Looking for the order of a system of arbitrary ordinary differential equations

De investigando ordine systematis æquationum differentialium vulgarium cujuscunque

Carl Gustav Jacob Jacobi (1804–1851)
Prof. ord. math. Regiom. (1832–1844) & Berol. (1844–1851)

Summarium

Hanc commentationem in medium protulerunt S. Cohn et C.W. Borchardt e manuscriptis posthumis Caroli G. J. Jacobi. Variæ formæ canonicæ quas datum systema æquationum differentialium vulgarium inducere potest considerantur. Investigatio ordinis systematis, sine formæ canonicæ auxilio, ad solvendum problema inæqualitatum reducitur: affectationum problema. Novum genus formularum, determinantia manca, introductum est. Cuiusmodi quantitas non evanescens indicio est, ordinem equalem esse solutioni H problematis inæqualitatum, quæ per algorithmum, Haroldi Kuhn methodi hungariæ similem, invenitur.

Abstract

This paper was edited by S. Cohn and C.W. Borchardt from posthumous manuscripts of C.G.J. Jacobi. The various canonical forms that a given system ordinary differential equations may take are considered. Looking for the order of the system, without using a normal form, is reduced to a problem of inequalities: the *affectation problem*. A new type of formulas, the truncated determinants, is introduced. The non vanishing of this quantity means that the order will be equal to the value H, solution of this inequalities problem, which is obtained by an algorithm similar to Harold Kuhn's Hungarian method.

Translated from the Latin by
François Ollivier (CNRS)
LIX UMR 7161
CNRS–École polytechnique
F-91128 Palaiseau CEDEX (France)
Email francois.ollivier@lix.polytechnique.fr

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Translator's comments

THIS TEXT is based on two fragments of manuscripts [II/13 b)] that were transcripted [II/13 c)] by Sigismund Cohn [II/13 a)], starting from 1859. After Cohn's death in 1861, the editing process was achieved by Borchardt [I/63] and the result [Jacobi 1] first appeared [Crelles 64] in 1865 and was later reproduced in volume V of Jacobi's works [GW V]. Cohn's version follows very closely the original material, but Borchardt made some more changes. He suppressed the first section, modified the presentation, rewrote many sentences. Most of his changes only concern the latin style and are not perceptible in the translation. I have tried to follow the original manuscript whenever it was possible. Changes made by Borchardt where kept when they can help the reader. [They are indicated by sans sherif letters, enclosed between brackets, *passages originally in italics are written in slanted letters.*]

I have also included some passages suppressed by Jacobi himself when they could help to understand the genesis of his ideas. [They appear in small letters and between square brackets.] Minor changes in typography introduced by Borchardt for better readability, or homogenization of notations that may differ in the two fragments are not indicated. I have kept almost all the material suppressed by Borchardt, only a couple of repetitions or lengthy passages are cut; this is indicated by [...].

I express my gratitude to Alexandre Sedoglavic for his kind help with a preliminary French translation of Jacobi's paper. Special thanks are due to Daniel J. Katz who made a very carefull rereading of the Latin transcription and of the English version, correcting many mistakes and inaccuracies. His patient work greatly improved the quality of the translation.

References

Primary material, manuscripts

The following manuscripts from Jacobis Nachlaß, Archiv der Berlin-Brandenburgische Akademie der Wissenschaft, were used to establish this translation. We thank the BBAW for permission to use this material and its staff for their efficiency and dedication.

[I/63] *De investigando ordine systematis æquationibus differentialium vulgarium cujuscunque*, final transcription by BORCHARD with indications for the composer. 9–28.

[II/13 a)] Sigismund COHN, *letter to* Carl Wilhelm BORCHARDT. Hirschberg, August, 25th 1859, 3 p.

[II/13 b)] Carl Gustav Jacob JACOBI, Manuscript *De ordine systematis æquationum differentialium canonici variisque formis quas inducere potest*, 2186–2189, 2191–2196 (§ 19), 2200–2206 (§ 21–23). 35 p. Basis of Cohn's transcription.

[II/13 c)] Sigismund COHN, transcription of [II/13 b)] with corrections and notes by Carl Wilhelm BORCHARDT, 39 p.

Publications

[Crelle 64] *Journal für die reine und angewandte Mathematik*, **64**, 4, 297–320, Berlin, Georg Reimer, 1865.

[GW V] *C.G.J. Jacobi's gesammelte Werke, V*, K. Weierstrass ed., Berlin, Georg Reimer, 1890.

[Jacobi 1] JACOBI (Carl Gustav Jacob),"De investigando ordine systematis æquationum differentialium vulgarium cujuscunque", [Crelle 64, p. 297-320], [GW V, p. 193–216].

[Jacobi 2] JACOBI (Carl Gustav Jacob), "De æquationum differentialium systemate non normali ad formam normalem revocando", [GW V, p. 485–513], english translation available in the special issue "Jacobi's Legacy" of *AAECC*, **20**, (1), 33–64, 2009.
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Translation

[Looking for the order of an arbitrary system of differential equations.]

[§. 1.

About the order of a canonical system of differential equations and the various forms it may take.[1]

BEING PROPOSED THE DIFFERENTIAL EQUATIONS

$$\frac{d^p x}{dt^p} = A, \frac{d^q y}{dt^q} = B, \text{ etc.} \tag{1)}$$

in which the derivatives appearing on the right are smaller than those appearing on the left—they constitute an explicit canonical form—one may express this system under other canonical forms,

$$\frac{d^m x}{dt^m} = M, \frac{d^n y}{dt^n} = N, \text{ etc.} \tag{2)}$$

in which derivatives of x greater than the $(m-1)^{\text{th}}$, of y greater than the $(n-1)^{\text{th}}$ etc. do not appear in M, N, etc. And the equations 2) will be formed in such a way that one may from them get back to the proposed equations 1), hence the systems 1) and 2) are mutually equivalent. If the derivatives of x up to the $(p-1)^{\text{th}}$, of y up to the $(q-1)^{\text{th}}$ are taken as new variables, one may substitute for the equations 1) $p+q$ etc. equations of the first order between $p+q+\cdots+1$ variables, so the complete integral equations must depend on $p+q+\cdots$ arbitrary constants. In the same way, if the derivatives of x up to the $(m-1)^{\text{th}}$, the derivatives of y up to the $(n-1)^{\text{th}}$ are taken as new variables, one may represent the equations 2) as $m+n+\cdots$ first order differential equations between $m+n+\cdots+1$ variables, the integral equations of which depend of $m+n+\cdots$ arbitrary constants. It must be that

$$m+n+\cdots = p+q+\cdots,$$

for the two systems 1) and 2) are equivalent, and their complete integration must produce the same number of arbitrary constants. I call the sum $m+n+\cdots = p+q+\cdots$ *the order of the system of differential equations*, so that whenever a system of differential equations is presented under a canonical

[1] The first title is due to Cohn and was intended as the title of the whole paper. The second is the title of a section 21 in Jacobi's manuscript.

form, its order is equal to the sum of the orders up to which the derivatives of each variable go, and the number of arbitrary constants on which depend its complete integral equations is the same.

If in the indicated way, by using new variables, we present the differential equations 1) and 2) as systems of first order differential equations, the transformation of one system into the other is obtained by the transformation of variables. In this way, knowing a multiplier of one system, a multiplier of the other appears. This is expressed by the proposition, *the two integrals*

$$\int \left\{ \frac{\partial A}{\partial \frac{d^{p-1}x}{dt^{p-1}}} + \frac{\partial B}{\partial \frac{d^{q-1}y}{dt^{q-1}}} \cdots \right\} dt,$$

$$\int \left\{ \frac{\partial M}{\partial \frac{d^{m-1}x}{dt^{m-1}}} + \frac{\partial N}{\partial \frac{d^{n-1}y}{dt^{n-1}}} \cdots \right\} dt,$$

depend one on the other.

It generally happens that the orders up to which go the derivatives of each variable in the transformed canonical differential equations may be arbitrarily chosen, provided that their sum remains equal to the system order. But, in all cases, one can eliminate the variables and their derivatives, with the exception of two, of which one can be the independent variable t, except if perhaps some of the proposed differential equations proceed from the others by differentiations and eliminations, so that the number of differential equations and of dependent variables is not the same. In general, the differential equations obtained in this way will be of the same order, whatever be these two variables, which order will be also the order of the proposed system of differential equations. In this case, if these two variables are t and x, the other equations of the transformed canonical system must provide the values of the remaining variables y, z etc. expressed by t, x and the derivatives of x. Indeed, if these equations would contain derivatives of variables y etc., the order *of this system* would exceed the order of the differential equation that takes place between t and x alone.

