Simultaneous proof of the dimensional conjecture and of Jacobi's bound

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Introduction

Jacobi's bound, probably formulated by Jacobi around 1840 [2] is an upper bound on the order of a system of n differential equations f_i in n variables x_j , which is expressed as the tropical determinant of the order matrix $A := (a_{i,j} := \mathrm{ord}_{x_j} f_i)$, with the convention $\mathrm{ord}_{x_j} f_i := -\infty$ when f_i is free from x_j and its derivatives :

$$\max_{\sigma\in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}$$

The result is known to hold for *quasi-regular* systems [3], but remains conjectural in the general case, i.e. for any prime component of an arbitrary system. Cohn was the first to relate it to the dimensional conjecture [1], which claims that the differential codimension of a component defined by a system of r equations is at most r, showing that Jacobi's bound implies the dimensional conjecture. During my talk at DART III, I proposed a scheme of proof for the bound, using first a proof of the dimensional conjecture, trying to generalize Ritt's proof for codimension 1 [6], which is based on Puiseux series computations. Then, some kind of reduction process was to be used, mostly based of Jacobi's reduction methods. The dimension properties was crucial there to withdraw components defined by two few equations, or equations satisfying relations. Working to complete this proof, I had trouble with the computation of Puiseux series that required to introduce some change of variables... allowing to get both results at the same time, thanks to the same reduction process. The result we prove is in fact a little more general. We define here the order of a component of positive codimension s as the order of its intersection with s generic hyperplanes [4]. Working with a system of r equations is not easy. In fact, it is better to consider a prime algebraic ideal of codimension r.

THEOREM 1. — *Let f_i, $1 \leq i \leq r$ be a characteristic set of an algebraic ideal $\mathscr{I}$ of codimension r in $\mathscr{F}\{x_1, \ldots, x_n\}$, where $\mathscr{F}$ is a differential field of chatacteristic* 0. *Let us denote by $S_{r,n}$ the set of injections from* $[1, r]$ *to* $[1, n]$.

If $\mathscr{P}$ be a prime component of $\{\mathscr{I}\}$, then :
i) the differential codimension s of $\mathscr{P}$ is at most r ;
ii) if the differential codimension of $\mathscr{P}$ is equal to r, the order of $\mathscr{P}$ is at most the strong Jacobi number

$$\mathscr{O} := \max_{\sigma \in S_{r,n}} \sum_{i=1}^{s} a_{i,\sigma(i)}.$$

If the f_i are just arbitrary equations, we easilly reduce to the hypotheses of the theorem by considering the prime components of $\sqrt{[f]}$.

Main ideas of the proof

The most concise way of presenting the proof is to replace the recursive reduction process by a *reductio ab absurdum*. Let us assume that i) or ii) is false. There exist counter-examples such that $n-r$ is minimal, and among them counter-examples with minimal Jacobi number. We will work out a contradiction.

Let $B := (\lambda_i + a_{i,j})$ be a minimal canon [2, 4] for the order matrix, meaning that (λ_i) is the smallest vector of integers such that B has elements maximal in their columns and located in all different lines and columns. We define $\Lambda := \max_i \lambda_i$, $\alpha_i := \Lambda - \lambda_i$ and $\beta_j := \max_i a_{i,j} - \alpha_i$. We say that some ordering $\prec$ on derivatives is a *Jacobi ordering* if $k_1 - \beta_{j_1} < k_2 - \beta_{j_2}$ implies $x_{j_1}^{(k_1)} \prec x_{j_2}^{(k_2)}$ $k - \beta_j$ is the *Jacobi order* of $x_j^{(k)}$

We may assume that the f_i are ordered by increasing α_i; let ϖ be the smallest integer such that:

A) $f_1, \ldots, f_{n-\varpi}$ is a characteristic set of a prime differential ideal $\mathscr{Q}$, for a Jacobi ordering ;

B) $\mathscr{J}_1 := \mathscr{Q} \cap \mathscr{F}[x_j^{(k)} | 1 \leq j \leq n; 0 \leq k \leq \beta_{j+\alpha_{n-\varpi}}] \subset \mathscr{P}$.

Now, we may assume further that the system f has been chosen with minimal ϖ, among those with minimal $n-r$ and Jacobi number.

Lemma 2. — *Under the above hypotheses, if ϖ is equal to* 0*, then $\mathcal{Q} \not\subset \mathcal{P}$; if $\varpi > 0$, then $\varpi < n$ and $\mathcal{J}_2 := \mathcal{Q} \cap \mathcal{F}[x_j^{(k)} | 1 \le j \le n; 0 \le k \le \beta_{j+\alpha_{n-\varpi+1}}] \not\subset \mathcal{P}$.*

PROOF. — As f_1 is a prime polynomial, it must be the char. set of a prime differential ideal, so $\varpi < n$.

Assume that $\varpi > 0$ and that $\mathcal{J}_2 \subset \mathcal{P}$. Let G be a Gröbner basis of $\mathcal{J}_1$, for the lexicographic ordering on monomials, considering a Jacobi ordering on derivatives, for which the main derivatives of $f_1^{(\alpha_1 - \alpha_{n-\varpi})}, \ldots, f_{n-\varpi}$ are the smallest with the same Jacobi order $\alpha_{n-\varpi}$. Then, the reduction of $\mathcal{J}_2$ by G must be a non trivial ideal (if not, $n-r$ would not be minimal) of which nonzero polynomials are of Jacobi order at least that $f_{n-\varpi+1}$ (if not, the Jacobi number of f would not be minimal). There exists a factor g_2 of such a polynomial g, modulo $\mathcal{J}_1$, that holds $\mathcal{P}$. Replacing $f_{n-\varpi+1}$ by g_2, we find a new system defining $\mathcal{P}$, with the same $n-r$ and Jacobi number, but smaller ϖ.

