Computing and Processing Correspondences with Functional Maps

SIGGRAPH 2017 course

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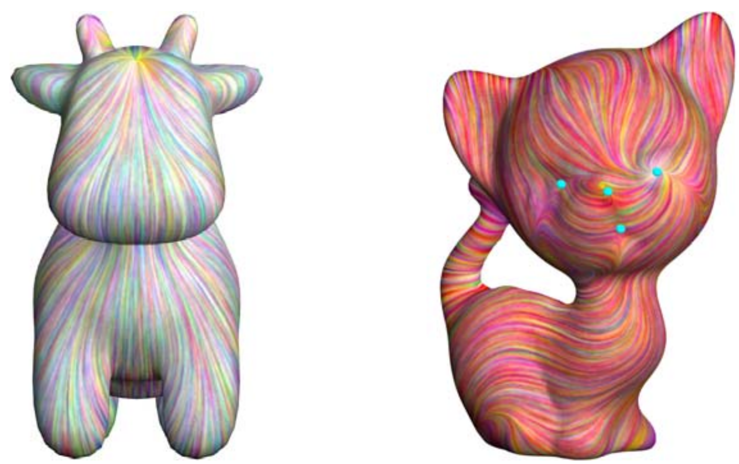
Functional Vector Fields



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European Research Council
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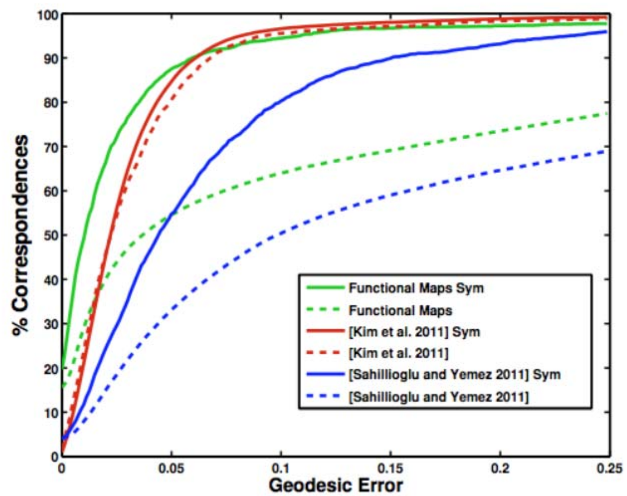
So far

Computing and analyzing FMAPs

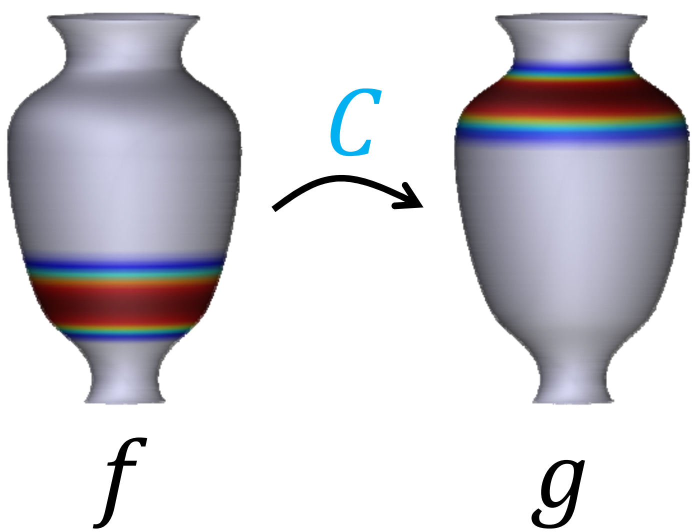


Using FMAPs for function transfer

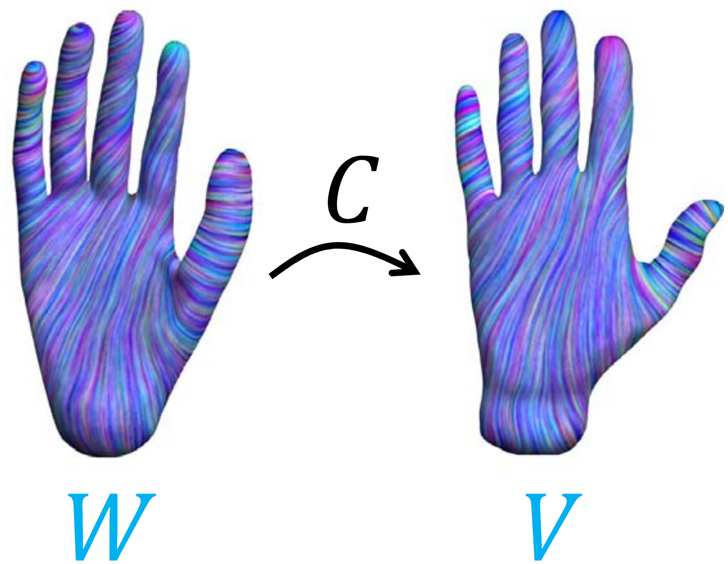
[Ovsjanikov, BC, Solomon, Butscher, Guibas, SIGGRAPH 2012]

This part

Computing FMAPs using vector fields



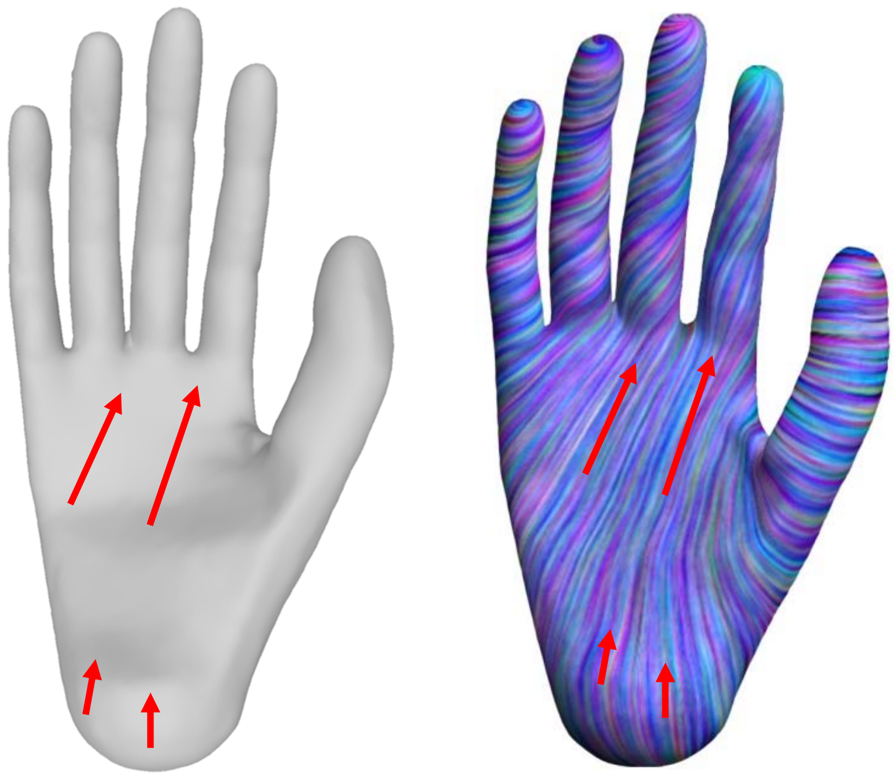
Using FMAPs for vector field transfer



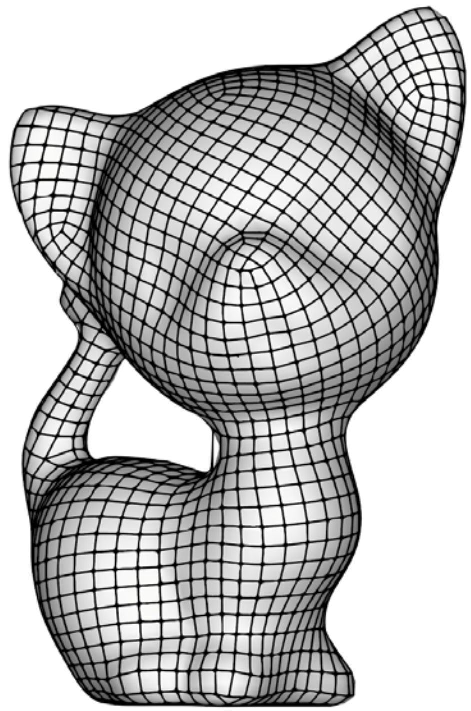
[Azencot, BC, Chazal, Ovsjanikov, SGP 2013]

What are vector fields?

- Smooth assignment of “arrow” per point
- Only tangent vector fields
- Visualize with texture
  - Can see direction but not length



Vector fields and maps – symbiosis

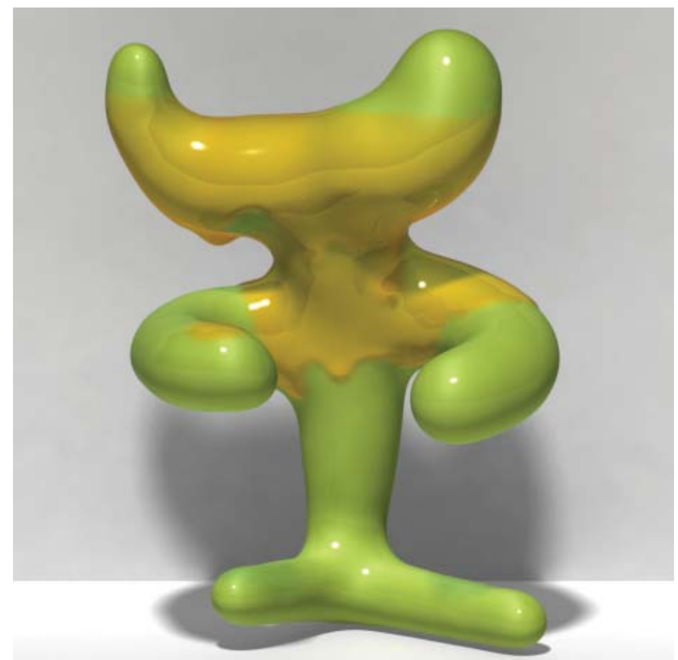


Quad remeshing

Azencot, Corman, BC, Ovsjanikov, SIGGRAPH 2017

Map Improvement

Corman, Ovsjanikov, Chambolle, SGP 2015



Fluid Simulation

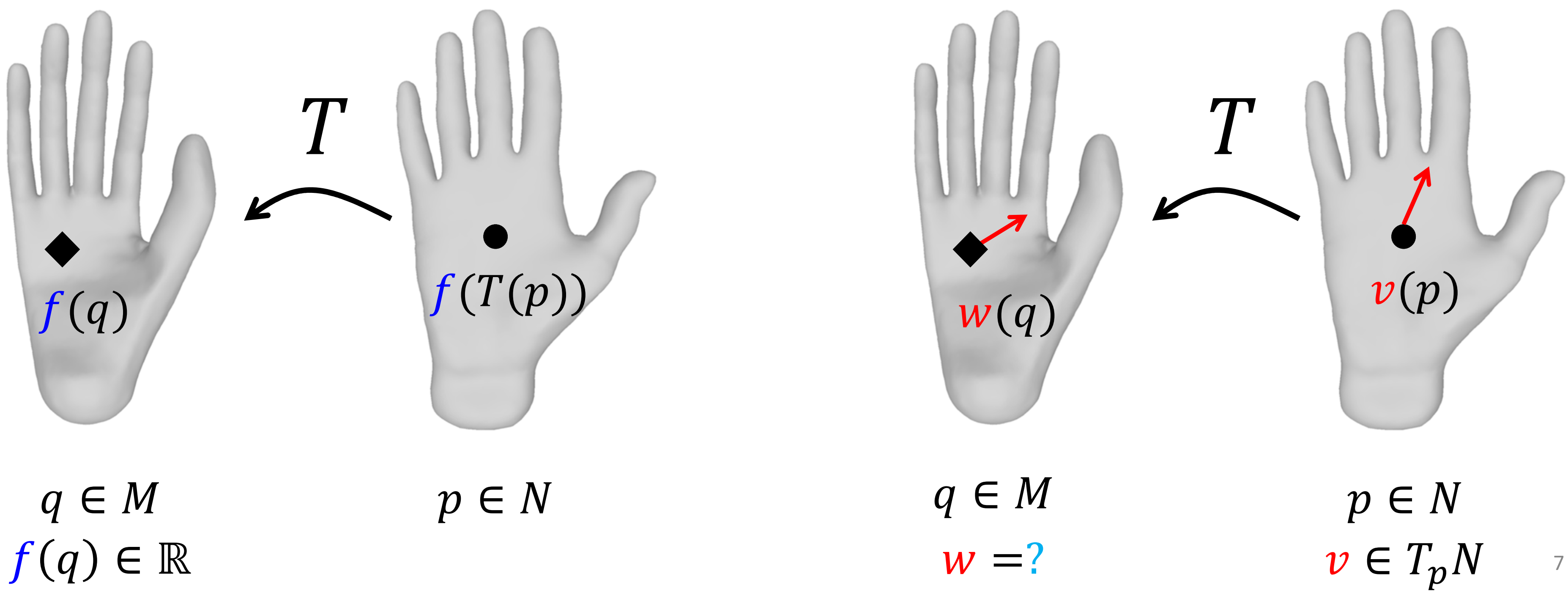
Azencot, Vantzos, Wardetzky, Rumpf, BC, SCA 2015

