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Cut Elimination in Nested Sequents for Intuitionistic Modal Logics

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Abstract. We present cut-free deductive systems without labels for the intuitionistic variants of the modal logics obtained by extending IK with a subset of the axioms d, t, b, 4, and 5. For this, we use the formalism of nested sequents, which allows us to give a uniform cut elimination argument for all 15 logic in the intuitionistic S5 cube.

1 Introduction

Intuitionistic modal logics are intuitionistic propositional logic extended with the modalities $\Box$ and $\Diamond$, obeying some variants of the k-axiom. Unlike for classical modal logic, there is no canonical choice, and many different versions of intuitionistic modal logics have been considered, e.g., [8, 23, 24, 21, 25, 2, 20]. For a survey see [25]. In this paper we consider the variant proposed in [24, 21] and studied in detail by Simpson [25], namely, we add the following axioms to intuitionistic propositional logic:

$$\begin{array}{ll} \mathsf{k}_1: & \Box(A \supset B) \supset (\Box A \supset \Box B) \\ \mathsf{k}_2: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \\ \mathsf{k}_3: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_4: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ \mathsf{k}_5: & \neg\Diamond\bot \end{array} \tag{1}$$

In a classical setting the axioms k_2–k_5 would follow from k_1 and the De Morgan laws. Recently, researchers have also studied the variant which allows only k_1 and k_2, and which is sometimes called *constructive modal logic* (e.g., [1, 18]). Since this leads to a different proof theory, it will not be discussed here. Independently from the chosen variant for the intuitionistic modal logic K, denoted by IK, one can add an arbitrary subset of the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$, shown in Figure 1. As in the classical setting, this yields 15 different modal logics. In [25], Simpson presents labeled natural deduction and labeled sequent calculus systems for all of them. In [11], Galmiche and Salhi present label-free natural deduction systems for the ones not using the d-axiom. In this paper we present label-free sequent calculus systems for all 15 logics in the "intuitionistic modal cube" (shown in Figure 2), together with a uniform syntactic cut-elimination proof. For this we use nested sequents [14, 3, 22] (in a variant already used in [11]).

The motivation for this work is twofold. First, sequent calculus is much better suited for automated proof search than natural deduction, and second, label-free

d: $\Box A \supset \Diamond A$	$\forall w.\ \exists v.\ wRv$	(serial)
t: $(A \supset \Diamond A) \wedge (\Box A \supset A)$	$\forall w.\ wRw$	(reflexive)
b: $(A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset A)$	$\forall w.\ \forall v.\ wRv \supset vRw$	(symmetric)
$\mathsf{4}$: $(\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A)$	$\forall w.\ \forall v.\ \forall u.\ wRv \wedge vRu \supset wRu$	(transitive)
$\mathsf{5}$: $(\Diamond A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset \Box A)$	$\forall w.\ \forall v.\ \forall u.\ wRv \wedge wRu \supset vRu$	(euclidean)

Fig. 1. Intuitionistic modal axioms d, t, b, $\mathsf{4}$, $\mathsf{5}$, with corresponding frame conditions

systems make it easier to study the theory of proof search and proof normalization. In fact, the sequent systems together with the cut-reduction procedure presented in this paper are the basis for ongoing research on the following two questions: (i) Is it possible to design a focussed system [16, 5, 17] yielding new normal forms for cut-free proofs and providing proof search mechanisms based on forward-chaining (program-directed search) and backward-chaining (goal-directed search) for intuitionistic modal logics? (ii) Can we give a term calculus (based on the λ-calculus in the style of [19]) for proofs, in order to provide a Curry-Howard-correspondence for intuitionistic modal logics (and not just the constructive modal logics mentioned above)?

There is a close relationship between the labeled and the label-free natural deduction systems of [25] and [11]. In fact, modulo the correspondence between (tree-)labeled systems and nested sequents [10], the basic systems for IK of [25] and [11] are identical. A similar correspondence can be observed between the labeled sequent systems of [25] and our systems, when restricted to the logic IK. However, the rules dealing with the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$ are very different from [25]. The shape of these rules is crucial for the internal cut-elimination proof.

Furthermore, note that our treatment of the "intuitionistic" in nested sequents is different from the one in [9] (which is two-sided inside each nesting and does not treat modalities), and the one in [13], (which focuses on variants of bi-intuitionistic tense logics, and does not cover all 15 logics in the $\mathsf{IS5}$-cube).

