How Much Geometry Lies in The Laplacian?

Encoding and recovering the discrete metric on triangle meshes

Distance Geometry Workshop in Bad Honnef, November 23, 2017

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What is Geometry Processing

Broad Goals:

To create mathematical models and practical tools for digital representation, manipulation and analysis of 3D shapes.
What is a Shape?

Continuous: a surface embedded in 3D.

Discrete: a graph embedded in 3D (triangle mesh).

Common assumptions:

• Connected.
• Manifold.
• Without Boundary.
Why triangle meshes

- Functions are piecewise linear inside triangles.
- Can compute gradients.
- Edge lengths correspond to (2x2) matrices inside triangles.

Piecewise-linear functions

\( f : \mathcal{V} \to \mathbb{R} \)

K. Crane and Botsch et al.
What is a Shape?

Continuous: a surface embedded in 3D.

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5k – 200k triangles

Shapes from the SCAPE, TOSCA and FAUST datasets
Shape **Comparison**

Given two 3D shapes, quantify if they are *similar*. 
Shape Matching

Given two 3D shapes, find corresponding points.
Shape Matching

Given two 3D shapes, find intrinsically isometric correspondences.
Why Shape Matching

Given a correspondence, we also can detect and measure the areas of change:

Data from: FunEvol group (CNRS, MNHN)
Today

- Encoding shape changes
- Recovering the shape from the Laplacian-based quantities.

Main observation:

- Many tasks can be formulated through manipulation of linear operators defined on (L2) function spaces.
- Can recover the metric even from noisy data.
Sources for the talk

Map-based Exploration of Intrinsic Shape Differences and Variability
*Rustamov, O., Azencot, Ben-Chen, Chazal, Guibas,* SIGGRAPH 2013

Functional Characterization of Intrinsic and Extrinsic Geometry
*Corman, Solomon, Ben-Chen, Guibas, O.*
Transactions on Graphics 2017
Background: Functional Maps

Rather than comparing \textit{points} on objects it is often easier to compare \textit{real-valued functions} defined on them.

\footnote{Functional Maps: A Flexible Representation of Maps Between Shapes, O., Ben-Chen, Solomon, Butscher. Guibas, SIGGRAPH 2012}

\footnote{Computing and Processing Correspondences with Functional Maps, O. et al., SIGGRAPH Courses 2017}
Background: Functional Maps

Rather than comparing points on objects it is often easier to compare real-valued functions defined on them. Such maps can be represented as matrices.

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3 Computing and Processing Correspondences with Functional Maps, O. et al., SIGGRAPH Courses 2017
Background: Functional Maps

Computing functional maps is often much easier (reduces to least squares) than point-to-point maps.

In practice, can think of a functional map as an matrix of size $n_{V_2} \times n_{V_1}$.

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3 Computing and Processing Correspondences with Functional Maps, O. et al., SIGGRAPH Courses 2017
Motivation

Given a pair of shapes and a *functional* map between them, detect similarities and *differences* (distortion) across them.

Do it in a *multi-scale* way (not be sensitive to *local* changes).

Accommodate approximate *soft* (functional) maps.

Map-Based Exploration of Intrinsic Shape Differences and Variability, Rustamov, O., Azencot, Ben-Chen, Chazal, Guibas, SIGGRAPH 2013.
Shape Differences Definition

Given a functional map $C_{MN} : \mathcal{F}(M) \rightarrow \mathcal{F}(N)$ and an inner product norm: $\|f\|_M^2 = \langle f, f \rangle_M$

Define a shape difference operator as linear operator $D$, s.t.

$$\langle f, D(g) \rangle_M = \langle C_{MN}(f), C_{MN}(g) \rangle_N \quad \forall f, g$$
Shape Differences Definition

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Existence and uniqueness of $D$ is guaranteed by the Riesz representation theorem.
Shape Differences Definition

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Define a shape difference operator as linear operator \( D \), s.t.
\[
\langle f, D(g) \rangle_M = \langle C_{MN}(f), C_{MN}(g) \rangle_N \quad \forall f, g
\]

