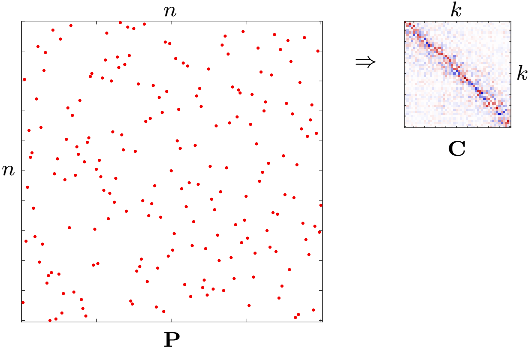
Point-wise map recovery

Task: Recover a point-to-point map from its functional representation



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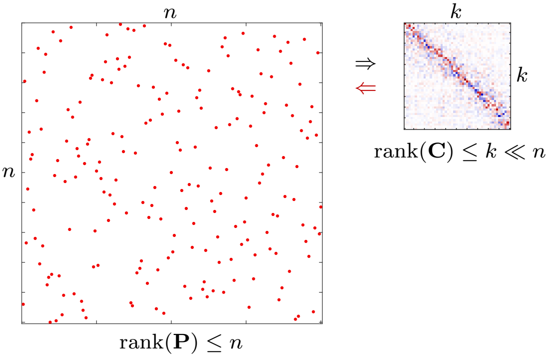

- The inverse problem is highly underdetermined

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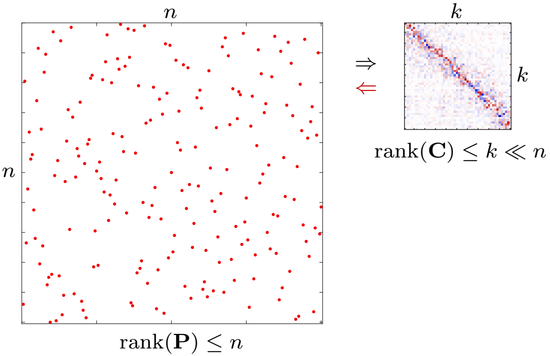
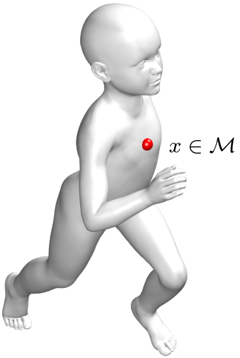


- The inverse problem is highly underdetermined
- Need to use priors on the expected structure of the underlying map (e.g. bijectivity, smoothness, partiality, etc.)

Mapping delta functions

Algorithm:

- For each $x \in \mathcal{M}$ construct the delta function $\delta_x : \mathcal{M} \to \mathbb{R}$



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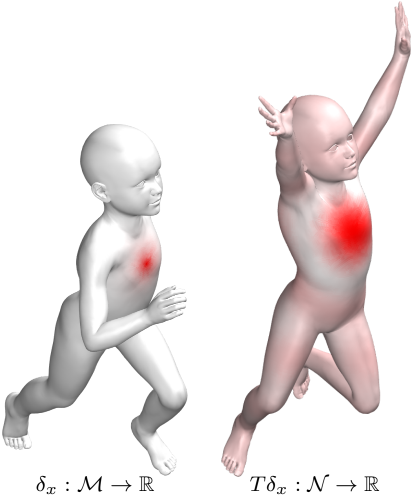

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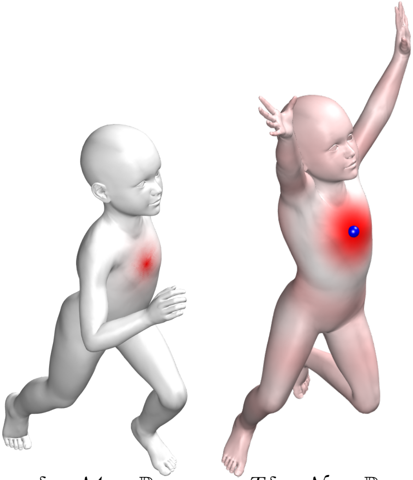

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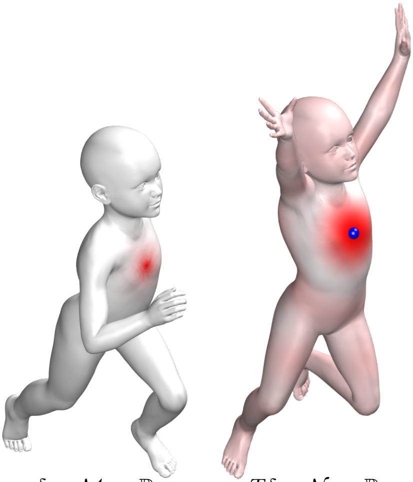
Doing this for all points is costly

The maximum is delocalized due to the band-limited approximation of T!