2 Preliminaries

The formulas of intuitionistic modal logic (IML) are generated by:

$$\mathcal{M} ::= \mathcal{A} \mid \bot \mid \mathcal{M} \wedge \mathcal{M} \mid \mathcal{M} \vee \mathcal{M} \mid \mathcal{M} \supset \mathcal{M} \mid \Box\mathcal{M} \mid \Diamond\mathcal{M} \tag{2}$$

where $\mathcal{A} = \{a, b, c, \ldots\}$ is a countable set of *propositional variables* (or *atoms*). We use $A, B, C, \ldots$ to denote formulas. Negation of formulas is defined as $\neg A = A \supset \bot$. The theorems of the intuitionistic modal logic IK are exactly those formulas that are derivable from the axioms of intuitionistic propositional logic and the axioms k_1–k_5 shown in (1) via the rules mp and nec shown below:

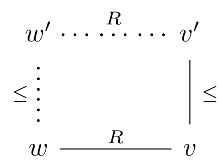
$$\mathsf{mp}\,\frac{A \quad A \supset B}{B} \qquad\qquad \mathsf{nec}\,\frac{A}{\Box A} \tag{3}$$

In the following, we recall the *birelational models* [21, 7] for IML, which are a combination of the Kripke semantics for propositional intuitionistic logic and the one for classical modal logic. A *frame* $\langle W, \leq, R\rangle$ is a non-empty set W of *worlds* together with two binary relations $\leq, R \subseteq W \times W$, where $\leq$ is a pre-order (i.e., reflexive and transitive), such that the following two conditions hold

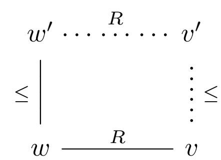
(F1) For all worlds w, v, v', if wRv and $v \leq v'$, then there is a w' such that $w \leq w'$ and $w'Rv'$.

(F2) For all worlds w', w, v, if $w \leq w'$ and wRv, then there is a v' such that $w'Rv'$ and $v \leq v'$.

These two conditions can be visualized as follows:



and



A *model* $\mathfrak{M}$ is a quadruple $\langle W, \leq, R, V\rangle$, where $\langle W, \leq, R\rangle$ is a frame, and V, called the *valuation*, is a monotone function $\langle W, \leq\rangle \to \langle 2^{\mathcal{A}}, \subseteq\rangle$ from the set of worlds to the set of subsets of propositional variables, mapping a world w to the set of propositional variables which are true in w. We write $w \Vdash a$ if $a \in V(w)$. The relation $\Vdash$ is extended to all formulas as follows:

$$
\begin{array}{lll}
w \Vdash A \wedge B & \text{iff} & w \Vdash A \text{ and } w \Vdash B \\
w \Vdash A \vee B & \text{iff} & w \Vdash A \text{ or } w \Vdash B \\
w \Vdash A \supset B & \text{iff} & \text{for all } w' \geq w\colon\ w' \Vdash A \text{ implies } w' \Vdash B \\
w \Vdash \Box A & \text{iff} & \text{for all } w', v' \in W\colon\ \text{if } w' \geq w \text{ and } w'Rv' \text{ then } v' \Vdash A \\
w \Vdash \Diamond A & \text{iff} & \text{there is a } v \in W \text{ such that } wRv \text{ and } v \Vdash A
\end{array}
\tag{4}
$$

We write $w \nVdash A$ if $w \Vdash A$ does not hold. In particular, note that $w \nVdash \bot$ for all worlds, and that we do *not* have that $w \Vdash \neg A$ iff $w \nVdash A$. However, we get the monotonicity property:

Lemma 2.1 (Monotonicity) *If $w \leq w'$ and $w \Vdash A$ then $w' \Vdash A$.*

Proof By induction on A, using (4), (F1), and the monotonicity of V. □

We say that a formula A *is valid in a model* $\mathfrak{M} = \langle W, \leq, R, V\rangle$, denoted by $\mathfrak{M} \Vdash A$, if for all $w \in W$ we have $w \Vdash A$. A formula A *is valid in a frame* $\langle W, \leq, R\rangle$, denoted by $\langle W, \leq, R\rangle \Vdash A$, if for all valuations V, we have $\langle W, \leq, R, V\rangle \Vdash A$. Finally, we say a formula is *valid*, if it is valid in all frames. As for classical modal logics, we can consider the axioms $\{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, whose intuitionistic versions are shown in Figure 1, and that we can add to the logic IK. For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ a frame is called an X-*frame* if the relation R obeys the corresponding frame conditions, which are also shown in Figure 1. For example, a $\{\mathsf{b}, \mathsf{4}\}$-frame is one in which R is symmetric and transitive. The following theorem is well-known:

Theorem 2.2 *A formula is derivable from $\mathsf{IK}+\mathsf{X}$ iff it is valid in all X-frames.*

