Fundamentals of 3D
Lecture 8: Introduction to meshes
Les maillages

Frank Nielsen
nielsen@lix.polytechnique.fr

23 Novembre 2011
Polygons

Polygonal chain

Regular (convex) polygon

Convex polygon

Nonsimple polygon

Nonconvex polygon with hole

Star-shaped polygon
Polygons: Star-shaped decomposition
Polygons: Star-shaped decomposition

Art gallery, illumination problems, robots' race, etc. Place guards...
3D : Orienting the fact edges for *outer normals*

Polyhedron, convex polyhedra
Platonic solids: 5 convex polyhedra

(a) Tetrahedron
(b) Hexahedron
(c) Octahedron
(d) Dodecahedron
(e) Icosahedron

<table>
<thead>
<tr>
<th>Platonic solid</th>
<th>Schlëfli symbol</th>
<th># Vertices</th>
<th># Faces</th>
<th># Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron (a)</td>
<td>(3, 3)</td>
<td>4</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Hexahedron (b)</td>
<td>(4, 3)</td>
<td>8</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>Octahedron (c)</td>
<td>(3, 4)</td>
<td>6</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>Dodecahedron (d)</td>
<td>(5, 3)</td>
<td>20</td>
<td>12</td>
<td>30</td>
</tr>
<tr>
<td>Icosahedron (e)</td>
<td>(3, 5)</td>
<td>12</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

Identical faces
(group of symmetry)
Uniform polyhedra

- **rhombicuboctahedron**
- **rhombicosidodecahedron**

- faces = regular polygons (not necessarily the same),
- isometry mapping of its vertices (=symmetry)
75 uniform finite polyhedra/ if you like origami
Meshes: Notations

\[ |\mathcal{V}(\mathcal{M})| = 9 \quad \mathcal{V}(\mathcal{M}) = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9\} \]

\[ |\mathcal{E}(\mathcal{M})| = 14 \quad \mathcal{E}(\mathcal{M}) = \{e_a, e_b, e_c, e_d, e_e, e_f, e_g, e_h, e_i, e_j, e_k, e_l, e_m, e_n\} \]

\[ |\mathcal{F}(\mathcal{M})| = 6 \quad \mathcal{F}(\mathcal{M}) = \{F_1, F_2, F_3, F_4, F_5, F_6\} \]

\[ E(p_8) = \{e_b, e_d, e_g, e_e, e_n\} \quad N(p_8) = \{p_1, p_2, p_3, p_4, p_9\} \]
Meshes: Connectivity (= structuring)

Triangle Soup

Triangle Strip

Triangle Fan (Umbrella)

Quad Soup

Quad Strip

Polygon
Meshes: Non-orientable surfaces

Möbius band

Klein bottle
Textured meshes: Realistic computer graphics
Osculating circles and curvature

Curvature is the inverse of the radius of the osculating circle.

\[ \rho(p) = \frac{1}{R(p)} \]
Mesh: **Sectional curvatures** and principal directions

Directions are **perpendicular** to each other

**sectional curvatures.**

Intersection of a surface $S$ with a plane containing point $p$ and its normal: 2D curve that can be analyzed using the osculating circles.
Mesh: Gaussian and mean curvatures

Gaussian curvature:

\[ \rho_G = \rho_{\text{max}} \times \rho_{\text{min}} \]

Mean curvature:

\[ \frac{\rho_{\text{max}} + \rho_{\text{min}}}{2} \]

→ In Riemannian geometry, many ways to define curvatures (and torsions)
Mesh: Integral Gaussian curvature/angle excess (Deviation from flatness)

\[ \int \int_{A \in T(v)} \rho_G(A) dA \simeq -\theta(v). \]

\[ \rho(v) = 2\pi - \sum_{i=1}^{\mid T(v) \mid} \theta_i. \]
Mesh: Ingredients of topology

Euler's formula is a **topological invariant**:

\[ \text{Vertices} - \text{Edges} + \text{Faces} = 2. \]