So that this thing be better understood, I will take two equations in three variables, that is one independent t, two dependent x and y. Let s be the system order, and let the equations be reduced, in the way I have told about, in a form such that one be a differential equation of the s^{th} order between t and x alone

3) $$\frac{d^s x}{dt^s} = S,$$

and the other express the value of y by t, x and the derivatives of x,

4) $$y = Y.$$

If Y contains *no* derivatives of x, that is to say if Y is a function of x and t alone, there will be no other canonical form of the equations

5) $$\frac{d^s x}{dt^s} = S, \quad y = Y,$$

except that, in some way inverse, for which one has a differential equation of the s^{th} order between t and y, and x is expressed by t and y. If the greatest derivative of x that contains Y is the i^{th}, the proposed system 5) will not give any other canonical system than those in which a derivative of y appears that equals or exceeds the $(s-i)^{\text{th}}$. We deduce from the equation $y = Y$ the following,

6) $$\frac{d^i x}{dt^i} = I,$$

the function I containing, besides t and y, only x and its derivatives not exceeding the $(i-1)^{\text{th}}$. Differentiating $s-i$ times the preceding equation, and eliminating with it $\frac{d^i x}{dt^i}$ as soon as it appears by differentiation, we successively produce the values of

$$\frac{d^i x}{dt^i}, \frac{d^{i+1} x}{dt^{i+1}}, \dots \frac{d^s x}{dt^s},$$

expressed by derivatives of x lower than the i^{th}, derivatives of y up to the $(s-i)^{\text{th}}$ and t. Substituting these ones in 3), there appears an equation in which the derivatives of y go up to the $(s-i)^{\text{th}}$, those of x up to the κ^{th}, where $\kappa \leq i-1$. This one constitutes with 6) an *other* canonical system that one may present in the following way,

7) $$\frac{d^i x}{dt^i} = I, \quad \frac{d^\kappa x}{dt^\kappa} = K,$$

the function K containing the $(s-i)^{\text{th}}$ derivative of y and derivatives of x that do no exceed the $(\kappa-1)^{\text{th}}$. In a similar way, differentiating $(i-\kappa)$ times the second equation and eliminating from the first equation the derivatives of x exceeding the $(\kappa-1)^{\text{th}}$, a *third* canonical system appears, that one may represent in the following way,

8) $$\frac{d^\kappa x}{dt^\kappa} = K, \quad \frac{d^\lambda x}{dt^\lambda} = \Lambda,$$

where $\lambda \leq \kappa - 1$, with the function Λ containing the $(s-\kappa)^{\text{th}}$ derivative of y and the derivatives of x that do not exceed the $(\lambda-1)^{\text{th}}$. Continuing in this way, we come at the end to a canonical system of the form

9) $$\frac{d^\nu x}{dt^\nu} = N, \quad x = X,$$

N containing derivatives of x lower than the ν^{th}, the $(s-\mu)^{\text{th}}$ derivative of y, where $\mu > \nu$ and X denoting a function free of x and its derivatives, containing the $(s-\nu)^{\text{th}}$ and lower derivatives of y. At last, differentiating ν times the equation $x = X$, we secure an ultimate canonical system, that one may express by the two equations

10) $$x = X, \frac{d^s y}{dt^s} = \Upsilon,$$

of which one is a differential equation of the s^{th} order between t and y alone.

In the same way, one produces all the systems presented in canonical form to which the system of differential equations 5) can be reduced. Similarly, it is obvious that one may come back from some system to the preceding one. In fact, the equation from which the auxiliary equations come by successive differentiations, is common to two systems, the one immediately following the other; so we may deduce from the transformed system the same auxiliary equations, so that the equivalence with the other system is obvious. If in the preceding systems

$$i = s-1, \quad \kappa = s-2, \quad \lambda = s-3, \quad \ldots \quad \nu = 1;$$

as it generally happens, we shall have $s+1$ canonical systems,

11) $$\frac{d^p x}{dt^p} = A, \quad \frac{d^q y}{dt^q} = B,$$

in which p and q may denote arbitrary numbers the sum of which is $= s$, the functions A and B containing only derivatives lower than those placed on the left.

In a general way, if one has an arbitrary canonical form 11), one gets to some other in this way. Let $\frac{d^m x}{dt^m}$ be the highest derivative of x that the function B contains, where $m \leq p-1$. Differentiating the second equation $p-m$ times and eliminating with this last the derivatives of x exceeding the $(m-1)^{\text{th}}$, one will get equations

$$\frac{d^q y}{dt^q} = B, \frac{d^{p+q-m} y}{dt^{p+q-m}} = B_{\prime}.$$

From the first of these ones, one may deduce the value $A_{\prime}$ of $\frac{d^m x}{dt^m}$, this being done, if $n = p+q-m$, we get another canonical system presented in explicit form,

$$\frac{d^m x}{dt^m} = A_{\prime}, \quad \frac{d^n y}{dt^n} = B_{\prime}$$

the function A, and B, containing no derivatives greater than the $(m-1)^{\text{th}}$ of x, the $(n-1)^{\text{th}}$ of y, the function A, not even derivatives of y greater than the q^{th}. And there will be no canonical system for which the order of the highest derivative of x will be contained between m and p or, which is the same, the order of the highest derivative of y between q and n.

Let us now assume to have between the independent variable t and n dependent variables $x_1, x_2, \dots\ x_n$, as many differential equations possessing an explicit canonical form. The general question arises to deduce from the proposed system of differential equations another one, enjoying a form in which the highest orders of some dependent variables derivatives decrease, of just as many others increase and of the remaining variables remain unchanged. Let respectively α_1, α_2 etc. be the highest orders up to which the derivatives of the variables x_1, x_2 etc. go in the proposed differential equations, so that the proposed differential equations be

$$\text{12)} \qquad \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} = u_1, \frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} = u_2, \dots\ \frac{d^{\alpha_n} x_n}{dt^{\alpha_n}} = u_n,$$

the functions u_1, u_2, etc. containing only derivatives lower that those placed on the left. If it is proposed to decrease the orders $\alpha_1, \alpha_2, \dots\ \alpha_m$, up to which go the derivatives of variables x_1, $x_2 \dots\ x_m$, the thing may be done in this way. We look in which of the functions u_{m+1}, u_{m+2} etc. are placed the highest derivatives of the variables x_1, x_2, $\dots\ x_m$, the orders of which are respectively $\beta_1, \beta_2, \dots\ \beta_m$, let $u_{m+1}, u_{m+2} \dots\ u_{2m}$ be functions in which they appear. If then from the equations

$$\text{13)} \qquad \frac{d^{\alpha_{m+1}} x_{m+1}}{dt^{\alpha_{m+1}}} = u_{m+1}, \frac{d^{\alpha_{m+2}} x_{m+2}}{dt^{\alpha_{m+2}}} = u_{m+2}, \dots\ \frac{d^{\alpha_{2m}} x_{2m}}{dt^{\alpha_{2m}}} = u_{2m},$$

one deduces the values

$$\text{14)} \qquad \frac{d^{\beta_1} x_1}{dt^{\beta_1}} = v_1, \frac{d^{\beta_2} x_2}{dt^{\beta_2}} = v_2, \dots\ \frac{d^{\beta_m} x_m}{dt^{\beta_m}} = v_m;$$

the derivatives of the variables x_1, x_2, $\dots\ x_m$ in the functions

$$v_1, v_2, \dots\ v_m, u_{2m+1}, u_{2m+2}, \dots\ u_n$$

will be respectively lower than the β_1^{th}, $\beta_2^{\text{th}} \dots\ \beta_m^{\text{th}}$, and β_1, $\beta_2 \dots\ \beta_m$ are smaller than the numbers α_1, $\alpha_2 \dots\ \alpha_m$. We differentiate the equations 14) respectively $\alpha_1 - \beta_1$, $\alpha_2 - \beta_2$, $\dots\ \alpha_m - \beta_m$ successive times and as soon as the $\alpha_{2m+1}^{\text{th}}$, $\alpha_{2m+2}^{\text{th}}$, $\dots$, α_n^{th} derivatives of variables $x_{2m+1}, x_{2m+2}, \dots\ x_n$ appear, one substitutes to them values coming from the $n-2m$ last [2204.a] proposed equations 12). This being done, if one considers the m first proposed equations above, one eliminates in them the derivatives

of x_1 from the β_1^{th} to the α_1^{th}
of x_2 from the β_2^{th} to the α_2^{th}
. . . .
of x_m from the β_m^{th} to the α_m^{th};

they provide m equations in which there are only derivatives of x_1, $x_2 \ldots x_m$ respectively lower than the β_1^{th}, β_2^{th}, $\ldots$ β_m^{th}; of x_{2m+1}, x_{2m+2}, x_n respectively lower than the $\alpha_{2m+1}^{\text{th}}$, $\alpha_{2m+2}^{\text{th}}$, $\ldots$ α_n^{th}; the derivatives of x_{m+1}, x_{m+2}, x_{2m} only go respectively up to the γ_1^{th}, γ_2^{th}, $\ldots$ α_m^{th} order, with $\gamma_1 = \alpha_{m+1} + \alpha_1 - \beta_1$, $\gamma_2 = \alpha_{m+2} + \alpha_2 - \beta_2$, $\ldots$ $\gamma_m = \alpha_{2m} + \alpha_{2m} - \beta_m$. So, from these equations come the values

$$\frac{d^{\gamma_1} x_{m+1}}{dt^{\gamma_1}} = w_1, \frac{d^{\gamma_2} x_{m+2}}{dt^{\gamma_2}} = w_2, \ldots \frac{d^{\gamma_m} x_{2m}}{dt^{\gamma_m}} = w_m.$$

Hence we get the transformed canonical system:

$$\begin{array}{llll}
\frac{d^{\beta_1} x_1}{dt^{\beta_1}} = v_1, & \frac{d^{\beta_2} x_2}{dt^{\beta_2}} = v_2, & \ldots & \frac{d^{\beta_m} x_m}{dt^{\beta_m}} = v_m \\
\frac{d^{\gamma_1} x_{m+1}}{dt^{\gamma_1}} = w_1, & \frac{d^{\gamma_2} x_{m+2}}{dt^{\gamma_2}} = w_2, & \ldots & \frac{d^{\gamma_m} x_{2m}}{dt^{\gamma_m}} = w_m \\
\frac{d^{\alpha_{2m+1}} x_{2m+1}}{dt^{\alpha_{2m+1}}} = u_{2m+1}, & \frac{d^{\alpha_{2m+2}} x_{2m+2}}{dt^{\alpha_{2m+2}}} = u_{2m+2}, & \ldots & \frac{d^{\alpha_n} x_n}{dt^{\alpha_n}} = u_n.
\end{array}$$

This satisfies what was requested, for the greatest orders of the derivatives of the variables x_1, x_2, $\ldots$ x_m are decreased, those of the variables x_{m+1}, x_{m+2}, $\ldots$ x_{2m} increased, those of the variables x_{2m+1}, x_{2m+2}, $\ldots$ x_n remain unchanged.