T_pN Tangent plane of N at p

Transporting data with point-to-point maps

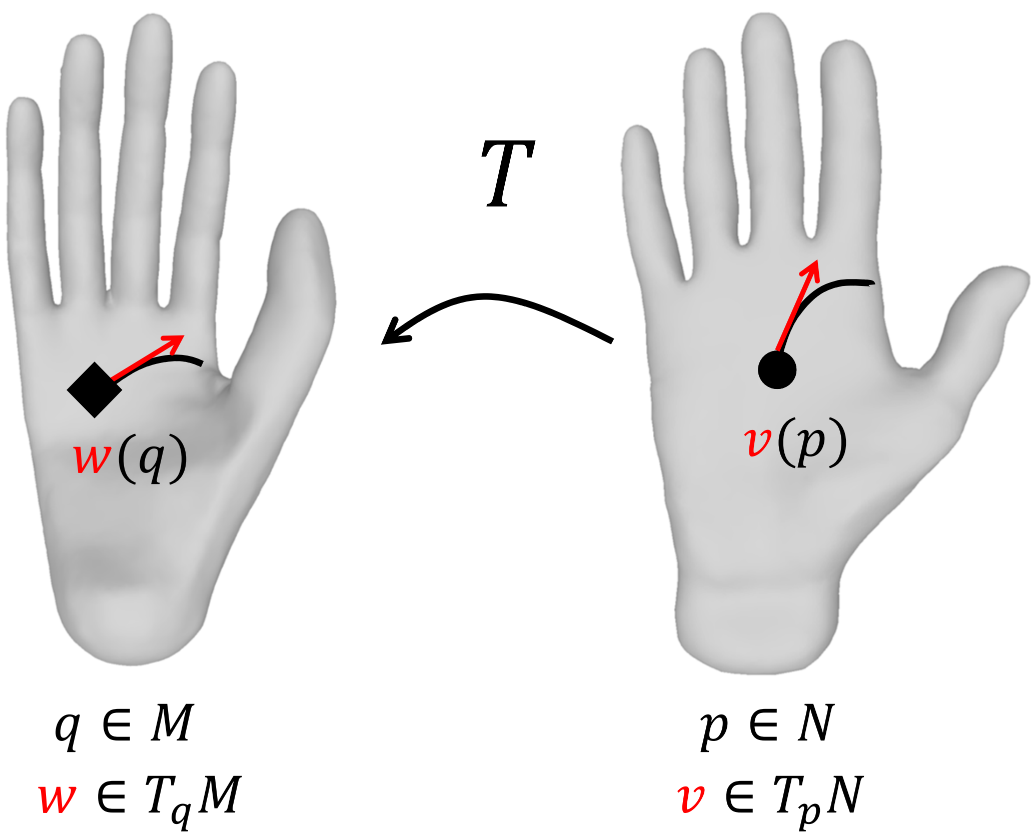
Mapping scalars

Mapping vectors



The Map Differential

$$w = dT_p(v)$$

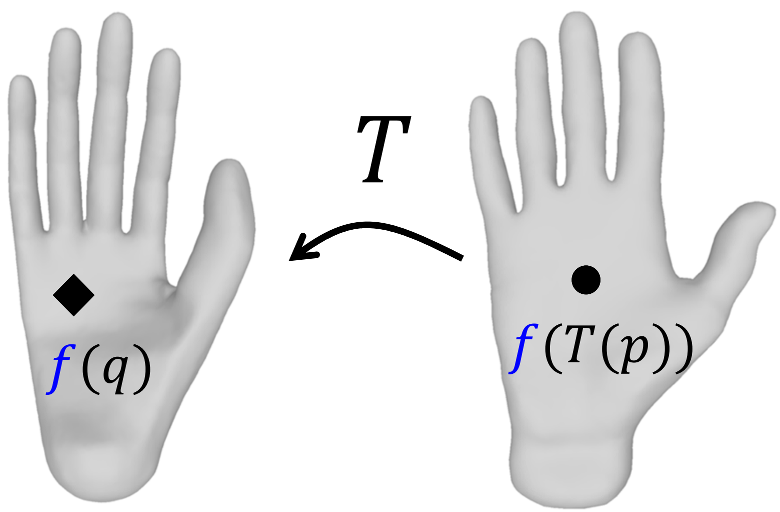


- Start a curve from p with tangent v
- Map the curve to M
- w is the tangent of the mapped curve at q

$T_p N$ Tangent plane of N at p

Transporting data with point-to-point maps

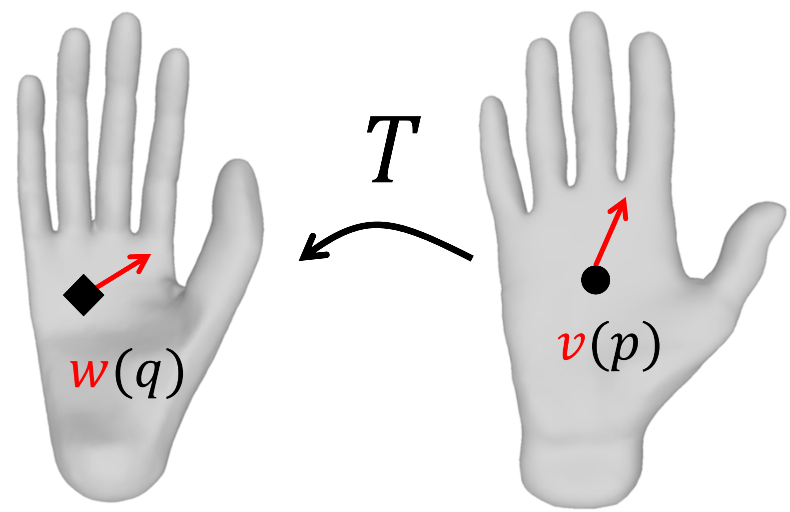
Mapping scalars



$q \in M$
$f(q) \in \mathbb{R}$

$p \in N$

Mapping vectors

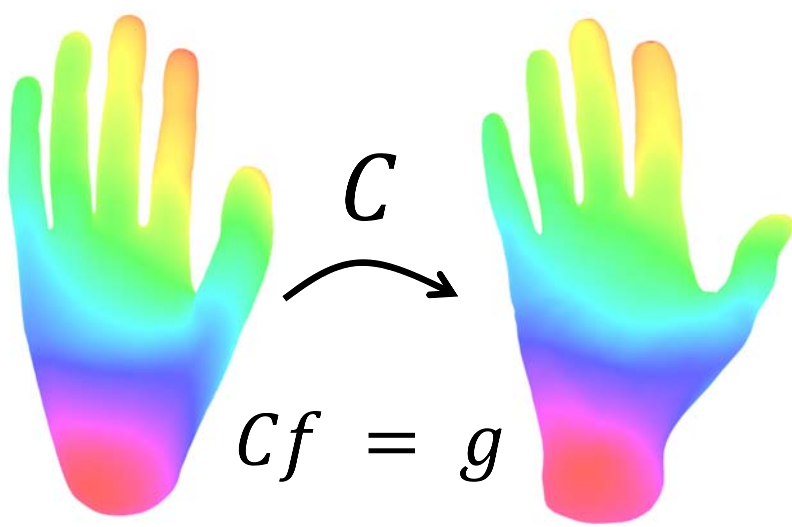


$q \in M$
$w = dT_p(v)$

$p \in N$
$v \in T_p N$

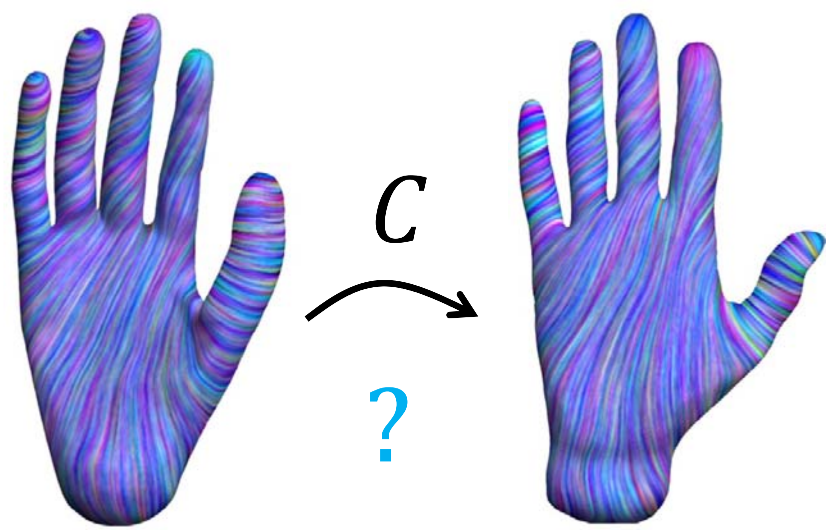
Transporting data with functional maps

Mapping functions



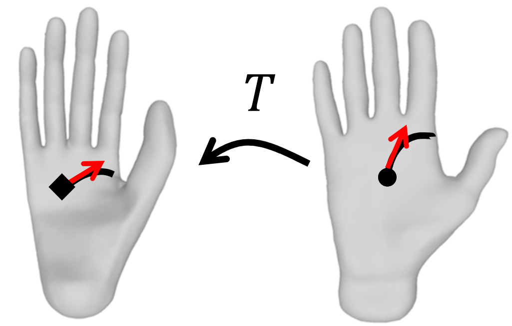
$f: M \to \mathbb{R}$ $\quad$ $g: N \to \mathbb{R}$

