# MPRI – cours 2.12.2

In order of apparition:

**F. Morain, R. Barbulescu, B. Smith**

**morain@lix.polytechnique.fr**

http://www.lix.polytechnique.fr/Labo/...
.../Francois.Morain/MPRI/2015



# I. Administrative details



## Schedule, etc.

**16 $\times$ 1.5 hour lectures:**

- F. Morain (2 lectures): groups for cryptology.
- R. Barbulescu (4 lectures): factorization and discrete logarithms.
- B. Smith (10 lectures):(hyper)elliptic curves and pairings.

See official MPRI page for more details, including dates, labs, etc.

**Internships:**

- F. MORAIN: primality proving with polynomials (AKS, Jacobi Sums, etc.);
- B. SMITH: algebraic curves, point counting algorithms.



## Expectations

- Algorithmic number theory is about algorithms of number theory and they need to be practiced (python/SAGE, Maple, Magma, pari-gp, etc.).
  Best way to realize that real computations take time and must be carefully implemented.
- Of course, we expect students to study and work/read/do exercises between lectures.

## Good reading

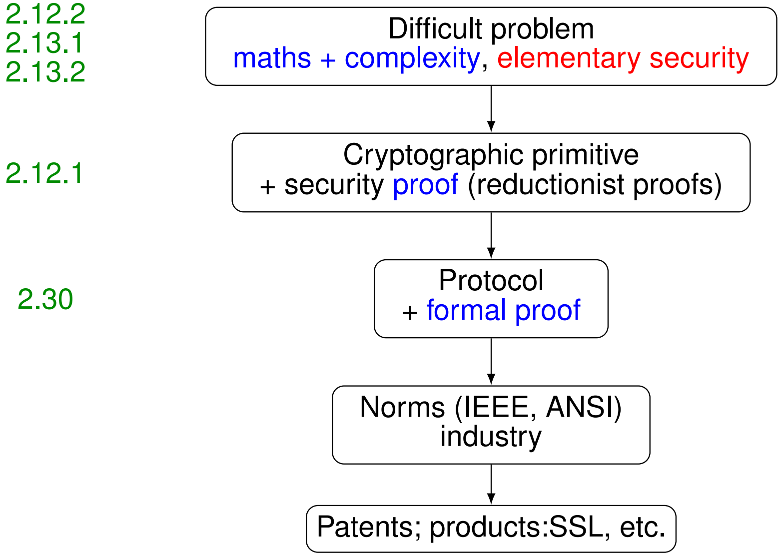

- G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Clarendon Press, 5th edition, 1985.
- D. E. Knuth. *The Art of Computer Programming: Seminumerical Algorithms*. Addison-Wesley, 2nd edition, 1981.
- H. Cohen. *A course in algorithmic algebraic number theory*, volume 138 of *Graduate Texts in Mathematics*. Springer–Verlag, 4th printing, 2000.
- P. Ribenboim. *The new book of prime number records*. Springer-Verlag, 1996.
- R. Crandall and C. Pomerance. *Primes – A Computational Perspective*. Springer Verlag, 2nd edition, 2005.
- FM. La primalité en temps polynomial [d'après Adleman, Huang; Agrawal, Kayal, Saxena]. Séminaire Bourbaki, Mars 2003.




# II. Overview of the lectures



## Goals






## Cryptographic motivations: two algorithms

### A) Diffie-Hellman

Public parameters: $p$ prime number, $g$ generator of $\mathbb{F}_p^*$.
Protocol:

$$A \overset{g^a \bmod p}{\longrightarrow} B$$

$$A \overset{g^b \bmod p}{\longleftarrow} B$$

$$A : K_{AB} = (g^b)^a \equiv g^{ab} \bmod p$$

$$B : K_{BA} = (g^a)^b \equiv g^{ab} \bmod p$$

DH problem: given $(p, g, g^a, g^b)$, compute $g^{ab}$.

DL problem: given $(p, g, g^a)$, find $a$.

**Thm.** DLP $\Rightarrow$ DHP; converse true for a large class of groups (Maurer & Wolf).

$\Rightarrow$ **Goal for us:** find a good resistant group.

## The difficulty of discrete logarithm computations

**Over finite fields:**

- $\mathbb{F}_p$:
  - Best algorithm so far: *à la* NFS $O(L_p[1/3, c'])$ (Gordon, Schirokauer).
  - record with $180dd$ (2014): Bouvier/Gaudry/Imbert/Jeljeli/Thomé (CADO-NFS), matrix $7.28 \cdot 10^6$ rows and columns.
- $\mathbb{F}_{p^n}$: Adleman-DeMarrais, function field sieve + optimizations.
  - $p = 2$: Coppersmith; $\mathbb{F}_{2^{809}}$: Gaudry *et alii* (2013).
  - record $\mathbb{F}_{3^{6\times 71}}$: Hayashi *et al.* (2010).
  - Medium $p$ case: Joux+Lercier; etc.; **lots of results in 2012-2013**; **Barbulescu/Gaudry/Thomé/Joux (2013): doable in quasipolynomial time** ⇒ see Barbulescu's part.
  - $\mathbb{F}_{p^2}$: $p$ with 90dd, Barbulescu/Gaudry/Guillevic/M. (2014)