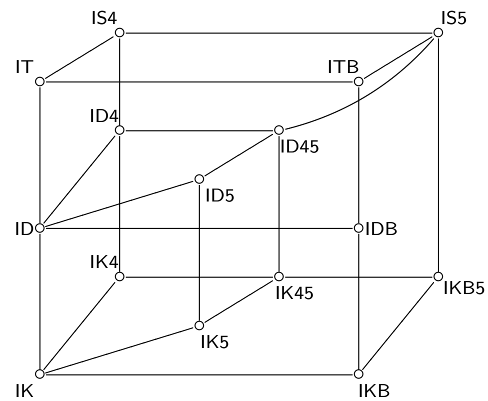


Fig. 2. The intuitionistic "modal cube"

Remark 2.3 Note that we do not have a true correspondence as for classical modal logics. For example, if t is valid in a frame $\langle W, \leq, R \rangle$ then R does not need to be reflexive (see [25, 21] for more details).

We will say a formula is X-*valid* iff it is valid in all X-frames. As in classical modal logic, we can, *a priori*, define 32 modal logics with the 5 axioms in Figure 1. But many of them coincide, for example, $\mathsf{IK} + \{\mathsf{t}, \mathsf{b}, \mathsf{4}\}$ and $\mathsf{IK} + \{\mathsf{t}, \mathsf{5}\}$ yield the same logic, called $\mathsf{IS5}$. There are, in fact, 15 different logics, which are shown in Figure 2, the intuitionistic version of the "modal cube" [12].

3 Nested Sequents for Intuitionistic Modal Logics

Let us now turn to nested sequents for IML. The data structure of a nested sequent for intuitionistic modal logics that we employ here has already been used in [11] and is almost the same as for classical modal logics [3, 4]: it is a tree whose nodes are multisets of formulas. The only difference is that in the intuitionistic case exactly one formula occurrence in the whole tree is special. We will mark it with a white circle $\circ$, while all other formulas are marked with a black circle $\bullet$. One can see this marking as a polarity assignment: $\bullet$ for *input polarity*, and $\circ$ for *output polarity*.[1] Formally, nested sequents for IML are generated by the grammar (where n and k can both be zero):

$$\Gamma ::= \Lambda, \Pi \qquad \Lambda ::= A_1^\bullet, \ldots, A_n^\bullet, [\Lambda_1], \ldots, [\Lambda_k] \qquad \Pi ::= A^\circ \mid [\Gamma] \tag{5}$$

Thus, a nested sequent consists of two parts: an *LHS-sequent* (denoted by Λ), in which all formulas have input polarity, and an *RHS-sequent* (denoted by Π), which is either a formula with output polarity or a bracketed sequent. A sequent of the shape as Γ in (5) is called a *full sequent*. The letters Δ and Σ

[1] We avoid the use of the "positive/negative" terminology because it is overloaded. For a thorough investigation into polarities as they are used here, see [15].

can stand for full sequents as well as LHS-sequents, depending on the context. Note that any RHS-sequent is also a full sequent, but not the other way around. As usual, we allow sequents to be empty, and we consider sequents to be equal modulo associativity and commutativity of the comma. Sometimes we write $\emptyset$ to denote the empty multiset, allowing us to write $[\emptyset]$, which is a well-formed LHS-sequent. If we forget the polarities, a nested sequent is of the shape $\Gamma = A_1, \ldots, A_k, [\Gamma_1], \ldots, [\Gamma_n]$.

The *corresponding formula* of a nested sequent is defined as follows:

$$\begin{aligned} \mathit{fm}(\Lambda, \Pi) &= \mathit{fm}(\Lambda) \supset \mathit{fm}(\Pi) \\ \mathit{fm}(A_1^\bullet, \ldots, A_n^\bullet, [\Lambda_1], \ldots, [\Lambda_k]) &= A_1 \wedge \cdots \wedge A_n \wedge \Diamond \mathit{fm}(\Lambda_1) \wedge \cdots \wedge \Diamond \mathit{fm}(\Lambda_k) \\ \mathit{fm}(A^\circ) &= A \\ \mathit{fm}([\Gamma]) &= \Box \mathit{fm}(\Gamma) \end{aligned}$$

We say a sequent is X-*valid* if its corresponding formula is.

As in the case of classical modal logics, we need the notion of *context* which is a nested sequent with a hole $\{\ \}$, taking the place of a formula. Since we have two polarities, input and output, there are also two kinds of contexts: *input contexts*, whose holes have to be filled with an input formula for obtaining a full sequent, and *output contexts*, whose holes have to be filled with an output formula for obtaining a full sequent. We also allow the holes in a context to be filled with sequents and not just formulas.

