+1)!$ and by the Proposition 3.1 (complements of) excited diagrams $D$ of $\delta_{n+2}/\delta_n$ correspond to Dyck paths $\gamma$ in $\mathsf{Dyck}(n)$. In this correspondence, if $u \in \overline{D}$ corresponds to point $(a,b)$ in $\gamma$ then $h(u) = 2b+1$ (see Figure 1). Translating from excited diagrams to Dyck paths, (3.3) becomes the desired Equation (1.6). □

Equation (1.6) can be generalized to thick strips $\delta_{n+2k}/\delta_n$.

**Corollary 3.8** *We have:*

$$\sum_{\substack{(\gamma_1,\ldots,\gamma_k)\in\mathsf{Dyck}(n)^k \\ \text{noncrossing}}} \prod_{r=1}^{k} \prod_{(a,b)\in\gamma_r} \frac{1}{2b+4r-3} = \left[\prod_{r=1}^{k-1}(4r-1)!!\right]^2 \det\left[\widehat{E}_{2(n+i+j)-3}\right]_{i,j=1}^{k}, \tag{3.4}$$

*where* $\widehat{E}_n = E_n/n!$ *and* $(a,b) \in \gamma$ *denotes a point of the Dyck path* $\gamma$.

**Proof:** For the LHS we use Corollary 3.7 to express $f^{\delta_{n+2k}/\delta_n}$ in terms of Euler numbers. For the RHS, we first use the NHLF to write $f^{\delta_{n+2k}/\delta_n}$ as a sum over excited diagrams $\mathcal{E}(\delta_{n+2k}/\delta_n)$ :

$$f^{\delta_{n+2k}/\delta_n} = |\delta_{n+2k}/\delta_n|! \sum_{D\in\mathcal{E}(\delta_{n+2k}/\delta_n)} \prod_{u\in\overline{D}} \frac{1}{h(u)},$$

where $\overline{D} = [\delta_{n+2k}/\delta_n] \setminus D$. By Proposition 3.1, excited diagrams of $\delta_{n+2k}/\delta_n$ correspond to $k$-tuples of noncrossing Dyck paths in $\mathsf{FanDyck}(k,n)$. Finally, one can check (see Figure 2: Center) that if $D \mapsto (\gamma_1,\ldots,\gamma_k)$ then

$$\prod_{u\in\overline{D}} h(u) = \left[\prod_{r=1}^{k-1}(4r-1)!!\right]^2 \prod_{(a,b)\in\gamma_r} (2b+4r-3),$$

which gives the desired RHS. □

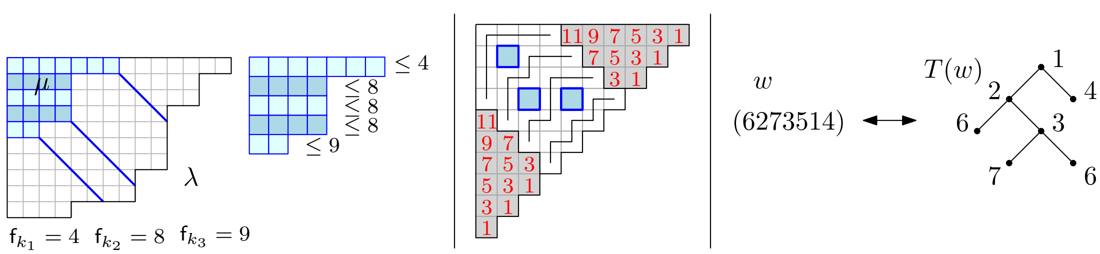


**Fig. 2:** Left: Given a skew shape $\lambda/\mu$, for each corner $k$ of $\mu$ we record the last row $f_k$ of $\lambda$ from diagonal $\mathsf{d}_{\mu_k - k}$. These row numbers give the bound for the flagged tableaux of shape $\mu$. Center: the hook-lengths of an excited diagram of $\delta_{3+8}/\delta_3$ corresponding to the 4-fan of Dyck paths on the right. Right: The full binary tree corresponding to the alternating permutation $w = (6273514)$.

## 3.4 Probabilistic variant of (1.6)

Here we present a new identity (3.6) which a close relative of the curious identity (1.6) we proved above.