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Linear assignment problem

For orthogonal bases $\mathbf{\Phi}_\mathcal{M}, \mathbf{\Phi}_\mathcal{N}$ we can write

$$\mathbf{C} = \mathbf{\Phi}_\mathcal{N}^\top \mathbf{P} \mathbf{\Phi}_\mathcal{M}$$

If the underlying map is known to be bijective, solve the LAP:

$$\min_{\mathbf{\Pi} \in \{0,1\}^{n \times n}} -\langle \mathbf{\Pi}, \mathbf{\Phi}_\mathcal{N} \mathbf{C} \mathbf{\Phi}_\mathcal{M}^\top \rangle \quad \text{s.t. } \mathbf{\Pi}^\top \mathbf{1} = \mathbf{1}\,,\ \mathbf{\Pi}\mathbf{1} = \mathbf{1}$$

Rodolà et al. 2015

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The delta function δ_x has coefficients $a_i = \phi_i^{\mathcal{M}}(x)$ (a column of $\mathbf{\Phi}_{\mathcal{M}}^{\top}$)
$\Rightarrow$ the image of all delta functions on $\mathcal{M}$ is simply $\mathbf{C}\mathbf{\Phi}_{\mathcal{M}}^{\top}$

Rodolà et al. 2015; Ovsjanikov et al. 2012

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Interpretation: Seek the permutation aligning the k-dimensional spectral embeddings $\mathbf{C}\mathbf{\Phi}_\mathcal{M}^\top$ and $\mathbf{\Phi}_\mathcal{M}^\top$ in the ℓ^2 sense

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- Inefficient for large shapes
- Lack of desirable properties on the recovered map (e.g. smoothness)

Rodolà et al. 2015; Ovsjanikov et al. 2012

Nearest neighbors

Relaxing bijectivity to stochasticity constraints:

$$\min_{\mathbf{P}\in\{0,1\}^{n\times m}} \|\mathbf{C}\mathbf{\Phi}_{\mathcal{M}}^{\top} - \mathbf{\Phi}_{\mathcal{N}}^{\top}\mathbf{P}\|_{\mathrm{F}}^{2} \quad \text{s.t. } \mathbf{P}^{\top}\mathbf{1} = \mathbf{1}$$

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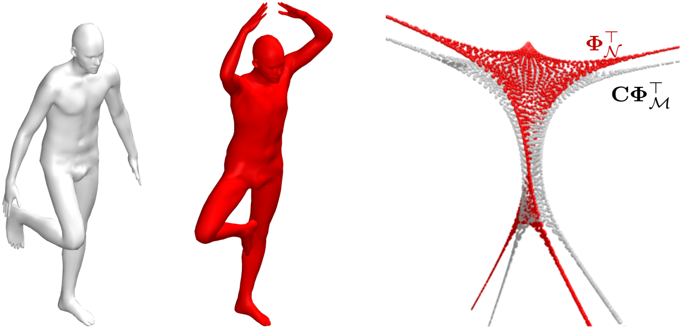
Can be solved efficiently by a nearest-neighbor search in $\mathbb{R}^k$



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Orthogonal refinement (ICP)

Orthogonal $\mathbf{C} \Leftrightarrow$ Area-preserving map

Idea: Treat $\mathbf{C}$ as a pre-alignment, do orthogonal refinement to improve map quality

Ovsjanikov et al. 2012

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Algorithm (**ICP**):

$\mathbf{P}$-step (nearest neighbors):

$$\mathbf{P}^* = \arg \min_{\mathbf{P} \in \{0,1\}^{n \times m}} \|\mathbf{C}^* \mathbf{\Phi}_{\mathcal{M}}^\top - \mathbf{\Phi}_{\mathcal{N}}^\top \mathbf{P}\|_{\mathrm{F}}^2 \quad \text{s.t. } \mathbf{P}^\top \mathbf{1} = \mathbf{1}$$

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$\mathbf{C}$-step (orthogonal Procrustes):

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Ovsjanikov et al. 2012

Non-orthogonal refinement (CPD)

For more general deformations (e.g. non-area preserving, non-isometric), do non-rigid refinement:

$$\min_{\mathbf{P}\in\{0,1\}^{n\times m}} \quad D_{\mathrm{KL}}(\mathbf{C}\boldsymbol{\Phi}_{\mathcal{M}}^\top, \boldsymbol{\Phi}_{\mathcal{N}}^\top\mathbf{P}) + \lambda \, \|\mathbf{C}\boldsymbol{\Phi}_{\mathcal{M}}^\top - \boldsymbol{\Phi}_{\mathcal{N}}^\top\mathbf{P}\|_\Omega^2$$