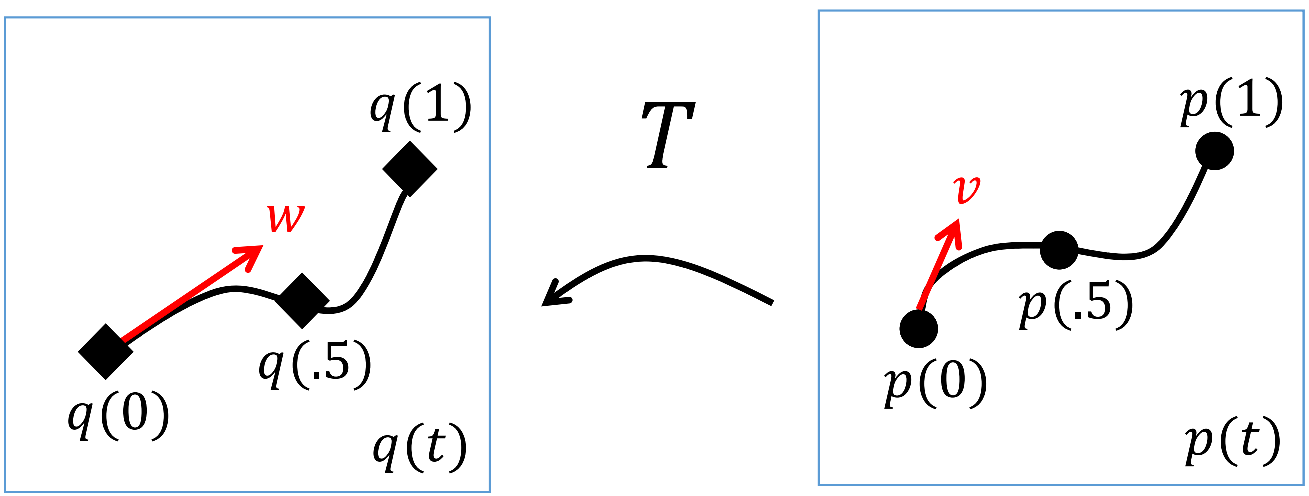
Mapping vector fields



W $\quad$ V

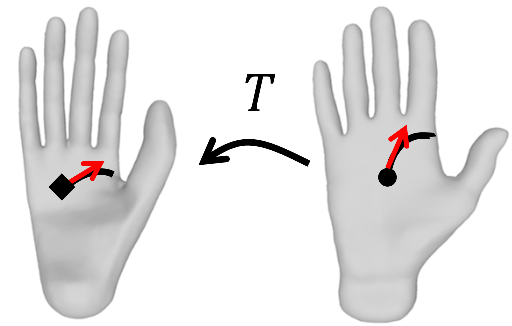
The Map Differential

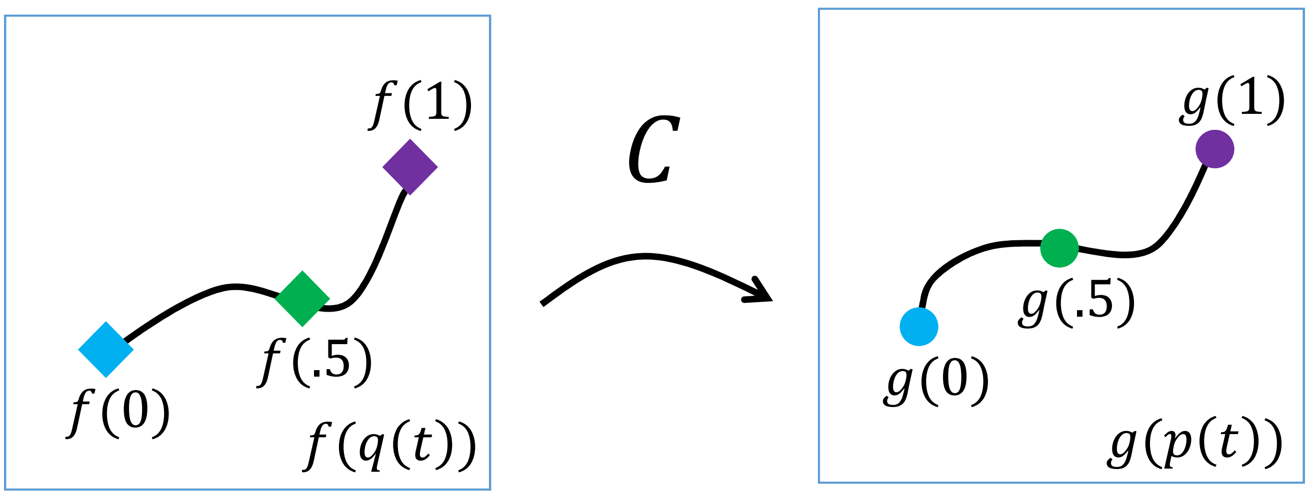




$q : [0,1] \to M$ $p : [0,1] \to N$

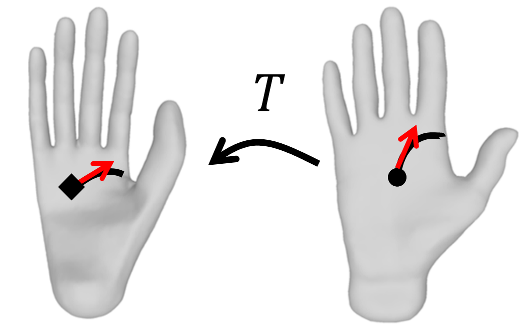
The functional Map Differential

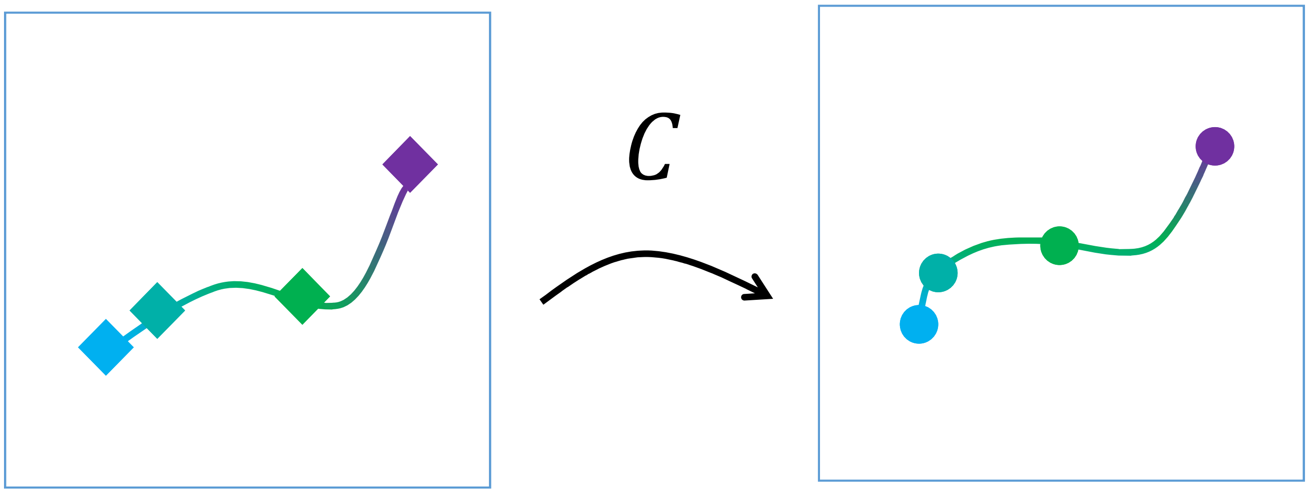




$f: [0,1] \rightarrow \mathbb{R}$ $\quad$ $g: [0,1] \rightarrow \mathbb{R}$

The functional Map Differential



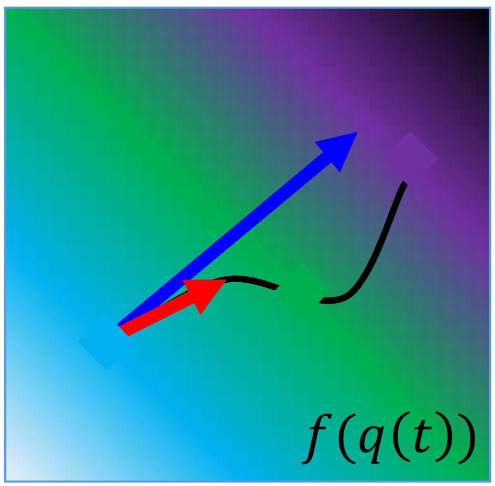


$f(t)$ $g(t)$

$$f'(0) = g'(0)$$

$$f'(0) = g'(0)$$

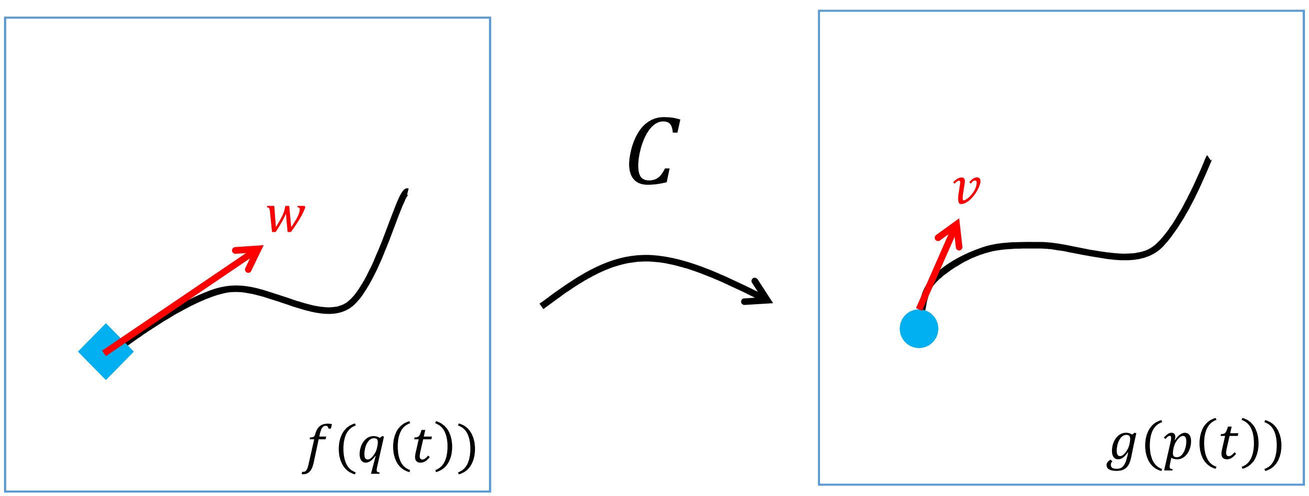
The functional Map Differential



$f: M \to \mathbb{R}$

$$f'(0) = \frac{df}{dt}\Big|_{\blacklozenge} = \frac{df}{dq}\Big|_{\blacklozenge}\frac{dq}{dt}\Big|_{\blacklozenge} = \langle \nabla f_{\blacklozenge}, w \rangle$$

The functional Map Differential

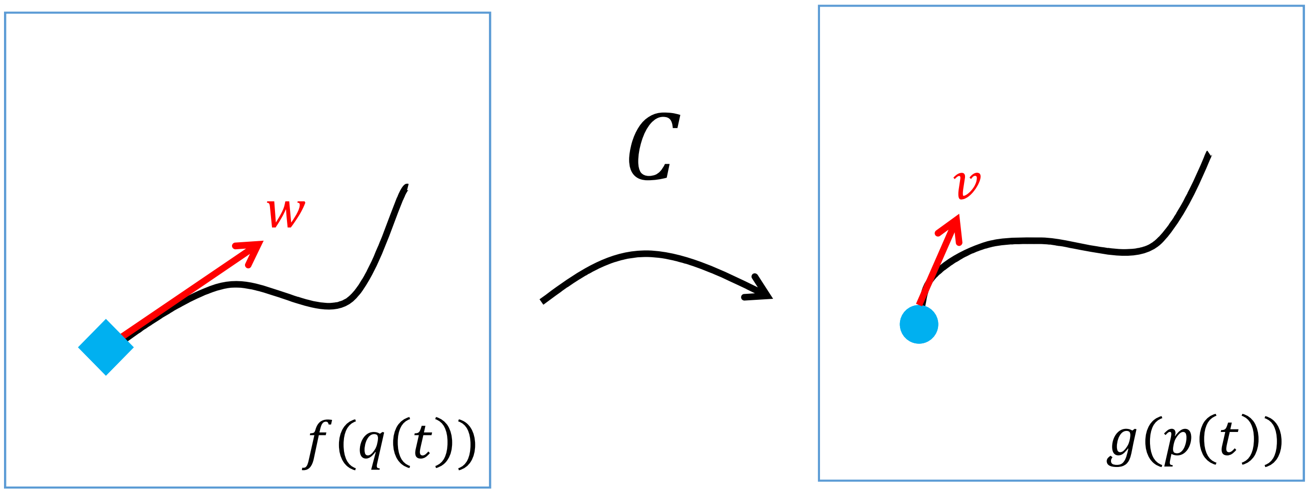


$f: M \to \mathbb{R}$ $\qquad$ $g: N \to \mathbb{R}$

$$f'(0) = g'(0)$$

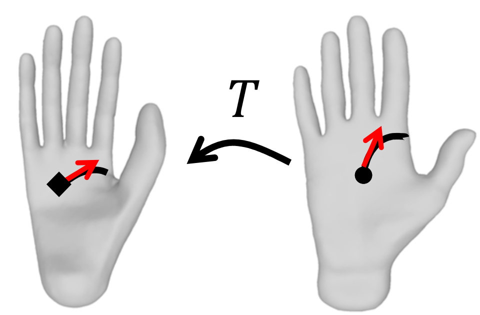
The functional Map Differential

$$f'(0) = g'(0)$$
$$f'(0) = \langle \nabla f_{\blacklozenge}, w \rangle$$
$$g'(0) = \langle \nabla g_{\bullet}, v \rangle$$