It may happen that the greatest derivatives of the variables x_1, x_2, $\ldots$ x_m appearing in the functions u_{m+1}, u_{m+2}, $\ldots$ u_{2m} do not appear in a number m of these functions, but perhaps only in one or two, and so that one cannot get the values 14) of their derivatives. Such questions require then a deeper investigation, that I will expose in some other occasion.

[§. 2.

The research is reduced to the resolution of a problem of inequalities.[2]]

It may happen that the values of the highest derivatives cannot be obtained from the proposed equations. For example, if there are among the equations some in which these derivatives do not appear, so that if we take them for *unknowns* the number of equations is not sufficient to determine them. *In that case, the number of arbitrary constants that makes appear a complete integration—that is the order of the system—is always less than the sum of the highest orders up to which go the derivatives of each variable in the proposed system.* We know the order of the system if we arrive by differentiations and eliminations to an equivalent canonical form, in such a way that one can go back from the canonical system to the proposed one. The sum of the highest orders up to which go the derivatives of each dependent variable in a canonical sytem will indeed be also the order of the non canonical sytem. But to find this order, the reduction to a canonical form is not necessary: the thing may also be achieved by the following considerations.

Assume that we have between the independent variable t and the n dependent variables $x_1,\ x_2,\ \ldots,\ x_n$, n differential equations

1) $$u_1 = 0,\ u_2 = 0,\ \ldots,\ u_n = 0,$$

and that ${h_k}^{(i)}$ is the highest order to which the derivatives of variable x_k go in the equation $u_i = 0$. I first observe that the question may be reduced to the more simple one where the proposed differential equations are linear. Indeed let $\alpha_1, \alpha_2, \ldots$ be the arbitrary constants that the *complete* values of the variables $x_1,\ x_2,\ \ldots,\ x_n$, depending on t require and let

2) $$\xi_k = \beta_1 \frac{\partial x_k}{\partial \alpha_1} + \beta_2 \frac{\partial x_k}{\partial \alpha_2} + \ldots,$$

denoting by $\beta_1,\ \beta_2$ etc. arbitrary constants. *Taking variations* of functions $x_1,\ x_2,\ \ldots,\ x_n$, we obtain from equations 1) *linear* equations between the variations $\delta x_1,\ \delta x_2,\ \ldots \delta x_n$

3) $$v_1 = 0,\ v_2 = 0, \ldots,\ v_n = 0,$$

and $h_k^{(i)}$ will be again the highest order, up to which the derivatives of $\xi_k = \delta x_k$ go in the equation $v_i = \delta u_i = 0$. A complete integration of the linear equations 3) is given by the formulas

[2] This was the beginning of the paper published by Borchardt, the section title is due to him.

$$\delta x_1 = \xi_1, \delta x_2 = \xi_2, \ldots, \delta x_n = \xi_n. \tag{4}$$

Hence the number of arbitrary constants in the complete integration of the proposed system 1) and of the linear system 3) is the same, meaning that the two systems have the same order.

[Looking for the order of the system, as one only considers the highest derivatives in the linear equations to which the proposed ones are reduced, one may assume coefficients to be constants. For differentiating the equations 3) iterated times in order to obtain new equations][3] When searching for the order of the linear differential system 3), we may assume that the coefficients are constants. In such a case, we secure a complete integration by a well-known method without any reduction to canonical form. Let us denote by the symbol $(\xi)_m$ an expression

$$A_0\xi + A_1\frac{d\xi}{dt} + A_2\frac{d^2\xi}{dt^2} + \cdots + A_m\frac{d^m\xi}{dt^m} = (\xi)_m,$$

in which $A_0, A_1, A_2, \ldots, A_m$ are constants; if we assume their coefficients to be constants, equations 3) will have the form:

$$5)\qquad \begin{cases} v_1 = (\xi_1)_{h_1'} + (\xi_2)_{h_2'} + \cdots + (\xi_n)_{h_n'} = 0, \\ v_2 = (\xi_1)_{h_1''} + (\xi_2)_{h_2''} + \cdots + (\xi_n)_{h_n''} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v_n = (\xi_1)_{h_1^{(n)}} + (\xi_2)_{h_2^{(n)}} + \cdots + (\xi_n)_{h_n^{(n)}} = 0. \end{cases}$$

I put in these equations $\xi_k = C_k e^{\lambda t}$, where C_k and λ denote constants; we get from 5)

$$6)\qquad \begin{cases} 0 = C_1[\lambda]_{h_1'} + C_2[\lambda]_{h_2'} + \cdots + C_n[\lambda]_{h_n'}, \\ 0 = C_1[\lambda]_{h_1''} + C_2[\lambda]_{h_2''} + \cdots + C_n[\lambda]_{h_n''}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = C_1[\lambda]_{h_1^{(n)}} + C_2[\lambda]_{h_2^{(n)}} + \cdots + C_n[\lambda]_{h_n^{(n)}}, \end{cases}$$

where $[\lambda]_m$ denotes an integer rational function of the m^{th} order of the quantity λ. Eliminating $C_1, C_2, \ldots, C_n$, we secure an algebraic equation whose roots give the values that λ can take, and to each root or value of λ corresponds a system of values C_1, C_2, $\ldots$, C_n that one may multiply by some arbitrary constant. Joining the values of each variable ξ_k thus obtained for each root, we get its complete value and, as the values coming from each variable are affected by corresponding arbitrary constants, the complete integration of equation 5) uses as many arbitrary constants as there are values of λ. So, the order of the system of linear equations 5), the same as that

[3]This was the beginning of an unachived proof.

of 3) or of the proposed differential equations 1) is equal to the degree of the algeraic equation defining λ. We can represent this equation in that way

7) $$0 = \Sigma \pm [\lambda]_{h'_1}[\lambda]_{h''_2} \cdots [\lambda]_{h_n^{(n)}},$$

and the degree of the right hand determinant will be the maximum of the 1.2.3 ... n sums of the sequence

$$h'_1 + h''_2 + \cdots + h_n^{(n)}$$

making the upper or lower indices vary in all possible ways. We have thus obtained the proposition:

Proposition I. *Let* $u_1 = 0,\ u_2 = 0,\ \ldots,\ u_n = 0,$ *be* n *differential equations between the independent variable* t *and the dependent variables* $x_1,\ x_2,\ \ldots,$ x_n *and let* $h_k^{(i)}$ *be [the order of] the maximal derivative of the variable* x_k *that appears in the equation* $u_i = 0$*. Then, calling* H *the maximum of sums*

$$h_1^{(i_1)} + h_2^{(i_2)} + \cdots + h_n^{(i_n)},$$

obtained when summing for indices $i_1, i_2, \ldots, i_n$*, all different the one from the other, among the indices* $1, 2, \ldots, n$*;* H *will be the order of the proposed system of differential equations, or also the number of arbitrary constants appearing in its complete integration.*

In what precedes, I call maximum a value that is not less than that of any other sum, so that many mutually equal maxima may occur, corresponding to different indices $i_1, i_2, \ldots, i_n$ of the system.

The degree of the algebraic equation 7) decreases if in the right side determinant the coefficient of the highest power of the quantity λ vanishes. And one gets the coefficient of the highest power of λ if, when forming the determinant, we substitute to each rationnal entire function $[\lambda]_{h_k^{(i)}}$ the coefficient of the highest, that is the $h_k^{(i)\text{th}}$ power that I will denote by $[c]_{h_k^{(i)}}$ and that among all the terms of the determinant