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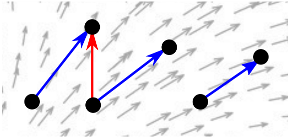
- $\|\cdot\|_{\Omega}^{2}$ promotes smooth displacements



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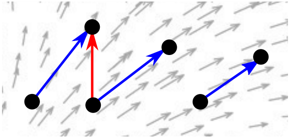
- $\|\cdot\|_\Omega^2$ promotes smooth displacements
- λ controls the regularity (rigid for $\lambda \to \infty$)



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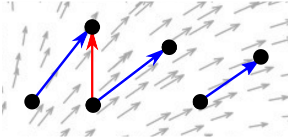
- $\|\cdot\|_{\Omega}^{2}$ promotes smooth displacements
- λ controls the regularity (rigid for $\lambda \to \infty$)
- Solved by coherent point drift



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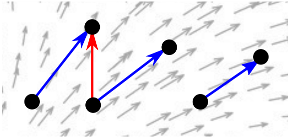
- $\|\cdot\|_\Omega^2$ promotes smooth displacements
- λ controls the regularity (rigid for $\lambda \to \infty$)
- Solved by coherent point drift
- Does not scale well



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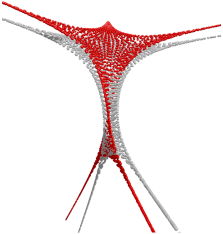
Comparison



Reference

Nearest neighbors

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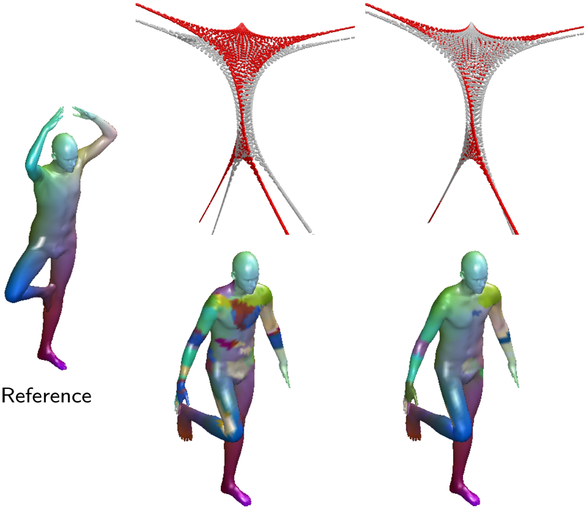

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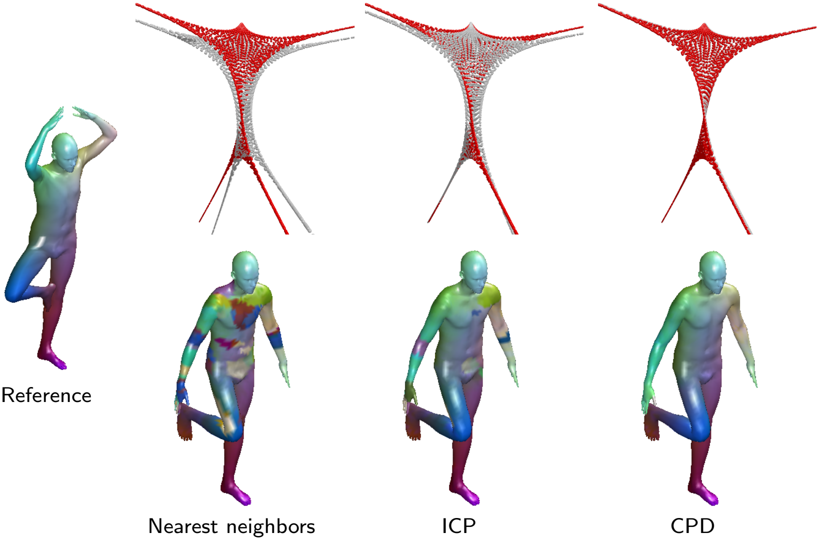

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Product manifold filter (PMF)

Given $\mathbf{P}_0$ point-to-point (e.g. from nearest-neighbors), consider the LAP:

$$\max_{\mathbf{\Pi}\in\{0,1\}^{n\times n}} \mathrm{trace}(\mathbf{\Pi}^\top \mathbf{K}_\mathcal{M} \mathbf{P}_0 \mathbf{K}_\mathcal{N}^\top) \quad \text{s.t. } \mathbf{\Pi}^\top \mathbf{1} = \mathbf{1}\,,\ \mathbf{\Pi}\mathbf{1} = \mathbf{1}$$

with $\mathbf{K}_\mathcal{M} = \exp(-\mathbf{D}^2_\mathcal{M}/\sigma^2)$ and $\mathbf{K}_\mathcal{N} = \exp(-\mathbf{D}^2_\mathcal{N}/\sigma^2)$

