Node-to-Node Disjoint Paths in $k$-ary $n$-cube with Faulty Edges

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Abstract—In a $k$-ary $n$-cube $Q^k_n$ with at most $2n - 2$ faulty edges, let $u$ and $v$ be any two given nodes. Suppose the number of healthy links of $u$ is no more than that of $v$, and denote by $m$. In this paper, we find $m$ disjoint paths between $u$ and $v$.

Keywords-interconnection networks; disjoint paths; $k$-ary $n$-cube; fault tolerance;

I. INTRODUCTION

$k$-ary $n$-cubes have been used as interconnection networks for distributed-memory parallel computers [1] and are popular choices for networks-on-chips [2]. $k$-ary $n$-cube has the following basic properties: it is vertex- and edge-symmetric [3]; it is Hamiltonian [4]; it has diameter $n\lfloor \frac{k}{2} \rfloor$; it has a recursive decomposition. Moreover, it has admirable properties in relation to routing, broadcasting and communication in general (see, for example, [3], [5]).

Of particular relevance to us is the one-to-one node-disjoint paths problem, which enable parallel communication between source and destination nodes, provide alternative routing paths when faults occurs or for the purpose of avoiding traffic jam. Whilst Menger’s Theorem [6] implies that there exist $2n$ node-disjoint paths between any two given nodes in a healthy $k$-ary $n$-cube. In particular, in a $k$-ary $n$-cube $Q^k_n$ with at most $2n - 2$ faulty edges, let $u$ and $v$ be any two given nodes; denote the number of healthy links of a node $u$ as $deg_H(u)$; we find $\min\{deg_H(u), deg_H(v)\}$ disjoint paths between $u$ and $v$.

As more and more processors are incorporated into parallel machines, faults become more common, be it faults in the processors or on the connections between processors. Given the significant cost of parallel machines, we would prefer to be able to tolerate (small numbers of) faults and still be able to use our parallel machine.

Bose et al [5] find $2n$ node-disjoint paths between any two given nodes in a healthy $k$-ary $n$-cube. In this paper, we are going to find node-disjoint paths between any two given nodes in a faulty $k$-ary $n$-cube. In particular, in a $k$-ary $n$-cube $Q^k_n$ with at most $2n - 2$ faulty edges, let $u$ and $v$ be any two given nodes; denote the number of healthy links of a node $u$ as $deg_H(u)$; we find $\min\{deg_H(u), deg_H(v)\}$ disjoint paths between $u$ and $v$.

The paper is organized as follows: the next section is the basic definition; we describe how to find disjoint paths in faulty $k$-ary 2-cube in Section III; by induction, we find disjoint paths for $k$-ary $n$-cube for $n \geq 3, k \geq 4$, and this is described in Section IV; Section V is the conclusion of the paper.

II. PRELIMINARIES

The $k$-ary $n$-cube, denoted by $Q^k_n$, for $k \geq 3$ and $n \geq 2$, has $k^n$ nodes indexed by $\{0, 1, \ldots, k-1\}^n$, and there is a link $\{(u_n, u_{n-1}, \ldots, u_1), (v_n, v_{n-1}, \ldots, v_1)\}$ if, and only if, there exists $d \in \{1, 2, \ldots, n\}$ such that $\min\{|u_d - v_d|, k - |u_d - v_d|\} = 1$, and $u_i = v_i$ for every $i \in \{1, 2, \ldots, n\} \setminus \{d\}$. Throughout, we
assume that addition on tuple elements is modulo $k$.

An index $d \in \{1, 2, \ldots, n\}$ is often referred to as a dimension. We can partition $Q^k_n$ over dimension $d$ by fixing the $d$th element of any node tuple at some value $v$, for every $v \in \{0, 1, \ldots, k-1\}$. This results in $k$ copies $Q_0, Q_1, \ldots, Q_{k-1}$ of $Q^k_{n-1}$ (with $Q_v$ obtained by fixing the $d$th element at $v$), with corresponding nodes in $Q_0, Q_1, \ldots, Q_{k-1}$ joined in a cycle of length $k$ (in dimension $d$) (we suppress $d$, $n$ and $k$ in the notation as they are always understood).

Denote $F$ the set of faulty edges in $Q^k_n$, $F_i$ the set of faulty edges in subgraph $Q_i$, $i \in \{0, 1, \ldots, k-1\}$. $F^v$ (resp. $H^v$) denotes the set of faulty (resp. healthy) edges that incident with vertex $v$. $F^v_i$ (resp. $H^v_i$) denotes the set of faulty (resp. healthy) edges that lie in $Q_i$ and incident with $v$. The set of edges incident with node $v$ is denoted by $N^v$. Denote the set of faulty edges that are not incident with $u$ and $v$ as $F^F$. Given a node $v = (v_n, v_{n-1}, \ldots, v_2, v_1)$, which lies in $Q_{v_1}$. Denote the node $(v_n, v_{n-1}, \ldots, v_2, v_1 + i)$ as $v^i$, $-(k - 1) \leq i \leq k - 1$. In addition, $v^i$ (resp. $v^{-i}$) is also denoted as $v^+$ (resp. $v^-$). The distance between two node $u$ and $v$ is denoted by $dist(u, v)$. A path can be written as a list of nodes or as a list of nodes and edges mixed or sub-paths with nodes and edges mixed. We put them in a pair of squared brackets to denote that is a path.

III. Node-to-Node Disjoint Paths in Faulty $k$-ary 2-Cube, $k \geq 4$

We first find disjoint paths in $Q^k_2$ with at most two faulty nodes. Based on this, we find disjoint paths for $Q^k_2$, $k \geq 5$ with at most two faulty nodes by mapping the graph $Q^k_2$ to $Q^4_2$.

As the procedure of finding disjoint paths for $Q^k_2$ with two faulty edges is long and tedious, we put it in the appendix A.

In $Q^k_2$, one node associates with one row and one column, and one edge associates with one row and two columns, or one column and two rows. Denote the two faulty edges as $f_1$ and $f_2$. If there are at most four rows and at most four columns associate with $u, v, f_1$, and $f_2$, it can be reduced to finding disjoint paths in $Q^4_2$ by removing some of the non-associated rows and columns, and then extend the disjoint paths by inserting back the rows and columns. So, we will only consider the cases that there are more than four rows or columns associated. We will consider the following cases:

1. six columns are associated;
2. five columns are associated.

The cases with more than four rows associated can be mapped to the above two cases by the symmetric property of $Q^k_n$.

If there are 6 columns associated, then there are at most 4 rows associated, and the configuration can be described as follows: $u = (0, 0), v = (i_1, j_1), f_1 = ((i_2, j_2), (i_2, j_3)), f_2 = ((i_3, j_4), (i_3, j_5))$, where $1 \leq i_1, i_2, i_3, j_1, j_2, j_3, j_4, j_5 \leq k$, $j_3 = j_2 + 1$, $j_5 = j_4 + 1$ and all the $j_t, 1 \leq t \leq 5$ are different from each other. To reduce the case to $Q^4_2$, we keep all of the associated rows and some other rows if necessary to have 4 rows; to have 4 columns, we set $j_1 = j_2 = 1, j_4 = 2$, if $j_1 < j_2 < j_4$, or we set $j_2 = 0, j_1 = j_4 = 2$, if $j_2 < j_1 < j_4$. The other cases can be easily mapped to these two cases by the symmetric property of $Q^k_2$.

If there are 5 columns associated, the configuration can be described as: $u = (0, 0), v = (i_1, j_1), f_1 = ((i_2, j_2), (i_2, j_3)), f_2 = ((i_3, j_4), (i_4, j_4))$, where $1 \leq i_1, i_t \leq k, i_s \neq i_t, j_s \neq j_t$ for all $1 \leq s, t \leq 4$. To reduce the case to $Q^4_2$ (to have 4 columns), we set $j_1 = 1, j_2 = 2, j_4 = 3$, if $j_1 < j_2 < j_4$, and set $j_2 = 0, j_1 = 2, j_4 = 3$, if $j_2 < j_1 < j_4$. By the symmetric property of $Q^k_2$, the four rows can be kept in the same way as the columns if there are five rows associated, or we just keep all of associated rows and some other rows if necessary to have four rows. The other cases can be easily mapped to these two cases.

Thus, we have the following lemma.

Lemma 1: In a $k$-ary 2-cube $Q^k_2$, when there are at most two faulty edges, for any two given nodes $u$ and $v$, there exist $\min\{deg_u, deg_v\}$ disjoint paths between.
IV. NODE-TO-NODE DISJOINT PATHS IN
FAULTY k-ARY n-CUBE, n ≥ 4, k ≥ 4

There are at most $2n - 2$ faulty edges in $Q_n^k$, and there are $n$ dimensions. Hence, we can partition $Q_n^k$ on a dimension, such that there is at most one faulty edge lies on that dimension.

Given an edge $e = (s, t)$ on some path, define $U$-$jump(e)$ as replacing the edge $e$ by a sub-path of length 3: $[s, s^-, t^-, t]$; similarly, $D$-$jump(e)$ is defined as replacing $e$ by another sub-path of length 3: $[s, s^+, t^+, t]$.

W.l.o.g., suppose we partition $Q_n^k$ through dimension one. So that there is either one faulty edge or none of the faulty edges lie in this dimension.

The proof is technical and complicated. There are a number of cases to be considered. For example, w.l.o.g., suppose $u = (0, 0, \ldots, 0)$, we need to consider several different cases depends on where the node $v = (v_n, v_{n-1}, \ldots, v_1)$ lies and whether there is a faulty edge lies on dimension 1 or not. Suppose there is one faulty edge lies on the dimension 1, then we have to distinguish the value of $v_1$:

1. $v_1 = 0$, which means $u$ and $v$ lie in the same sub-graph $Q_0$;
2. $v_1 \neq 0$, which means $u$ and $v$ lie in different sub-graphs;

The technique in building disjoint paths for these two cases are very different. For example, for case (1), we can use induction on $Q_0$ by assuming some of the faulty edges are healthy if necessary so as to have at most $2n - 4$ faulty edges (if a path includes an edge, say $e$, that is originally fault, we need to do adjustment to avoid the faulty edge $e$, for example, do a $U$-$jump(e)$ or $D$-$jump(e)$), and then find some more paths to link the dimension 0 edges between them; while for case (2), as $u$ and $v$ lie in different sub-graphs, we can apply induction on each sub-graph by making some assumptions; however, we need to link these disjoint paths in different sub-graphs together, which makes the procedure complicated.