Let $\mathcal{BT}(n)$ be the set of *plane full binary trees* $\tau$ with $2n+1$ vertices, i.e. plane binary trees where every vertex is a leaf or has two descendants. These trees are counted by $|\mathcal{BT}(n)| = C_n$ (see e.g. [S4, §2]). Given a vertex $v$ in a tree $\tau \in \mathcal{BT}(n)$, $h(v)$ denotes the number of descendants of $v$ (including itself). An *increasing* labelling of $\tau$ is a labelling $\omega(\cdot)$ of the vertices of $\tau$ with $\{1, 2, \ldots, 2n+1\}$ such that if $u$ is a descendant of $v$ then $\omega(v) \leq \omega(u)$. By abuse of notation, let $f^\tau$ is the number of increasing labelings of $\tau$. By the HLF for trees (see e.g. [Sag3]), we have:

$$f^\tau = \frac{(2n+1)!}{\prod_{v\in\tau} h(v)}. \tag{3.5}$$

**Proposition 3.9** *We have:*

$$\sum_{\tau\in\mathcal{BT}(n)} \prod_{v\in\tau} \frac{1}{h(v)} = \frac{E_{2n+1}}{(2n+1)!}. \tag{3.6}$$

**Proof:** The RHS of (3.6) gives the probability $E_{2n+1}/(2n+1)!$ that a permutation $w \in S_{2n+1}$ is alternating. We use the representation of a permutation $w$ as an *increasing binary tree* $T(w)$ with $2n+1$ vertices (see e.g. [S2, §1.5]). It is well-known that $w$ is a *down-up* permutation (equinumerous with up-down/alternating permutations) if and only if $T(w)$ is an increasing full binary tree [S2, Prop. 1.5.3]. See Figure 2: Right for an example. We conclude that the probability $p$ that an increasing binary tree is a full binary tree is given by $p = E_{2n+1}/(2n+1)!$.

On the other hand, we have:

$$p = \sum_{\tau\in\mathcal{BT}(n)} \frac{f^\tau}{(2n+1)!},$$

where $f^\tau/(2n+1)!$ is the probability that a labelling of a full binary tree $\tau$ is increasing. By (3.5), the result follows. □

**Remark 3.10** *Note the similarities between* (3.6) *and* (1.6). *They have the same RHS, both are sums over the same number $C_n$ of Catalan objects of products of $n$ terms, and both are variations on the (usual) HLF* (1.1) *for other posets. As the next example shows, these equations are quite different.*

**Example 3.11** *For $n = 2$ there are $C_2 = 2$ full binary trees with 5 vertices and $E_5 = 16$. By Equation* (3.6) $1/(3\cdot 5) + 1/(3\cdot 5) = 16/5!$. *On the other hand, for the two Dyck paths in* $\mathsf{Dyck}(2)$, *Equation* (1.6) *gives* $1/(3\cdot 3) + 1/(3\cdot 3\cdot 5) = 16/5!$.

## 3.5 $q$-analogue of Euler numbers via SSYT

We use our first $q$-analogue of NHLF (Theorem 1.5) to obtain identities for $s_{\delta_{n+2k}/\delta_n}(1, q, q^2, \ldots)$ in terms of Dyck paths.

**Proof of Corollary 1.8:** By Theorem 1.5 for the skew shape $\delta_{n+2}/\delta_n$ and (3.2) we have

$$\frac{E_{2n+1}(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} = \sum_{D\in\mathcal{E}(\delta_{n+2}/\delta_n)} \prod_{(i,j)\in[\delta_{n+2}]\setminus D} \frac{q^{\lambda'_j - i}}{1-q^{h(i,j)}}. \tag{3.7}$$

Let $D$ in $\mathcal{E}(\delta_{n+2}/\delta_n)$ corresponds to the Dyck path $\gamma$ and cell $(i,j)$ in $D$ corresponds to point $(a,b)$ in $\gamma$ then $h(i,j) = 2b+1$ and $\lambda'_j - i = b$. Using this correspondence, the LHS of (3.7) becomes the LHS of the desired expression. □

**Corollary 3.12**

$$\sum_{\substack{(\gamma_1,\ldots,\gamma_k)\in \mathsf{Dyck}(n)^k \\ \textit{noncrossing}}} \prod_{r=1}^{k} \prod_{(a,b)\in\gamma_r} \frac{q^{b+2r-2}}{1-q^{2b+4r-3}} = \left(\prod_{r=1}^{k-1}[4r-1]!!\right)^2 \det\left[\widetilde{E}_{2(n+i+j)-3}(q)\right]_{i,j=1}^{k}$$

*where* $\widetilde{E}_n(q) := E_n(q)/\big((1-q)(1-q^2)\cdots(1-q^n)\big)$ *and* $[2m-1]!! := (1-q)(1-q^3)\cdots(1-q^{2m-1})$.

**Proof:** For the LHS, use Corollary 3.6 to express $s_{\delta_{n+2k}/\delta_n}(1,q,q^2,\ldots)$ in terms of $q$-Euler polynomials $\widetilde{E}_m(q)$. For the RHS, first use Theorem 1.5 for the skew shape $\delta_{n+2k}/\delta_n$ and then follow the same argument as that of Corollary 3.8. □

**Remark 3.13** *Let us emphasize that the only known proof of Corollary 3.12 that we have, uses both the algebraic proof of Theorem 1.5 in [MPP], properties of the Hillman-Grassl bijection, and the Lascoux–Pragacz theorem (Theorem 3.5). As such, this is the most technical result of the paper.*

# 4 Pleasant diagrams and RPP of border strips and thick strips

In this section we study pleasant diagrams in $\mathcal{P}(\delta_{n+2}/\delta_n)$ and our second $q$-analogue of NHLF (Theorem 1.6) for RPP of shape $\delta_{n+2}/\delta_n$.