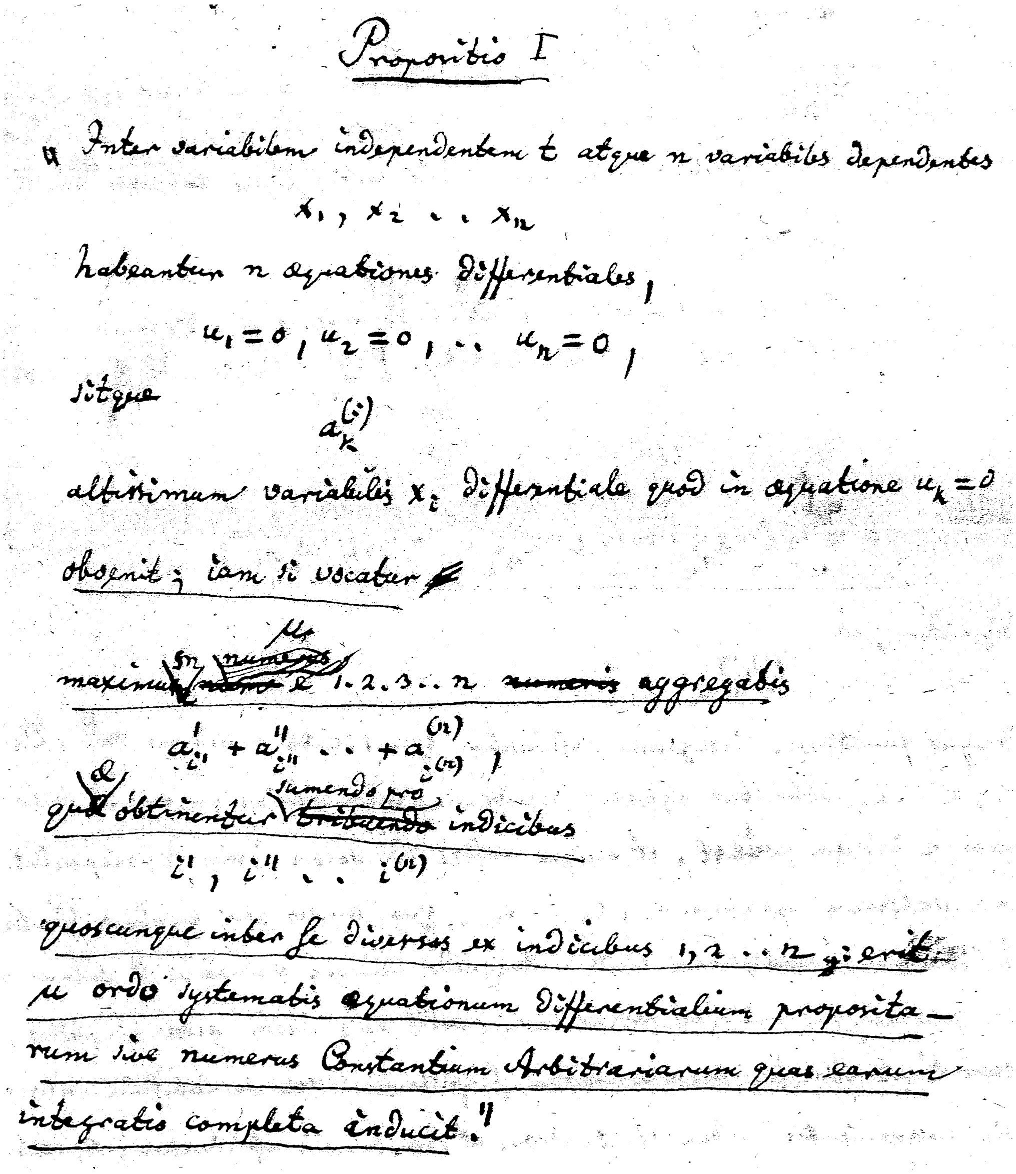
$$\pm[c]_{h_1^{(i_1)}}[c]_{h_2^{(i_2)}} \ldots [c]_{h_n^{(i_n)}}$$

we only keep those in which the sum of indices

$$h_1^{(i_1)} + h_2^{(i_2)} + \cdots + h_n^{(i_n)}$$

reaches the maximal value H. About this, the reduction of the degree will only happen in those cases where for two or more of the indices $i_1, i_2, \ldots, i_n$ of the system, the preceding sum reaches the same value and the sum of products

$$\pm[c]_{h_1^{(i_1)}}[c]_{h_2^{(i_2)}} \ldots [c]_{h_n^{(i_n)}}$$



Propositio I

„Inter variabilem independentem t atque n variabiles dependentes

$x_1, x_2 \ldots x_n$

habeantur n aequationes differentiales,

$u_1 = 0,\ u_2 = 0,\ \ldots\ u_n = 0,$

sitque

$a_k^{(i)}$

altissimum variabilis x_i differentiale quod in aequatione $u_k = 0$ obvenit; iam si vocetur μ

maximum e 1.2.3..n aggregatis

$a'_{i'} + a''_{i''} \ldots + a^{(n)}_{i^{(n)}},$

quae obtinentur sumendo pro indicibus

$i', i'' \ldots i^{(n)}$

quoscunque inter se diversos ex indicibus 1, 2 .. n; erit μ ordo systematis aequationum differentialium propositarum sive numerus Constantium Arbitrariarum quas earum integratio completa inducit."

Proposition I., original manuscript,
[II/13 b) fos 2202r and 2202v] (photomontage, scale 1).
The notation has been unified by Borchardt: in the first part of the manuscript, the bound H is denoted by "μ".

corresponding to these sets of indices added with the same signs vanishes. In what precedes, $[c]_{h_k^{(i)}}$ will be equal to the coefficient of the term $\delta \frac{d^{h_k^{(i)}} x_k}{dt^{h_k^{(i)}}}$ coming from the variation of the function u_i, that is

$$[c]_{h_k^{(i)}} = \frac{\partial u_i}{\partial \dfrac{d^{h_k^{(i)}} x_k}{dt^{h_k^{(i)}}}}.$$

Taking this in account, the next proposition appears, which completes the first one.

Proposition II. *We call $u_k^{(i)}$ the partial derivative of u_i taken with respect to the highest derivative of x_k contained in the function u_i. Among all terms of the determinant $\Sigma \pm u_1^{(i_1)} u_2^{(i_2)} \dots u_n^{(i_n)}$, we only keep those in which the sum of orders of derivatives of each variable, according to which in every*

$$u_1^{(i_1)}, u_2^{(i_2)}, \dots, u_n^{(i_n)}$$

partial differentiation is accomplished, reaches the maximal value that we call H. Then, if the sum of the remaining terms of the determinant is denoted in this way by a determinant sign between parentheses

$$\left(\Sigma \pm u_1' u_2'' \dots u_n^{(n)}\right),$$

the order of the system of differential equations $u_1 = 0,\ u_2 = 0,\ \dots,\ u_n = 0$ will be less than the maximum H precisely if

$$\left(\Sigma \pm u_1' u_2'' \dots u_n^{(n)}\right) = 0,$$

where this equality does not hold, the order of the system is always equal to the maximal value H.

We get by what precedes a new kind of formula, the *truncated* determinants

$$\left(\Sigma \pm u_1' u_2'' \dots u_n^{(n)}\right).$$

The vanishing of this quantity is the sign that the order of the system of differential equations

$$u_1 = 0,\ u_2 = 0,\ \dots,\ u_n = 0$$

decreases, due to their particular structure.

Having searched for the order of a system of arbitrary differential equations, a way is paved to find a method for performing their reduction to canonical form. Which provides at the same time a direct proof of the formula found. But we need first to study carefully the nature of the maximum considered here and how it is easilly found.

[§. 3.]

About the resolution of the problem of inequalities on which the research of the order of the system of arbitrary differential equations is supported[4]. [Considering a table, we define a canon. An arbitrary canon being given, we find a simplest one.]

By what precedes, the research of the order of a system of ordinary differential equations is reduced to the following problem of inequalities, which is also worth to be considered for itself:

Problem.

We dispose nn arbitrary quantities $h_k^{(i)}$ in a square table in such a way that we have n horizontal series and n vertical series having each one n terms. Among these quantities, to chose n being transversal, that is all disposed in different horizontal and vertical series, which may be done in $1.2\dots n$ ways; and among these ways, to research one that gives the maximum of the sum of the n chosen numbers.

[...it may occur that all combinations lead to the same sum. For example if, as it happens for the isoperimetrical problem, this table

$$\begin{array}{llll} 2m_1 & m_1+m_2 & \dots & m_1+m_n \\ m_2+m_1 & 2m_2 & \dots & m_2+m_n \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ m_n+m_1 & m_n+m_2 & \dots & 2m_n, \end{array}$$

is given. If m_1 is the greatest of the quantities $m_1, m_2, \dots, m_n$, the terms of each verticals will be made equal by increasing respectively the lines of the horizontal series by the positive values $0, m_1-m_2, \dots, m_1-m_n, \dots$][5]

[6]The quantities $h_k^{(i)}$ being disposed in a square figure

[4]This was a § 23 in the manuscript.

[5]The remaining of [II/13 b) fo 2200] has been ruled out by Jacobi. I extract this part which shows how the ideas developped in the last section of the paper could have arisen from his work on the isoperimetrical problem. T.N.

[6]Cohn's transcription continues here with a second fragment of a § 19 entitled by Jacobi *About the differentiations and eliminations by which the shortest reduction* (see [Jacobi 2]) *to canonical form is done. A problem of inequalities that must be solved to perform this reduction.* T.N.

$$\begin{matrix} h'_1 & h'_2 & \dots & h'_n \\ h''_1 & h''_2 & \dots & h''_n \\ \dots & \dots & \dots & \dots \\ h_1^{(n)} & h_2^{(n)} & \dots & h_n^{(n)}, \end{matrix}$$

we can add to each term of the same horizontal series a same quantity, and we call $\ell^{(i)}$ the quantity added to the terms of the i^{th} horizontal series. This being done, each of the $1.2\dots n$ transversal sums among which we need to find a maximum is increased by the same quantity

$$\ell' + \ell'' + \cdots + \ell^{(n)} = L,$$

because, in order to form these sums, we need to pick a term in each horizontal series. Hence, if we pose

$$h_k^{(i)} + \ell^{(i)} = p_k^{(i)}$$

and that the maximal transversal sum of the terms $h_k^{(i)}$ is

$$h_1^{(i_1)} + h_2^{(i_2)} + \cdots + h_n^{(i_n)} = H,$$

this makes that the value of the maximal sum formed with the $p_k^{(i)}$ is

$$p_1^{(i_1)} + p_2^{(i_2)} + \cdots + p_n^{(i_n)} = H + L,$$

and reciprocally. So that finding the proposed maximum for the quantities $h_k^{(i)}$ or $p_k^{(i)}$ is equivalent.