Furthermore, if we go into details for case (2), we have the following sub-cases to consider:

(2.1) $dist(u, v^{-v_1}) = 0$, which means the node $u$ and the node $v$ lie on the same dimension 1 circle. Node $u$ or its neighbor can reach node $v$ or its corresponding neighbor along the dimension 1 circle.

(2.2) $dist(u, v^{-v_1}) = 1$, which means node $u$ and node $v$ lie on different dimension 1 circles, but they have one overlap incident edge $(u, v^{-v_1})$.

(2.3) $dist(u, v^{-v_1}) \geq 2$, which means node $u$ and node $v$ lie on different dimension 1 circles, and they have no overlap incident edges.

These cases are similar to each other. However, the complexity of each case varies much. Due to the space limited, we will not include all of the proof in the paper, but the representative ones and the remainder will be left in the appendix.

We claim that case (2.2) is representative as most of the techniques used in the other cases will be used here. For example, use induction on $Q_0$ to find disjoint paths is similar to case (1); to extend these disjoint paths to $Q_{v_1}$ is similar to case (2.1) when $\exists e_1 \in F_u, e_1^{v_1} \notin F^v$ and $\exists e_2 \notin F^u, e_2^{v_1} \in F^v$, and case (2.3). Also, similar discussion on the case that the dimension one faulty edge does not block any path can be applied to the case when there are no dimension one faulty edges.

The condition for case (2.2) is:

1. There is one dimension one faulty edge $f = (r^h, r^{h+1})$;
2. $dist(u, v^{-v_1}) = 1$

In what follows, we will discuss case (2.2) in details.

**Case 2.2** $dist(u, v^{-v_1}) = 1$, i.e., There exists some $j \in \{2, 3, \ldots, n\}, v_j = 1$, and for all $i \in \{2, 3, \ldots, n\} \setminus \{j\}, v_i = 0$. W.l.o.g., suppose $j = 2$.

Let $e \in N^t$. For a given $i \in \{1, 2, \ldots, k-1\}$, we define $map(t, i)$ as temporarily mark $e^i$ as faulty if $e^i$ is faulty, or healthy otherwise.

Suppose $r, s$ lie in a same sub-graph $Q_i, i \in \{0, 1, \ldots, k-1\}$, and $|F_i| \leq 2n - 4$. Denote $DP(r, s)$ as a procedure of finding disjoint paths between $r$ and $s$ in the sub-graph $Q_i$ by induction.

Please note, if the following two paths exist, and are not blocked, we will always keep them
in the set of disjoint paths, and thus we may not mention them in the following discussion unless necessary:

\[ \sigma_1 : [u, v^-i, v]; \quad \text{and} \]
\[ \sigma_2 : [u, u^+, v]. \]

(A22-1) Assume \( f \) is healthy.

As there are now at most \( 2n - 3 \) faulty edges in \( Q_n \), we have the following cases to consider. Recall that we suppose \( |F_u| \geq |F^v| \).

Suppose \( v_1 = 1 \).

(A22-1.1) \( |F_u| = 2n - 3 \).

We only need to find 3 disjoint paths. Let \((u, s)\) be the only healthy edge in \( Q_0 \) that incident with \( u \). The two more paths are:

(a) \([u, u^-, v^2, v^-, v];\) and
(b) \([u, s, s^+, \ldots, s^2, u^2, v^2, v^+, v];\)

(A22-1.2) After doing map\((v, 0)\), we have \( |F_0| = 2n - 3 \).

If \( |F_u \cup F^v| = 2n - 3 \), we must have \( |F^v| \geq |F^v| + 1 \). Assume a faulty edge \((u, s)\) is healthy, and do DP\((u, v^-)\). Remove the node \( v^- \) from each disjoint path, and extend to \( v \). For simplicity, this procedure after doing DP\((u, v^-)\) will be called “extend” the paths to \( v \) in throughout the remainder part of this section. Two more paths can be:

(a) \([u, u^+, u^2, v^+, v];\) and
(b) \([u, u^-, v^2, v^-, v];\)

If there is a faulty edge \( e \) that is neither incident with \( u \) nor with \( v^- \), assume \( e \) is healthy and do DP\((u, v^-)\); extend the paths to \( v \). Do U-jump\((e)\) if \( e \) lies on some of the disjoint paths. Two more paths are:

(a) \([u, u^+, u^2, v^+, v];\) and
(b) \([u, u^-, v^2, v^-, v];\)

(A22-1.3) \( |F_1| \leq 2n - 4 \), and after doing map\((v, 0)\), we have \( |F_0| \leq 2n - 4 \).

Do map\((v, 0)\) and DP\((u, v^-)\). Extend the paths to \( v \). Two more paths are:

(a) \([u, u^+, u^2, v^+, v];\) and
(b) \([u, u^-, v^2, v^-, v];\)

From our above construction method, in a similar way to the case \( v_1 = 1 \) but extending the dimension 1 edges, we will be able to find disjoint paths for the case \( v_1 \geq 2 \).

If the dimension 1 edge \( f \) lies on some of the above path, we will rebuild disjoint paths as follows.

(A22-2) \( f \) is blocking some path.

Suppose \( v_1 = 1 \).

(A22-2.1) \( f = (u, u^+) \).

(A22-2.1.1) \( |F_u| = |F^v| \).

We first do map\((v, 0)\). But \((u, v^-)\) is marked as faulty if either \((u, v^-)\) is originally faulty or \((u^+, v)\) is faulty.

If \((u, v^-) \in F\), and \((u^+, v) \notin F\), assume two of \( v^- \)'s faulty links \( e_1, e_2 \) are healthy. Do DP\((u, v^-)\). Let edge \((u, w)\) lies on the path that \( e_1 \) lies on, and \((u, z)\) lies on the path that \( e_2 \) lies on. Remove the paths that contains \( e_1 \) or \( e_2 \). For all of the other paths, extend them to \( v \). Build the following three paths:

(a) \([u, u^-, \rho_1, v^2, v^-, v];\)
(b) \([u, z; z^2, \rho_2, v^+, v];\) and
(c) \([u, w, w^+, \rho_3, u^+];\)

where \( \rho_1 \in Q_{k-1}, \rho_2 \in Q_2 \) and \( \rho_3 \in Q_1 \), and they can be found by induction if not straightforward. Specifically, \( \rho_3 \) can be build by finding disjoint paths between \( w^+ \) and \( v \) (note that there are at most \( 2n - 4 \) faulty edges lie in \( Q_1 \)).

If \((u, v^-) \notin F\), and \((u^+, v) \notin F\), do map\((v, 0)\).

If there are more than \( 2n - 4 \) faulty edges in \( Q_0 \), we assume \((u, v^-)\) is healthy. Do DP\((u, v^-)\); remove path \([u, v^-]\) extend the paths to \( v \). Build one more path:

\([u, u^-, \ldots, u^{v-1}, \rho, v^+, v],\)

where \( \rho \in Q_{v_1+1} \).

If either \((u, v^-) \in F\) and \((u^+, v) \in F\), or \((u, v^-) \notin F\) and \((u^+, v) \notin F\), do map\((v, 0)\); assume \( e \in F^v \) is healthy. Do DP\((u, v^-)\). Let \((u, w)\) is on the path containing \( e \). Remove this path, and extend all other paths to \( v \). Build the following two paths:

(a) \([u, w, w^+, \rho_1, u^+, v];\) where \( \rho_1 \) can be chosen from the disjoint paths between \( w^+ \) and \( v \) in \( Q_1 \); and
(b) \([u, u^{k-1}, \ldots, u^{v-1}, \rho_2, v^+, v];\) where \( \rho_2 \in Q_{v_1+1} \) and can be found by induction.

If \((u, v^-) \in F\) and \((u^+, v) \in F\), path (a) is not needed.
(A22-2.1.2) $|F^u| > |F^v|$

(A22-2.1.2.1) $(u, v^-) \in F$ and $(u^+, v) \notin F$.

If $|F^u| = |F^v| + 1$, then after $map(v, 0)$, we have $|F^0_v| = |F^0_v^-| + 1$. Assume faulty edge $e$ that incident with $v^-$ is healthy. Do $DP(u, v^-)$. Let $(u, w)$ lie on the path that passes the edge $e$. Remove this path and extend all other paths to $v$.

Build the following two paths:
(a) $[u, w, w^-, \rho_1, v^-, v, \rho_1, v^-; e, v^-, v], \rho_1 \in Q_{k-1}$
(b) $[u, u^-, \ldots, u^{n+1}, \rho_2, v^+, v], \rho_2 \in Q_{v+1}$

If $|F^u| \geq |F^v| + 2$, then after $map(v, 0)$, assume one of the faulty edges $e$ that incidents with $u$ is healthy. Do $DP(u, v^-)$. Remove the path that passes the edge $e$ and extend all other paths to $v$.

Build the following path:
$[u, u^-; \ldots, u^{n+1}, \rho_2, v^+, v], \rho_2 \in Q_{v+1}$

(A22-2.1.2.2) $(u, v^-) \notin F$ and $(u^+, v) \in F$; or $(u, v^-) \in F$ and $(u^+, v) \in F$; or $(u, v^-) \notin F$ and $(u^+, v) \notin F$.