Closed triangulated manifold:

\[ \text{Vertices} \leq \frac{2}{3} \text{Edges}, \quad \text{and} \quad \text{Vertices} \leq 2 \text{Faces} - 4. \]

\[ \text{Edges} \leq 3 \text{Vertices} - 6, \quad \text{and} \quad \text{Edges} \leq 3 \text{Faces} - 6. \]

\[ \text{Faces} \leq \frac{2}{3} \text{Edges}, \quad \text{and} \quad \text{Faces} \leq 2 \text{Vertices} - 4. \]
Mesh topology: Genus, polyhedra with holes (topology=global property)

\[ \text{#Vertices} - \text{#Edges} + \text{#Faces} = 2 - 2\text{#Genus} \]
Mesh: Primal/Dual graph representations
Mesh: Primal/Dual graph representations

Primal Voronoi/Dual Delaunay triangulation
Simplicial complexes: Building blocks (LEGO-type)

(a) Simplicial complex

(b) Nonsimplicial complex
Sketching meshes: Pen computing
Meshes for the masses.
→ Difficult to design

http://www.sony CSL.co.jp/person/nielsen/PT/vteddy/vteddy-desc.html
© Frank Nielsen 2011
**Triangulation meshes:**

**Always possible in 2D (but difficult)**

**NOT Always possible in 3D!!! (require additional Steiner points)**

**SCHONHARDT’S POLYHEDRON  CHAZELLE’S POLYHEDRON**
Meshes: Procedural modeling/ city(buildings)
L-system (Lindenmayer) process

\[
\begin{align*}
\text{START} & \rightarrow A \\
A & \rightarrow B \\
B & \rightarrow AB
\end{align*}
\]

Fibonacci's sequences

<table>
<thead>
<tr>
<th>Step</th>
<th>String</th>
<th>String Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>B</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>AB</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>BAB</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>ABBAB</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>BABABBAB</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>ABBABBABABABBABABBAB</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>BABABBABABBABABBABABBAB</td>
<td>21</td>
</tr>
</tbody>
</table>

Grammar, language, parsing, etc.
L-system (Lindenmayer) process / LOGO

\[
\text{START} \rightarrow F \rightarrow F + F - -F + F
\]
Data-structures for meshes: Indexed face list

(This is the PPM equivalent for 3D objects)

Object Oriented Graphics Library (OOGL) / OFF format

```
1 OFF
2 # Geomview OOGL format cube.off
3 # # Vertices # Faces # Edge
4 8 6 0
5 # Vertex table
6 -0.500000 -0.500000 0.500000
7 0.500000 0.500000 0.500000
8 -0.500000 0.500000 0.500000
9 0.500000 0.500000 0.500000
10 -0.500000 0.500000 -0.500000
11 0.500000 0.500000 -0.500000
12 -0.500000 -0.500000 -0.500000
13 0.500000 -0.500000 -0.500000
14 # Face index table (first vertex index: 0)
15 4 0 1 3 2
16 4 2 3 5 4
17 4 4 5 7 6
18 4 6 7 1 0
19 4 1 7 5 3
20 4 6 0 2 4
```
Data-structures for meshes: Indexed face list

→ If we change vertex coordinates, all faces updated simultaneously!
Optimizing bandwidth: Triangle/quad strips

Compress mesh vertices
→ Compress mesh connectivity

\textbf{GreedyStripMesh}(\mathcal{M})
1. \textit{\langle Overview of the greedy stripping method \rangle}
2. \textbf{while} there remains triangles in \(\mathcal{M}\)
3. \textbf{do} Pick a triangle \(T\) of \(\mathcal{M}\) that has minimum number of adjacent triangles
4. \textbf{for each edge} \(e\) of \(T\), build the strip passing through \(T\) and \(e\)
5. Choose the longest strip and remove its triangles from \(\mathcal{M}\)
Many data-structures for meshes....