Let us bring it about that the quantities ℓ', ℓ'', ..., $\ell^{(n)}$ be determined in such a way that, the quantities $p_k^{(i)}$ being disposed in square in the same way as the quantities $h_k^{(i)}$ and chosing a maximum in each vertical series, these maxima be placed in all different horizontal series. If we call $p_k^{(i_k)}$ the maximum of terms

$$p'_k, p''_k, \dots, p_k^{(n)},$$

the sum

$$p_1^{(i_1)} + p_2^{(i_2)} + \cdots + p_n^{(i_n)}$$

will be the maximum among all the transversal sums formed with the quantities $p_k^{(i)}$. [...] Indeed, in this case, we have without trouble the maximal transversal sum formed with the proposed quantities $h_k^{(i)}$

$$h_1^{(i_1)} + h_2^{(i_2)} + \cdots + h_n^{(i_n)}.$$

So that we solve the proposed problem when we find quantities $\ell', \ell'', \dots, \ell^{(n)}$ [satisfying the given condition].

For short, I will call canon a square figure in which the maxima of the various vertical series are in all different horizontal series. It is clear that in such a *canon*, we can increase or decrease all terms by the same quantity, so that among the quantities $\ell', \ell'', \dots, \ell^{(n)}$ one or more may be made equal to 0, the others being positive. If $\ell_i = 0$, the series $p_1^{(i)}$, $p_2^{(i)}$, $\dots$, $p_n^{(i)}$ is the same as the orignal series $h_1^{(i)}, h_2^{(i)}, \dots, h_n^{(i)}$, so that I will call *unchanged* series a series of the canon that corresponds to a quantity ℓ being zero. Among all solutions, there will be a simplest one, meaning that the quantities $\ell^{(i)}$ will take minimal values, so that we will find no others for which some quantities $\ell^{(i)}$ will take smaller values, the remaining staying unchanged. I will call the canon corresponding to that solution a *simplest canon*, the structure of which I will consider bellow.

In what follows, by *series*, I will always mean a *horizontal* series; dealing with a vertical one, it will be expressly stated. As in the following only the maxima of terms placed in the same *vertical* come in consideration, I will always mean by *maximum* a term maximal among all those of the same vertical. So, I will call *maximum of a series*, a term of a horizontal series being maximal among all those placed in the same vertical as itself. It may happen that a series has no maximum or many different ones. But if the figure is constituted like a canon, each series certainly possesses a maximum, for if many are present in the same series, we can always select so that all maxima [of different series] belong to different verticals, [i.e. *they form a complete system of transversal maxima*]. We consider in a simplest canon the system of these maxima and if there are many such ones, we chose an arbitrary one. *Let us now sort all the series into two parts: series J and K, in any manner so that no series K be unchanged, meaning that none of the quantities ℓ corresponding to the series $K = 0$. I say that*

I. *[in a simplest canon] there is at least one of the maxima of series K that is equal to a term located in the same vertical and belonging to a series J.*

If not, we could decrease all the quantities ℓ related to series K by a same quantity until one of these quantities, or one of the maxima of series K become equal to a term placed on the same vertical and belonging to a series J. In this way indeed, the maxima will remain maxima and the canon structure will not be perturbed. So, the proposed quantities ℓ would not be minimal positive values nor the canon a simplest one.

If K contains a single series, then the preceding theorem implies this other.

II. *In a simplest canon, the maximum of some non unchanged series is equal to another term in the same vertical.*

Being given a simplest canon, we chose again a complete system of transversal maxima. In an arbitrary series α_1, to which corresponds a non zero quantity ℓ, there is a *maximum* to which is equal, according to II, a term [in the same vertical] located in a series α_2 where there is again a *maximum* being equal to a term in the same vertical from a series α_3, [...] and so on. If many terms of the same vertical are equal to a given *maximum*, the decribed process may be performed in various ways, but I say that

[III. *In a simplest canon,*] *among these various ways there is always one by which one reaches a series to which corresponds the value $\ell = 0$, i.e. an unchanged series.*

Obviously, it is not necessary that one gets to all series of the canon by this process, but it is possible that one gets back to the same one, in which case the same series will come back in the same order. To prove the proposition, starting from a given maximum, let us consider all the remaining maxima to which one may go by the indicated process used in all the possible ways. As these maxima are in different series, the set of which is denoted by K, I say that that none of them is equal to a term of another series. Otherwise, we could go by our process from the given maximum to that term and, as this term is in a new series, a new maximum is reached, for in a new series there is a new maximum. This is against the assumption made that all maxima to which one may get from the given one are in series K. Now, according to I, there cannot exist in a simplest canon a set of series among which no one be unchanged and no maximum of which equal a term placed in another series. Hence among the series K, one must always find at least an unchanged one. q.e.d.

I will now prove the following auxiliary theorem

IV. *The simplest canon is unique, or equivalently the quantities ℓ', ℓ'', ... $\ell^{(n)}$ that provide it.*

For brevity, I will call, in what follows, canon $(m', m'', \dots, m^{(n)})$ an arbitrary canon in which quantities ℓ', ℓ'', ..., $\ell^{(n)}$ take respectively the values m', m'', ..., $m^{(n)}$, that I assume to be always positive or zero. This being defined, we shall have about two canons the

Theorem. *Two canons being given, $(f', f'', \dots, f^{(n)})$ et $(g', g'', \dots, g^{(n)})$, let the quantities $g^{(\alpha+1)}$, $g^{(\alpha+2)}$ ... $g^{(n)}$ be respectively greater than $f^{(\alpha+1)}$, $f^{(\alpha+2)}$ ... $f^{(n)}$ and the remaining g', g'' ... $g^{(\alpha)}$ resp. equal or smaller than f', f'' ... $f^{(\alpha)}$, there will always be another canon $(g', g'', \dots, g^{(\alpha)}, h^{(\alpha+1)}, h^{(\alpha+2)}, \dots, h^{(m)})$ in which quantitities $h^{(\alpha+1)}, h^{(\alpha+2)} \dots$ $h^{(m)}$ are equal or smaller than the quantities $f^{(\alpha+1)}$, $f^{(\alpha+2)}$ ... $f^{(n)}$.*

We call respectively $q_k^{(i)}$ and $r_k^{(i)}$ the quantities that constitute the first and the second canon, with

$$r_k^{(i)} = q_k^{(i)} + g^{(i)} - f^{(i)},$$

and let again the system of transversal maxima in the first canon be

$$q_1^{(i_1)}, q_2^{(i_2)}, \dots, q_n^{(i_n)},$$

where the $i_1, i_2, \dots, i_n$ are all different; in the second canon one also has the system of transversal maxima

$$r_1^{(i_1)}, r_2^{(i_2)}, \dots, r_n^{(i_n)}.$$

For in fact, all the transversal sums of the second canon exceed the corresponding ones of the first by the same quantity

$$g' + g'' + \cdots + g^{(n)} - \left\{ f' + f'' + \ldots + f^{(n)} \right\},$$

so, as the sum

$$q_1^{(i_1)} + q_2^{(i_2)} + \cdots + q_n^{(i_n)}$$

is maximal, the sum

$$r_1^{(i_1)} + r_2^{(i_2)} + \cdots + r_n^{(i_n)}$$

must be maximal too. And, as in any canon we have by definition a maximal transversal sum, of which each term is maximal among all those of its vertical, the terms

$$r_1^{(i_1)}, r_2^{(i_2)}, \cdots, r_n^{(i_n)}$$

must be respectively equal to the maxima of the first, second, ..., n^{th} verticals, so that their sum could be maximal. So, as $i_1, i_2, \dots, i_n$ are all different one from the other, these terms constitute themselves a system of transversal maxima. **q.d.e.**

As the quantities $g^{(\alpha+1)}$, $g^{(\alpha+2)}$, ..., $g^{(n)}$ are respectively greater than $f^{(\alpha+1)}$, $f^{(\alpha+2)}$, ..., $f^{(n)}$, quantities themselves all assumed positive or zero, the quantities $g^{(\alpha+1)}$, $g^{(\alpha+2)}$, ..., $g^{(n)}$ are all positives. I observe then that it cannot happen that, in the second canon, one finds a maximum belonging to the series $\alpha+1$, $\alpha+2$,..., n, to which is equal a term placed in the same vertical, but belonging to one of the remaining series. Let in fact this maximum be in the series i_k, so that

$$r_k^{(i_k)} = r_k^{(i)},$$

where i is one of the numbers $1, 2, \dots, \alpha$ and i_k one of the numbers $\alpha+1$, $\alpha+2$, ..., n: we shall have according to the formula given above

$$q_k^{(i_k)} + g^{(i_k)} - f^{(i_k)} = q_k^{(i)} + g^{(i)} - f^{(i)},$$

where according to the assumption made $g^{(i_k)} - f^{(i_k)}$ is positive and $g^{(i)} - f^{(i)}$ or $= 0$ or negative. Hence we have

$$q_k^{(i_k)} < q_k^{(i)},$$

which is absurd for $q_k^{(i_k)}$ is a maximum among the terms of the same vertical $(q'_k, q''_k, \ldots, q_k^{(n)})$. So, as in the second canon a term of the same vertical located in one of the remaining series cannot be equal to maxima placed in the $(\alpha+1)^{\text{th}}$, $(\alpha+2)^{\text{th}}$,..., n^{th} series, the quantities $g^{(\alpha+1)}$, $g^{(\alpha+2)}$, ..., $g^{(n)}$ may all be decreased by a same quantity, the others staying unchanged, until in one of the series $(\alpha+1)$, $(\alpha+2)$, ..., n one finds a maximum not greater than the value of another term located in the same vertical, belonging to one of the remaining series or until one of the quantities $g^{(\alpha+1)}$, $g^{(\alpha+2)}$, ..., $g^{(n)}$ vanishes. By this decreasing, no maximum, nor the nature of the canon will be destroyed. If by these means we get

$$\left(g', g'', \ldots, g^{(\alpha)}, g_1^{(\alpha+1)}, g_1^{(\alpha+2)}, \ldots, g_1^{(n)}\right)$$

and $g_1^{(\beta+1)}$, $g_1^{(\beta+2)}$, ... are still greater than the corresponding quantities $f_1^{(\beta+1)}$, $f_1^{(\beta+2)}$, ..., we get by the same method a new canon in which these quantities will get smaller values and one can go on like this until one reaches a canon

$$\left(m', m'', \ldots, m^{(\alpha)}, m^{(\alpha+1)}, m^{(\alpha+2)}, \ldots, m^{(n)}\right)$$

where all the included quantities are respectively equal or smaller than f', f'', ..., $f^{(n)}$) [and g', g'', ..., $g^{(n)}$)]. **q.d.e.**

It follows from IV that

V. *There is no canon for which one of the quantities ℓ', ℓ'', ..., $\ell^{(n)}$ takes a smaller value than for the most simple canon.*

Let us assume to be given such a canon, by the former method we could obtain another one for which at least one of the quantities $\ell', \ell'', \ldots, \ell^{(n)}$ would take a smaller value than in the simplest canon, the others being not greater, which is contrary to the definition of a simplest canon.