Vestner, Rodolà, Litman, Bronstein, Cremers 2016

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- Interpretation as an inference problem from stochastic data

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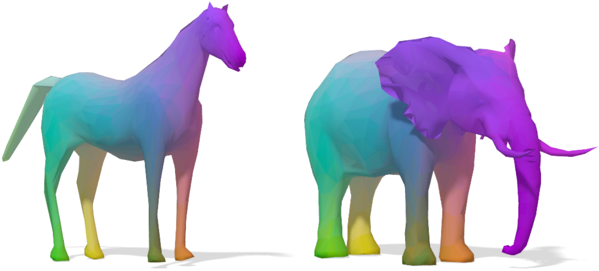
- Interpretation as an inference problem from stochastic data
- The similarity induces smooth maps



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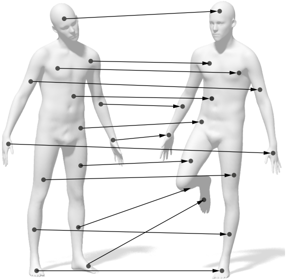

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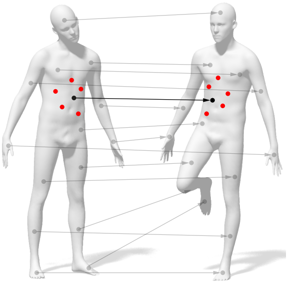

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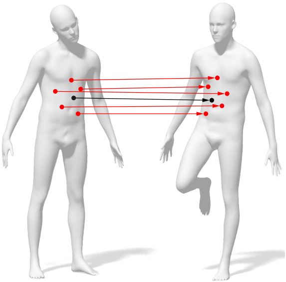

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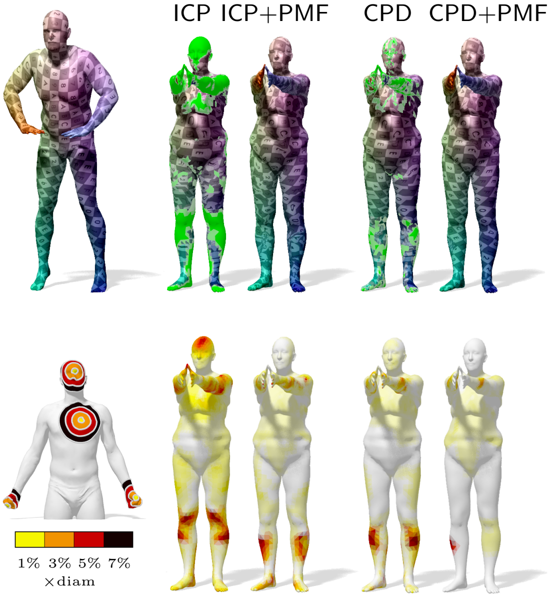
Examples

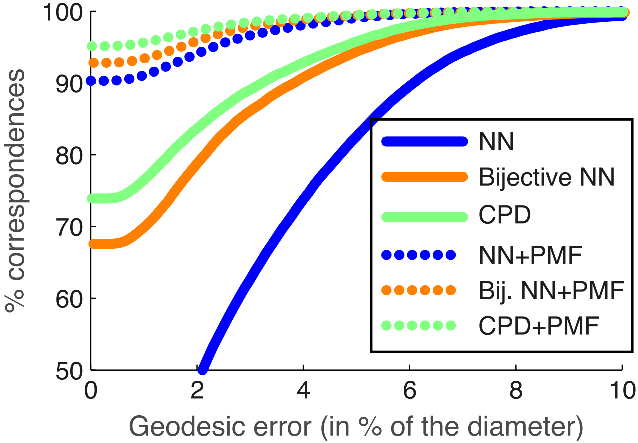
Vestner, Rodolà, Litman, Bronstein, Cremers 2016