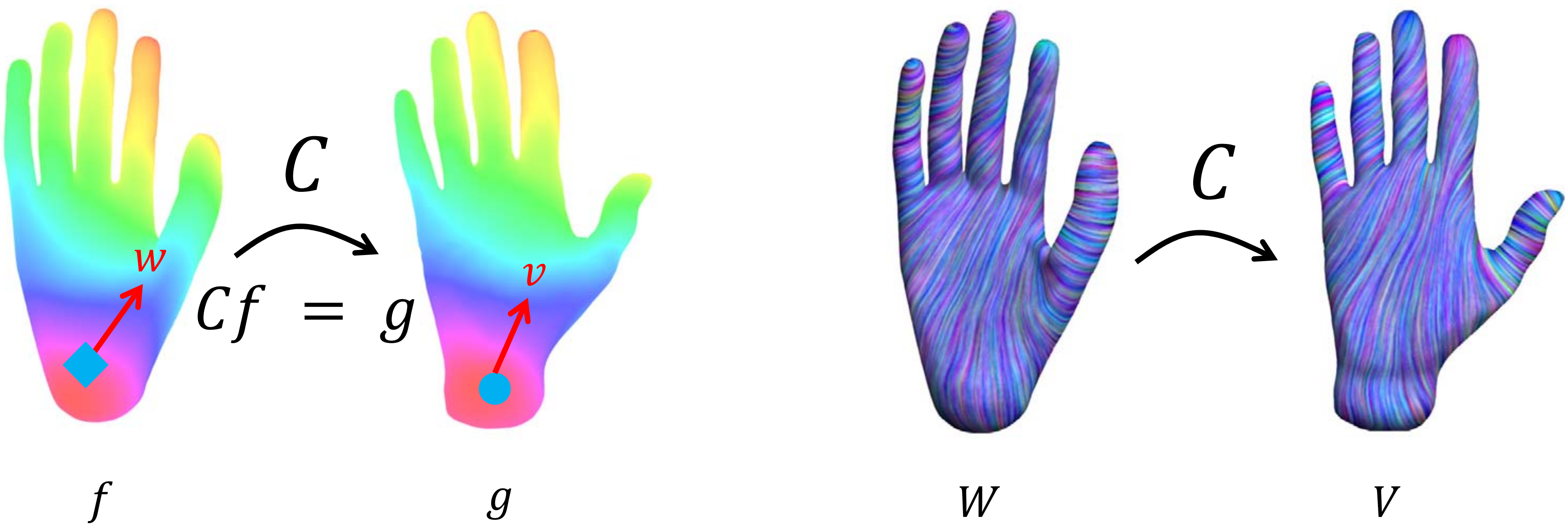


$f : M \to \mathbb{R}$ $g : N \to \mathbb{R}$

$$\langle \nabla f_{\blacklozenge}, w \rangle = \langle \nabla g_{\bullet}, v \rangle$$



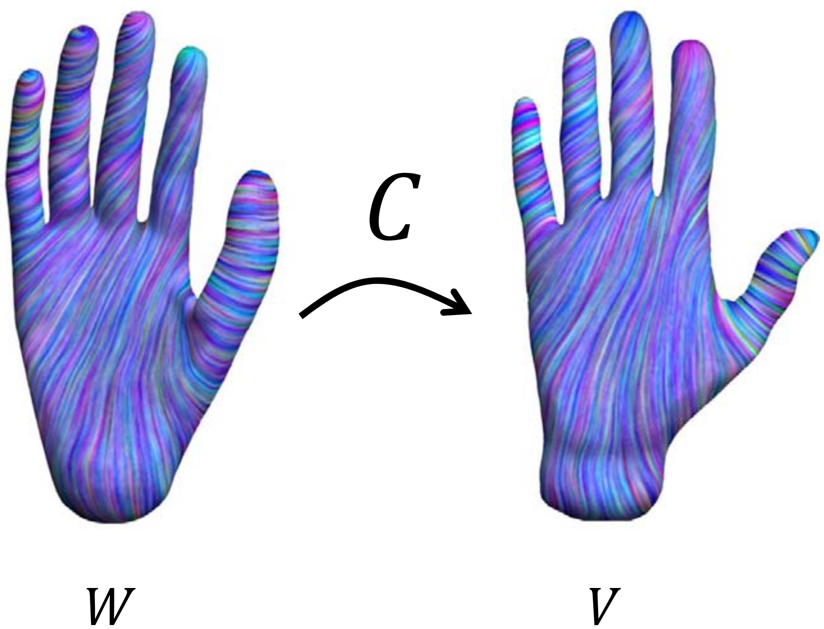
The functional Map Differential



$$\langle \nabla f_{\blacklozenge}, w \rangle = \langle \nabla g_{\bullet}, v \rangle$$