As a corollary of proposition V,

VI. *There is no unchanged series in some canon that is not also found in the simplest one.*

Obviously, the smallest values that the quantities $\ell', \ell'', \ldots, \ell^{(n)}$ can take is $= 0$.

In order to know whether some canon is or not the simplest, we can add this proposition.

VII. *A canon being given and having chosen in it a system of transversal maxima, we first denote A the unchanged series, then B the series whose maxima are equal to a term of a series A located in the same vertical, then C the series whose maxima are equal to a term of a series B located in the same vertical, and so on. If, continuing this process, we exhaust all the series of the canon, it will be the simplest.*

Let the quantities $\ell', \ell'', \ldots, \ell^{(n)}$ correspond to the proposed canon and the quantities ℓ_1', ℓ_1'', $\ldots$, $\ell_1^{(n)}$ to some other canon. We assume to be chosen the same system of transversal maxima as in the proposed theorem, to which corresponds a system of transversal maxima in the other canon.

If $\ell_1^{(\gamma)} < \ell^{(\gamma)}$, the maximum of the series γ in the other canon will possess a smaller value than in the proposed canon. If the series γ belongs to the set C, so that in the proposed canon, the maximum of the series γ is equal to a term of the series β belonging to the set B, then $\ell^{(\beta)}$ must be smaller in the other canon than in the proposed one. For indeed, calling $p_k^{(i)}$ the terms of the proposed canon and $q_k^{(i)}$ those of the other, we shall have

$$q_k^{(\beta)} = p_k^{(\beta)} + \ell_1^{(\beta)} - \ell^{(\beta)},$$

hence, if $p_k^{(\gamma)} = p_k^{(\beta)}$ is the maximum of the series γ, one will have

$$q_k^{(\beta)} = p_k^{(\beta)} + \ell_1^{(\beta)} - \ell^{(\beta)} = q_k^{(\gamma)} + \ell_1^{(\beta)} - \ell^{(\beta)} - \left\{\ell_1^{(\gamma)} - \ell^{(\gamma)}\right\}.$$

Then, as $q_k^{(\gamma)}$ is the maximum of the k^{th} vertical, so that $q_k^{(\gamma)} \geq q_k^{(\beta)}$ and $\ell_1^{(\gamma)} < \ell^{(\gamma)}$, we must have $\ell_1^{(\beta)} < \ell^{(\beta)}$.

Next, in the proposed canon, the maximum of the series β is equal to a term of the series α belonging to the set A and we show in the same way that we must have $\ell_1^{(\alpha)} < \ell^{(\alpha)}$, which is absurd for, according to the assumption made, $\ell^{(\alpha)} = 0$ and ℓ_1', ℓ_1'', $\ldots$, $\ell_1^{(n)}$ are positive or zero. The reduction to absurdity proceeds in the same way, to whatever set A, B, C, D, $\ldots$ may belong the series γ to which corresponds in the other canon the quantity $\ell_1^{(\gamma)}$ less than that of the considered canon $\ell^{(\gamma)}$. So, if the canon is as assumed in VI[I], the values ℓ cannot take any other smaller values; in other words, the proposed canon is the simplest.

What precedes contains the solution of the problem, *an arbitrary canon being given, find the simplest one*. Let us assume that, in the given canon, one or more series are unchanged, which is obtained, if it is not the case, by decreasing all the ℓ of the same quantity. Let us call A the set of unchanged series, B the set of series, the maximum of which is equal to a term of the same vertical belonging to A [...] and so on. If by this process we exhaust all series, the canon is, according to VII, already the simplest. Let us assume that there remain series, in which there are no *maxima* equal to terms of the same vertical belonging to the built sets. Then, the terms of the remaining series are all decreased of a same quantity, until one of their quantities ℓ become zero or one of their *maxima* decrease as far as being equal to a term in the same vertical and belonging to the built sets. That done, we get another canon, in which is increased the number of series that enter the sets that one can successively form according to the indicated rule. If in the new canon, all series come in these sets, then it will be the simplest. If not, new canons are to be constructed by repeating the same process, always fewer series remaining outside the sets that can be built, until we secure a canon in which these sets will exhaust all the series and which is the requested simplest canon. I observe that in the new canon, one may omit vertical series in which the *maxima* that belong to sets A, B etc. are placed. In this way, one spares a lot of writing work.[7]

Example.

Proposed table.

	I	II	III	IV	V	VI	VII
I	7	7	4	15	14	6	1
II	3	8	7	6	11	14	10
III	6	11	15	16	15	23	10
IV	4	11	14	25	20	21	27
V	5	2	8	10	23	18	30
VI	1	8	3	9	6	20	17
VII	11	12	8	22	24	21	40

Proposed canon.

	I	II	III	IV	V	VI	VII	ℓ
I	12*	12	9	20	19	11	6	5
II	11	16*	15	14	19	24	18	8
III	9	14	18*	19	18	26	13	3
IV	5	12	15	26*	21	22	28	1
V	10	7	13	15	28*	23	35	5
VI	7	14	9	15	12	26*	23	6
VII	11	12	8	22	24	21	40*	0

Starting from the proposed table, adding to the terms of the various series the respective numbers 5, 8, 3, 1, 5, 6, 0, we get a new table, in which some maximal terms among all those located in the same vertical are placed in different horizontal series, which is the characteristic property of a canon. It is proposed to find the simplest canon. The series VII constitutes in the given canon the set A. I subtract unity from the terms of the remaining series, which produces the canon I.

[7]Efficiency considerations in Jacobi's manuscript always remain at this informal level, which is very much in the spirit of computers work in observatories at that time. See, e.g. Grier (David Alan), *When computers were humans*, Princeton university press, Princeton, 2005.

I.

	I	II	III	IV	V	VI	l
I	11*	11	8	19	18	10	4
II	10	15*	14	13	18	21	7
III	8	13	17*	18	17	25	2
IV	4	11	14	25*	20	21	0
V	9	6	12	14	27*	22	4
VI	6	13	8	14	11	25*	5
VII	11	12	8	22	24	21	0

I have omitted the last vertical, where stands the maximum of series VII. In the canon I, the series IV and VII constitute the set A, the series I the set B. I subtract 2 from the others terms, rejecting the first and fourth verticals, where stand the maxima of series IV and I, which produces II.

II.

	II	III	V	VI	ℓ
I	11	8	18	10	4
II	13*	12	16	19	5
III	11	15*	15	23	0
IV	11	14	20	21	0
V	4	10	25*	20	2
VI	11	6	9	23*	3
VII	12	8	24	21	0

In the canon II, the series III, IV, VII constitute the set A, the series I and VI the set B; I subtract unity from the second and fifth series, producing the last canon, that is the simplest canon, corresponding to the values of ℓ 4, 4, 0, 0, 1, 3, 0.

Adding these to the terms of the various series of the proposed table, we get the

Simplest canon.

	I	II	III	IV	V	VI	VII	ℓ
I	11*	11	8	19	18	10	5	4
II	7	12*	11	10	15	18	14	4
III	6	11	15*	16	15	23	10	0
IV	4	11	14	25*	20	21	27	0
V	6	3	9	11	24*	19	31	1
VI	4	11	6	12	9	23*	20	3
VII	11	12	8	22	24	21	40*	0

The series III, IV, VII constitute the set A, the series I, II, V, VI the set B; we see that these sets exhaust all series, which is the characteristic property of the simplest canon.

If we do not give ourselves a canon, but only the terms of the table constituting a maximal transversal sum, we reach the simplest canon by adding to each series the smallest quantity such that the term of this series belonging to the minimal transversal sum be made equal to the maximum of its vertical. Having applied this process to every series and having repeated it if necessary, we must get a canon that will be the simplest, for we do not add to the series any increment greater than what is necessary for making the given terms maximal in their respective verticals.

Example.

[Proposed table.]