Examples



Vestner, Rodolà, Litman, Bronstein, Cremers 2016

Comparison



Vestner, Rodolà, Litman, Bronstein, Cremers 2016

Application: Point-to-point map improvement

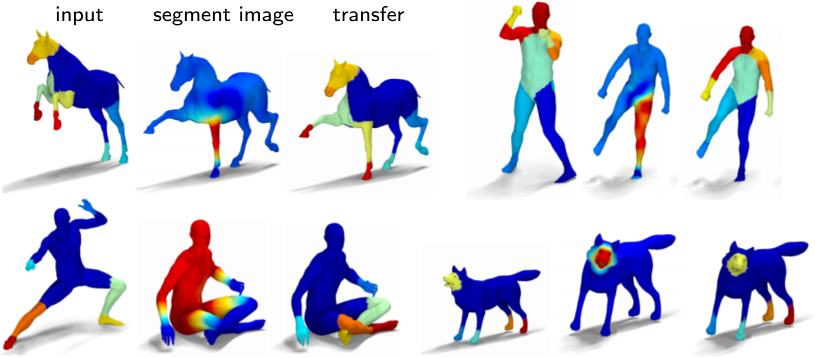
Refinement can be used to improve noisy maps obtained with any point-wise matching pipeline

reference input improved

Kim et al. 2011; Ovsjanikov et al. 2012

Application: Segmentation transfer

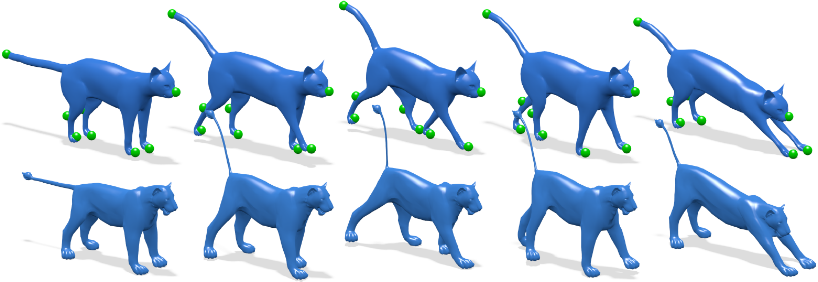
Transfer indicator functions for each segment, without resorting to a point-to-point correspondence



Ovsjanikov et al. 2012

Application: Simultaneous shape editing

Coupled bases allow to solve for the deformation field in the functional domain, and transfer pose to multiple shapes simultaneously



Kovnatsky, Bronstein, Bronstein, Glashoff, Kimmel 2013; Rong et al. 2008

Application: Partial scanning

Litany, Rodolà, Bronstein, Bronstein, Cremers 2016

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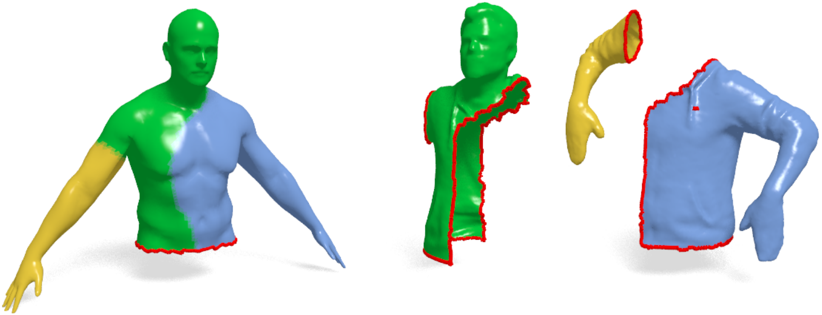

Litany, Rodolà, Bronstein, Bronstein, Cremers 2016

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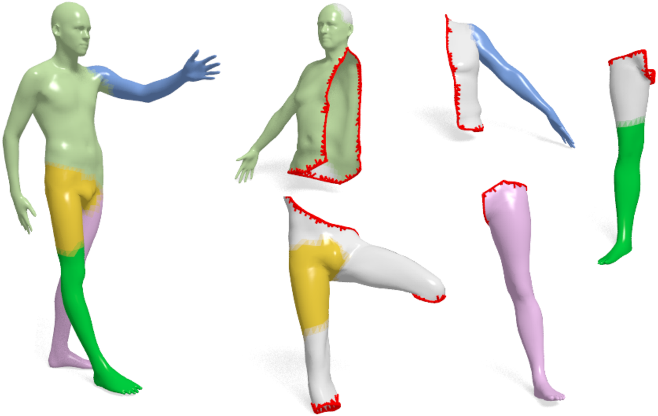