Do $map(v, 0)$. If there are at most $2n - 4$ faulty edges in $Q_0$, do $DP(u, v^-)$, and extend the paths to $v$. Build another path:
$[u, u^-; \ldots, u^{n+1}, \rho_2, v^+, v], \rho_2 \in Q_{v+1}$

If there are $2n - 3$ faulty edges in $Q_0$ and $(u, v^-) \in F$ after $map(v, 0)$, we assume $(u, v^-)$ is healthy and do $DP(u, v^-)$. Remove the path $[u, v^-]$. Extend all other paths to $v$, and build one more path as above.

If there are $2n - 3$ faulty edges in $Q_0$ and $(u, v^-) \notin F$ after $map(v, 0)$, we have $(u, v^-) \notin F$ and $(u^+, v) \notin F$. Hence, we assume one of $v^-$’s faulty link $e$ is healthy and do $DP(u, v^-)$. Remove the path that passes edge $e$ and extend all other path to $v$. Build one more path as above.

(A22-2.2) $f = (u, u^-)$

(A22-2.2.1) $|F^u| = |F^v|

(A22-2.2.1.1) $(u, v^-) \in F$ and $(u^+, v) \notin F$.

Do $map(v, 0)$, and assume faulty edges $e_1, e_2$ incident with $v^-$ are healthy. Do $DP(u, v^-)$. Let $(u, w)$ lies on the path that passes $e_1$, and $(u, z)$ lies on the path that passes $e_2$. Remove these two paths and extend all other paths to $v$. Build the following two paths:
(a) $[u, w, w^-, \rho_1, v^-, v, \rho_1, v^-, v], \rho_2 \in Q_{k-1}$, and
(b) $[u, z^+, z^2, \rho_2, v^+, v], \rho_2 \in Q_2$.

$\rho_1$ and $\rho_2$ can be built by induction in their corresponding sub-graph.

(A22-2.2.1.2) $(u, v^-) \notin F$ and $(u^+, v) \in F$.

Do $map(v, 0)$ but mark $(u, v^-)$ as healthy, and do $DP(u, v^-)$. Extend the paths to $v$. Build one more path:
$[u, u^+, u^2, \rho_2, v^+, v], \rho_2 \in Q_2$.

(A22-2.2.1.3) $(u, v^-) \in F$ and $(u^+, v) \in F$.

Do $map(v, 0)$, and $DP(u, v^-)$. Extend the disjoint paths to $v$. Build the same one more path as in case (A22-2.2.1.2).

(A22-2.2.1.4) $(u, v^-) \notin F$ and $(u^+, v) \notin F$.

Do $map(v, 0)$. There exists one faulty edge $e$ incident with $v^-$. We assume $e$ is healthy, and find $DP(u, v^-)$. Remove the path that passes the edge $e$ (if such path exists), and extend all other paths to $v$.

(A22-2.2.2) $|F^u| > |F^v|

(A22-2.2.2.1) $(u, v^-) \in F$ and $(u^+, v) \notin F$.

Do $map(v, 0)$. If $|F^u| = |F^v| + 1$, assume edge $e$, a faulty edge incident with $v^-$, is healthy, and do $DP(u, v^-)$. Let $(u, w)$ lies on the path passes $e$. Remove this path and expand all other paths to reach $v$. Build one more path:
$[u, w, w^-, \rho, v^-, v^-, v, \rho, v^-; e, v^-; v], \rho \in Q_0$

If $|F^u| \geq |F^v| + 2$, assume edge $(u, v^-)$ is healthy, and do $DP(u, v^-)$. Remove the path $[u, v^-]$, and extend all other paths to $v$.

(A22-2.2.2.2) $(u, v^-) \notin F$ and $(u^+, v) \notin F$.

Do $map(v, 0)$. If there are no more than $2n - 4$ faulty edges in $Q_0$, do $DP(u, v^-)$, and extend the disjoint paths to $v$.

If there are $2n - 3$ faulty edges in $Q_0$ and $|F^u_0| = |F^v_0|$, then there must exist $e$ that is neither incident with $u$ nor incident with $v^-$. Assume $e$ is healthy and do $DP(u, v^-)$. If some path passes edge $e$, do $U\text{-}jump(e)$. Extend these paths to $v$. 
(A22-2.2.3) \( (u^+, v) \in F \).
No matter whether \( (u, v^-) \) is faulty or not, we do \( map(v, 0) \), and assume \( (u, v^-) \) is healthy and do \( DP(u, v^-) \). Extend the disjoint paths to \( v \).
Build one more path:
\[ [u, u^+, u^2, \rho, v^+, v]. \]

(A22-2.3) \( f = (v, v^-) \).
If \( (u, v^-) \notin F \) and \( (u^+, v) \in F \), we assume two of \( u \)'s incident faulty edges \( e_1, e_2 \) (lie in \( Q_0 \)) are healthy. Do \( DP(u, v^-) \). Suppose \( w_1 \) (resp. \( w_2 \)) lies on the path that edge \( e_1 \) (resp. \( e_2 \)) lies on, where \( w_1 \) and \( w_2 \) are neighbors of \( v^- \). Remove these two paths, extend all other paths to \( v \), and build the following three paths:

(a) \[ [u, v^-, v^2, \ldots, v^+, v]; \]
(b) \[ [u, u^+, \rho_1, w^{-1}, w_1, w^+_1, v], \rho_1 \in Q_{k-1}; \] and
(c) \[ [u, u^+, u^2, \rho_2, w^+_2, w^+_2, v], \rho_2 \in Q_2. \]

Based on the above paths, if \( (u^+, v) \notin F \), no need to build the above path (c) as \( u, u^+, v \) is already one of the disjoint paths; if \( (u, v^-) \notin F \), no need to build the above path (a).

(A22-2.4) \( f = (v, v^+). \)

(A22-2.4.1) \( (u, v^-) \notin F \) and \( (u^+, v) \in F \).
There exist faulty edges \( e_1, e_2 \) incident with node \( u \). Assume they are healthy, and do \( DP(u, v^-) \). Let \( w_i \) lies on the path that passes edge \( e_i \), for \( i = 1, 2 \). Remove path that passes \( e_1 \) and \( e_2 \). Extend all of the other paths to \( v \). Build the following two more paths:

(a) \[ [u, u^-, \rho_1, w^{-1}, w_1, w^+_1, v], \rho_1 \in Q_{k-1}; \]
(b) \[ [u, u^+, \rho_2, w^-2, w^2, w^+_2, v], \rho_2 \in Q_2. \]

(A22-2.4.2) \( (u, v^-) \notin F \) and \( (u^+, v) \notin F \).
Similar to case (A22-2.4.1), but only assume one faulty edge \( e_1 \) is healthy, and only need to find one more path (a) as in case (A22-2.4.1).

(A22-2.4.3) \( (u, v^-) \in F \) and \( (u^+, v) \in F \).
There exists \( e_1 \in F^u \). Assume \( e_1 \) is healthy, and do \( DP(u, v^-) \). Let \( w_1 \) lies on the path that passes the edge \( e_1 \). Remove this path and extend all other paths to \( v \). Build the following two paths:

(a) \[ [u, v^-, \rho_1, v^2, v_-, v]; \]
(b) \[ [u, u^+, \rho_2, w^-2, w^2, w^+_2, v]. \]

(A22-2.4.4) \( (u, v^-) \in F \) and \( (u^+, v) \notin F \).
Similar to case (A22-2.4.3), but no need to make any assumption, except when \( |F_0| = 2n - 3 \) after \( map(v, 0) \). If the exceptional case happens, we assume \( (u, v^-) \) is healthy and then do \( DP(u, v^-) \). Extend the paths to \( v \).

(A22-2.5) \( f = (u^h, u^{h+1}) \) or \( f = (v^h, v^{h+1}), h \neq 0, -1 \).
In this case, the two paths below are included in the disjoint paths set:

(a) \[ [u, v^-, v^-]; \] and
(b) \[ [u, u^+, v]. \]

We need to find a path linking edge \( (u, u^-) \) and edge \( (v, v^+) \) but avoid the faulty edge \( f \). It can be either

(a) \[ [u, u^-, \rho_1, v^-2, v_-, v], \rho_1 \in Q_{k-1}, \]
if \( f = (u^h, u^{h+1}) \) for some \( h \neq 0, -1 \), or
(b) \[ [u, u^+, \rho_2, w^-2, w^2, v], \rho_2 \in Q_2, \]
if \( f = (v^h, v^{h+1}) \) for some \( h \neq 0, -1 \).

The other paths can be build similar to case (A22-1).

As we assume \( v_1 = 1 \) at the beginning of case (A22-2), we now consider \( v_1 > 1 \).

Compare to the case (A22-1), where we assume \( f \) is healthy and when \( v_1 > 1 \), here we only need to care about the case when \( f = (r^h, r^{h+1}) \) with \( 0 \leq h \leq v_1 \) but \( f \) is neither incident with \( u \) nor incident with \( v \), and blocks some of the paths as constructed in case (A22-1) (when \( v_1 > 1 \)). We consider the following cases:

- If \( r = u \), there exists \( s \), a neighbor of \( u \), such that \( (u^s, s^j) \) is healthy for every \( j = 0, 1, \ldots, k - 1 \). This is true, as there are at most \( 2n - 3 \) faulty edges in all \( Q_i, i = 0, 1, \ldots, k - 1 \), and each node in \( Q_i \) has \( 2n - 2 \) neighbors inside \( Q_i \). Thus, we adjust the blocked path by replacing the faulty edge with a sub-path: \([r^h, s^h, s^{h+1}, r^{h+1}] \). Note that if \( h = v_1 - 1 \) or \( h = 0 \), some slight adjustment is necessary, which will be simple and is not stated here.