I	11*	7	6	4	6	4	11
II	11	12*	11	11	3	11	12
III	8	11	15*	14	9	6	8
IV	19	10	16	25*	11	12	22
V	18	15	15	20	24*	9	24
VI	10	18	23	21	19	23*	21
VII	5	14	10	27	31	20	40*

The terms marked with an asterisk form a maximal transversal sum, it appears that I got the proposed table from the preceding one by changing the vertical series in horizontal ones and the verticals in horizontals; doing so, the same terms constitute a maximal transversal sum, but the table is no longer an canon. According to the given rule, I add respectively $8, 6, 8, 2, 7$ to the series I, II, III, IV, V, which gives:

[Derived table.]

I	19*	15	14	12	14	12	19
II	17	18*	17	17	9	17	18
III	16	19	23*	22	17	14	16
IV	21	12	18	27*	13	14	24
V	25	22	22	27	31*	16	31
VI	10	18	23	21	19	23*	21
VII	5	14	10	27	31	20	40*

I add respectively $6, 4$ to the series I, II, which produces the requested simplest canon

I	25*	21	20	18	20	18	25
II	21	22*	21	21	13	21	22

while the remaining stay the same as in the table beside. In the obtained canon, the series VI et VII constitute the set A, the series III, IV, V the set B, the series I, II the set C, as these sets contain all series, we have the proof that the canon is the simplest.

A canon being given, we also know a maximum transversal sum of the proposed table, so we can by what precedes reduce this other problem, *being given an arbitrary canon, to look for the simplest*, to the problem solved. Hence, it will have two solutions, one by successive subtractions, as above, the other by successive additions, meaning that if we deduce from the given canon a maximal transversal sum of the proposed table, we apply, knowing it, the preceding method.

[§. 4.

We finish the solution of the inequality problem considered in the preceding paragraph. A table being given, we find a canon.[8]]

We still have to show how to find an arbitray canon; having found one, we have seen various ways to obtain the simplest. So, we propose the following inequality problem that must be our starting point.

Problem.

Being given nn quantities $h_k^{(i)}$ where the indices i and k take the values 1, 2, …, *n, to find n minimal positive quantities*

$$\ell', \ell'', \dots, \ell^{(n)}$$

such that, having set $h_k^{(i)} + \ell^{(i)} = p_k^{(i)}$, and having chosen for each k a maximum among the terms

$$p'_k, p''_k, \dots, p_k^{(n)},$$

which we denote by $p_k^{(i_k)}$, the indices i_1, i_2, …, i_n be all different from each other.

Solution.

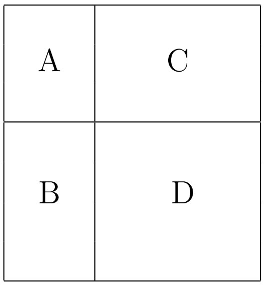
[…][9] If there are in the table series in which no *maximum* exists, a first and in some way preparatory operation consists in this: I increase them by the minimal quantity so that one of their terms become equal to a *maximum* placed in the same vertical. We get thus a new table [that I call prepared table], in which every series possesses one or more maxima, but all maxima of the different series do not necessarily belong to different verticals. But, at least, one has *two* series whose maxima belong to two verticals, which only appears when all maxima are placed in the same series and all the terms of a same vertical are equal; if not, the number of transversal maxima is always > 2. If $n = 2$, the problem is solved by this preliminary operation.

In the new table, I look for the maximal number of transversal maxima. If there are many possible choices, it is enough to consider at least one system. Having chosen it, I solve the proposed problem by successively increasing the number of transversal maxima until we get a table equipped with a system of n transversal maxima that will be the canon sought. So, we only have to

[8] This new section and its title are due to Borchardt.

[9] Borchardt has suppressed definitions that were already given above.

show that one can augment by one the number of transversal maxima with a suitable increasing of series.



I divide the table in four parts as in the figure in the margin. We assume that the chosen transversal maxima are all in part A, so that the series where they are fill the parts A and C; the verticals to which they belong fill the parts A and B. I call *upper* the series filling parts A and C and *lower* these filling parts B and D. I then call *left* the verticals filling parts A and B and *right* the verticals filling parts C and D. Then, in part D there is no maximum. If so, the number of transversal maxima would be increased, contradicting the hypothesis that it is maximal. So, the right verticals have all their maxima in C; the maximal terms in their own verticals of the lower series are in B, and every one of them will be equal to a maximum of the same vertical located in A, for in the space A are placed the maxima of all the left verticals as well as those of all the upper series.

Granting this, I divide all the series in three classes, defined as follows.

I choose those of the upper series that, besides maxima in A, possess also others, placed in C, so that at least one of these series exists. Let us assume that one of the maxima of these series placed in A be equal to some other term of the same vertical; we look for a maximum placed in the same series as this term and, if it is equal to another term in the same vertical, we look again for a maximum placed in the same series as that term, and so on. All the series that one may reach in this way, from the starting series, constitute the *first class.*

I say that, *among the series of the first class, there is neither lower series*, nor upper series from which one may go to a lower series by the indicated process. For in fact, starting from a series having besides a maximum in A another one in C, we consider a system of maxima placed in A to which we have come by the indicated method, and whose last, if possible, is equal to a term in the same vertical placed in B. All these maxima placed in A are, by hypothesis, transversal maxima and we shall get in their own place a new system of transversal maxima if we substitute to each of them the equal term placed in the same vertical. In this way, we substitute to the last maximum the term placed in B, without using the first series, from which we started. So, adjoining the *maximum* of this series placed in C, in order to form a new system of maxima, the number of transversal maxima will be increased by one, which contradicts the assumption that this number was maximal.

Namely a lower series, in which is a term equal to the last maximum, enters the upper series and a right series, in which is some maximum of the series from which we started, enters the left series.

The upper series that do not belong to the first class and from which one cannot go to a lower series by the method indicated above belong to the *second class*. It may happen that this class does not exist.

At last, all the lower series and all the upper series from which one can go to lower series by the described method belong to the *third class*. *So, if a term of a lower series is equal to a maximum of an upper series in the same vertical—which is always the case— this upper series will belong to the third class.* The third class, unless the table is already a canon, contains at least two series, one upper and one lower.

[I will express again what I have demonstrated about the first class by saying that, *among the upper series of the third class, there is none that possess a maximum placed in* C.]

The observations made on this occasion also help to find the maximal number of transversal maxima in the prepared table. In fact, having posed a system of transversal maxima, the first one to present itself, this classification indicates if this number may be increased.

The described classification being done, *all the third class is increased by the same quantity and the smallest that makes a term of one of the series of this class reach a maximal term placed in the same vertical and belonging to a series of the first or second class.*

So, if the maximum belongs to the first class, the number of transversal maxima may be increased. Let there in fact be an upper series that possesses, besides a maximum in A another one in C and from where one may go in the indicated way to a lower series. That series is to be counted in the number of upper series whereas we need to increase that of left verticals with the right vertical where stands that maximum placed in C. If the term of a series in the third class, equal to a maximum of a series of the first, is located in D, the transversal maxima remain unchanged: we only have to add this term. But if that term is in B, we need change all the maxima forming that chain by which we get down to the lower series from the series containing the maximum in C. Namely, each of these transversal maxima is to be replaced by the term in the same vertical that is equal to it, and the last by the term in B, new transversal maximum appearing by adding at the beginning the maximum of the first series placed in C, as I have remarked concerning the first class.

If the maximum to which a term of the third class is equal lies in a series of the second, nothing changes, except that these series will go to the third class together with all the remaining series of the second class from which, by the indicated chain, one goes to that series. Repeating this operation again, whether we increase the number of transversal maxima or we decrease that of the second class series, so that we get a table deprived of second class series, because they all went to the third, if the number of transversal maxima is not increased before that. But then, by the given process, we get undoubtedly an increasing of transversal maxima. Having obtained it, we need to operate a new repartition of transversal maxima in the assigned three classes, for the different cases that may arise and that would be too long to enumerate, and, this being done, to repeat the operation until we get a canon in which all lower series will become upper and right verticals left.

And by the method previously described, we get non only a canon, but the simplest one. To prove it, I will show that the quantities by which are increased the series are minimal, because they are required to produce any canon. And first, as regards the preparatory process, I notice that each term of the canon is greater or equal to the corresponding term of the given table, the canon being obtained by adding to each series of the table only positive or zero quantities. So, the maximum in each vertical of the canon will be greater or equal to the maximum in the same vertical of the given table. But in the canon, there will be in each series a maximum, so a term that is greater or equal to the maximum of the given table placed in the same vertical; so, we need to increase each series of the given table without a maximum by a quantity such that one of its terms becomes greater of equal to the maximum of the same vertical. Hence, if we consider the quantities by which the terms of a series differ from the maxima of the same vertical, the quantity by which the series must be increased cannot be less than the minimum of these quantities. So, increasing each series deprived of a maximum of the minimal quantity that will make one of its term equal to the maximum of the same vertical, these series will certainly not be increased by a quantity greater than what is required to build the canon.

The preparation being done, if it produces already a canon by itself, this one is certainly the simplest; we have seen in fact that positive quantities, minimal to produce a canon, are added to the series of the given table. But if a canon did not yet arise, we had to proceed to the three classes partition. I will show now that, to produce a canon, *it cannot be that any series of the third class remains unchanged.*

In this demonstration, I will call S the prepared table, K the obtained canon. I always assume what was already required for the classification of series: to have before one's eyes a system of transversal maxima in S, in the space A, so that if there are many such systems in A, any of them is to be chosen. Likewise in K, if many systems of tranversal maxima arise, I assume that one has been chosen.