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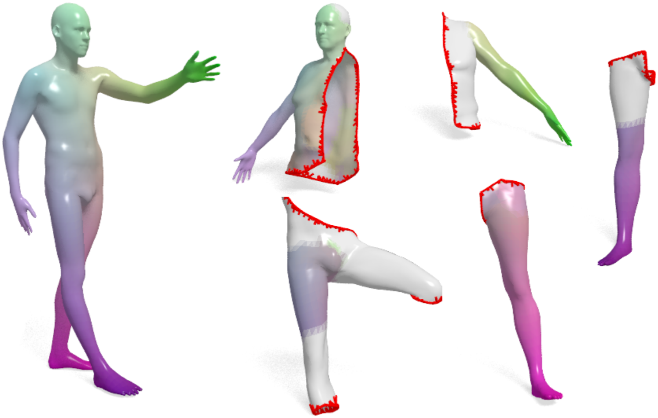
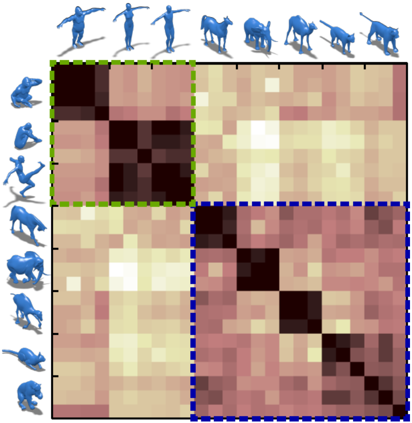


Litany, Rodolà, Bronstein, Bronstein, Cremers 2016

Application: Shape retrieval

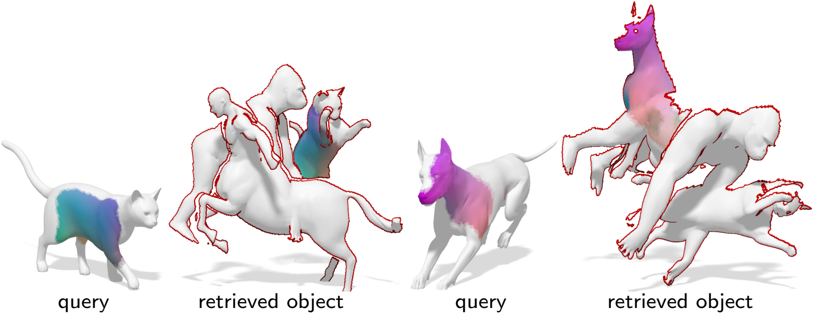
The average ratio of the norms of the diagonal and off-diagonal elements of $\mathbf{C}$ can be used as a global similarity criterion



Kovnatsky, Bronstein, Bronstein, Glashoff, Kimmel 2013

Application: Object detection and recognition

The final energy can be used as an indicator that the object is present in the scene; localization does not require a point-wise correspondence



Cosmo, Rodolà, Masci, Torsello, Bronstein 2016

Application: Improving map collections

- The pairwise maps can be improved by considering the context

Nguyen et al. 2011; Ovsjanikov, Ben-Chen, Solomon, Butscher, Guibas 2012; Kovnatsky, Glashoff, Bronstein 2016

Application: Improving map collections

- The pairwise maps can be improved by considering the context
- Compose maps along cycles

$$m_{X,Y} = m_{Z,Y} \circ m_{X,Z} = \mathbf{C}_{Z,Y}\mathbf{C}_{X,Z}$$

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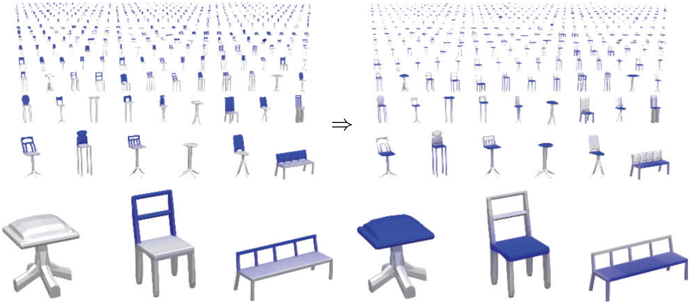
- Replace faulty maps with composites along shortest paths
- Optimize over cycle-consistent functional maps

$$\min_{\mathbf{C}} \|\mathbf{C}\|_* + \lambda \sum_{(i,j)\in\mathcal{G}} \|\mathbf{C}_{ij}\mathbf{A}_{ij} - \mathbf{B}_{ij}\|_{2,1}$$

Nguyen et al. 2011; Ovsjanikov, Ben-Chen, Solomon, Butscher, Guibas 2012; Kovnatsky, Glashoff, Bronstein 2016; Huang, Wang, Guibas 2014

Application: Analysis of shape collections

Find structural similarities in heterogeneous shape collections



consistent basis functions

Huang, Wang, Guibas 2014

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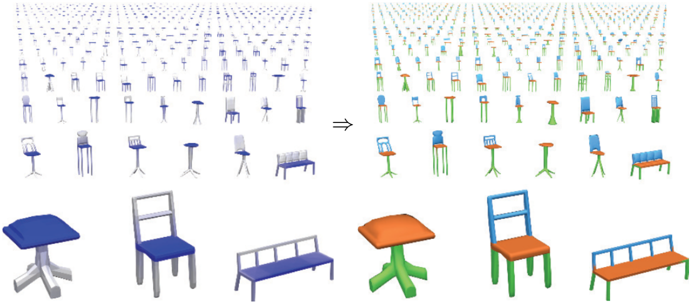