We let \( V \) and \( R \), be operators associated with \( L_2 \) and \( H_1 \) inner products:
\[
V : \langle f, g \rangle_{L_2} = \int f(x)g(x) d\mu
\]
\[
R : \langle f, g \rangle_{H_1} = \int \langle \nabla f(x), \nabla g(x) \rangle d\mu
\]
Shape Differences Definition

Given a functional map \( C_{MN} : \mathcal{F}(M) \rightarrow \mathcal{F}(N) \)
and an inner product norm: \( \|f\|_M^2 = <f, f>_M \)

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We let \( V \) and \( R \), be operators associated with \( L_2 \) and \( H_1 \) inner products:
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V : <f, g>_{L_2} = \int f(x)g(x)d\mu \\
R : <f, g>_{H_1} = \int \langle \nabla f(x), \nabla g(x) \rangle d\mu = <f, \Delta g>_{L_2}
\]
Shape Differences Definition

Given a functional map $C_{MN} : \mathcal{F}(M) \to \mathcal{F}(N)$
and an inner product norm: $\|f\|_M^2 = < f, f >_M$

Define a shape difference operator as linear operator $D$, s.t.

$$\langle f, D(g) \rangle_M = \langle C_{MN}(f), C_{MN}(g) \rangle_N \quad \forall f, g$$

We let $V$ and $R$, be operators associated with $L_2$ and $H_1$
inner products. In the discrete setting, reduces to simply
matrix transposes and inverses:

$$< f, g >_{L_2} = f^T A g$$
$$< f, g >_{H_1} = f^T L g$$
Shape Differences Properties

Theorem:

If $C_{MN}$ comes from a point to point map, then:

1) $V = Id$ if and only if the map is area-preserving.
2) $R = Id$ if and only if the map is conformal.

1) $\langle f, g \rangle_{L^2(M)} = \langle C_{MN}(f), C_{MN}(g) \rangle_{L^2(N)} \quad \forall f, g$

2) $\langle f, g \rangle_{H^1(M)} = \langle C_{MN}(f), C_{MN}(g) \rangle_{H^1(N)} \quad \forall f, g$
Shape Differences in Collections

Since shape differences $D_{M,N1}, D_{M,N2}$ are operators with the same domain/range, we can compare distortion on multiple shapes.
Since shape differences $D_{M,N_1}, D_{M,N_2}$ are operators with the same domain/range, we can compare distortion on multiple shapes.
Comparing Shape Differences

Find a shape $D_i$, such that the difference between shapes $B$ and $D_i$ is as close as possible to the difference between $A$ and $C_i$. 

Map-Based Exploration of Intrinsic Shape Differences and Variability, Rustamov, O., Azencot, Ben-Chen, Chazal, Guibas, SIGGRAPH 2013
Recap

Shape differences represent the distortion as a pair of linear operators, defined via:

\[
< f, D(g) >_M = < F(f), F(g) >_N \quad \forall \ f, g
\]
Question

How much information is contained in these operators?

Theorem:

If $F$ comes from a point map:

$$R = Id, \quad \text{and} \quad V = Id$$

If and only if the map is an intrinsic isometry.

$$V : \int_M f g d\mu^M = \int_N f g d\mu^N$$

$$R : \int_M <\nabla f, \nabla g> \mu^M = \int_N <\nabla f, \nabla g> \mu^N$$
Can we recover the metric?

Given a base shape $M$ and two shape difference operators, can we recover the target shape?

Functional Characterization of Intrinsic and Extrinsic Geometry
Corman, Solomon, Ben-Chen, Guibas, O. TOG 2017
Can we recover the metric?

Given a base shape M and two shape difference operators, can we recover the target shape?

Possible limitation:

Shape difference operators are blind to isometric deformations.

\[ V : <f, g> = \int f(x)g(x)d\mu(x) \]
\[ R : <f, g> = \int \langle \nabla f(x), \nabla g(x) \rangle d\mu(x) \]

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Possible limitation:
Shape difference operators are blind to isometric deformations.

Best hope:
Recover the metric and solve for the pose.