We will consider in S the set of all the unchanged upper series of the third class, if any, that is those to which nothing is added to form the canon K, or also those being the same in S and K. We will call H the set of these series and we consider transversal maxima of these, chosen in S and K. I say that *the systems of these maxima in S and K will be in the same verticals.* Let in fact M be one of these maxima in K placed in an unchanged series, an equal term of the same series, itself maximal in its vertical, will correspond to it in S. For, as we go from S to K by positive additions, the terms of this vertical in S are smaller of equal to the corresponding terms in K; so if their maximum in K is equal to a term of S in the same vertical, this one must be all the more maximal among the terms in the same vertical in S. As, according to the properties of the classes, an upper series of the third class has no term maximal in its own vertical in C, the term M must belong to the space A. We call V the set of verticals in which stand the maxima of the series of H in S and we assume that the vertical in which is M does not belong to the verticals of V. There will exist in S in this vertical a maximum $N = M$ belonging to the transversal maxima chosen in space A and that is why this maximum N will be placed in a series that does not belong to H. The transversal maxima chosen in the series H are themselves in the verticals of V, whereas N is assumed to be in a vertical not belonging to V. This new series must be an upper series belonging to the third class; the maximum N belongs in fact to the space A and from the given definition of classes, if there is in the same vertical maximal terms all equal the one with the other, the series in which they are placed belong to the same class. Then, if in order to form the canon K we would add to the series a nonzero quantity, the term of K corresponding to N would be greater than N, and also greater than the term M placed in the same vertical, which cannot happen for M is maximal in its vertical. So, this series must be itself unchanged, which is absurd for we have assumed that the series of H are *the set of all* the unchaged series of the third class. So M itself is necessarily placed in a vertical of V; as this is true for every maxima, it follows that the system of transversal maxima of the series of H chosen in K are in the same verticals than the system of transversal maxima of these same series chosen in S, **q.d.e.**

If we take in S terms corresponding and equal to the maxima of the series of H in K, these will form in S another system of transversal maxima which are in the same horizontal and vertical series. That cannot be done, unless the terms of the two systems placed in the same verticals are equal. Whence we get this other proposition: *if we take in S, in some unchanged series of the third class, a maximum, we will have in K an equal maximum in the same vertical, in an upper series of the same class.* I always assume that the maxima in S, as in K, are taken in the chosen systems of transversal maxima.

Besides, the last proposition is proved in the same way if H stands for the set of series of the second class; on the other hand it is only for these that the proposition is strong and significant. I will now prove that there is no unchanged series of the third class.

It appears first that *there is no unchanged lower series.* If in fact there is some unchanged lower series, let M be its maximum in K, taken from the chosen system of transversal maxima; this same term will in S be maximal among all those of the same vertical and for that reason it is equal to a maximum from a series of the third class placed in the same vertical and belonging to transversal maxima (see above the definition of the third class). But, according to the preceding corollary, there must be in K, in the same vertical, a maximum of an upper series belonging to the transversal maxima, whence we shall have in K, in the same vertical two transversal maxima, one in an upper series, the other M in a lower one, what is contrary to the notion of transversal maxima.

I will now show that *if there is an unchanged upper series of the third class, there is a lower one unchanged*; as it is impossible, it will be proved that there is no unchanged series of the third class, neither lower nor upper.

Assume to be given an upper series of the third class, that I will denote by s. According to the definition of the third class, we shall have series s, s_1, s_2, …, s_{m-1} such that their maxima $M, M_1, M_2, \ldots, M_{m-1}$ that are taken from the chosen system of transversal maxima have each of them in the same vertical an equal term N_i in the following series, the last M_{m-1} being equal to a term N_{m-1} of the same vertical in a lower series[10]; so that N_i and M_{i+1} are both in the same series and that M_i and N_i are both equal and in the same vertical. Then, if an upper series s of the third class is unchanged, we shall have, according to the preceding corollary a maximum in K equal to M itself and placed in the same vertical; so when forming the canon, it will be impossible to increase the series s_1, for then one would increase the term N and the maximum M itself, placed in the same vertical, would disappear.

[10]This is series s_m. T.N.

So, the series s_1 must remain unchanged, and one proves in the same way that each one of the series $s_2, s_3, \ldots, s_{m-1}$, as well as the lower series s_m, are unchanged, which we have seen to be impossible. As, in order to form the

canon no series of the third class may remain unchanged, let f be the smallest quantity by which these series must be increased, so that, being increased by f, there is in the new table at least one that, in order to form the canon does not need to be increased more, but will stay *unchanged.* Let g be the minimal quantity by which the series S of the third class are increased, so that one of its terms becomes equal to a maximum of a series of the first or second class [placed in the same vertical]. If $f' < g$ and that every series of the third class S are increased by f', we see that in the new table, the partition of series in classes is not modified, and that each one belongs to the same class as in S. So, it cannot be that $f < g$; for if so, we would have a table in which would be some *unchanged* series of the third class, which cannot be. Hence we see that the minimal quantity by which the series of the third class must be increased, so that one of their terms reaches a maximum of a series of the first or second class [placed in the same vertical] is smaller or equal to the smallest of the quantities by which the series of the third class must be increased to form the canon. From which follows that, according to the given rule, we never employ additions greater than what is necessary to form the canon, so that *the canon obtained by our rule is the simplest.*

Example I.

Proposed table.

11	7	6	4	6	4	11
11	12	11	11	3	11	12
8	11	15	14	9	6	8
19	10	16	25	11	12	22
18	15	15	20	24	9	24
10	18	23	21	19	23	21
5	14	10	27	31	20	40

Prepared table.

$\underline{19}^*$	15	14	12	14	12	19	*t*
17	$\underline{18}^*$	17	17	9	17	18	*t*
15	$\underline{18}$	22	21	16	13	15	*t*
$\underline{19}$	10	16	25	11	12	22	*t*
$\underline{19}$	16	16	21	25	10	25	*t*
10	$\underline{18}$	$\underline{23}$	21	19	$\underline{23}^*$	21	
5	14	10	$\underline{27}$	$\underline{31}$	20	$\underline{40}^*$	

I.

$\underline{20}^*$	16	15	13	15	13	20	*t*
18	$\underline{19}^*$	18	18	10	18	19	*t*
16	$\underline{19}$	$\underline{23}^*$	22	17	14	16	
$\underline{20}$	11	17	26	12	13	23	
$\underline{20}$	17	17	22	26	11	26	*t*
10	18	$\underline{23}$	21	19	$\underline{23}^*$	21	
5	14	10	$\underline{27}$	$\underline{31}$	20	$\underline{40}^*$	

II.

$\underline{21}^*$	17	16	14	16	14	21	*t*
18	$\underline{19}^*$	18	18	10	18	19	
16	$\underline{19}$	$\underline{23}^*$	22	17	14	16	
$\underline{21}$	12	18	$\underline{27}^*$	13	14	24	
$\underline{21}$	18	18	23	27	12	27	*t*
10	18	$\underline{23}$	21	19	$\underline{23}^*$	21	
5	14	10	$\underline{27}$	$\underline{31}$	20	$\underline{40}^*$	

III.

$\underline{22}^*$	18	17	15	17	15	22	*t*
18	$\underline{19}^*$	18	18	10	18	19	*t*
16	$\underline{19}$	$\underline{23}^*$	22	17	14	16	
21	12	18	$\underline{27}^*$	13	14	24	
$\underline{22}$	$\underline{19}$	19	24	28	13	28	*t*
10	18	$\underline{23}$	21	19	$\underline{23}^*$	21	
5	14	10	$\underline{27}$	$\underline{31}$	20	$\underline{40}^*$	

Simplest canon.

$\underline{25}^*$	21	20	18	20	18	25
21	$\underline{22}^*$	21	21	13	21	22
16	$\underline{19}$	$\underline{23}^*$	22	17	14	16
21	12	18	$\underline{27}^*$	13	14	24
$\underline{25}$	$\underline{22}$	22	$\underline{27}$	$\underline{31}^*$	16	31
10	18	$\underline{23}$	21	19	$\underline{23}^*$	21
5	14	10	$\underline{27}$	$\underline{31}$	20	$\underline{40}^*$

In the given table, the three first series and the fifth have no maximal terms. We need to add to these series the minimal numbers 8, 6, 7, 1, by which we can make one of their terms become maximal. In the table prepared in this way, I have underlined all the maximal terms of each vertical and put a star in the exponent of the chosen transversal maxima (denoted by an asterisk). At last, I have noted with a t the series of the third class that we find in this way. First belong to it all the series α that have no starred term, that above I have called lower series; then the series β that have a starred term in a vertical where a term of a series α has already been underlined; if, besides starred terms, the series β have other underlined terms, we search in the same verticals new starred terms that belong to series γ, and so on: all the easily found series α, β, γ etc. form the third class. It also is apparent that in order to fully apply the rule, it is only requested that the third class series be known and that the partition into first and second classes is unnecessary. For in fact the rule requires nothing more than to increase together all series of the third class by a minimal quantity such that one of their terms becomes equal to one of the maximal starred terms of other series located in the same vertical. All the work is actually reduced to the increasing of series, the choice of transversal maxima and the determination of third class series, after which a new increasing is performed. Which is to be continued until all terms can be denoted with asterisks, in which case we have reached the simplest canon.

One may, by various artefacts, spare the work of rewriting the table after any change. Namely, to go from a table to the next, it is not necessary to have other terms before one's eyes than those being maximal in each vertical and those just lower, and it is enough to write only these ones. Then, it is not necessary to respect the series order, it is enough to rule out the series to be increased and to rewrite them under the unchanged ones. But these means and others that are easily used for a great amount of numbers are left to each one's choice. In the next example, for a better readability we avoid such abreviations. [...][11]

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[11] This example is to be found in [Jacobi 2].