$$C\langle \nabla f, W \rangle = \langle \nabla g, V \rangle$$

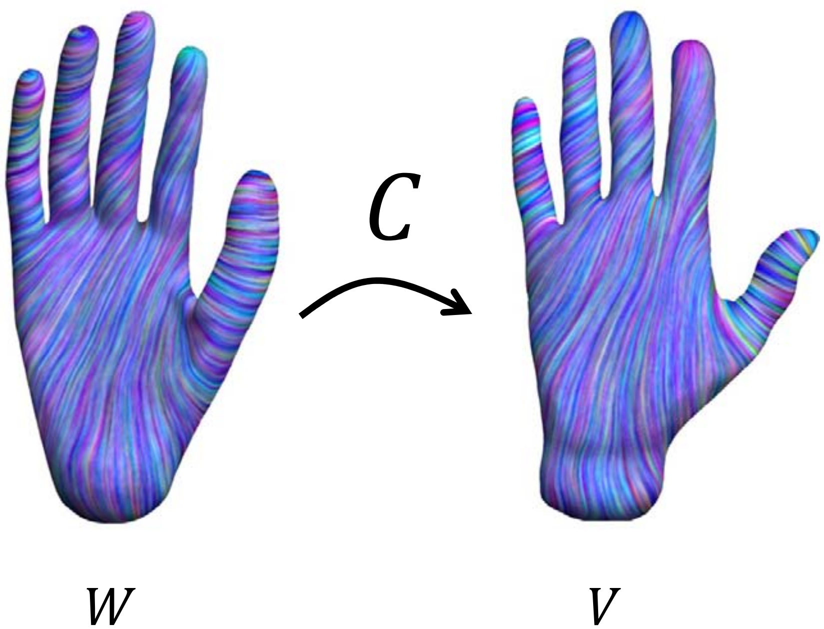
Corresponding vector fields



For all f

$$C\langle W, \nabla f\rangle = \langle V, \nabla(Cf)\rangle$$

Corresponding vector fields

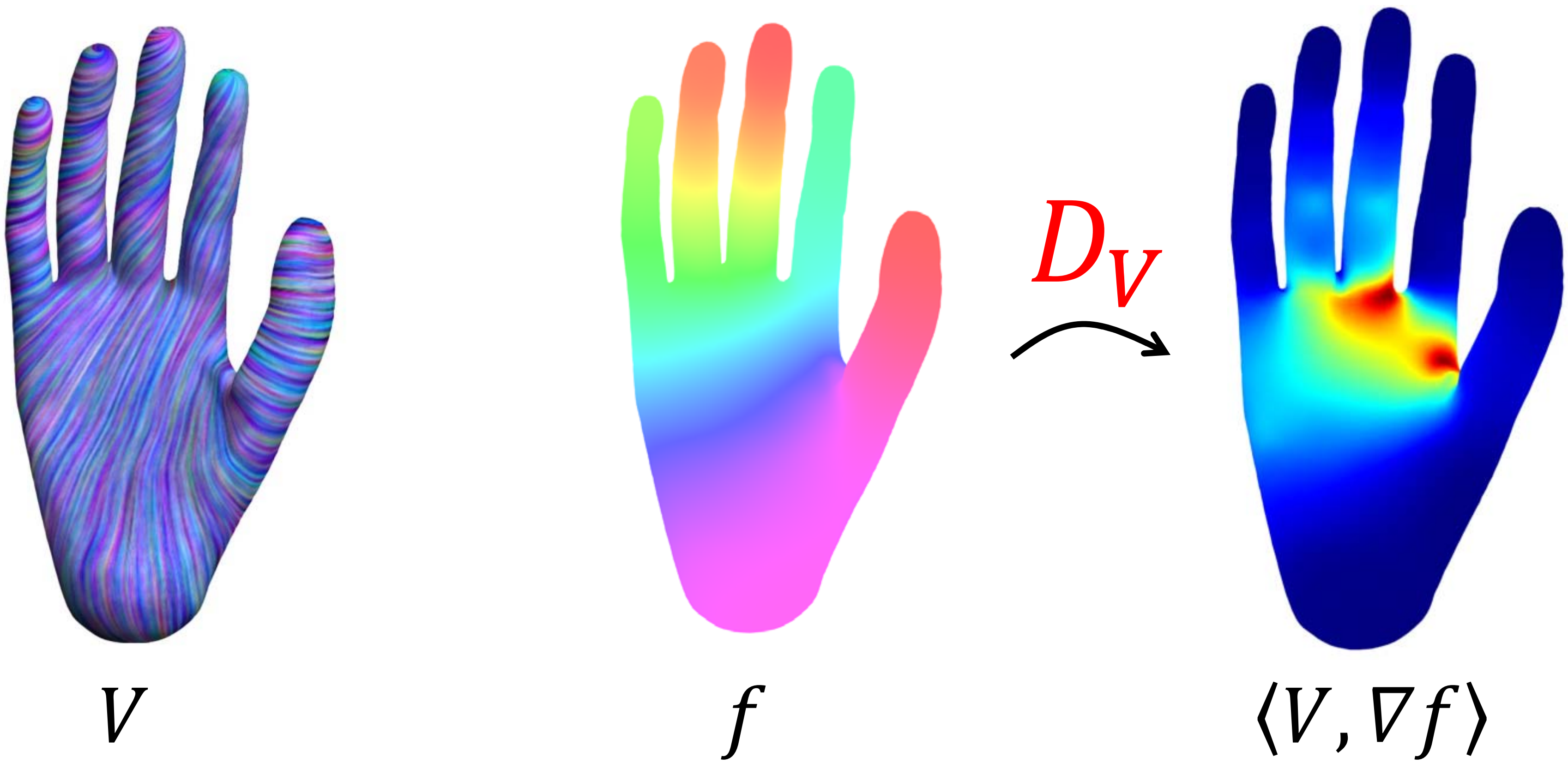


For all f

$$C\langle W, \nabla f\rangle = \langle V, \nabla (Cf)\rangle$$

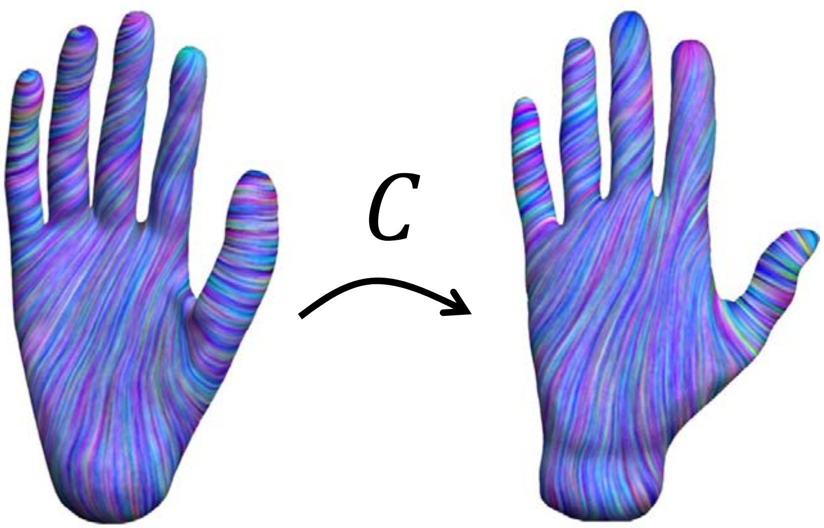
Functional Vector Fields

Linear
Complete*



$$C\langle W, \nabla f\rangle = \langle V, \nabla(Cf)\rangle$$

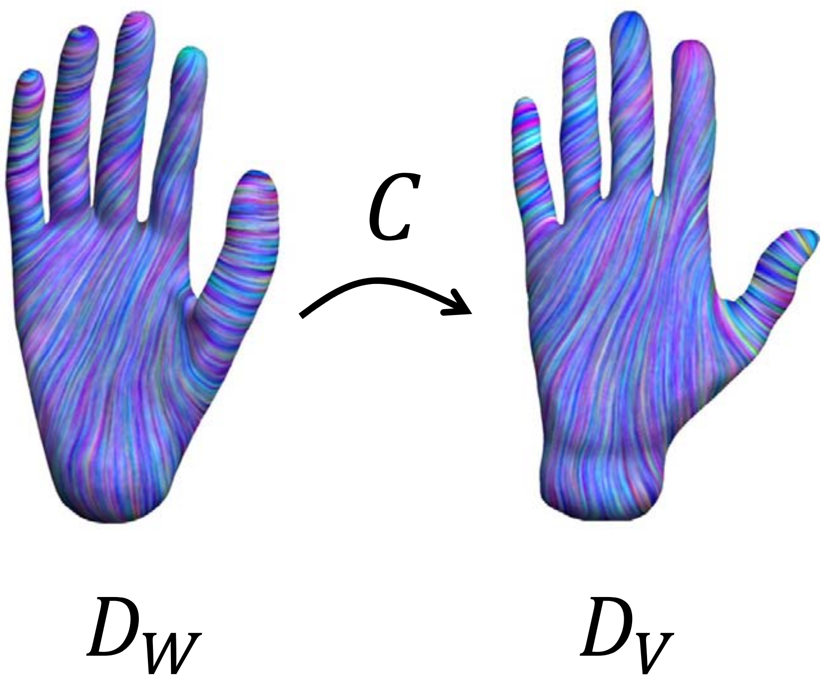
Corresponding FVFs



D_W D_V

$$C D_W f = D_V C f$$

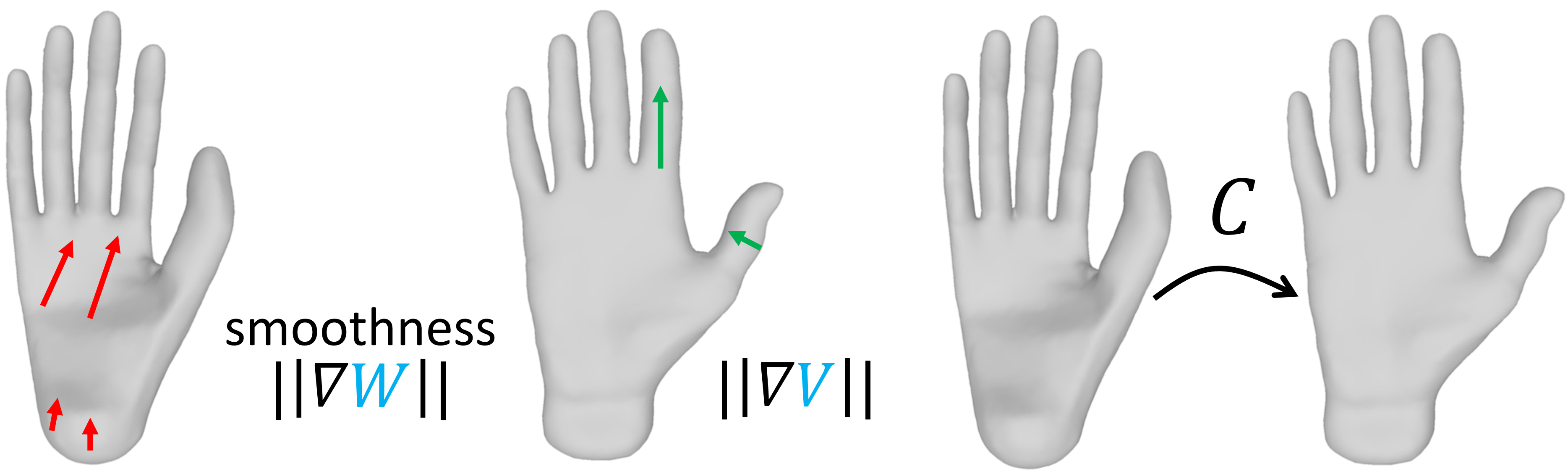
Corresponding FVFs commute with FMap



$$C D_W = D_V C$$

Application – Joint vector field design

$$\min_{V,W}$$

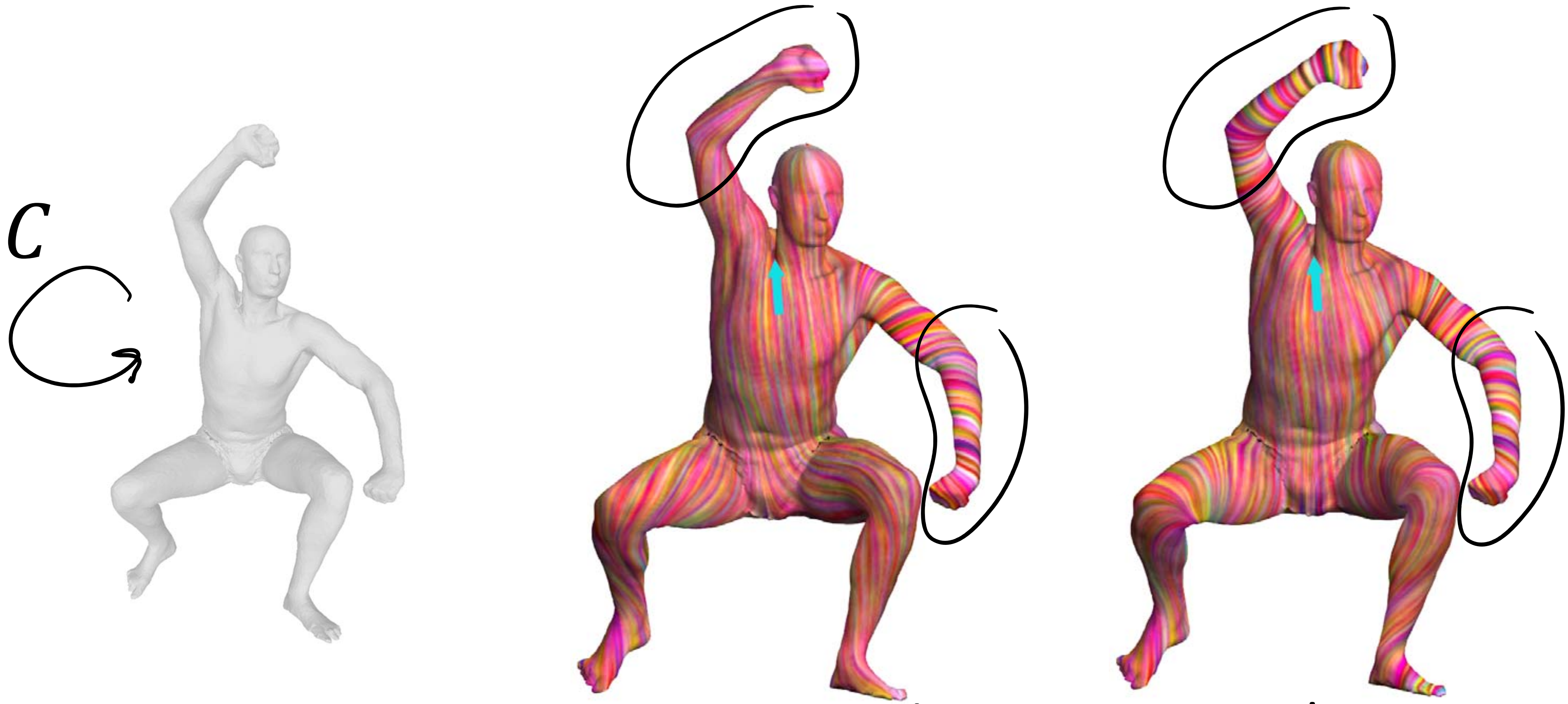


$||W_i - w_i||$ constraints

$||V_i - v_i||$

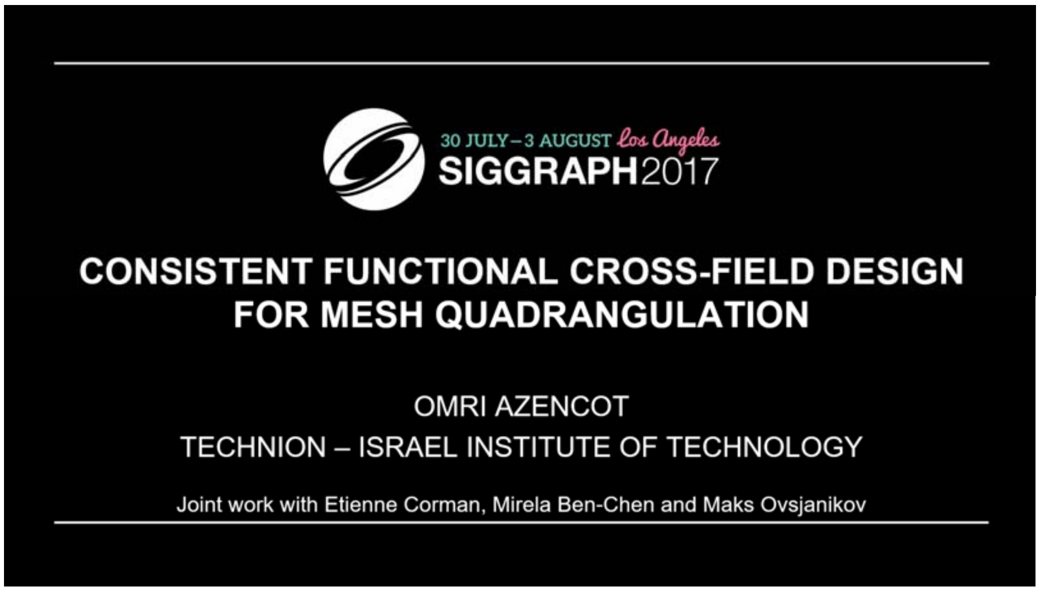
$||C D_W - D_V C||$ map consistency

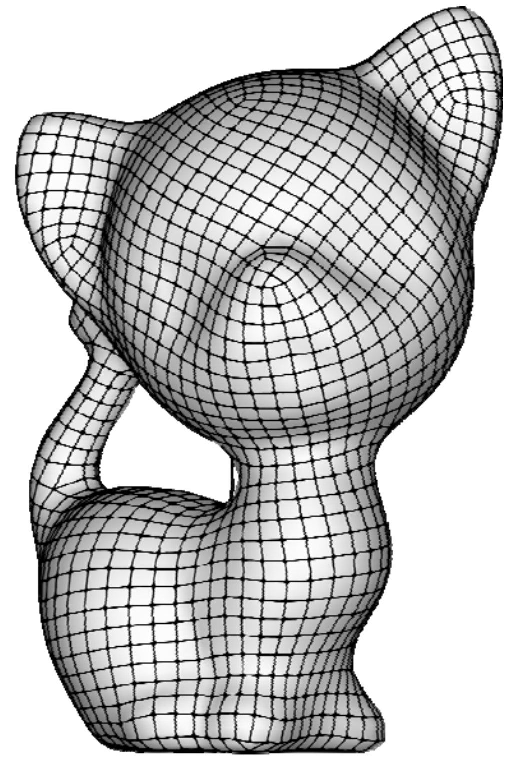
Application – Joint vector field design



constraints + smoothness + consistency

Application - Joint Quad-remeshing



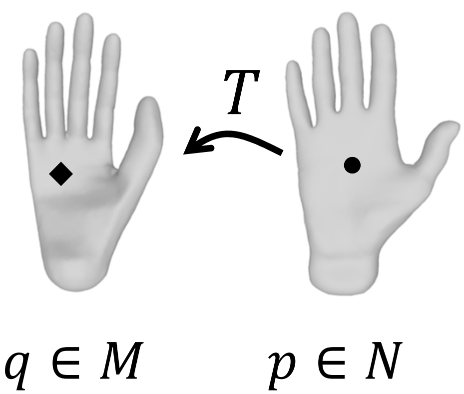


Wed 9am, Room 152

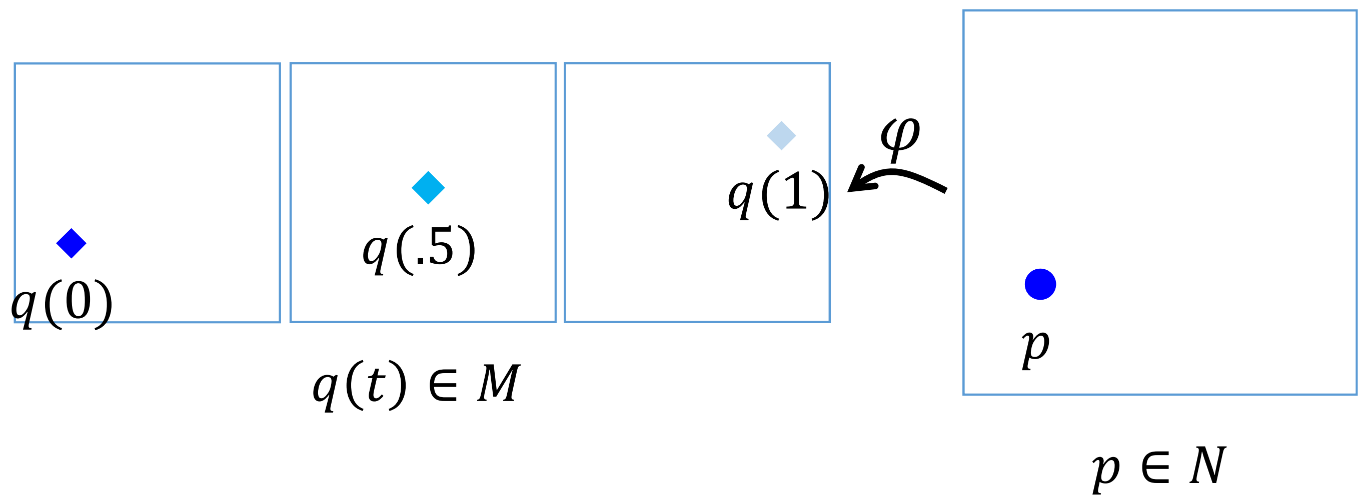
[Azencot, Corman, BC, Ovsjanikov, SIGGRAPH 2017]