## 4.1 Pleasant diagrams and Schröder numbers

Let $s_n$ be the $n$-th *Schröder number* [OEIS, A001003] which counts lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$, $(1,-1)$, and $(2,0)$ that never go below the $x$-axis and no steps $(2,0)$ on the $x$-axis.

**Theorem 4.1** *We have:* $|\mathcal{P}(\delta_{n+2}/\delta_n)| = 2^{n+2}s_n$, *for all* $n \geq 1$.

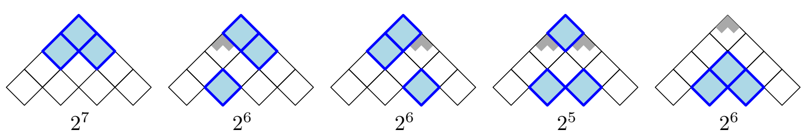


**Fig. 3:** Each Dyck path $\gamma$ of size $n$ with $k$ high peaks (denoted in gray) yields $2^{2n-k+2}$ pleasant diagrams. For $n=3$, we have $C_5 = 5$ and $s_3 = 11$. Thus, there are $|\mathcal{E}(\delta_{3+2}/\delta_3)| = C_3 = 5$ excited diagrams and $|\mathcal{P}(\delta_{3+2}/\delta_3)| = 2^5 s_3 = 352$ pleasant diagrams.

The proof of Theorem 4.1 is based on the following lemma which in turn is a special case of a more general characterization of pleasant diagrams in [MPP, §6.3]. Recall that *high peak* of a Dyck path $\gamma$ is a peak of height strictly greater than one. We denote by $\mathcal{HP}(\gamma)$ the set of high peaks of $\gamma$, and by $\mathcal{NP}(\gamma)$ the points of the path that are not high peaks. We use $2^{\mathcal{S}}$ denote the set of subsets of $\mathcal{S}$.

**Lemma 4.2** *The pleasant diagrams in $\mathcal{P}(\delta_{n+2}/\delta_n)$ are in bijection with $\bigcup_{\gamma\in\mathsf{Dyck}(n)}\Big(\mathcal{HP}(\gamma)\times 2^{\mathcal{NP}(\gamma)}\Big)$.*

**Proof:** By the characterization of pleasant diagrams in [MPP, §6.3],

$$\mathcal{P}(\delta_{n+2}/\delta_n) = \bigcup_{D\in\mathcal{E}(\delta_{n+2}/\delta_n)} \Lambda(D)\times 2^{[\delta_{n+2}]\setminus(D\cup\Lambda(D))}.$$

By the proof of Proposition 3.1 excited diagrams $D$ of shape $\delta_{n+2}/\delta_n$ are in correspondence with Dyck paths $\gamma$ in $\mathsf{Dyck}(n)$. Under this correspondence $D\mapsto\gamma$, excited peaks $\Lambda(D)$ are identified with high peaks $\mathcal{HP}(\gamma)$ and the rest $[\delta_{n+2}]\setminus(D\cup\Lambda(D))$ is identified with $\mathcal{NP}(\gamma)$, defined as set of points in $\gamma$ that are not high peaks. □

**Proof of Theorem 4.1:** It is known (see [Deu]), that the number of Dyck paths of size $n$ with $k-1$ high peaks equals the *Narayana number* $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. On the other hand, Schröder numbers $s_n$ can be written as

$$s_n \,=\, \sum_{k=1}^{n} N(n,k)\,2^{k-1}. \tag{4.1}$$

. By Lemma 4.2, we have:

$$|\mathcal{P}(\delta_{n+2}/\delta_n)| \,=\, \sum_{\gamma\in\mathsf{Dyck}(n)} 2^{|\mathcal{NP}(\gamma)|}\,. \tag{4.2}$$

Suppose Dyck path $\gamma$ has $k-1$ peaks, $1\le k\le n$. Then $|\mathcal{NP}(\gamma)| = 2n+1-(k-1)$. Therefore, equation (4.2) becomes

$$|\mathcal{P}(\delta_{n+2}/\delta_n)| \,=\, 2^{n+2}\sum_{k=1}^{n} N(n,k)2^{n-k} \,=\, 2^{n+2}\sum_{k=1}^{n} N(n,n-k+1)2^{n-k} \,=\, 2^{n+2}s_n\,,$$

where we use the symmetry $N(n,k) = N(n,n-k+1)$ and (4.1). □

In the same way as $|\mathcal{E}(\delta_{n+2k}/\delta_n)|$ is given by a determinant of Catalan numbers, the preliminary computations suggests that $|\mathcal{P}(\delta_{n+2k}/\delta_n)|$ is given by a determinant of Schröder numbers.