We define the *depth* of a context inductively as follows:

$$\begin{aligned} \mathit{depth}(\{\ \}) &= 0 \\ \mathit{depth}(\Delta, \Gamma\{\ \}) &= \mathit{depth}(\Gamma\{\ \}) \\ \mathit{depth}([\Gamma\{\ \}]) &= 1 + \mathit{depth}(\Gamma\{\ \}) \end{aligned}$$

Example 3.1 Let $\Gamma_1\{\ \} = C^\bullet, [\{\ \}, [B^\bullet, C^\bullet]]$ and $\Delta_1 = A^\bullet, [B^\circ]$ and $\Gamma_2\{\ \} = C^\bullet, [\{\ \}, [B^\bullet, C^\circ]]$ and $\Delta_2 = A^\bullet, [B^\bullet]$. Then $\mathit{depth}(\Gamma_1\{\ \}) = \mathit{depth}(\Gamma_2\{\ \}) = 1$. Furthermore, $\Gamma_1\{\Delta_2\}$ and $\Gamma_2\{\Delta_1\}$ are not well-formed full sequents, because the former would contain no output formula, and the latter would contain two. However, we can form $\Gamma_1\{\Delta_1\} = C^\bullet, [A^\bullet, [B^\circ], [B^\bullet, C^\bullet]]$ and $\Gamma_2\{\Delta_2\} = C^\bullet, [A^\bullet, [B^\bullet], [B^\bullet, C^\circ]]$. Their corresponding formulas are $\mathit{fm}(\Gamma_1\{\Delta_1\}) = C \supset \Box(A \wedge \Diamond(B \wedge C) \supset \Box B)$ and $\mathit{fm}(\Gamma_2\{\Delta_2\}) = C \supset \Box(A \wedge \Diamond B \supset \Box(B \supset C))$, respectively.

Observation 3.2 Note that every output context $\Gamma\{\ \}$ is of the shape

$$\Lambda_1, [\Lambda_2, [\ldots, [\Lambda_n, \{\ \}] \ldots]] \tag{6}$$

for some $n \geq 0$, where all Λ_i are LHS-sequents. Filling the hole of an output context with a full sequent yields a full sequent, and filling it with an LHS-sequent yields an LHS-sequent. Every input context $\Gamma\{\ \}$ is of the shape $\Gamma'\{\Lambda\{\ \}, \Pi\}$ where $\Gamma'\{\ \}$ and $\Lambda\{\ \}$ are output contexts (i.e., are of the shape (6) above) and Π is a RHS-sequent. Furthermore, $\Gamma'\{\ \}$ and $\Lambda\{\ \}$ and Π are uniquely defined by the position of the hole $\{\ \}$ in $\Gamma\{\ \}$.

$$\bot^\bullet \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{id} \frac{}{\Gamma\{a^\bullet, a^\circ\}}$$

$$\wedge^\bullet \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \frac{\Gamma\{A^\circ\} \qquad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet \frac{\Gamma\{A^\bullet\} \qquad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad \vee^\circ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset^\bullet \frac{\Gamma^\downarrow\{A \supset B^\bullet, A^\circ\} \qquad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box^\bullet \frac{\Gamma\{\Box A^\bullet, [A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \qquad \Box^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}}$$

$$\Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Diamond^\circ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

Fig. 3. System NIK

We can chose to fill the hole of a context $\Gamma\{\ \}$ with nothing, which means we simply remove the $\{\ \}$. This is denoted by $\Gamma\{\emptyset\}$. In Example 3.1 above, $\Gamma_1\{\emptyset\} = C^\bullet, [[B^\bullet, C^\bullet]]$ is an LHS-sequent and $\Gamma_2\{\emptyset\} = C^\bullet, [[B^\bullet, C^\circ]]$ is a full sequent. More generally, whenever $\Gamma\{\emptyset\}$ is a full sequent, then $\Gamma\{\ \}$ is an input context. Sometimes we also need a context with many holes, denoted by $\Gamma\{\ \}\cdots\{\ \}$.

Definition 3.3 For every input context $\Gamma\{\ \}$ (resp. full sequent Δ), we define its *output pruning* $\Gamma^\downarrow\{\ \}$ (resp. $\Delta^\downarrow$) to be the same context (resp. sequent) with the unique output formula removed. Thus, $\Gamma^\downarrow\{\ \}$ is an output context (resp. $\Delta^\downarrow$ is an LHS-sequent). If $\Gamma\{\ \}$ is already an output context (resp. if Δ is already an LHS-sequent), then $\Gamma^\downarrow\{\ \} = \Gamma\{\ \}$ (resp. $\Delta^\downarrow = \Delta$).