Map "animation"

One map



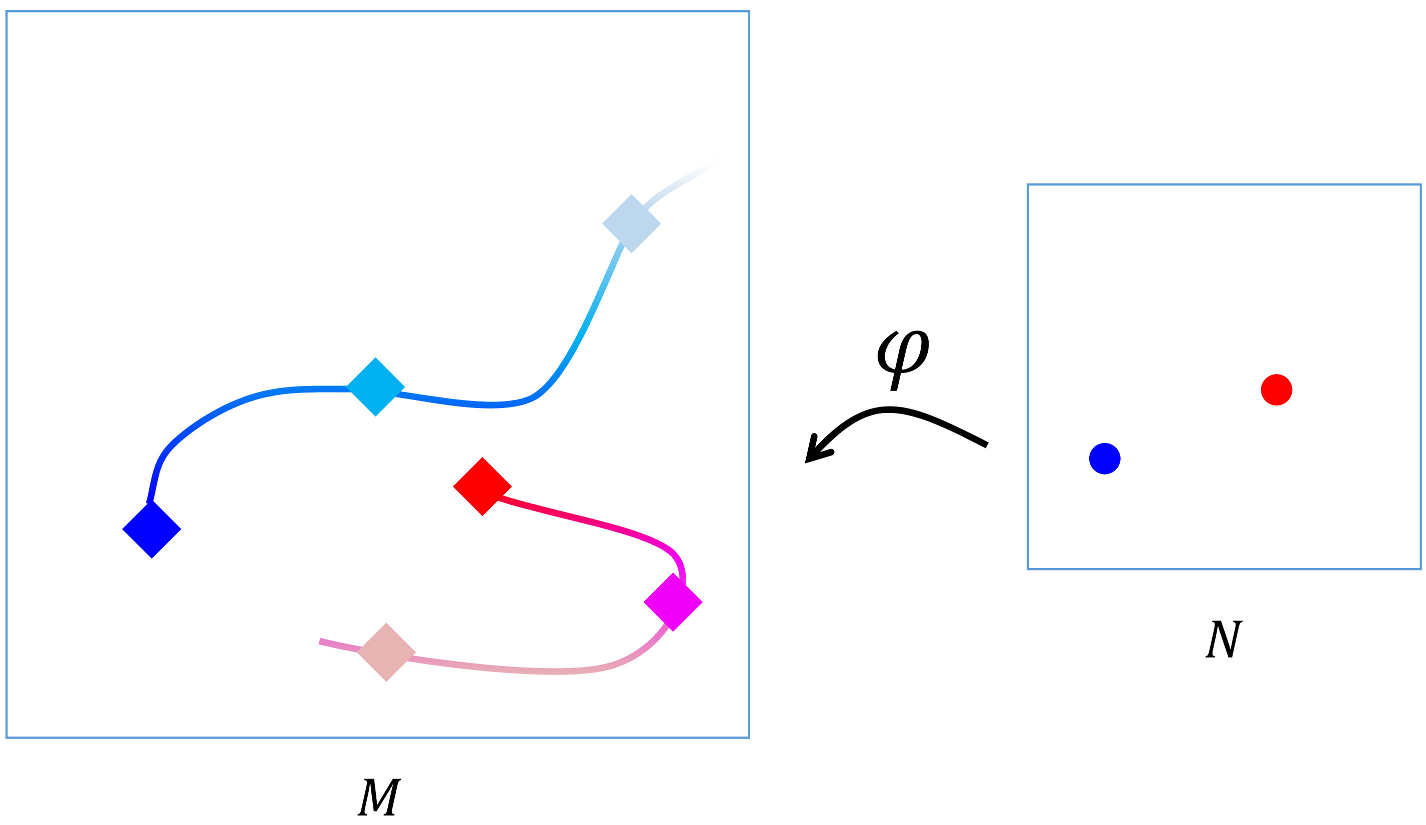
Map sequence



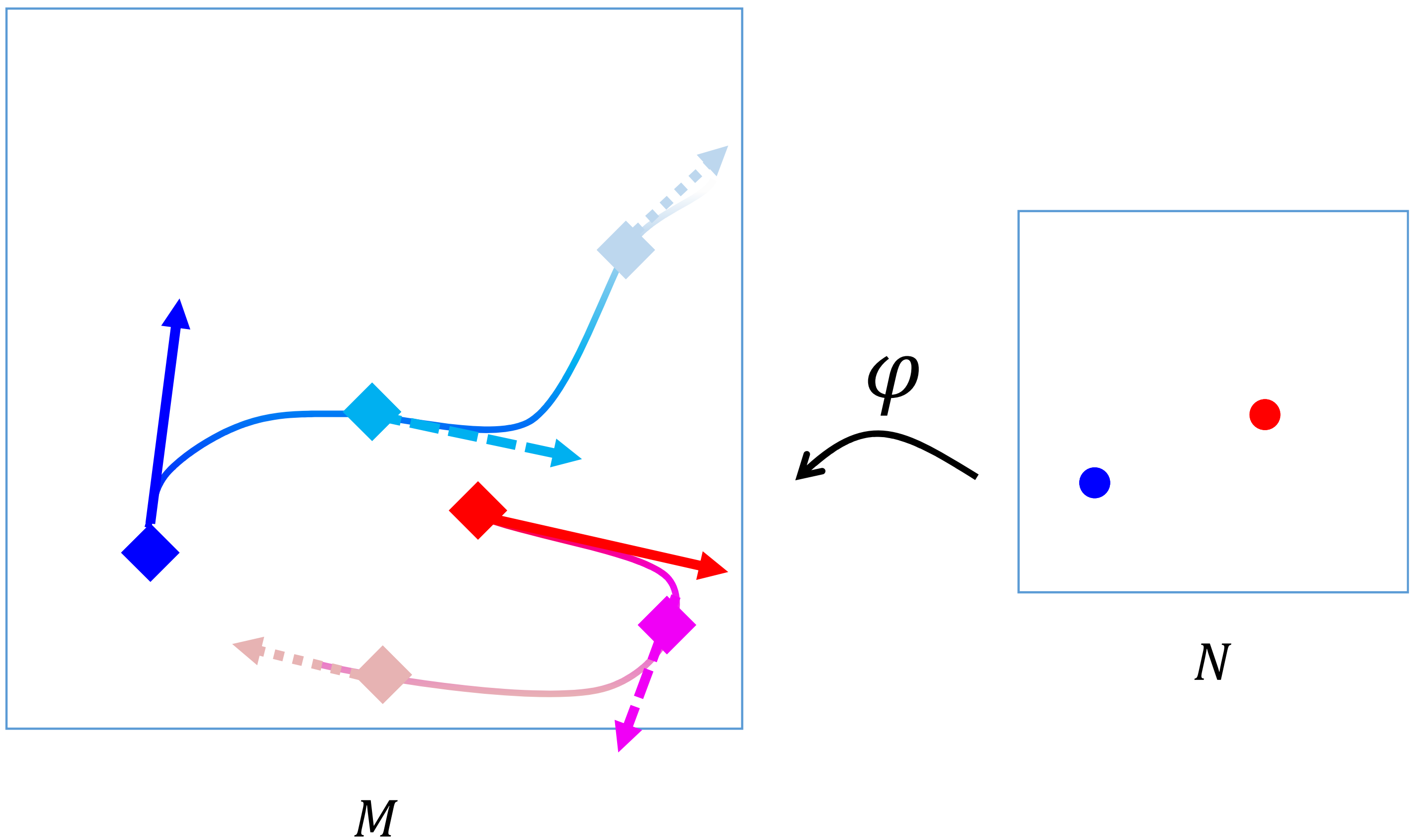
$$\varphi: \mathbb{R} \times N \to M$$

$$\varphi: \mathbb{R} \times N \rightarrow M$$

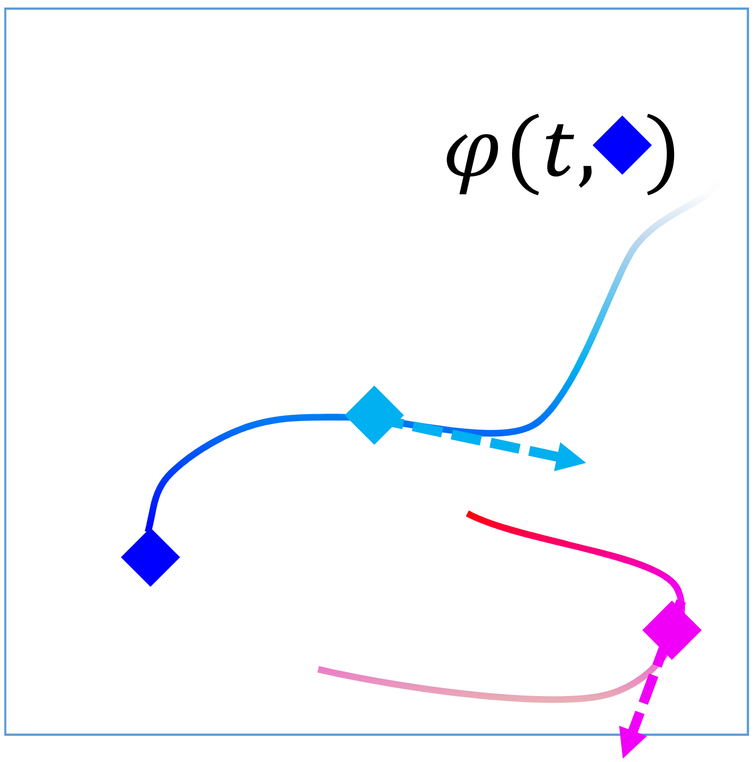
A 1-parameter family of maps



Back to vector fields



Back to vector fields, self maps



$V(.5) \in TM$

$$N = M, \qquad \varphi(0) = Id$$

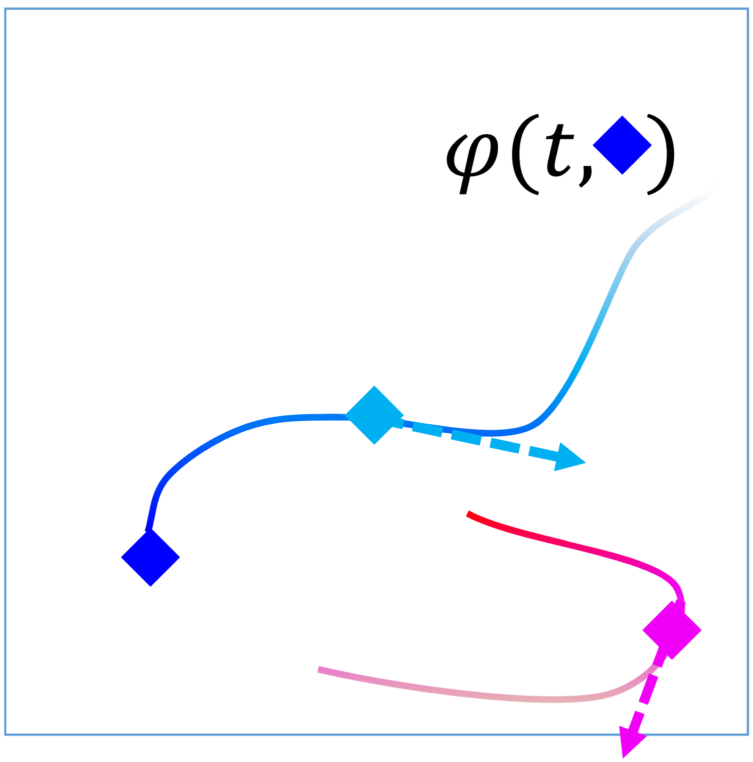
$$V(.5, \blacklozenge) = \frac{\partial \varphi}{\partial t}(.5, \blacklozenge)$$