Functional Characterization of Intrinsic and Extrinsic Geometry
Corman, Solomon, Ben-Chen, Guibas, O. TOG 2017
A metric on the triangle mesh

From metric to inner products on a triangle mesh:

Given the inner product between every pair of functions can we recover the metric? Probably\(^1,2\)

When the information is exact

\(^1\text{Zeng et al.} \textit{Discrete heat kernel determines discrete Riemannian metric.} \text{Graph. Models,} 2012\)
\(^2\text{De Goes et al.} \textit{Weighted triangulations for geometry processing,} \text{TOG,} 2014\)
A metric on the triangle mesh

From metric to inner products on a triangle mesh:

\[ L_{ij} = \langle \nabla e_i, \nabla e_j \rangle = \frac{1}{2} \cot(\alpha_{ij}) + \frac{1}{2} \cot(\beta_{ij}) \]

Given the Laplacian of a shape can we recover the metric?

- What if it is known approximately?
- Using Shape Difference Operators?

De Goes et al. Weighted triangulations for geometry processing, TOG, 2014
Recovering the metric

From metric to inner products on a triangle mesh:

Theorem:

Given the two shape difference operators, the discrete metric can be recovered by solving 2 linear systems that are "almost always" full-rank.

Functional Characterization of Intrinsic and Extrinsic Geometry
Solomon, Corman, Ben-Chen, Guibas, O. TOG 2017
A metric on the triangle mesh

Alternative expression for the cotangent weights:

\[
\langle \nabla e_i, \nabla e_j \rangle = \frac{1}{2} \cot(\alpha_{ij}) + \frac{1}{2} \cot(\beta_{ij}) = \frac{1}{8A_1}(d_0^2 - d_1^2 - d_2^2) + \frac{1}{8A_2}(d_0^2 - d_3^2 - d_4^2)
\]

Re-write the weights in terms of edge lengths.

Boscaini et al. *Shape-from-operator: Recovering shapes from intrinsic operators*, CGF, 2015
Corman et al. *Functional Characterization of Intrinsic and Extrinsic Geometry*, TOG 2017
Recovering the metric

Theorem:

\[ <e_i, e_j> = \frac{1}{12} (A_1 + A_2) \]

\[ <\nabla e_i, \nabla e_j> = \frac{1}{8A_1} (d_0^2 - d_1^2 - d_2^2) \]
\[ + \frac{1}{8A_2} (d_0^2 - d_3^2 - d_4^2) \]

- The areas are linear in the L2 inner product and for fixed areas, the squared edge lengths are linear in the H1 inner product.
- The resulting linear systems are generically invertible.
Recovering the metric

From Laplacian to the metric:

Theorem:

The edge lengths can be recovered via two linear systems from two matrices of inner products (functions and gradients = cotangent weights), Both are generically invertible.
Recovering the metric

\[ <e_i, e_j> = \frac{1}{12} (A_1 + A_2) \]
\[ <\nabla e_i, \nabla e_j> = \frac{1}{8A_1} (d_0^2 - d_1^2 - d_2^2) + \frac{1}{8A_2} (d_0^2 - d_3^2 - d_4^2) \]

- The areas are linear in the L2 inner product and for fixed areas, the *squared edge lengths* are linear in the H1 inner product.
- The resulting linear systems are *generically* invertible.
- Can be phrased as a least squares problem even if matrices are noisy/functions are in a different basis.
Recovering the metric

The areas are linear in the L2 inner product and for fixed areas, the squared edge lengths are linear in the H1 inner product.

The resulting linear systems are generically invertible.

\[ \langle \nabla e_i, \nabla e_j \rangle = \frac{1}{8A_1}(d_0^2 - d_1^2 - d_2^2) + \frac{1}{8A_2}(d_0^2 - d_3^2 - d_4^2) \]

A mesh for which \( C(l^2; \mu) \) is not invertible when \( \mu = 1 \).