We are now ready to see the inference rules. Figure 3 shows system NIK, a nested sequent system for intuitionistic modal logic IK. There are more rules than in the classical version [3] because for each connective we need two rules, one for the input polarity, and one for the output polarity. Note how the $\supset^\bullet$-rule makes use of the output pruning. This is necessary because we allow only one output formula in the sequent. Without this restriction, we would collapse into the classical case.

In the course of this paper we will make use of the additional structural rules

$$\mathsf{nec}^{[]} \frac{\Gamma}{[\Gamma]} \quad \mathsf{w} \frac{\Gamma\{\emptyset\}}{\Gamma\{\Lambda\}} \quad \mathsf{c} \frac{\Gamma\{A^\bullet, A^\bullet\}}{\Gamma\{A^\bullet\}} \quad \mathsf{m}^{[]} \frac{\Gamma\{[\Delta_1], [\Delta_2]\}}{\Gamma\{[\Delta_1, \Delta_2]\}} \quad \mathsf{cut} \frac{\Gamma^\downarrow\{A^\circ\} \qquad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}} \tag{7}$$

called *necessitation*, *weakening*, *contraction*, *box-medial*, and *cut*, respectively. These rules are not part of the system, but we will see later that they are all admissible. Note that in the weakening rule Λ has to be an LHS-sequent, and the contraction rule can only be applied to input formulas. For the $\mathsf{m}^{[]}$-rule it is not relevant where in $\Gamma\{[\Delta_1, \Delta_2]\}$ the output formula is located. The cut-rule makes use of the output pruning, in the same way as the $\supset^\bullet$-rule. Explicit

$$\mathsf{d}^\circ\ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Diamond A^\circ\}} \qquad \mathsf{t}^\circ\ \frac{\Gamma\{A^\circ\}}{\Gamma\{\Diamond A^\circ\}} \qquad \mathsf{b}^\circ\ \frac{\Gamma\{[\Delta], A^\circ\}}{\Gamma\{[\Delta, \Diamond A^\circ]\}}$$

$$\mathsf{d}^\bullet\ \frac{\Gamma\{\Box A^\bullet, [A^\bullet]\}}{\Gamma\{\Box A^\bullet\}} \qquad \mathsf{t}^\bullet\ \frac{\Gamma\{\Box A^\bullet, A^\bullet\}}{\Gamma\{\Box A^\bullet\}} \qquad \mathsf{b}^\bullet\ \frac{\Gamma\{[\Delta, \Box A^\bullet], A^\bullet\}}{\Gamma\{[\Delta, \Box A^\bullet]\}}$$

$$\mathsf{4}^\circ\ \frac{\Gamma\{[\Diamond A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}} \qquad \mathsf{5}^\circ\ \frac{\Gamma\{\emptyset\}\{\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}\{\emptyset\}}\ \mathrm{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

$$\mathsf{4}^\bullet\ \frac{\Gamma\{\Box A^\bullet, [\Box A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \qquad \mathsf{5}^\bullet\ \frac{\Gamma\{\Box A^\bullet\}\{\Box A^\bullet\}}{\Gamma\{\Box A^\bullet\}\{\emptyset\}}\ \mathrm{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

Fig. 4. Intuitionistic $\Diamond^\circ$- and $\Box^\bullet$-rules for the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$.

contraction is not needed in NIK because contraction is implicitly present in the $\supset^\bullet$- and $\Box^\bullet$-rules [6]. Note that the id-rule applies only to atomic formulas. But as usual with sequent style system, the general form is derivable:

Proposition 3.4 *The rule* $\mathsf{id}\ \overline{\Gamma\{A^\bullet, A^\circ\}}$ *is derivable in* NIK.

Figure 4 shows the intuitionistic versions for the rules for the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$. They are almost the same as the corresponding rules in the classical case [3]. The only difference is that here we need two rules for each axiom: a $\Diamond^\circ$-rule and a $\Box^\bullet$-rule. Note that contraction is implicitly present in the $\Box^\bullet$-rules but not in the $\Diamond^\circ$-rules. For a subset $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, we denote by $\mathsf{X}^\bullet$ and X° the corresponding sets of $\Box^\bullet$-rules and $\Diamond^\circ$-rules, respectively.

4 Soundness

In this section we will show that all rules presented in Figures 3 and 4 are indeed sound. More precisely, we prove the following theorem:

Theorem 4.1 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, and let* $\mathsf{r}\ \frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma}$ *(for* $n \in \{0, 1, 2\}$*) be an instance of a rule in* $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ$*. Then:*

(i) the formula $fm(\Gamma_1) \wedge \cdots \wedge fm(\Gamma_n) \supset fm(\Gamma)$ *is* X*-valid, and*

(ii) whenever a sequent Γ *is provable in* $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ$*, then* Γ *is* X*-valid.*

Clearly, (ii) follows almost immediately from (i). But for proving (i), we need a series of lemmas. We begin by showing that the deep inference principle used in all rules is sound.