However, it might be the case that the only healthy edge in \( Q_h \) for node \( r^h \) is \( v^h v_1 \), and \( v^h v_1 \) is already on some path. In this case, we change to partition the graph \( Q_n \) through dimension 2. This will result in one dimension two faulty edge. For convenience,
we also call a sub-graph \( Q_i \) if the dimension two number is \( i \); we also denote the a node \((v_n, v_{n-1}, \ldots, v_2 + i, v_1)\) as \( v^i \) if \( v = (v_n, v_{n-1}, \ldots, v_2, v_1) \), for \(-k + 1 \leq i \leq k - 1\). Now we have \( \text{dist}(u, v^{-i}) > 1 \), and the only dimension 2 faulty edge is: \((0, 0, \ldots, 1, i)\). As there are no faulty edges in the sub-graph \( Q_0 \), we find disjoint paths by induction between \( u \) and \((0, 0, \ldots, 0, v_1)\), and extend these paths to \( v \). Thus we have found \(2n - 4\) disjoint paths.

Two more paths can be:

(a) \[ [u, u^-, p_1, v^-2, v^-, v], p_1 \in Q_{k-1} \text{ can be found by induction; } \]
(b) \[ [u, u^+, u^2, p_2, v^+, v], p_2 \in Q_2 \text{ can be found by induction. } \]

This is essentially a small case of Case 2.3: \( \text{dist}(u, v^{-i}) > 1 \) in appendix B. For the case that node \( r \) is a neighbor of \( v^- \), it can be done similar to the case \( r = u \).

- If \( r = v^{-v_1} \), and the only healthy edge incident with \( r^h \) is \( u^h \), then repartition the graph similar to the case above. Otherwise, there exists \((r^h, s^h) \notin F \) and \((r^{h+1}, s^{h+1}) \notin F \), replace the faulty edge \( f \) by a sub-path of length three: \([r^h, s^h, s^{h+1}, r^{h+1}]\). If \((s^h, s^{h+1})\) is already on some path and whose dimension one part is \( s^x, s^{x+1}, \ldots, s^y \), we replace these edges by \( s^{x-1}, \ldots, s^{y+1}, s^y \).

Based on Lemma 1 and the above discussion include the sections in the appendix, we obtain the following result.

**Theorem 1:** In a \( k \)-ary \( n \)-cube with at most \(2n - 2\) faulty edges, there exist \( \min\{\deg_H(u), \deg_H(v)\} \) disjoint paths between any two given nodes \( u \) and \( v \).

V. CONCLUSION

We have proved that there exist \( \min\{\deg_H(u), \deg_H(v)\} \) node-disjoint paths in \( k \)-ary \( n \)-cube when there exist at most \(2n - 2\) faulty edges. This will make the \( k \)-ary \( n \)-cube interconnection network robust in communicating and routing even though there have up to \(2n - 2\) faulty edges. Our future research will focus on finding shortest node-disjoint paths for \( k \)-ary \( n \)-cubes with faulty edges and/or faulty nodes.

**REFERENCES**


To the referee: the appendix part will be removed if the paper being accepted by the conference, but we will put the related information on our website: http://www.dur.ac.uk/yonghong.xiang/ for public access, and we will direct the reader to our website in the final version of the paper as well.

APPENDIX A.

DISJOINT PATHS IN $Q_2^4$

Without lose of generality (W.l.o.g.), suppose $u = (0, 0)$. By the vertex symmetric property of $Q_n^k$, for node $v$, only the following five cases need to be considered: $(0, 1), (0, 2), (1, 1), (2, 1), (2, 2)$.

When there are no faulty edges in $Q_2^4$, if a path passes node $(0, 1)$ (resp. $(1, 0), (3, 0)$, and $(0, 3)$), then we name the path $\rho_{10}$ (resp. $\rho_{20}, \rho_{30}$, and $\rho_{40}$). We call these four of $u$'s neighbors the residual nodes. Denote $\rho_{ij}$ as the $j$th alternative path to path $\rho_{i0}$ if it passes the same residual vertex as $\rho_{i0}$.

1) $v = (0, 1)$: Suppose there are no faulty edges in $Q_2^4$, then we can build four disjoint paths between $u$ and $v$ as follows (see Fig.1(a)).
\[
\begin{align*}
\rho_{10} & : [u, v]; \\
\rho_{20} & : [u, (1, 0), (1, 1), v]; \\
\rho_{30} & : [u, (3, 0), (3, 1), v]; \\
\rho_{40} & : [u, (0, 3), (0, 2), v].
\end{align*}
\]

Let $e_1 = ((1, 0), (1, 1)), e_2 = ((3, 0), (3, 1)), e_3 = ((0, 2), (0, 3))$. From Fig.1(b), we can see that if any two of $e_1, e_2, e_3$ are faulty, we have the following two sets of alternative paths $\rho_{21}, \rho_{31}$ and $\rho_{41}$ (see Fig. 1(b)) and $\rho_{22}, \rho_{32}$ and $\rho_{42}$.
\[
\begin{align*}
\rho_{21} & : [u, (1, 0), (2, 0), (2, 1), (1, 1), v]; \\
\rho_{31} & : [u, (3, 0), (3, 3), (3, 2), (3, 1), v]; \\
\rho_{41} & : [u, (0, 3), (1, 3), (1, 2), (0, 2), v];
\end{align*}
\]
\[
\begin{align*}
\rho_{22} & : [u, (1, 0), (1, 3), (1, 2), (1, 1), v]; \\
\rho_{32} & : [u, (3, 0), (2, 0), (2, 1), (3, 1), v]; \\
\rho_{42} & : [u, (0, 3), (3, 3), (3, 2), (0, 2), v].
\end{align*}
\]

$\rho_{ij}$ is disjoint with $\rho_{it}, t \neq j$. So, if $e_i$ is faulty for some $i \in \{1, 2, 3\}$ and the other faulty edge lies on one of its alternative path, then we can use the other alternative path.

Suppose $\{f_1, f_2\} \cap (F^u \cup F^v) \neq \emptyset$. (a) If $(u, v) \in F$, then, the problem reduces to the case discussed above (just remove the path $[u,v]$). If only one of the faulty edges linked with $u$ or $v$, then we only need to build 3 disjoint paths and this again reduces to the case discussed above. (b) If the both faulty edges are linked with $u$ or both linked with $v$, or one links with $u$ and one links with $v$ but they both lie on $\rho_{i0}$ for some $i \in \{1, 2, 3, 4\}$. All we need to do is to remove the path(s) that contain faulty edges. (c) If the both faulty edges are linked with different nodes and are not lying on the same path, then we need to adjust some of the $\rho_{i0}$ to obtain 3 disjoint paths. When $(u, (1, 0))$ and $(v, (0, 2))$ are faulty, we keep path $\rho_{10}$ and $\rho_{30}$, and build the following path (see Fig.1(c)):
\[
\rho_{43} : [u, (0, 3), (0, 2), (1, 2), (1, 1), v].
\]

When $(u, (1, 0))$ and $(v, (3, 1))$ are faulty, keep path $\rho_{10}$ and $\rho_{40}$, and build a new path (see Fig.1(d)):
\[
\rho_{33} : [u, (3, 0), (3, 1), (2, 1), (1, 1), v].
\]

2) $v \in \{(0, 2), (1, 1), (1, 2), (2, 2)\}$: If there are no faulty edges in $Q_2^4$, we can easily build four disjoint paths $\rho_{10}, \rho_{20}, \rho_{30}, \rho_{40}$ between $u$ and $v$.

For $v = (0, 2)$, we have: (See Fig.2(a))
For $v = (1, 1)$, we have: (See Fig.2(c))

- $\rho_{10} : [u, (0, 1), v]$;
- $\rho_{20} : [u, (1, 0), (1, 1), (1, 2), v]$;
- $\rho_{30} : [u, (3, 0), (3, 1), (3, 2), v]$;
- $\rho_{40} : [u, (0, 3), v]$;
- $\rho_{21} : [u, (1, 0), (1, 1), (1, 2), v]$;
- $\rho_{31} : [u, (3, 0), (3, 3), (3, 2), v]$.

For $v = (1, 2)$, we have: (See Fig.3(a))

- $\rho_{10} : [u, (0, 1), (0, 2), v]$;
- $\rho_{20} : [u, (1, 0), (1, 1), v]$;
- $\rho_{30} : [u, (3, 0), (3, 1), (2, 1), v]$;
- $\rho_{40} : [u, (0, 3), (1, 3), (1, 2), v]$;
- $\rho_{31} : [u, (3, 0), (2, 0), (2, 1), v]$;
- $\rho_{41} : [u, (0, 3), (0, 2), (1, 2), v]$.

For $v = (2, 2)$, we have: (See Fig.3(c))

- $\rho_{10} : [u, (0, 1), (0, 2), (1, 2), v]$;
- $\rho_{20} : [u, (1, 0), (2, 0), (2, 1), v]$;
- $\rho_{30} : [u, (3, 0), (3, 1), (3, 2), v]$;
- $\rho_{40} : [u, (0, 3), (1, 3), (2, 3), v]$;
- $\rho_{11} : [u, (0, 1), (3, 1), (2, 1), v]$;
- $\rho_{21} : [u, (1, 0), (1, 3), (1, 2), v]$;
- $\rho_{31} : [u, (3, 0), (2, 0), (2, 3), v]$;
- $\rho_{41} : [u, (0, 3), (0, 2), (3, 2), v]$.

If none of the $F^u \cup F^v = \emptyset$, and one faulty edge is on some path, one not, then we can find an alternative path for the blocked path as shown above. Hence, we only need to consider when one faulty edge lies on some path, say $f_i \in \rho_{i0}, i \in \{1, 2, 3, 4\}$, and the other one lies on its alternative path $f_2 \in \rho_{i1}$.

1) $v = (0, 2)$ (please refer to Fig.2(a)): if $\rho_{20}$ and $\rho_{21}$ are both blocked, i.e., one of $e_1 = ((1, 0), (1, 1))$ and $e_2 = ((1, 1), (1, 2))$, and one of $((1, 0), (1, 3))$ and $((1, 2), (1, 3))$ are faulty, then define $\rho_{22} : [u, (1, 0), (2, 0), (2, 1), (2, 2), v]$; if $\rho_{30}$ and $\rho_{31}$ are both blocked, then define $\rho_{32} : [u, (3, 0), (2, 0), (2, 1), (2, 2), (3, 2), v]$.