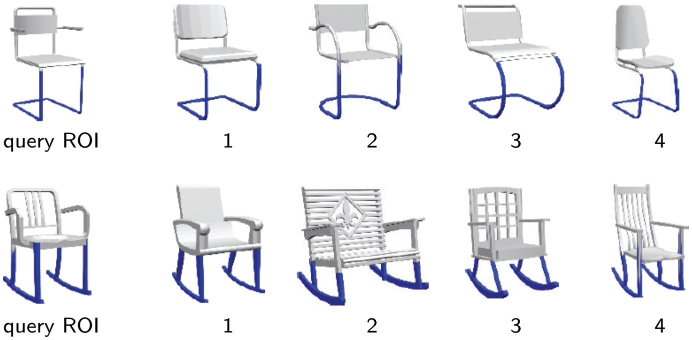
co-segmentation

Huang, Wang, Guibas 2014

Application: Analysis of shape collections

With shape differences we can compare shapes as well as deformations, and therefore

- find similar shapes



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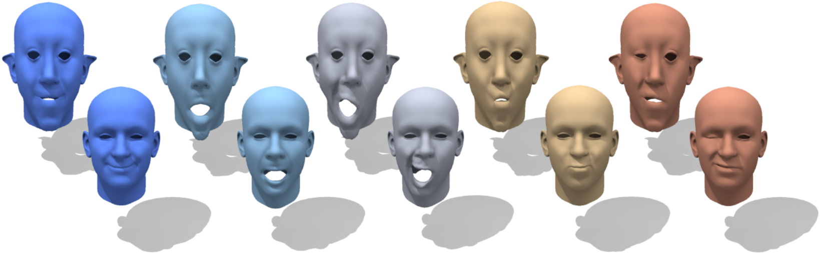

Huang, Wang, Guibas 2014; Rustamov, Ovsjanikov, Azencot, Ben-Chen, Chazal, Guibas 2013

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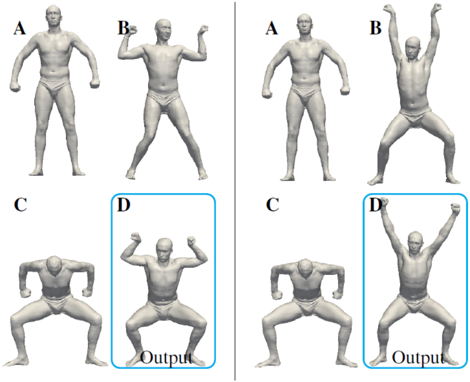

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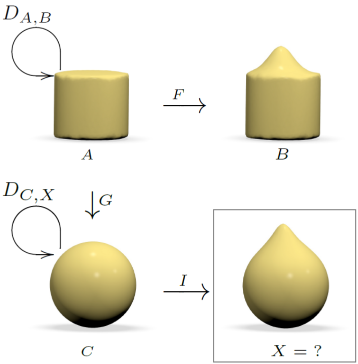

Huang, Wang, Guibas 2014; Rustamov, Ovsjanikov, Azencot, Ben-Chen, Chazal, Guibas 2013; Boscaini, Eynard, Kourounis, Bronstein 2015

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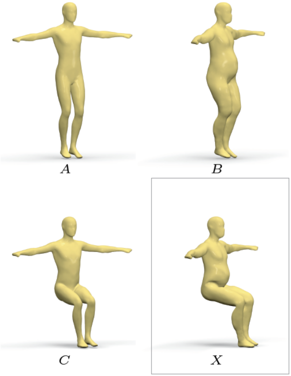
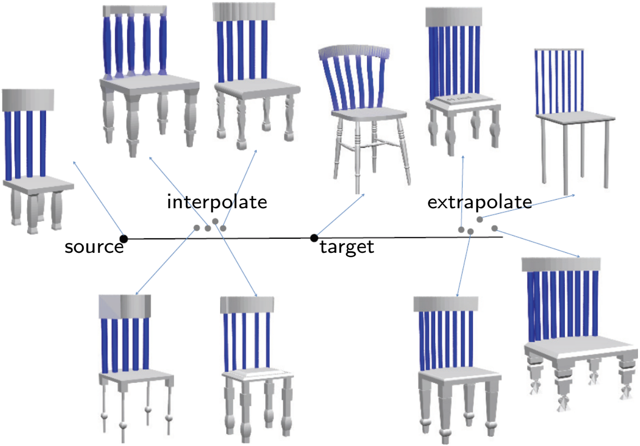