Lemma 4.2 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, and let* A*,* B*, and* C *be formulas.*

(i) If $A \supset B$ *is* X*-valid, then so is* $(C \supset A) \supset (C \supset B)$*.*

(ii) If $A \supset B$ *is* X*-valid, then so is* $\Box A \supset \Box B$*.*

(iii) If $A \supset B$ *is* X*-valid, then so is* $(C \wedge A) \supset (C \wedge B)$*.*

(iv) If $A \supset B$ *is* X*-valid, then so is* $\Diamond A \supset \Diamond B$*.*

(v) If $A \supset B$ *is* X*-valid, then so is* $(B \supset C) \supset (A \supset C)$*.*

Proof This follows immediately from (4) and Lemma 2.1. □

Lemma 4.3 *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, let Δ and Σ be full sequents, and let $\Gamma\{\ \}$ be an output context. If $fm(\Delta) \supset fm(\Sigma)$ is X-valid, then so is $fm(\Gamma\{\Delta\}) \supset fm(\Gamma\{\Sigma\})$.*

Proof Induction on $\Gamma\{\ \}$ (see Obs. 3.2), using Lemma 4.2.(i) and (ii). □

Lemma 4.4 *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, let Δ and Σ be LHS-sequents, and $\Gamma\{\ \}$ an input context. If $fm(\Sigma) \supset fm(\Delta)$ is X-valid, then so is $fm(\Gamma\{\Delta\}) \supset fm(\Gamma\{\Sigma\})$.*

Proof By Observation 3.2, we have that $\Gamma\{\ \} = \Gamma'\{\Lambda\{\ \}, \Pi\}$ for some $\Gamma'\{\ \}$ and $\Lambda\{\ \}$ and Π. By induction on $\Lambda\{\ \}$, using Lemma 4.2.(iii) and (iv), we get that $fm(\Lambda\{\Sigma\}) \supset fm(\Lambda\{\Delta\})$ is X-valid. From Lemma 4.2.(v) it then follows that $(fm(\Lambda\{\Delta\}) \supset fm(\Pi)) \supset (fm(\Lambda\{\Sigma\}) \supset fm(\Pi))$, i.e., $fm(\Lambda\{\Delta\}, \Pi) \supset fm(\Lambda\{\Sigma\}, \Pi)$ is X-valid. Now the statement follows from Lemma 4.3. □

Lemma 4.5 *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. Then any full sequent of the shape $\Gamma\{a^\bullet, a^\circ\}$ or $\Gamma\{\bot^\bullet\}$ is X-valid.*

Proof If a formula A is X-valid, then so are $\Box A$ and $C \supset A$ for an arbitrary formula C. Since $a \supset a$ is trivially X-valid, the validity of $\Gamma\{a^\bullet, a^\circ\}$ follows by induction on $\Gamma\{\ \}$ (which is of shape (6)). For $\Gamma\{\bot^\bullet\}$, note that this sequent is of shape $\Gamma'\{\Lambda\{\bot^\bullet\}, \Pi\}$ (by Observation 3.2). By an easy induction on $\Lambda\{\ \}$, we can can show that $fm(\Lambda\{\bot^\bullet\}) \supset \bot$ is X-valid. Since $\bot \supset A$ is X-valid for any formula A, we can conclude that $fm(\Lambda\{\bot^\bullet\}) \supset fm(\Pi)$ is X-valid, and therefore $fm(\Lambda\{\bot^\bullet\}, \Pi)$. Now, X-validity of $\Gamma\{\bot^\bullet\}$ follows by induction on $\Gamma'\{\ \}$. □

Lemma 4.6 *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, and let $\mathsf{r}\,\dfrac{\Gamma_1}{\Gamma_2}$ be an instance of w, c, $\mathsf{m}^{[]}$, $\vee^\circ$, $\Box^\circ$, $\Diamond^\circ$, $\supset^\circ$, $\wedge^\bullet$, $\Diamond^\bullet$, or $\Box^\bullet$. Then $fm(\Gamma_1) \supset fm(\Gamma_2)$ is X-valid.*