2) $v = (1, 1)$ (please refer to Fig.2(c)): if $\rho_{30}$ and $\rho_{31}$ are both blocked, then define $\rho_{32} : [u, (3, 0), (3, 3), (2, 3), (2, 2), (2, 1), v]$; if $\rho_{40}$ and $\rho_{41}$ are both blocked, then define $\rho_{42} : [u, (0, 3), (3, 3), (3, 2), (2, 2), (1, 2), v]$;
3) $v = (2, 1)$ (please refer to Fig.3(a)): if $\rho_{10}$ and $\rho_{11}$ are both blocked, then define $\rho_{22} : [u, (0, 1), (1, 1), (1, 1), v]$; if $\rho_{20}$ and $\rho_{21}$ are both blocked, then define $\rho_{22} : [u, (1, 0), (1, 3), (3, 2), (2, 0), v]$ and $\rho_{42}$ be $u, (0, 3), (0, 2), (1, 2), (2, 2), v$; if $\rho_{30}$ and $\rho_{31}$ are both blocked, then define $\rho_{32} : [u, (3, 0), (3, 3), (3, 2), (3, 1), v]$; if $\rho_{40}$ and $\rho_{41}$ are both blocked, then define $\rho_{42} : [u, (0, 3), (3, 3), (3, 2), (2, 2), v]$.

4) $v = (2, 2)$ (please refer to Fig.3(c)): if $\rho_{10}$ and $\rho_{11}$ are both blocked, then define $\rho_{22} : [u, (0, 1), (1, 1), v]$; if $\rho_{20}$ and $\rho_{21}$ are both blocked, then define $\rho_{22} : [u, (1, 0), (1, 1), v]$; if $\rho_{30}$ and $\rho_{31}$ are both blocked, then define $\rho_{32} : [u, (0, 3), (3, 3), (2, 3), v]$; if $\rho_{40}$ and $\rho_{41}$ are both blocked, then define $\rho_{42} : [u, (3, 0), (3, 3), (3, 2), v]$.

If only one faulty edge is linked with $u$ or $v$, then we only need to build 3 disjoint paths between $u$ and $v$, and this reduces to what we have discussed above.

If both faulty edges are of $u$ and/or $v$’s neighbors, then we have the following several cases.

1) both belong to either $u$ or $v$’s neighbors, then we only need to find two disjoint paths and it has been done;
2) one is $u$’s neighbor, and the other is $v$’s neighbor, and they are on the same path as shown in Fig.1~Fig.3, then we have done as well;
3) one is $u$’s neighbor, and the other is $v$’s neighbor, and they are not on the same path as shown in the above Figures, we need to rebuild some paths.

- $v = (0, 1)$ (refer to Fig.1(a)): if $\rho_{20}$ and $\rho_{30}$ are blocked, define $\rho_{32} : [u, (3, 0), (3, 1), (2, 1), (1, 1), v]$.
  if $\rho_{20}$ and $\rho_{40}$ are blocked, then define $\rho_{42} : [u, (0, 3), (1, 3), (1, 2), (1, 1), v]$.
  if $\rho_{30}$ and $\rho_{40}$ are blocked, then define $\rho_{42} : [u, (0, 3), (3, 3), (3, 2), (3, 1), (0, 1), v]$
- $v = (0, 2)$ (refer to Fig.2(a)): if $\rho_{10}$ and $\rho_{20}$ are blocked, define $\rho_{22} : [u, (1, 0), (1, 1), (0, 1), v]$; if $\rho_{10}$ and $\rho_{30}$ are blocked, then define $\rho_{32} : [u, (3, 0), (3, 1), (0, 1), v]$; if $\rho_{10}$ and $\rho_{40}$ are blocked, then define $\rho_{42} : [u, (1, 0), (1, 1), (0, 1), v]$.
  if $\rho_{20}$ and $\rho_{30}$ are blocked, then define $\rho_{32} : [u, (3, 0), (3, 1), (0, 1), v]$; if $\rho_{20}$ and $\rho_{40}$ are blocked, then define $\rho_{42} : [u, (0, 3), (1, 3), (1, 2), v]$; if $\rho_{30}$ and $\rho_{40}$ are blocked, then define $\rho_{42} : [u, (0, 3), (3, 3), (3, 2), v]$.

- $v = (1, 1)$ (refer to Fig.2(c)): if $\rho_{10}$ and $\rho_{20}$ are blocked, then define $\rho_{32} : [u, (3, 0), (3, 1), (0, 1), v]$; and $\rho_{32} = [u, (1, 0), (2, 0), (2, 1), v]$; if $\rho_{10}$ and $\rho_{30}$ are blocked, then define $\rho_{32} : [u, (3, 0), (3, 1), (0, 1), v]$; if $\rho_{10}$ and $\rho_{40}$ are blocked, then define $\rho_{42} : [u, (0, 3), (3, 3), (3, 2), (2, 2), v]$; if $\rho_{30}$ and $\rho_{40}$ are blocked, then define $\rho_{42} = [u, (0, 3), (1, 3), (1, 2), v]$.

- $v = (2, 1)$ (refer to Fig.3(a)): if $\rho_{10}$ and $\rho_{20}$ are blocked, define $\rho_{22} = [u, (1, 0), (1, 1), v]$; if $\rho_{10}$ and $\rho_{30}$ are blocked, define $\rho_{32} = [u, (3, 0), (3, 1), (0, 1), v]$; if $\rho_{10}$ and $\rho_{40}$ are blocked, define $\rho_{42} = [u, (0, 3), (0, 2), (0, 1), v]$; if $\rho_{20}$ and $\rho_{30}$ are blocked, define $\rho_{32} = [u, (3, 0), (2, 0), (2, 1), v]$; if $\rho_{20}$ and $\rho_{40}$ are blocked, then define $\rho_{42} = [u, (0, 3), (1, 3), (1, 0), v]$; if $\rho_{30}$ and $\rho_{40}$ are blocked, define $\rho_{42} = [u, (0, 3), (3, 3), (3, 2), (3, 1), v]$.

- $v = (2, 2)$ (refer to Fig.3(c)), we have the following cases to consider: if $\rho_{10}$ and $\rho_{20}$ are blocked, define $\rho_{22} = [u, (1, 0), (1, 1), (1, 2), v]$; if $\rho_{10}$ and $\rho_{30}$ are blocked, define $\rho_{32} = [u, (1, 0), (1, 1), (1, 2), v]$, and
We have three cases to consider:

Case 1.1 $v_1 = 0$. This means that $u$ and $v$ lie in $Q_0$.

**Case 1.1** $|F_0| = 2n - 3$.

(A11-1) $f \in F^u \cup F^v$. W.l.o.g., let $f = (u, u^-)$.

(A11-1.1) $|F^u| \geq |F^v| + 1$.

If there exists $e \in F^F$, we assume $e$ is healthy, and find disjoint paths in $Q_0$ by induction. If $e$ lies on some path, then $e$ is a $U$-jump. Build one more path as follows: $[u, u^+, \rho, v^+, v]$, where the sub-path $\rho$ is a path in $Q_1$ that links with $u^+$ and $v^+$.

Please note that $\rho$ can be easily obtained by induction. For example, to find $\rho$ in $Q_1$, even if $|F_1| > 2n - 4$, we can still find such a path by assume some of the faulty edges are healthy until $|F_1| \leq 2n - 4$ is satisfied and then use induction to find disjoint paths. Let $\rho$ be one path that doesn’t include originally faulty edges. This is true, as there are at most $2n - 3$ faulty edges ($f$ lies outside any $Q_i$, for $i = 0, 1, \ldots, k - 1$) in $Q_1$, and each node has $2n - 2$ neighbors.

If $F^F = \emptyset$, and $(u, v) \in F$, assume $(u, v)$ is healthy. Find disjoint paths in $Q_0$ by induction. Remove the path $[u, v]$; find a path: $[u, u^+, \rho, v^+, v]$, where $\rho$ is a path in $Q_1$, and we have done.

If $F^F = \emptyset$, and $(u, v) \notin F$, we have $|F^u| \geq |F^v| + 2$. Assume faulty edge $(u, w)$ is healthy, find disjoint path by induction in $Q_0$, and remove the path that passes $w$; find a path: $[u, u^+, \rho, v^+, v]$, where $\rho$ is a path in $Q_1$. We have done.

(A11-1.2) $|F^u| \leq |F^v|$. We have $|F^u| \leq |F^v| - 1$. Assume faulty edge $(w, v)$ is healthy, and find disjoint paths in $Q_0$ by induction. There is a path that passes the node $w$, do a $U$-jump. Find one more path: $[u, u^+, \rho, v^+, v]$.

(A11-2) $f \notin F^u \cup F^v$.

(A11-2.1) $|F^F| \geq 1$.

Let $e = (s, t) \in F^F$, such that $(s, s^+) \notin F$ and $(t, t^+) \notin F$, find disjoint paths in $Q_0$ by induction. If $e$ lies on some path, do a $D$-jump. Find a path in $Q_1$ that links with $u^+$ and $v^+$ and avoid $e$; find a path in $Q_{k-1}$ that links with $u^-$ and $v^-$.

(A11-2.2) $|F^F| = 0$.

If $|F^u| > |F^v|$, assume faulty edge $(u, w)$ is healthy, and find disjoint paths by induction in $Q_0$. Remove the path that passes the node $w$. Find two paths: $[u, u^-, \rho_1, v^-, v]$, and $[u, u^+, \rho_2, v^+, v]$, where $\rho_1 \in Q_{k-1}$ and $\rho_2 \in Q_1$.

If $|F^u| = |F^v|$, as $|F^F| = 0$, we must have $(u, v) \notin F$; assume it is healthy, and find disjoint paths in $Q_0$ by induction. Remove the path $[u, v]$. Find two paths: $[u, u^-, \rho_1, v^-, v]$, and $[u, u^+, \rho_2, v^+, v]$, where $\rho_1 \in Q_{k-1}$ and $\rho_2 \in Q_1$.