$$\blacklozenge = \varphi(.5, \blacklozenge)$$

Back to vector fields, self maps

$$V(.5, {\color{cyan}\blacklozenge}) = \frac{\partial \varphi}{\partial t}(.5, {\color{blue}\blacklozenge})$$

$${\color{cyan}\blacklozenge} = \varphi(.5, {\color{blue}\blacklozenge})$$



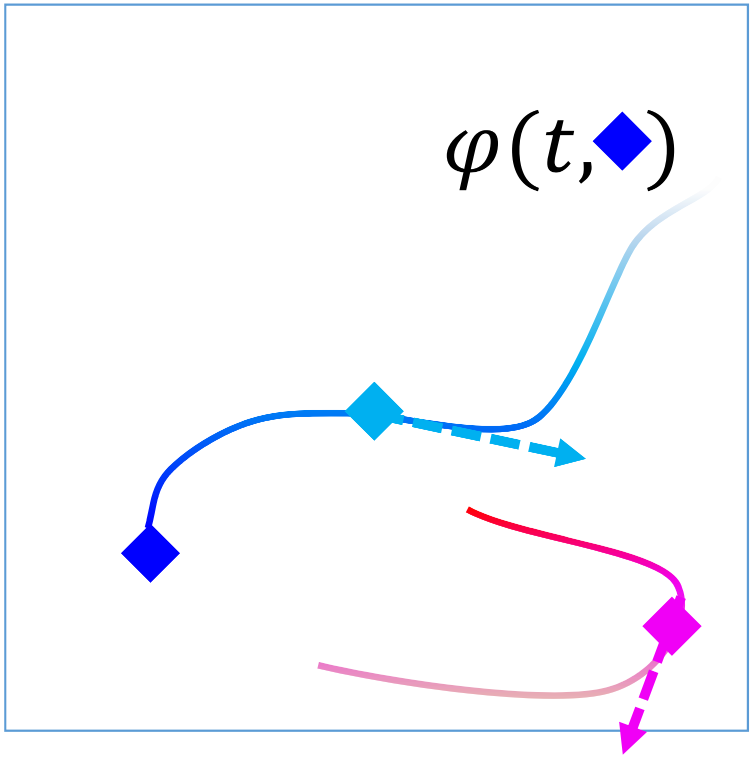
$V(t) \in TM$

$$V(t, \varphi(t, {\color{blue}\blacklozenge})) = \frac{\partial \varphi}{\partial t}(t, {\color{blue}\blacklozenge})$$

$$\boxed{V(t, \varphi(t)) = \frac{\partial \varphi}{\partial t}(t)}$$

$$V(t, \varphi(t)) = \frac{\partial \varphi}{\partial t}(t)$$

From maps to vector fields

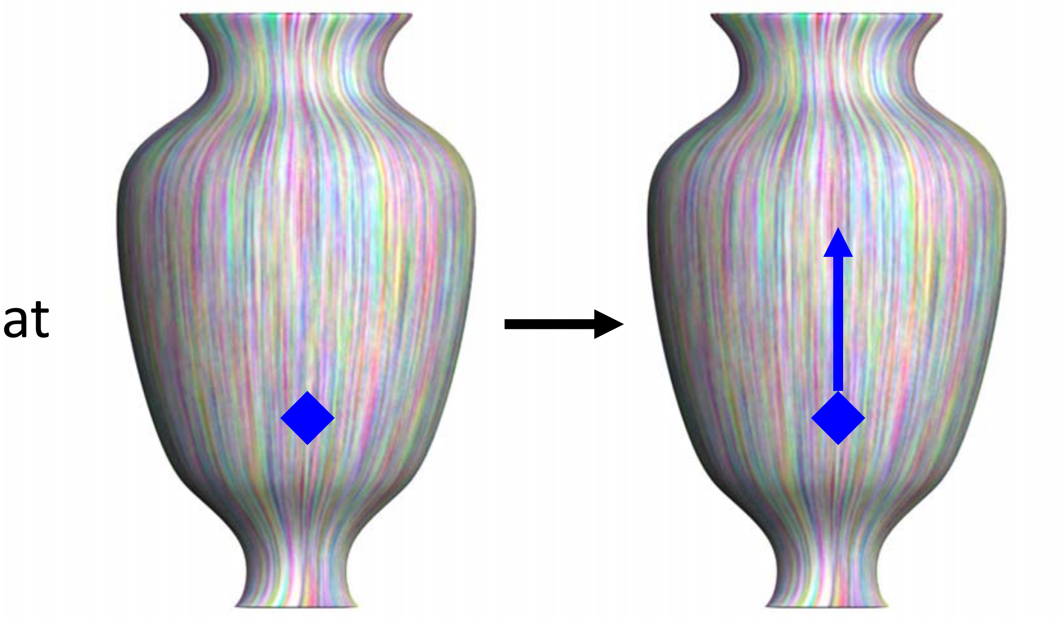


$V(t) \in TM$

$$V \circ \varphi = \frac{\partial \varphi}{\partial t}$$

From vector fields to trajectories

- We know: map → vector fields
- How to do: vector fields → map?
- Given $V(t)$ solve for $\varphi(t)$ such that



$$\varphi(0) = Id$$

The flow PDE $$\frac{\partial \varphi}{\partial t} = V \circ \varphi$$

From vector fields to functional trajectories

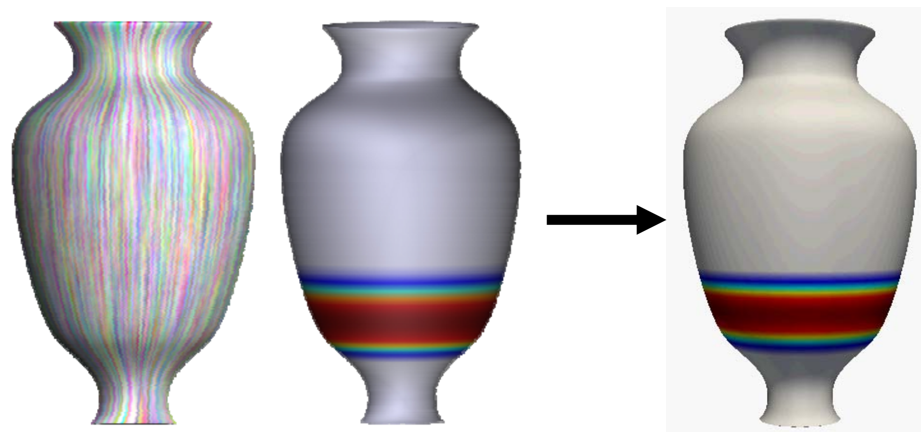
- Given (stationary) V, f_0 solve for $f(t)$ such that

The flow:

$$\varphi(0) = Id \quad \frac{\partial\varphi}{\partial t} = V \circ \varphi(t)$$

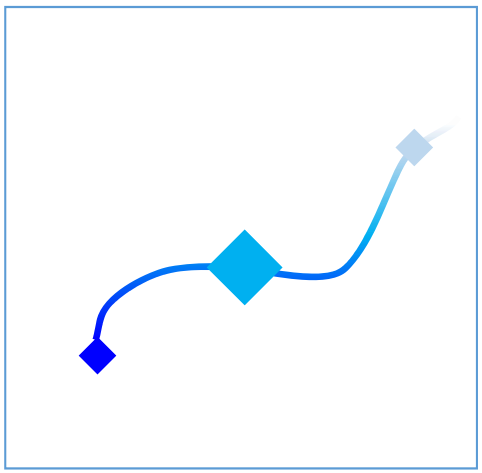
The functional flow:

$$f(t) = f_0 \circ \varphi^{-1}(t)$$

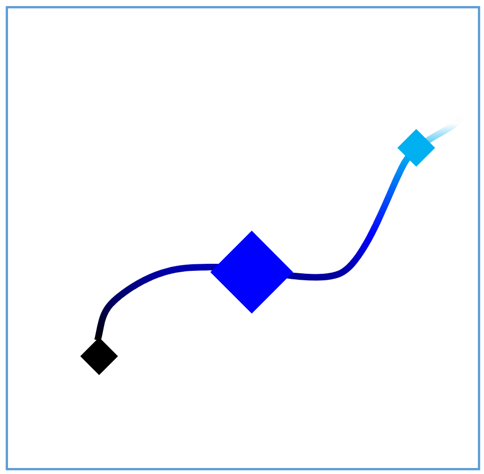


V f_0 $f(t)$

The functional flow



$f(0)$

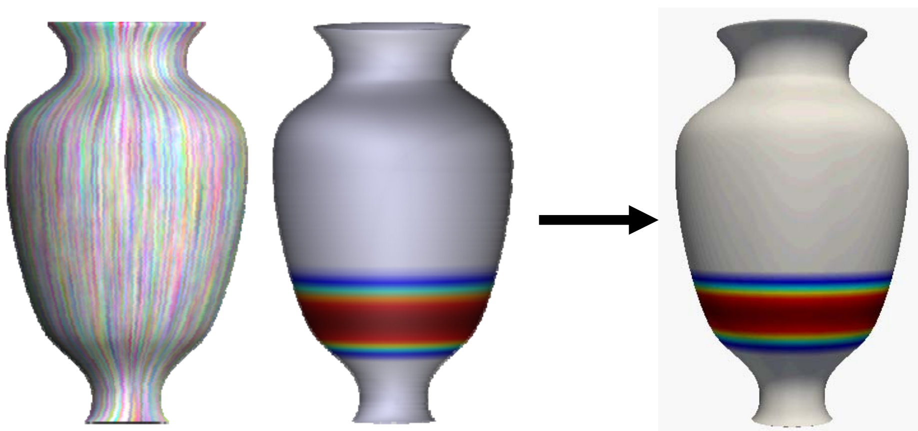


$f(.5)$

$$f'(t) = -\langle \nabla f(t), V \rangle$$

$$f'(t) = -\langle \nabla f(t), V \rangle$$

The functional flow



V $\quad$ f_0 $\quad$ $f(t)$

$$f'(t) = -D_V f(t)$$

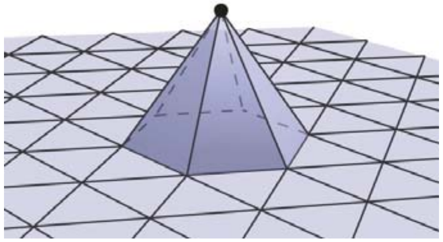
$$f'(t) = -D_V f(t)$$

The discrete functional flow

- The surface M is a triangle mesh
- The function f is represented using a linear basis $\{\gamma_i\}$, e.g. hat basis
- $\mathbf{f}$ is a vector, $\mathbf{D_V}$ is a matrix
- A matrix ODE

$$\mathbf{f} = \sum_i \mathbf{f}_i \gamma_i$$

$$\frac{d\mathbf{f}(t)}{dt} = -\mathbf{D_V}\mathbf{f}(t)$$



The discrete functional flow

- A matrix ODE

$$\frac{d\mathbf{f}(t)}{dt} = -\mathbf{D_V}\mathbf{f}(t)$$

- Closed form solution

$$\mathbf{f}(t) = \exp(-t\mathbf{D_V})\mathbf{f}(0)$$

Implementation

- Mesh M: n vertices, m faces.
- Represent f_0 in the hat basis - $\mathbf{f}_0 \in \mathbb{R}^n$
- Represent D_V in the hat basis – a sparse matrix $\mathbf{D_V} \in \mathbb{R}^{n\times n}$

$$\mathbf{D_V} = \left[\quad I_{\mathcal{V}}^{\mathcal{F}} \quad\right]_{n\times m} \left[\quad V \quad\right]_{m\times 3m} \left[\ \mathrm{grad}\ \right]_{3m\times n}$$