- The areas are linear in the L2 inner product and for fixed areas, the squared edge lengths are linear in the H1 inner product.
- The resulting linear systems are generically invertible.
Enforcing the Triangle Inequality

\[ \langle \nabla e_i, \nabla e_j \rangle = \frac{1}{8 A_1} (d_0^2 - d_1^2 - d_2^2) + \frac{1}{8 A_2} (d_0^2 - d_3^2 - d_4^2) \]

- Regularization, for noisy/incomplete linear systems:
  \[
  E = \frac{1}{2} \begin{pmatrix}
    2x_1 & x_3 - x_1 - x_2 & x_2 - x_1 - x_3 \\
    x_3 - x_1 - x_2 & 2x_2 & x_1 - x_2 - x_3 \\
    x_2 - x_1 - x_3 & x_1 - x_2 - x_3 & 2x_3
  \end{pmatrix}
  \]
  Is positive semi-definite if and only if \( x_1, x_2, x_3 \) are non-negative and their square roots satisfy the triangle inequality.
Problem:
Given a triangle mesh with approximate edge lengths
Recover the embedding.

Main idea: deform the triangles to match the target metric.

$$\mathcal{E}(p') = \sum_{t \in \mathcal{M}} \min_{Q_t \in SO(3)} A_t \left\| J_t(p') - Q_t \tilde{W}_t^{-1} \right\|_F^2$$

Iterate between computing $p'$ and $Q_t$.

Panozzo et al., *Frame Fields: Anisotropic and Non-Orthogonal Cross Fields*, SIGGRAPH 2014
Recovering the shape

With only the edge-lengths, there are multiple near-isometries. Recovering the exact pose is hard.

Source | Target | Intrinsic

Functional Characterization of Intrinsic and Extrinsic Geometry
Solomon, Corman, Ben-Chen, Guibas, O. Conditionally accepted at TOG 2016
Extrinsic Information

Can we add additional extrinsic information? Encode the *second fundamental form*?

**One Option:**
Use dihedral angles to represent encode principal curvatures.

**Difficulty:**
Angle-based values are both unstable and difficult to recover in the presence of noise.

Second Fundamental Form is a *quadratic form*, not an angle.
Extrinsic Information

Can we add additional extrinsic information? Encode the second fundamental form?

Main idea: offset surfaces.

Given a family of immersions, where each point follows the outward normal direction:

\[
\frac{\partial g}{\partial t} \bigg|_{t=0} = 2h \bigg|_{t=0} \quad \text{and} \quad \frac{\partial \mu}{\partial t} \bigg|_{t=0} = H \mu,
\]

- \(g\): Metric (first fundamental form)
- \(h\): Second fundamental form
- \(\mu\): Local area
- \(H\): Mean curvature
Shape Differences Based on Offset Surfaces

Given two shapes, compute four difference operators: two between the shapes, and two between their offsets.

\[ V_{M^o, N^o}, R_{M^o, N^o} \]

\[ V_{M, N}, R_{M, N} \]

\[ V_{M, N}, R_{M, N} \] encode change in metric,

\[ V_{M^o, N^o}, R_{M^o, N^o} \] encode change in curvature
Exploring shapes with extrinsic information

Theorem: PCA of various shape difference operators

PCoA of various shape difference operators
Reconstruction from shape differences

Consequence:

Given the *four* shape difference operators, the shape can be recovered by solving 4 linear systems of equations.

Shape reconstruction can be phrased as reconstruction based on lengths of tetrahedra.
Reconstruction from shape differences

Consequence:

An operator view:
The shape is fully encoded by two operators for the first and two for the second fundamental forms.

A coherent, parallel theory in the continuous and discrete case.
Shape Recovery from operators
Shape Recovery from operators

Can use the pipeline for interpolation/extrapolation, even with different connectivity.
Shape Recovery from operators

Source  | Target  | $d_H = 0.119$  | $k_M = 20$  | $d_H = 0.069$  | $k_M = 40$  | $d_H = 0.064$  | $k_M = 60$  | $d_H = 0.036$  | $k_M = 80$  | $d_H = 0.023$  | $k_M = 100$

|       |         |         |         |         |         |         |         |         |         |         |         |
Laplacian-based methods can be used for both similarity and difference (distortion).

Can recover the metric from a Laplacian even in a noisy/approximate case.

Shapes can be represented as sets of linear operators and recovered via “simple” optimization problems.

Second fundamental form encoded via offsets.
Thank you!

Questions?