**Case 1.2** $|F_0| \leq 2n - 4$.

(A12-1) $f \in F^u \cup F^v$. W.l.o.g., let $f = (u, u^-)$.

(A12-1.1) $|F^u| \geq |F^v| + 1$.

By induction, find disjoint paths in $Q_0$, and find a path $[u, u^+, \rho, v^+, v]$, where $\rho \in Q_1$.

(A12-1.2) $|F^u| \leq |F^v|$.

Assume faulty edge $(w, v)$ is healthy, and by induction, find disjoint paths in $Q_0$. Suppose $(u, s)$ lies on the path that passes node $w$. Remove this path. Find two more paths: $[u, s, s^-, \rho_1, v^-, v]$, and $[u, u^+, \rho_2, v^+, v]$, where $\rho_2 \in Q_1$.

(A12-2) $f \notin F^u \cup F^v$.

By induction, find disjoint paths in $Q_0$. Now, we need to find another two disjoint paths.

(A12-2.1) It is possible that there exist $i = 1$ or $i = k - 1$, such that $|F_i| > 2n - 4$. W.l.o.g., suppose $F_{k-1} > 2n - 4$. Hence, $F_1 < 2$. So, in $Q_1$, by induction, we find disjoint paths between $u^+$ and $v^+$, and choose the shortest one as the one
we need. To find a path between \( u^- \) and \( v^- \), we have the following two cases.

(A12-2.1.1) \(|F_{k-1}| = 2n - 2\).

A special case is that \(|F_{k-1}^u| = 2n - 2\) or \(|F_{k-1}^v| = 2n - 2\). It can be solved by finding a shortest path between \( u^- \) and \( v^- \), and do a \( U\)-jump to avoid the faulty edge on the path.

If \(|F^F| \geq 2\), we assume two of \( F^F \) edges to be healthy, and by induction find disjoint paths between \( u^- \) and \( v^- \) in \( Q_{k-1} \). Choose one of the shortest path that has no original faulty edges on as the one we need.

If \(|F^F| \leq 1\), W.l.o.g., suppose \(|F_{k-1}^u| \geq |F_{k-1}^v|\). Choose two of \( u^- \)'s faulty links to be healthy, and find disjoint paths by induction in \( Q_{k-1} \). Choose one of the paths that has no original faulty edges on as the one we need.

(A12-2.1.2) \(|F_{k-1}| = 2n - 3\).

If \(|F_{k-1}^u| \geq 1\), we assume one \( F_{k-1}^u \) edge to be healthy, and by induction find disjoint paths between \( u^- \) and \( v^- \) in \( Q_{k-1} \). Choose one of the paths that has no original faulty edges on as the one we need.

\( F_{k-1}^v \neq \emptyset \), W.l.o.g., suppose \(|F_{k-1}^u| \geq |F_{k-1}^v|\). Choose one of \( u^- \)'s faulty links to be healthy, and find disjoint paths by induction in \( Q_{k-1} \). Choose one of the paths that has no original faulty edges on as the one we need.

(A12-2.2) \(|F_{k-1}| \leq 2n - 4\) and \(|F_1| \leq 2n - 4\).

By induction, find disjoint paths in \( Q_{k-1} \) and \( Q_1 \), and each choose a shortest path as the one we need.

**Case 2** \( v_1 \neq 0\).

W.l.o.g., suppose \(|F^u| \geq |F^v|\), and \( v_1 \leq \left\lfloor \frac{k}{2} \right\rfloor \).

**Case 2.1** \( \text{dist}(u, v^{-v_1}) = 0 \).

(A21-1) \( \forall e \in N_u \), if \( e \notin F \), then \( e^{v_1} \notin F \).

The paths can be built as follows: start from \( u \), followed by a healthy path and then straight down to a neighbor of \( v \) then reach \( v \). Find another two paths: \( \sigma_1 : [u, u^+, u^2, \ldots, v^-, v] \) and \( \sigma_2 : [u, u^- k, u^k - 2, \ldots, v^-, v] \).

The faulty edge \( f = (r, r^+) \) may block one of the disjoint paths. If \( v_1 = 1 \) and \( f = (u, v) \), this is a trivial case as we can just remove the path \([u, v]\). In what follows, we assume \( f \neq (u, v) \), or if \( f \) is any of \((u, u^+)\) and \((v, v^-)\), we have \( v_1 \geq 2 \).

(A21-1.1) \( r = u^h, h \in \{1, 2, \ldots, v_1 - 2\}(v_1 - 2 \geq 1) \).

There must exists \( t \in N_0 \), \((u, t) \notin F \), such that all the edges incident to \( t^i \) are healthy, for \( i = 0, 1, \ldots, k - 1 \). This is true as there are at most \(|F_0| + |F_{v_1}| \leq 2n - 3\), and there are \( 2n - 2 \) neighbors for each of the node in \( Q_i \), for \( i = 0, 1, 2, \ldots, k - 1 \).

We remove the path that passes \( t \), and build the following path: \([u, t, s, s^+, s^2, \ldots, s^n, t^{v_1}, v]\), where \( s \) is \( t \)'s neighbor. The length of this path is at most \( \left\lceil \frac{k}{2} \right\rceil + 4 \). Another path can be built as: \([u, u^+, v^-, t^{h+1}, u^{h+1}, v^{h+2}, \ldots, v^-, v]\).

The length of this path is at most \( \left\lceil \frac{k}{2} \right\rceil + 2 \).

(A21-1.2) \( r = u^h, h \in \{v_1 + 1, v_1 + 2, \ldots, k - 2\} \).

For this case, we need to modify the path \( \sigma_2 \) to be: \([u^{v_1} (= v_1), u^{v_1+1}, \ldots, u^h, t^{h+1}, u^{h+1}, v^{h+2}, \ldots, v^{k-1}, u]\), where \( t \) is a neighbor of \( u \), such that \( (u^h, t^h) \notin F \) and \( (u^{h+1}, t^{h+1}) \notin F \). This is also true as there are at most \( 2n - 3 \) faulty edges, and each node has \( 2n - 2 \) links in each subgraph \( Q_i, i = 0, 1, 2, \ldots, k - 1 \).

(A21-1.3) \( r = u \). In this case, we must have \(|F^u| > |F^v|\). There is no need to do any change but remove the path \([u, u^+, \ldots, v]\).

(A21-1.4) \( r = u^- \). In this case, we must have \(|F^u| > |F^v|\). There is no need to do any change but remove the path \([u, u^+, \ldots, v]\).

(A21-1.5) \( r = v^+ \). If there exists \((u, t) \in F\), such that \((v, t^{v_1}) \notin F\), adjust the path \( \sigma_2 \) as follows: \([u, u^-, \ldots, v^+, \rho, t^{v_1}, t^{v_1}, v]\), where \( \rho \) is a path between \( t^{v_1+1} \) and \( v^+ \). This path can be easily build by induction, and the length is at most \( L_{n-1} \). Hence the total length of path \( \sigma_2 \) is \( L_{n-1} + k + 1 \).

(A21-1.6) \( r = v^- \). Similar to (A21-1.2) above, suppose \( t \) is such a node that links with \( u \), and all its neighbor links and their parallel edges are healthy. If \( v_1 \geq 2 \), and there exists \((u, s) \in F\), such that \((v, s^{v_1}) \notin F\), remove the path that passes \( s \) and the path that passes \( t \). Build the following two paths: \([u, t, t^-, s^-, s^+, \ldots, s^n, v]\), and \([u, t^+, r^+, s^2, \ldots, r^{v_1}, v]\), where \( \rho \in Q_{k-1} \) can be build by finding disjoint paths between \( u^- \) and \( r^- \) by induction.
we only need to build one more path: \((P1)\): 

\[ u, s, s^1, \ldots, r, t^h, t^{h+1}, \ldots, t^v, s^v, v. \]

(A21-2) \( \exists e_1 = (u, t) \notin F^u \) and \( e^{v_1} \in F \), and \( \exists e_2 = (u, s) \in F^u \) and \( e^{v_1} \notin F \). In this case, there are at most 2\( n - 4 \) faulty edges in each subgraph \( Q_i, i = 0, 1, \ldots, k - 1 \).

Find a node \( w \) in \( Q_0 \), such that \( r \neq w^1 \), for every \( i = 0, 1, \ldots, k - 1 \); \( f \notin N^w \), for \( i = 0, 1, \ldots, k - 1 \). Such \( w \) exists. There are \( k^{n-1} \) nodes in \( Q_0 \), and there are at most \( 2n - 2 \) faulty edges in \( Q^n_k \). Node \( u, r \) will block \( 2n - 1 \) nodes each. As \( u \) and \( v \) each incidents with at least one faulty link and \( f \) has already been considered, we have at most \( 2n - 2 - 3 \) faulty edges left. with each block two nodes. Hence the number of available nodes is at least \( k^{n-1} - 2(2n - 1) - 2(2n - 5) \geq 4 \).

So, there are at least 4 choices for \( w \).

The steps of building disjoint paths in this case are as follows.

Step 1: Find disjoint paths between \( u \) and \( w \) in \( Q_0 \) by induction.

Step 2: Remove \( w \) from each path, and extend the path by straight going down until reach a neighbor of \( w^{v_1} \).

Step 3: Assume the edges are faulty if they are (1) incident with \( w^{v_1} \), (2) not incident with any path built in step 2, and (3) lie inside \( Q_{v_1} \).

If there are at most \( 2n - 4 \) faulty edges in \( Q_{v_1} \), we find disjoint paths between \( v \) and \( w^{v_1} \) by induction, and link these paths with the paths built in step 2.