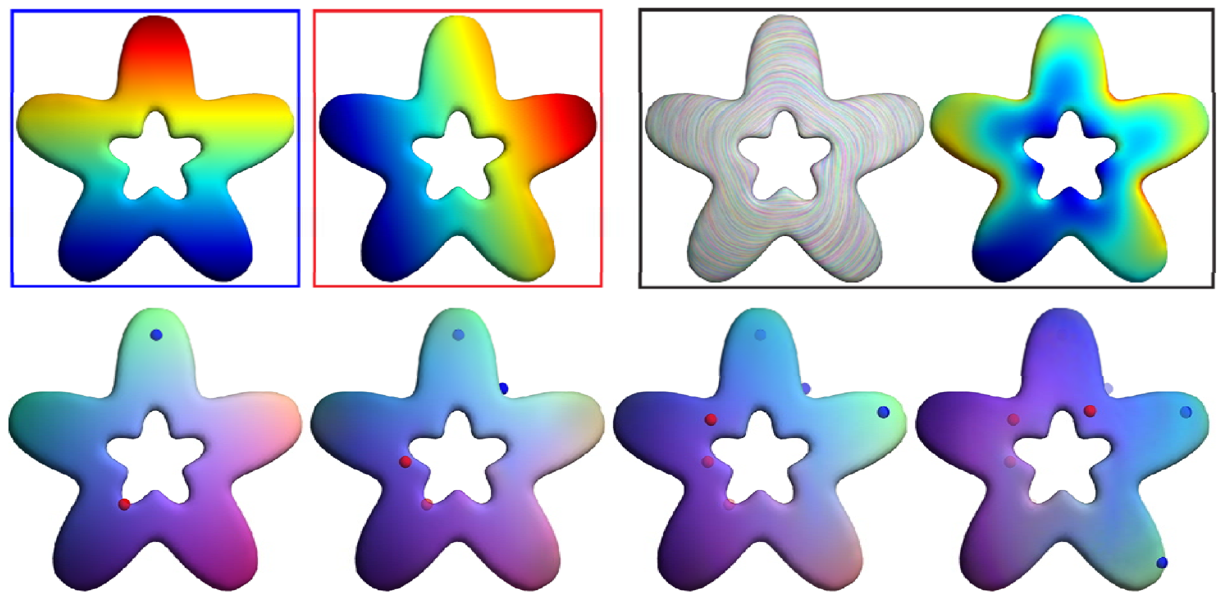
Interpolate to vertices *vector per face* *gradient operator*

- Compute $\mathbf{f}(\mathrm{t}) = \mathrm{expmv}(t, \mathbf{D_V}, \mathbf{f}_0)$

expmv: https://www.mathworks.com/matlabcentral/fileexchange/29576-matrix-exponential-times-a-vector?focused=5172371&tab=function

Applications
Maps from vector fields

$$\min_{\mathbf{V}} || \exp(\mathbf{D_V})\, \mathbf{f} - \mathbf{g} ||$$

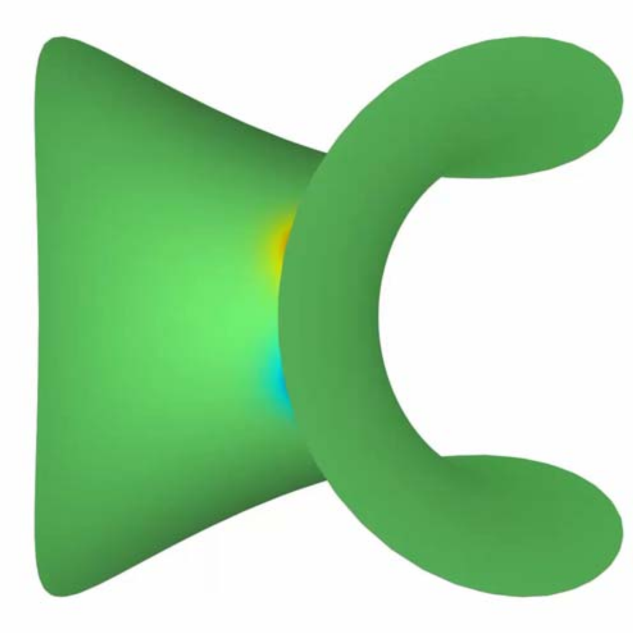


[Azencot, Vantzos, BC, SGP 2016]

Applications
Incompressible flow on surfaces

- Numerically simulate the equation

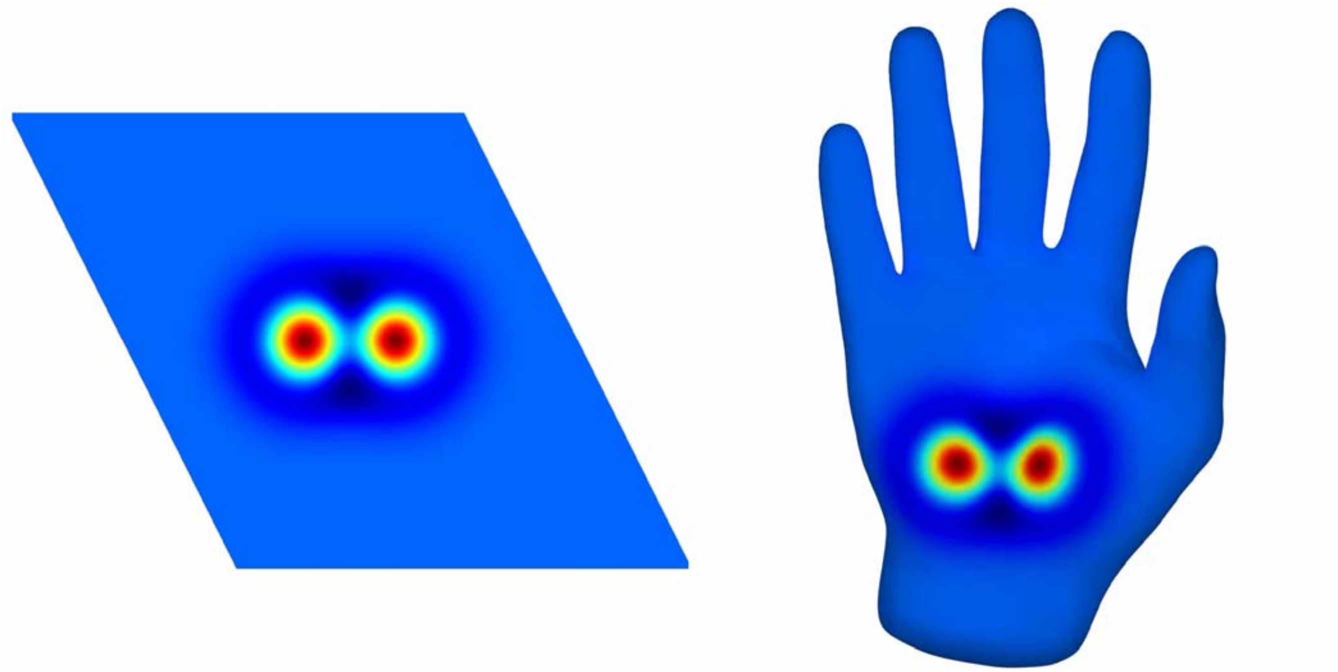
$$\frac{\partial \omega_t}{\partial t} = -D_{v_t} \omega_t$$



- Vector field changes, more complicated
- Total vorticity ω_t conserved by construction

[Azencot, Weißmann, Ovsjanikov, Wardetzky, BC, SGP 2014]

Applications
Incompressible flow on surfaces



Applications
Viscous thin films on surfaces

Total volume of fluid conserved by construction

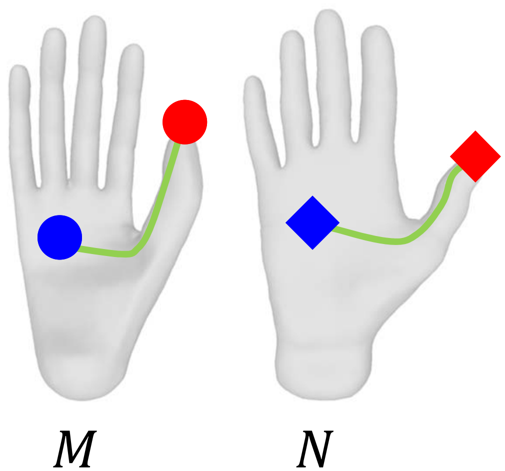
[Azencot, Vantzos, Wardetzky, Rumpf, BC, SCA 2015]

$\mathrm{C} = \exp(\mathbf{D_V})$

Applications
Inferring vector fields

T is distance preserving

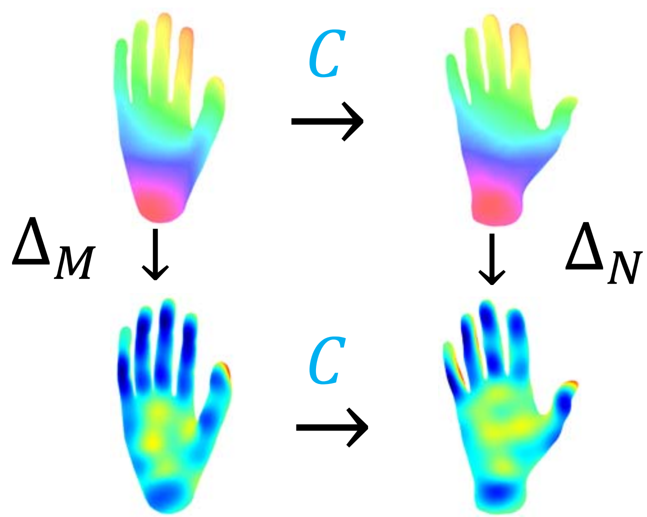
$$\int_{M\times M} \left(d_M(\bullet,\bullet) - d_N(\blacklozenge,\blacklozenge)\right)^2$$



[Azencot, BC, Chazal, Ovsjanikov, SGP 2013]

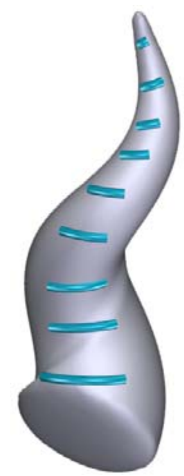
C *commutes* with Δ

$$||C\Delta_M - \Delta_N C||^2_{\mathrm{Fro}}$$



D_V *commutes* with Δ

$$||D_V\Delta - \Delta D_V||^2_{\mathrm{Fro}}$$

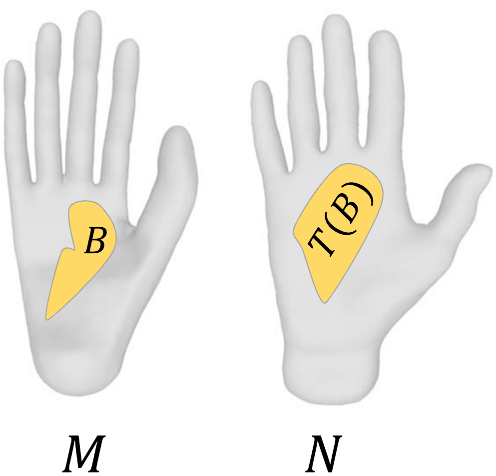


$$\mathrm{C} = \exp(\mathbf{D_V})$$

Applications
Inferring vector fields

T is area preserving

$$\int_B d\mu_M = \int_{T(B)} d\mu_N$$

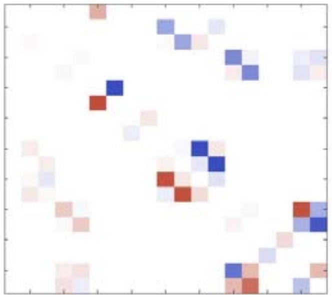


C is *orthogonal*

$$\int_M fg = \int_N (Cf)(Cg)$$

C is a *rotation* in functional space

D_V is *anti-symmetric*



V is *divergence free*

(some) conclusions

- Vector fields and maps relate through tangents to trajectories
- Can be used to transport vector fields with maps or to compute maps from vector fields
- The functional approach allows to use linear algebra instead of geometric tracing for trajectory related problems

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