$$L_N[\alpha, c] = \exp((c + o(1))(\log N)^\alpha (\log\log N)^{1-\alpha}).$$



## ECDLP

**ECC2K-108:** (Harley *et al.*, taken from `http://cristal.inria.fr/~harley/`)

- 1300 individuals, 9500 machines, dec 1999 until april 2000.
- 200,000 days on a 450 MHz PC with MMX, i.e. more than 500 years. For comparison, cracking a 56-bit DES key by exhaustive search would take about 110,000 days.
- $2.8 \times 10^{15}$ elliptic-curve operations of which $2.3 \times 10^{15}$ led to distinguished points recorded at INRIA; 2.05 million distinguished points in 1.3 Gigabytes of email.

**ECC112b:** taken from `http://lacal.epfl.ch/page81774.html`, Bos/Kaihara/Kleinjung/Lenstra/Montgomery (EPFL/Alcatel-Lucent Bell Laboratories/MSR)

$p = (2^{128} - 3)/(11 \cdot 6949)$, curve secp112r1

- 3.5 months on 200 PS3; $8.5 \times 10^{16}$ ec additions ($\approx$ 14 full 56-bit DES key searches); started on January 13, 2009, and finished on July 8, 2009.
- half a billion distinguished points using 0.6 Terabyte of disk space.



## ECDLP – cont'd

**ECC2K-113:** *Solving the discrete logarithm of a 113-bit Koblitz curve with an FPGA cluster*, E. Wenger & P. Wolfger, 2014.

24 days on an 18-core Virtex-6 FPGA cluster.

Hardware is fun:

- 165 MHz instead of maximum 275 MHz.
- (more or less related) one ECC-breaker per FPGA.



## As a quick comparison

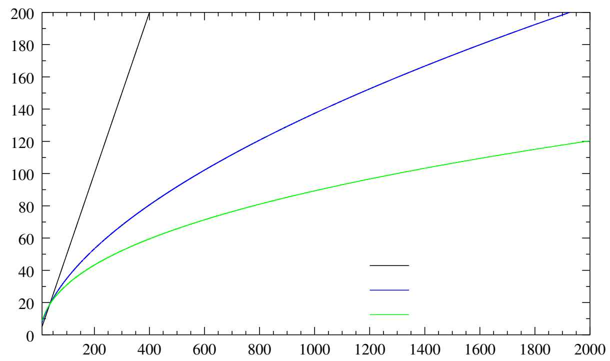



Figure: (Log of) Security vs. bit size of key (exponential, $L(1/2)$, $L(1/3)$)

$$L_x[\alpha, c] = \exp\left((c + o(1))(\log x)^\alpha (\log\log x)^{1-\alpha}\right).$$

# B) RSA

Key generation: Alice chooses two primes $p$ and $q$, $p \neq q$, $N = pq$, $e$ s.t. $\gcd(e, \lambda(N)) = 1$, $d \equiv 1/e \bmod \lambda(N)$.

Public key: $(N, e)$.

Private key: $d$ (or $(p, q)$).

Encryption: Bob recovers the authenticated public key of Alice; sends $y = x^e \bmod N$.

Decryption: Alice computes $y^d \bmod N \equiv x \bmod N$.

**Rem.** of course, in real life, more has to be done, but this has already been told somewhere else.

$\Rightarrow$ **Goal for us:** what size should $N$ have, in order not to be factored?