If there are \( 2n - 3 \) faulty edges in \( Q_{v_1} \), there must exist \( e \in Q_{v_1} \), such that it is not incident with \( v \) and \( w^{v_1} \). Obviously, there are no faulty edges in \( Q_{u} \) for \( i \neq 0, v_{1} \). Assume \( e \) is healthy, and find disjoint paths between \( v \) and \( w^{v_1} \) by induction. If \( e \) lies on some path, do \( D\text{-jump}(e) \).

Step 4: Build one or two more paths.

(A21-2.1) \( f = (u, u^+) \). If \( |F^u| > |F^v| \), then we only need to build one more path: \((P1)\): \([u, u^+, \ldots, v]\). If \( |F^u| = |F^v| \), then we need to build two more paths: one is path \((P1)\) as above. There have an healthy edge \((u, s)\), that is not on any path. We build the following path: \((P2)\): \([u, s, s^1, \rho_1, u^-, u^{-1}, \ldots, u^{v_1}, v]\), where \( \rho_1 \) is a path lies in \( Q_{k-1} \) that can be found by induction.

(A21-2.2) \( f = (u, u^+) \). Path \((P2)\) as built above is kept for this case.

If \( v_1 = 1 \), or \( v_1 > 1 \) and \( |F^u| > |F^v| \), then no need to find more path.

If \( v_1 > 1 \) and \( |F^u| = |F^v| \), then we go back to Step 3. Before finding disjoint paths in \( Q_{v_1} \) by induction, we assume one of \( v \)'s faulty edge is healthy. After finding disjoint paths, name the path passes this faulty edge as \( \rho \). Suppose \( \rho \cap Q_{v_1-1} = s^{v_1-1} \). In \( Q_{v_1-1} \), find a path to link \( v^- \) and \( s^{v_1-1} \) and disjoint with all other paths. This can be done by carefully setting some of \( w^{v_1-1} \)'s links as faulty and using induction to find disjoint paths between \( v^- \) and \( w^{v_1-1} \). Path \( \rho \) is now changed to be: \( \rho : [u, \ldots, s, s^{v_1-1}, \ldots, v^-] \).

(A21-2.3) \( f = (v, v^-) \). If \( |F^u| = |F^v| \), similar to above (A21-2.2) to build a new path. Otherwise, we don’t need any more path.

(A21-2.4) \( f = (v, v^+) \). If \( |F^u| = |F^v| \), similar to above (A21-2.1) to build a new path. Otherwise, we don’t need any more path.

Case 2.2 \( dist(u, v^{-v_1}) = 1 \) has been discussed in the main body of the paper.

Case 2.3 \( dist(u, v^{-v_1}) > 1 \).

We expand the definition for \( FF \): if we are considering finding disjoint paths between node \( w \) and node \( z \) in a sub-graph \( Q_i \), \( FF \) is defined as the set of faulty edges that are neither incident with \( w \) nor incident with \( z \). This expansion also applies to appendix C.

General steps in building disjoint paths:

1. Do \( map(v, 0) \).
2. Do \( DP(u, v^-) \).
3. Extend the paths to \( v \).
4. Find two more paths: (Ga): \([u, u^-, \rho_1, v^{-v_1-1}, v^{-v_1}, \ldots, v]\); (Gb): \([u, u^+, \ldots, u^{v_1}, u^{v_1+1}, \rho_2, v^+, v]\).

If all of these paths are not blocked, then we have done. However, there will have some case that we need to avoid faults. Below is the detailed discussion.
(A23-1) \( v_1 = 1 \).

(A23-1.1) \( f = (u, u^+) \)

(A23-1.1.1) \( |F^u| = |F^v| \)

Do \( \text{map}(v, 0) \). There exists a faulty edge \( e \), incident with \( v^- \). Assume \( e \) is healthy, and do \( DP(u, v^-) \). Suppose \( w \) lies on the path that \( e \) lies on. Remove this path and extend all other paths to \( v \). Build the following two more paths: (a): \([u, w, w^-, \rho_1, v^-, v^-] \), where \( \rho_1 \) can be found by induction in \( Q_{k-1} \) on \( w^- \) and \( v^- \) to avoid \( u^- \); (b): \([u, u^-, \ldots, u^{v+1}, \rho_2, v^+, v] \).

(A23-1.2) \( |F^u| \geq |F^v| + 1 \)

Do \( \text{map}(v, 0) \).

If there are at most \( 2n - 4 \) faulty edges in \( Q_0 \), do \( DP(u, v^-) \), and extend the disjoint paths to \( v \). Build one more path: (a): \([u, u^-, \rho_1, v^-, v^-, v], \rho_1 \in Q_{k-1} \)

If there are \( 2n - 3 \) faulty edges in \( Q_0 \), we have two cases to consider: (i) \( |F^u| = |F^v| + 1 \). This means that \( F^F \geq 1 \). Let \( e \in F^F \), and assume \( e \) is healthy, do \( DP(u, v^-) \). If \( e \) lies on some path, do a \( U\)-jump. Build path (a) as above but need to avoid existing paths (the jumped up edge) and node \( u^- \). (ii) \( |F^u| \geq |F^v| + 2 \). Assume \( e \), an faulty incident with \( u \), is healthy. Do \( DP(u, v^-) \). Remove this path and extend all of the other paths to \( v \).

(A23-1.2) \( f = (u, u^-) \)

(A23-1.2.1) \( |F^u| = |F^v| \)

Do \( \text{map}(v, 0) \). There exists \( e \), an faulty edge incident with \( v^- \). We assume it is healthy, and do \( DP(u, v^-) \). Suppose \( w \) lies on the path that passes \( e \). Remove this path and extend all other paths to \( v \). Build the following two paths: (a): \([u, w, w^-, \rho_1, v^-, v^-, v], \rho_1 \in Q_{k-1} \); (b): \([u, u^+, u^2, \rho_2, v^+, v] \).

(A23-1.2) \( |F^u| \geq |F^v| + 1 \)

Do \( \text{map}(v, 0) \). If there are at most \( 2n - 4 \) faulty edges in \( Q_0 \), do \( DP(u, v^-) \), and extend all paths to \( v \). Find one more path as follows: (a): \([u, u^+, u^2, \rho_2, v^+, v] \).

If there are \( 2n - 3 \) faulty edges in \( Q_0 \), and \( |F^u| = |F^v| + 1 \), there exists \( e \in F^F \). Assume \( e \) is healthy, and do \( DP(u, v^-) \). If \( e \) lies on some path, do a \( U\)-jump(\( e \)). One more path is built as above (a).

If there are \( 2n - 3 \) faulty edges in \( Q_0 \), and \( |F^u| \geq |F^v| + 2 \), assume \( e \), a faulty edge incident with \( u \), is healthy, and do \( DP(u, v^-) \). Remove the path that \( e \) lies on, and extend all other paths to \( v \). One more paths is built as above (a).

(A23-1.3) \( f = (v, v^-) \)

Do \( \text{map}(v, 0) \). Assume \( e \in F \), incident with \( u \), is healthy. Do \( DP(u, v^-) \). Suppose \( w \), a neighbor of \( v^- \) lies on the path that passes edge \( e \). Remove this path, and extend all other paths to \( v \). Build two more paths: (a): \([u, u^-, \rho_1, w^-, w, w^+, v] \); (b): \([u, u^+, u^2, \rho_2, v^+, v] \).

(A23-1.4) \( f = (v, v^+) \)

Compare to case (A23-1.3), we only need to adjust paths (a) and (b) as follows: (a): \([u, u^-, \rho_1, v^-, v^-1, v] \); (b): \([u, u^+, u^2, \rho_2, w^2, w^+, v] \).

(A23-1.5) \( f = (v^-, v^-2) \)

After the four general steps G1, G2, G3, and G4, the path (Ga) is blocked by \( f \). To avoid the faulty edge \( f \), we choose one of the disjoint paths, such that it includes the following sub-path: \([u, w, \rho_1, t, \rho_1^+, \rho_1^+ \) t \], where \( t \) is a neighbor of \( v^- \), and \( \rho_1^+, \rho_1^- \) are faulty free. Remove this path and build another three paths: (a): \([u, u^-, \rho_1^-, t^-, t, v^-, v] \); (b): \([u, w, w^+, \rho_1^+, t^+, v] \); (c): \([u, u^+, u^2, \rho_2, v^+, v] \).

(A23-1.6) \( f = (u^+, u^2) \)

After the 4 general steps, path (Gb) is blocked by \( f \). There must exist \((u^+, w^+) \in Q_1 \) is healthy. An alternative path is built as follows: \([u, u^+, w^+, \rho_2, v^+, v] \), where \( \rho_2 \in Q_1 \).

(A23-2) \((v_1 \geq 2) \)

This case can be analyzed similarly to case (A23-1), and will not discussed here.

APPENDIX C.

WHERE IS NO DIMENSION ONE FAULTY EDGE

W.l.o.g., suppose \( |F^u| = |F^v| \)

Case 1 \( v_1 = 0 \).

Case 1.1 \( |F^u| = 2n - 2 \).

(B11-1) \( |F^F| \geq 2 \).

Let \( e_1, e_2 \in F^F \), and assume them are healthy. Find disjoint paths by induction in \( Q_0 \). If \( e_1 \) (or \( e_2 \)) is on some path, do \( U\)-jump(\( e_1 \)) (or \( U\)-jump(\( e_2 \))). Build two more paths as follows: (a):
If \( (u, v) \in F \), assume it is healthy, and build disjoint paths by induction. Remove the path \([u, v]\) and build two more paths \(\sigma_1\), and \(\sigma_2\) similar to (B11-2) above; but need to avoid faulty edge \(f\).

If \( (u, v) \notin F \), then assume \(e \in F^u\) is healthy. Find disjoint paths by induction in \(Q_0\). Remove the path if it include \(e\). Find two more paths \(\sigma_1\), and \(\sigma_2\) similar to (B11-2) above; but need to avoid faulty edge \(f\).