Proof For the rules $\vee^\circ$, $\Box^\circ$, $\Diamond^\circ$, $\supset^\circ$ this follows immediately from Lemma 4.3, where for $\Diamond^\circ$ we need the k_2-axiom. For the rules $\wedge^\bullet$, $\Diamond^\bullet$, w, and c, the lemma follows immediately from Lemma 4.4. The $\Box^\bullet$-rule can be decomposed into c and the rule $\tilde{\Box}^\bullet\,\dfrac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}}$, for which we need a case distinction: If the output formula occurs inside Δ, then we use the validity of axiom k_1 and Lemma 4.3. If the output formula occurs inside $\Gamma\{\ \}$, then we need the validity of the formula $(\Box A \wedge \Diamond B) \supset \Diamond(A \wedge B)$ for all A and B. This can easily be shown using the definition of $\Vdash$. Then the lemma follows from Lemma 4.4. Finally, for the $\mathsf{m}^{[]}$-rule we also make a case distinction: If the output formula is inside $\Gamma\{\ \}$, we need the validity of the formula $\Diamond(A \wedge B) \supset \Diamond A \wedge \Diamond B$ for all A and B, which can easily be shown using the definition of $\Vdash$. Then the the statement of the lemma follows from Lemma 4.4. If the output formula occurs inside Δ_1 or Δ_2, then we use the validity of axiom k_4 and Lemma 4.3. □

Consider now the rules in Fig. 5, which are special cases of the rules $\mathsf{5}^\circ$ and $\mathsf{5}^\bullet$.

Proposition 4.7 *The rule $\mathsf{5}^\circ$ is derivable in $\{\mathsf{5}_1^\circ, \mathsf{5}_2^\circ, \mathsf{5}_3^\circ\}$, and the rule $\mathsf{5}^\bullet$ is derivable in $\{\mathsf{5}_1^\bullet, \mathsf{5}_2^\bullet, \mathsf{5}_3^\bullet, \mathsf{c}\}$.*

$$\mathsf{5}_1^\circ\ \frac{\Gamma\{[\Delta],\Diamond A^\circ\}}{\Gamma\{[\Delta,\Diamond A^\circ]\}}\qquad \mathsf{5}_2^\circ\ \frac{\Gamma\{[\Delta],[\Diamond A^\circ,\Sigma]\}}{\Gamma\{[\Delta,\Diamond A^\circ],[\Sigma]\}}\qquad \mathsf{5}_3^\circ\ \frac{\Gamma\{[\Delta,[\Diamond A^\circ,\Sigma]]\}}{\Gamma\{[\Delta,\Diamond A^\circ,[\Sigma]]\}}$$

$$\mathsf{5}_1^\bullet\ \frac{\Gamma\{[\Delta],\Box A^\bullet\}}{\Gamma\{[\Delta,\Box A^\bullet]\}}\qquad \mathsf{5}_2^\bullet\ \frac{\Gamma\{[\Delta],[\Box A^\bullet,\Sigma]\}}{\Gamma\{[\Delta,\Box A^\bullet],[\Sigma]\}}\qquad \mathsf{5}_3^\bullet\ \frac{\Gamma\{[\Delta,[\Box A^\bullet,\Sigma]]\}}{\Gamma\{[\Delta,\Box A^\bullet,[\Sigma]]\}}$$

Fig. 5. Variants of the rules for the 5-axiom

Proof The rule $\mathsf{5}^\circ$ allows to move an output $\Diamond^\circ$-formula from anywhere in the sequent tree, except the root, to any other place in the sequent tree. The same can be achieved with the rules $\mathsf{5}_1^\circ, \mathsf{5}_2^\circ, \mathsf{5}_3^\circ$, and similarly for $\mathsf{5}^\bullet$. □

Lemma 4.8 *Let* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$*, let* $\mathsf{x} \in \mathsf{X}$*, and let* $\mathsf{r}\,\frac{\Gamma_1}{\Gamma_2}$ *be an instance of* x° *or* $\mathsf{x}^\bullet$*. Then* $\mathit{fm}(\Gamma_1) \supset \mathit{fm}(\Gamma_2)$ *is* X*-valid.*

Proof For the rules d°, t°, b°, and $\mathsf{4}^\circ$ this follows immediately from Lemma 4.3 and the validity of the corresponding axioms, shown in Fig. 1 (note that b° can be decomposed into $\mathsf{m}^{[]}$ and $\tilde{\mathsf{b}}^\circ\ \frac{\Gamma\{A^\circ\}}{\Gamma\{[\Diamond A^\circ]\}}$, and $\mathsf{4}^\circ$ into $\Diamond^\circ$ and $\tilde{\mathsf{4}}^\circ\ \frac{\Gamma\{\Diamond\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}}$). For $\mathsf{5}^\circ$ we use Proposition 4.7, where soundness of $\mathsf{5}_1^\circ$, $\mathsf{5}_2^\circ$, and $\mathsf{5}_3^\circ$ is shown as for b° and $\mathsf{4}^\circ$ (using that axiom 5 implies $\Diamond\cdots\Diamond A \supset \Box\Diamond A$). For the rules $\mathsf{d}^\bullet$, $\mathsf{t}^\bullet$, $\mathsf{b}^\bullet$, $\mathsf{4}^\bullet$, and $\mathsf{5}^\bullet$ we proceed similarly, using soundness of the c-rule and Lemma 4.4 instead of Lemma 4.3. □

Let us now turn to showing the soundness of the branching rules $\wedge^\circ$, $\vee^\bullet$, $\supset^\bullet$, and cut. For this, we start with the binary versions of Lemmas 4.2, 4.3, and 4.4.