Winged edges
Half edges
Quad edges

A Winged Edge

Pointer References
(Discrete) Laplacian smoothing on meshes

\[ \mathbf{v} \leftarrow \mathbf{v} + \frac{\lambda}{|N(\mathbf{v})|} \sum_{i=1}^{|N(\mathbf{v})|} (N(\mathbf{v}, i) - \mathbf{v}) \]
Surface subdivision: Refining

→ Limit surface is smooth
Surface subdivision: Dr Loop scheme

New vertex creation

\[ p = \frac{1}{8} p_1 + \frac{3}{8} p_2 + \frac{1}{8} p_3 + \frac{3}{8} p_4 \]

Vertex relocation

\[ p = (1 - k\lambda)p + \lambda \sum_{i=1}^{k} N(p, i) \]

\[ \lambda = \frac{1}{k} \left( \frac{5}{8} - \left( \frac{3}{8} + \frac{1}{4} \cos \frac{2\pi}{k} \right)^2 \right) \]

Simplified to

\[ \lambda = \begin{cases} \frac{3}{16} & k = 3, \\ \frac{3}{8k} & k > 3. \end{cases} \]
**Subdivision**($\mathcal{M}, k$)

1. ◀ General subdivision framework ▶
2. ◀ Depend on whether the scheme is approximating/interpolating and primal/dual ▶
3. for $e \in \mathcal{M}$
4. do ◀ for each edge ▶
5. Create new vertex using the vertex creation mask
6. ◀ Could be several for $n$-adic subdivision ▶
7. for $v \in \mathcal{M}$
8. do Move original vertex using the vertex displacement mask
9. Reconnect all vertices as a triangular mesh based on $\mathcal{M}$

**Limit position, smoothness properties**

$$\lambda_{\infty} = \frac{1}{\frac{3}{8}\lambda + k}$$
Kobbelt's subdivision (+edge flipping)

\[ p = (1 - \lambda n)p + \lambda \sum_{i=1}^{\left| N(p) \right|} N(p, i). \]

......New Edges

\[ \lambda = \frac{4 - 2 \cos \frac{2\pi}{n}}{9n} \]

**FIGURE 5.41** The face and vertex masks for the \( \sqrt{3} \)-subdivision.
Mesh subdivision: Combinatorial explosion!

<table>
<thead>
<tr>
<th>Level</th>
<th>#Vertices</th>
<th>#Faces</th>
<th>#Edges</th>
<th>#Vertices</th>
<th>#Faces</th>
<th>#Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
<td>6</td>
<td>12</td>
<td>4</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>26</td>
<td>24</td>
<td>48</td>
<td>10</td>
<td>16</td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td>98</td>
<td>96</td>
<td>192</td>
<td>34</td>
<td>64</td>
<td>96</td>
</tr>
<tr>
<td>3</td>
<td>386</td>
<td>384</td>
<td>768</td>
<td>130</td>
<td>256</td>
<td>384</td>
</tr>
<tr>
<td>4</td>
<td>1538</td>
<td>1536</td>
<td>3072</td>
<td>514</td>
<td>1024</td>
<td>1536</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>8</td>
<td>393218</td>
<td>393216</td>
<td>786432</td>
<td>131074</td>
<td>262144</td>
<td>393216</td>
</tr>
</tbody>
</table>
Remeshing
Decimating mesh
Mesh simplification
Progressive mesh representations

Level of details/streaming...
Parameterization and texture mapping

Image texture is 2D
Parameterization and texture mapping

(a)  (b)  (c)  (d)  (e)

Minimize distortion

Atlas

Conformal mapping (preserve angles)
H- and V- representations of polytopes

Half-spaces representation

\[ P = \{ x \in \mathbb{R}^n : Ax \leq b \} \]

Vertex representation (convex hull)

\[ \bar{P} = \{ x \in \mathbb{R}^n : x = \sum_i \lambda_i x_i, \sum_i \lambda_i = 1, \lambda_i \geq 0 \} \]