**Case 1.3** \(|F_0| \leq 2n - 4\).

Find disjoint paths by induction in \(Q_0\). Find two more paths \(\sigma_1\) and \(\sigma_2\) similar to (B11-2) above; but need to avoid faulty edges. \(\sigma_1\) and \(\sigma_2\) can be obtained by induction even though there may have more than \(2n - 4\) faulty edges lie in \(Q_{k-1}\) or \(Q_1\). An exceptional case is, for example, when all of the \(2n - 2\) edges in \(N^u\) are faulty. We can not find \(\sigma_1\) in \(Q_{k-1}\). This can be solved by doing a \(U\)-jump\(e\) on \(\sigma_1\), where \(e \in N^u\).

**Case 2** \(v_1 \neq 0\)

**Case 2.1** \(\text{dist}(u, v^{-v_1}) = 0\).

We will always include the following two paths in the disjoint paths set unless otherwise stated: \(\sigma_1 : [u, u^-, \ldots, u^{v_1-1}, v]\), and \(\sigma_2 : [u, u^+, \ldots, u^{v_1-1}, v]\).

(B21-1) \(\forall e \in N_u\), if \(e \notin F\), then \(e^{v_1} \notin F\).

To find disjoint paths for this case is straightforward.

(B21-2) \(\exists e_1 \in N_u, e_2 \in N_v\), such that \(e_1 \in F, e_2 \notin F, \) and \(e^{v_1} \notin F\).

If \(|F_u| = 2n - 3\), we can easily find 3 disjoint paths between \(u\) and \(v\) as follows: \([u, u^+, \ldots, v]\); \([u, u^-, r^-, r^{-2}, \ldots, r^{v_1}, v]\) (suppose \(e_1 = (u, r)\)); and \([u, s, s^-, u^-, r^{-2}, \ldots, u^{v_1+1}, v]\) (suppose \(e_2 = (v, s^{v_1})\)). Similarly, we can find 3 paths for the case when \(|F_v| = 2n - 3\).

Now, we suppose \(|F_u| \leq 2n - 4\) and \(|F_v| \leq 2n - 4\).

Find a node \(w \in Q_0\), such that \(N^w \cap F = \emptyset\). By a simple calculation \((k^{n-1} - [(2n - 2) + 1 + 2(n - 2) - 1] \geq 4)\), we claim that such \(w\) exists.

Step 1.

If \(|F_0| = 2n - 3\) (we have \(F^F \neq \emptyset\)), assume \(e \in F^F\) is healthy; otherwise, no need to make any assumption. Find disjoint paths by induction
in \(Q_0\) between \(u\) and \(w\). If \(e\) lies on some path, do a \(U\)-jump\((e)\).

Step 2. Remove \(w\) and extend these paths each to a neighbor of \(w^{v_1}\).

Step 3. Assume those edges in \(N^{w^{v_1}}\) that are not incident with any of the paths built in Step 2 are faulty.

(B21-2.1) If this results in at most \(2n-4\) faulty edges in \(Q_{v_1}\), find disjoint paths between \(v\) and \(w^{v_1}\) in \(Q_{v_1}\) by induction. Remove the node \(w^{v_1}\) from each of the path and thus links with the path built in Step 2.

(B21-2.2) If this results in \(2n-3\) faulty edges in \(Q_{v_1}\), and all faulty edges are incident with one of \(v\) and \(w^{v_1}\), then assume \(e_3 \in F^{w^{v_1}}\) is healthy. Otherwise, there exists \(e_4 \in F\) not incident with any of \(v\) and \(w^{v_1}\), and assume \(e_4\) is healthy. Find disjoint paths in \(Q_{v_1}\) by induction. Remove the path if it passes edge \(e_3\), or do \(D\)-jump\((e_4)\) if it passes edge \(e_4\). All remained paths are incident with the paths built in step 2.

(B21-2.3) If this results in \(2n-2\) faulty edges in \(Q_{v_1}\), we will have several cases. (1) All faulty edges are incident with either \(v\) or \(w^{v_1}\). If \(|F^v| = |F^{w^{v_1}}|\), for each of \(v\) and \(w^{v_1}\), we assume \(e_3 \in F^v\) and \(e_4 \in F^{w^{v_1}}\) are healthy and find disjoint paths by induction. Do \(D\)-jump\((e_3)\) or \(D\)-jump\((e_4)\) to avoid the faulty edge \(e_3\) or \(e_4\) if it lies on some path. If they lie on a same path, then remove that path. All remained paths are incident with the paths built in step 2.

(2) One of the faulty edge \(e_3\) is neither incident with \(v\) nor \(w^{v_1}\). We must have \(|F^v| + 1 \leq |F^{w^{v_1}}|\). Thus we assume \(e_3\) and \(e_4 \in F^{w^{v_1}}\) are healthy, and find disjoint paths between \(w^{v_1}\) and \(v\) by induction. If \(e_3\) lies on some path, then do \(D\)-jump\((e_3)\); if \(e_4\) lies on some path, then remove this path. All remained paths are incident with the paths built in step 2.

(3) At least two faulty edges \(e_3\) and \(e_4\) are neither incident with \(v\) nor \(w^{v_1}\). Assume \(e_3\) and \(e_4\) are healthy, and find disjoint paths between \(w^{v_1}\) and \(v\) by induction. If \(e_i\) lies on some path, then do a \(D\)-jump\((e_i)\), for \(i = 3, 4\).

Case 2.2 \(v_1 \neq 0, \text{dist}(u, v^{v_1}) = 1\).

Need to consider that if \((u, v^-)\) and/or \((u^+, v)\) are faulty.

If \((u, v^-)\) is healthy but \((u^+, v)\) is faulty, we have the following cases.

If \((u, v^-)\) is faulty but \((u^+, v)\) is healthy, we have the following cases.

If they are both faulty, we will discuss it in the following cases.

(B22-1) \(v_1 = 1\).

Define two paths as: \(\sigma_1 : [u, u^+, v]; \sigma_2 : [u, v^-, v]; \sigma_3 : [u, u^+, v^k-2, v^{k-3}, \ldots, v^+, v]; \sigma_4 : [u, u^+, u^2, v^+, v]; \sigma_5 : [u, u^-, v^-, v^-]\).

(B22-1.1) After map\((v, 0)\), we have \(|F_0| = 2n - 2\).

We have either at most one of \((u, v^-)\) and \((u^+, v)\) is faulty.

We choose two faulty edges \(e_1, e_2\) by the following rules:

(1) if \(|F^F| \geq 2\), let \(e_1, e_2 \in F^F\); (2) if \(|F^F| = 1\), let \(e_1 \in F^F\), and \(e_2 \in F^u \setminus F^{v^-}\); (3) if \(|F^F| = 0\) and \(|F^u| > |F^u|\), we choose \(e_1, e_2 \in F^u \setminus F^{v^-}\); (4) if \(|F^F| = 0\) and \(|F^u| = |F^v-|\), we choose \(e_1 \in F^u \setminus F^{v^-}, e_2 \in F^v- \setminus F^u\).

Assume \(e_1, e_2\) are healthy. Do \(DP(u, v^-)\), and extend the disjoint paths to \(v\).

If \(e_1, e_2\) are chosen from above case (4), and they both lie on a same path, just remove this path and include \(\sigma_1, \sigma_2\), if \(\sigma_i, i = 1, 2\) is faulty free, and \(\sigma_3\) in disjoint paths; if they lie on different paths, do \(U\)-jump\((e_i)\), for \(i = 1, 2\) and include \(\sigma_1\) and \(\sigma_2\), if \(\sigma_i, i = 1, 2\) is faulty free, in disjoint paths.

Otherwise, \(e_1, e_2\) are chosen from above case (1–3) if \(e_i\) lies on a path and \(e_i \in F^F\), do \(U\)-jump\((e_i)\), \(i \in \{1, 2\}\); if \(e_i\) lies on a path and \(e_i \in F^u, i \in \{1, 2\}\), remove the path. Include \(\sigma_1, \sigma_2\) and \(\sigma_3\), if \(\sigma_i, i \in \{1, 2, 3\}\) exists, in disjoint paths.

(B22-1.2) After doing map\((v, 0)\), \(|F_0| = 2n - 3\).

If \(\exists e \notin F^u \cup F^{v^-}\), assume \(e\) is healthy, and do \(DP(u, v^-)\). Extend the disjoint paths to \(v\). Do \(U\)-jump\((e)\) or \(D\)-jump\((e)\) to avoid faults if necessary. Two more paths can be \(\sigma_1\) and \(\sigma_2\). However, if there is one faulty edge \(e\) lies on \(\sigma_1\) or \(\sigma_2\), just do \(U\)-jump\((e)\) or \(D\)-jump\((e)\).

If \(|F^u \cup F^{v^-}| = 2n - 3\), let choose a faulty edge \(e\) from \(F^u\), and assume it is healthy. Do \(DP(u, v^-)\). Extend the disjoint paths to \(v\). If \(e\) lies on some path, just remove the path. Include
$\sigma_1$ and $\sigma_2$ as disjoint paths. Similarly, if there is one faulty edge $e$ lies on $\sigma_1$ (or $\sigma_2$), just do $U$-$jump(e)$ (or $D$-$jump(e)$).

(B22-1.3) After doing $map(v, 0)$, $|F_0| \leq 2n-2$.

Do $DP(u, v^-)$. Extend the disjoint paths to $v$. Include $\sigma_1$ and $\sigma_2$, if exist, as disjoint paths. However, if $\sigma_1$ is blocked from outside $Q_0$ and $Q_1$, we can easily find an alternative path that is disjoint from all of existing one by replacing the sub-path in $Q_{k-1}$ with a path that can be found by induction. The case that $\sigma_2$ is blocked from outside $Q_0$ and $Q_1$ can be dealt with similarly.

(B22-2) $v_1 \geq 2$.

This case can be analyzed similar to case (B22-1), and we will not go into details.

**Case 2.3** $v_1 \neq 0, dist(u, v^{-v_1}) \geq 2$.

This can be dealt with similar to **Case 2.2**.