Assume now that ϖ is equal to 0. If $\mathcal{Q} \subset \mathcal{P}$, the prime component $\mathcal{P}$ would be equal to $\mathcal{Q}$, of which f is a characteristic set for a Jacobi ordering. This would imply that i) the order of $\mathcal{P}$ is $\mathcal{O}$ and ii) its dimension equal to $n-r$. So, $\mathcal{Q} \not\subset \mathcal{P}$. □

We have shown that possible couter-examples are related to *singular* components of the system $\mathcal{I}$, i.e. components for which $H_f = 0$.

The singular case

The idea used to achieve the proof in the singular case is to reduce to the regular one by a suitable change of variables. It may be illustrated by the most simple example of equation $f(x) = x'^2 - 4x = 0$. We introduce a change of variables defined by $y = x'$. The new system $x - y^2 = 0$, $y(y'-1) = 0$ is equivalent to $f(x) = 0$, but the main and singular components of this system are now respectively associated to the factors $y' - 1 = 0$ and $y = 0$.

As $\mathcal{P}$ is a component that does not contain $\mathcal{Q}$, we may find a minimal n-uple of integers μ_i such that $[f^{(\nu)} | 1 \le i \le n-\varpi, 0 \le \nu \le \mu_i] : H^\infty$ (Where H is the product of initials and separants of $f_1, \ldots, f_{n-\varpi}$) contains a polynomial that does not belong to $\mathcal{P}$.

We may now assume that the system has been chosen among those with minimal $n-r$, then minimal Jacobi number, then minimal ϖ, in such a way that $\gamma := \max_{i=1}^{r} \mu_i$ is minimal. Obviously, it must be greater than 0. We may assume that $\mu_1 > 0$ for $1 \le i \le s$ and that the leading derivatives of f_i is $x_i^{(\alpha_i + \beta_i)}$. We increase the ground field with s arbitrary function ζ_i : the new ground field is $\mathcal{F}\langle\zeta_i\rangle$, which does not change the number, orders and dimensions of the components. We introduce new variables $y_i = x_i' + \zeta_i x_i$.

We start with the prime ideal $\mathscr{I} + [y_i - x_i' - \zeta_i x_i]$ for which f_i, $1 \le i \le n$ and $y_i - x_i' - \zeta_i x_i$, $1 \le i \le r$ is a characteristic set.

The first step is to rewrite the equation defining the new variables as $x_i' = y_i - \zeta_i x_i$, to differentiate them, so that each derivatives $x_i', \ldots, x_i^{(\Lambda+\beta_i)}$ is expressed as a linear combination of the $y_i, \ldots, y_i^{(\Lambda+\beta_i-1)}$ and the x_i, that may be substituted in the elements of $\mathscr{I}$ to get a new ideal $\mathscr{I}_1$, with a standard basis $\tilde{f}_i$, obtained by sustitution from the f_i.

The next step is to eliminate the x_i', $1 \le i \le s$ in $x_i' - y_i + \zeta_i x_i$. Thanks to the arbitrary function ζ_i, $\Delta := |\partial \tilde{f}_i / \partial x_j; 1 \le i, j \le s| \notin \mathscr{P}$. We get new equations g_i by pseudo reduction and $(g_i, \mathscr{I}_1) : \Delta^\infty$ generates the same perfect ideal as before.

This ideal need not be prime. If $\gamma > 1$, then we just need to keep some component that is included in $\mathscr{P}$. The values of $n - r$, the Jacobi number and ϖ are preserved, but it is easilly seen that new value of γ is $\gamma - 1$, by construction.

If $\gamma = 1$, then the components of $\sqrt{(g_i, \mathscr{I}_1) : \Delta^\infty}$ that are included in $\mathscr{P}$ have the same $n - r$, but strictly smaller Jacobi number, as they correspond to the former singular components, where H did vanish: a final contradiction that completes the proof. $\square$

References

[1] COHN (Richard M.), "Order and dimension", *Proc. Amer. Math. Soc.* **87** (1983), n$^{\text{o}}$ 1, 1–6.

[2] JACOBI (Carl Gustav Jacob), "Looking for the order of a system of arbitrary ordinary differential equations", *AAECC* **20**, (1), 7–32, 2009.

[3] KONDRATIEVA (Marina Vladimirovna), MIKHALEV (Aleksandr Vasil'evich), PANKRATIEV (Evgeniĭ Vasil'evich), "Jacobi's bound for independent systems of algebraic partial differential equations", *AAECC*, **20**, (1), 65–71, 2009.

[4] OLLIVIER (François) and SADIK (Brahim), "La borne de Jacobi pour une diffiété définie par un système quasi régulier", *Comptes rendus Mathématique*, **345**, 3, 2007, 139–144.

[5] OLLIVIER (François), "Jacobi's Bound and Normal Forms Computations", *Differential Algebra and Related Topics*, Li Guo et William Y. Sit éd., World Scientific, Singapour, to appear.

[6] RITT (Joseph Fels), *Differential Algebra*, Amer. Math. Soc. Colloq. Publ., vol. 33, A.M.S., New-York, 1950.