# Rules of the game

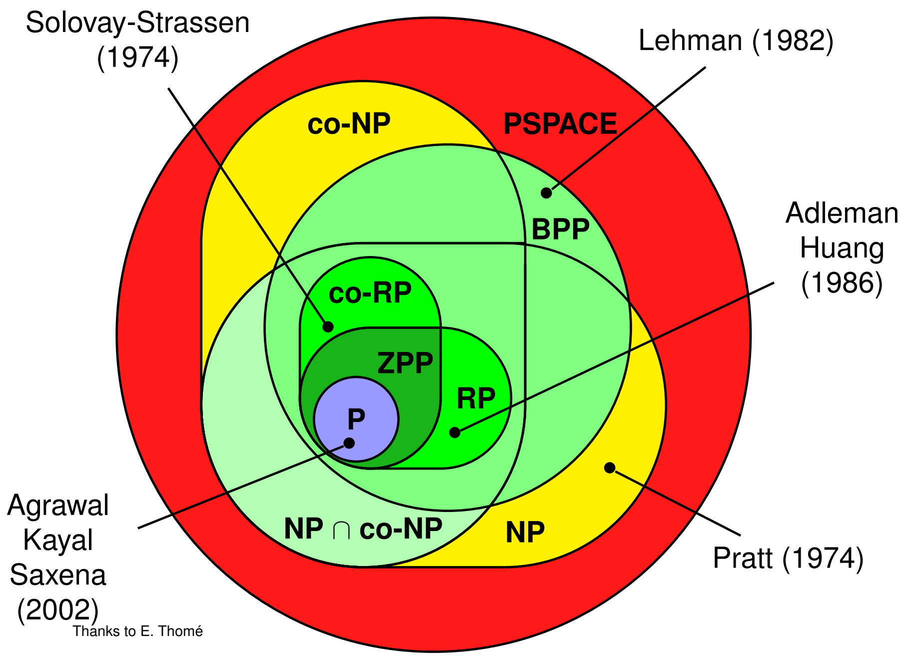
$$N = \prod_{i=1}^{k} p_i^{\alpha_i}.$$

- What do we do in practice? Which size is doable?
  Factorization : number field sieve
  $O(\exp(c(\log N)^{1/3}(\log\log N)^{2/3}))$ ; 768 bits (a lot of people, 2010).
  Primality: hopefully without too much factoring, past some easy trial division; 25,000 decimal digits.
- Complexity question: to which class does **isPrime?** belong?

Best : **P** (e.g., integer multiplication).
At least : **RP**.
And: what about a proof?



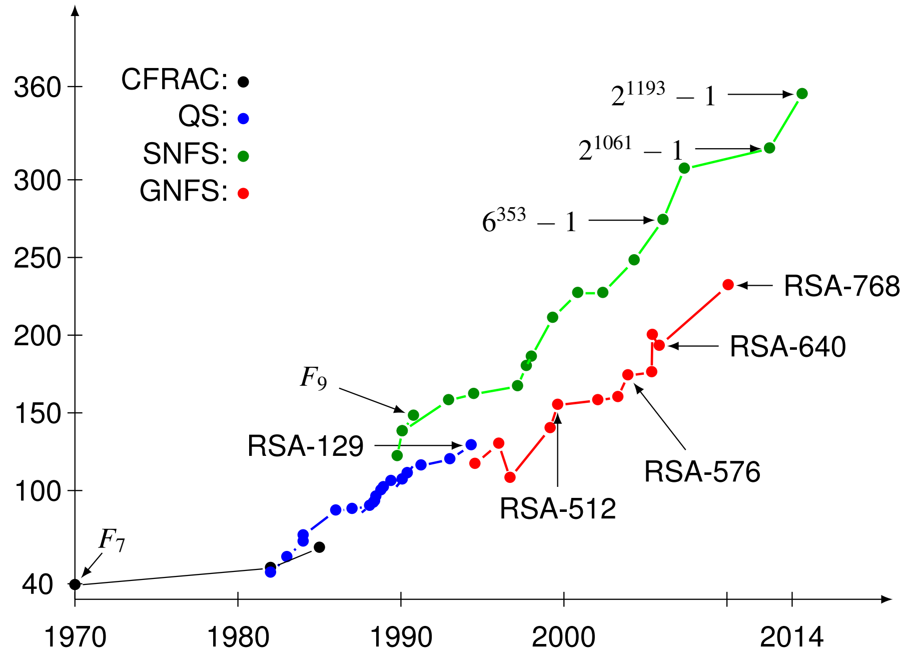
# Complexity classes



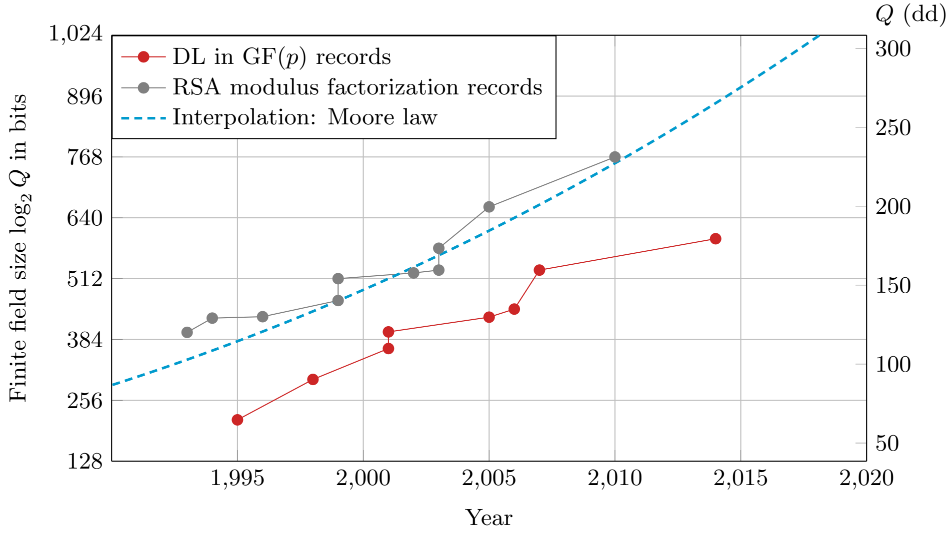

Thanks to E. Thomé



# How difficult is factoring?



# DL vs. IF


Q (dd)
DL in GF(p) records
RSA modulus factorization records
Interpolation: Moore law
Finite field size log2 Q in bits
1,024
896
768
640
512
384
256
128
300
250
200
150
100
50
1,995
2,000
2,005
2,010
2,015
2,020
Year