**Conjecture 4.3** *We have: $|\mathcal{P}(\delta_{n+4}/\delta_n)| = 2^{2n+5}(s_n s_{n+2} - s_{n+1}^2)$. More generally, for all $k\ge 1$, we have:*

$$|\mathcal{P}(\delta_{n+2k}/\delta_n)| \,=\, 2^{\binom{k}{2}}\det\big[\mathfrak{s}_{n-2+i+j}\big]_{i,j=1}^{k}\,, \quad \textit{where}\ \ \mathfrak{s}_n = 2^{n+2}s_n\,.$$

Here we use $\mathfrak{s}_n = |\mathcal{P}(\delta_{n+2}/\delta_n)|$ in place of $s_n$ in the determinant to make the formula more elegant. In fact, the power of 2 can be factored out.

## 4.2 $q$-analogue of Euler numbers via RPP

We use our second $q$-analogue of the NHLF (Theorem 1.6) and Lemma 4.2 to obtain identities for the generating function of RPP of shape $\delta_{n+2}/\delta_n$ in terms of Dyck paths. Recall the definition of $E_n^*(q)$ from the introduction:

$$E_n^*(q) \,=\, \sum_{\sigma\in\mathsf{Alt}(n)} q^{\mathrm{maj}(\sigma^{-1}\kappa)}\,,$$

where $\kappa = (13254\ldots)$. Note that $\mathrm{maj}(\sigma\kappa)$ is the sum of the descents of $\sigma\in S_n$ not involving both $2i+1$ and $2i$.

**Example 4.4** *To complement Example 3.4, we have:* $E^*_1(q) = E^*_2(q) = 1$, $E^*_3(q) = q+1$, $E^*_4(q) = q^4+q^3+q^2+q+1$, *and* $E^*_5(q) = q^7+2q^6+2q^5+3q^4+3q^3+2q^2+2q+1$.

**Proof of Corollary 1.9:** By the theory of $P$-partitions, the generating series of RPP of shape $\delta_{n+2}/\delta_n$ equals

$$\sum_{\pi\in \mathrm{RPP}(\delta_{n+2}/\delta_n)} q^{|\pi|} \;=\; \frac{\sum_{u\in\mathcal{L}(P_{\delta_{n+2}/\delta_n})} q^{\mathrm{maj}(u)}}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}\,,$$

where the sum in the numerator is over linear extensions $\mathcal{L}(P_{\delta_{n+2}/\delta_n})$ of the *zigzag* poset $P_{\delta_{n+2}/\delta_n}$ with a natural labelling. These linear extensions are in bijection with alternating permutations of size $2n+1$ and

$$E^*_{2n+1}(q) \;=\; \sum_{\sigma\in\mathsf{Alt}_{2n+1}} q^{\mathrm{maj}(\sigma^{-1}\kappa)} \;=\; \sum_{u\in\mathcal{L}(P_{\delta_{n+2}/\delta_n})} q^{\mathrm{maj}(u)}\,.$$

Thus

$$\sum_{\pi\in \mathrm{RPP}(\delta_{n+2}/\delta_n)} q^{|\pi|} \;=\; \frac{E^*_{2n+1}(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}\,. \tag{4.3}$$

By Lemma 4.2,

$$\sum_{D\in\mathcal{P}(\delta_{n+2}/\delta_n)} \prod_{u\in D} \frac{q^{h(u)}}{1-q^{h(u)}} = \sum_{\gamma\in\mathsf{Dyck}(n)} \left[\prod_{(c,d)\in\mathcal{HP}(\gamma)} q^{2d+1} \prod_{(a,b)\in\gamma} \frac{1}{1-q^{2b+1}}\right] \tag{4.4}$$

Combining (4.4) with Theorem 1.6 for the skew shape $\delta_{n+2}/\delta_n$ yields the desired expression. □

Finally, preliminary computations suggest the following analogue of Corollary 3.6.

**Conjecture 4.5** *We have:*

$$\sum_{\pi\in \mathrm{RPP}(\delta_{n+2k}/\delta_n)} q^{|\pi|} = q^{-N} \det\left[\widetilde{E}^*_{2(n+i+j)-3}(q)\right]_{i,j=1}^{k}\,,$$

*where* $N = k(k-1)(6n+8k-1)/6$ *and* $\widetilde{E}^*_k(q) = E^*_k(q)/(1-q)\cdots(1-q^k)$.

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