Huang, Wang, Guibas 2014; Rustamov, Ovsjanikov, Azencot, Ben-Chen, Chazal, Guibas 2013; Boscaini, Eynard, Kourounis, Bronstein 2015

Application: Shape exploration

Using shape differences we can interpolate/extrapolate the difference between corresponding regions of a shape pair



Huang, Wang, Guibas 2014

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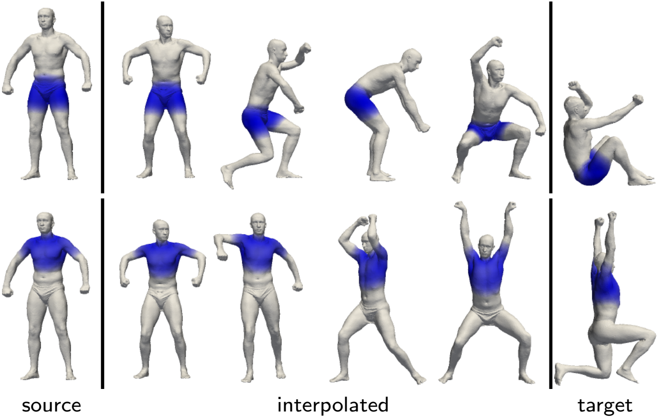
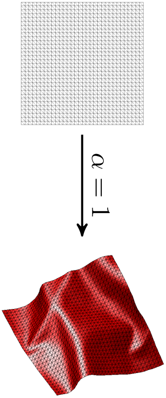


Huang, Wang, Guibas 2014; Rustamov, Ovsjanikov, Azencot, Ben-Chen, Chazal, Guibas 2013

Application: Shape interpolation

Linear interpolation in shape differences space:

$$D_\alpha = (1-\alpha)I + \alpha D$$



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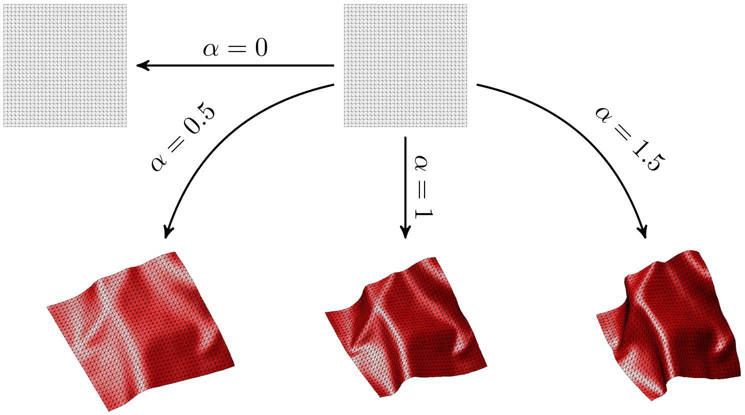
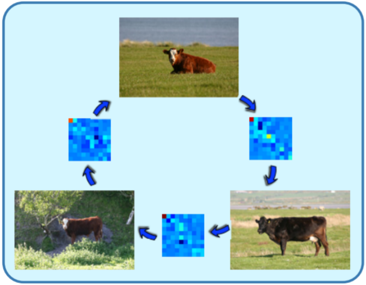
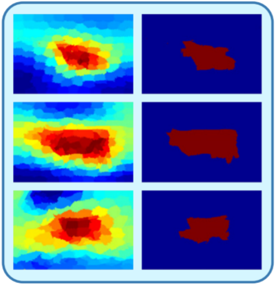


Application: Image co-segmentation

Network of maps can be used for co-segmentation of images



consistent functional maps



segmentations

Wang, Huang, Guibas 2013; Wang, Huang, Ovsjanikov, Guibas 2014

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Wang, Huang, Guibas 2013; Wang, Huang, Ovsjanikov, Guibas 2014

Things we did not cover

- Point-wise map recovery by vector field flow
- Point-wise map recovery for partial functional maps
- Functional fluids
- Visualization and analysis of functional maps
- Matching via quotient spaces
- Permuted sparse coding
- Functional correspondence via matrix completion
- Coupled functional maps
- Functional maps for image data
- ...

Corman et al. 2015; Rodolà et al. 2016; Azencot et al. 2015; Vantzos et al. 2016; Ovsjanikov et al. 2013; Pokrass et al. 2013; Kovnatsky et al. 2015; Eynard et al. 2016; Wang et al. 2013

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Summary

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- Very easy to pick up!

Summary

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Code and course notes available at:
`http://www.lix.polytechnique.fr/~maks/fmaps_course/`

Thank you!