Lemma 4.9 *Let* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$*, and let A, B, C, and D be formulas.*
(i) If $A \wedge B \supset C$ *is* X*-valid, then so is* $(D \supset A) \wedge (D \supset B) \supset (D \supset C)$*.*
(ii) If $A \wedge B \supset C$ *is* X*-valid, then so is* $\Box A \wedge \Box B \supset \Box C$*.*
(iii) If $C \supset A \vee B$ *is* X*-valid, then so is* $(D \wedge C) \supset (D \wedge A) \vee (D \wedge B)$*.*
(iv) If $C \supset A \vee B$ *is* X*-valid, then so is* $\Diamond C \supset \Diamond A \vee \Diamond B$*.*
(v) If $C \supset A \vee B$ *is* X*-valid, then so is* $(A \supset D) \wedge (B \supset D) \supset (C \supset D)$*.*

Proof As Lemma 4.2, this follows immediately from (4) and Lemma 2.1. □

Lemma 4.10 *Let* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$*, let* Δ_1*,* Δ_2*, and* Σ *be full sequents, and let* $\Gamma\{\ \}$ *be an output context. If* $\mathit{fm}(\Delta_1) \wedge \mathit{fm}(\Delta_2) \supset \mathit{fm}(\Sigma)$ *is* X*-valid, then so is* $\mathit{fm}(\Gamma\{\Delta_1\}) \wedge \mathit{fm}(\Gamma\{\Delta_2\}) \supset \mathit{fm}(\Gamma\{\Sigma\})$*.*

Proof Induction on $\Gamma\{\ \}$, using Lemma 4.9.(i) and (ii). □

Lemma 4.11 *Let* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$*, let* Δ_1*,* Δ_2*, and* Σ *be LHS-sequents, and let* $\Gamma\{\ \}$ *be an input context. If* $\mathit{fm}(\Sigma) \supset \mathit{fm}(\Delta_1) \vee \mathit{fm}(\Delta_2)$ *is* X*-valid, then so is* $\mathit{fm}(\Gamma\{\Delta_1\}) \wedge \mathit{fm}(\Gamma\{\Delta_2\}) \supset \mathit{fm}(\Gamma\{\Sigma\})$*.*

Proof By Observation 3.2, we have that $\Gamma\{\ \} = \Gamma'\{\Lambda\{\ \},\Pi\}$ for some $\Gamma'\{\ \}$ and $\Lambda\{\ \}$ and Π. By induction on $\Lambda\{\ \}$, using Lemma 4.9.(iii) and (iv), we get that $\mathit{fm}(\Lambda\{\Sigma\}) \supset \mathit{fm}(\Lambda\{\Delta_1\}) \vee \mathit{fm}(\Lambda\{\Delta_2\})$ is X-valid. From Lemma 4.9.(v) it then follows that $\mathit{fm}(\Lambda\{\Delta_1\},\Pi) \wedge \mathit{fm}(\Lambda\{\Delta_2,\Pi\}) \supset \mathit{fm}(\Lambda\{\Sigma\},\Pi)$ is X-valid. Now the statement follows from Lemma 4.10. □

Lemma 4.12 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*, and let* $\mathsf{r}\,\dfrac{\Gamma_1 \quad \Gamma_2}{\Gamma_3}$ *be an instance of* $\wedge^\circ$*,* $\vee^\bullet$*,* $\supset^\bullet$*, or* cut*. Then* $\mathit{fm}(\Gamma_1) \wedge \mathit{fm}(\Gamma_2) \supset \mathit{fm}(\Gamma_3)$ *is* X*-valid.*

Proof For the $\wedge^\circ$- and $\vee^\bullet$-rules, this follows immediately from Lemmas 4.10 and 4.11. For $\supset^\bullet$ and cut, it suffices to show the statement for the rule

$$\tilde{\supset}^\bullet \, \frac{\Gamma^{\downarrow}\{A^\circ\} \qquad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \tag{8}$$

By Observation 3.2 and Definition 3.3, this rule is of shape

$$\tilde{\supset}^\bullet \, \frac{\Gamma'\{\Lambda\{A^\circ\}, [\Pi\{\emptyset\}]\} \qquad \Gamma'\{\Lambda\{B^\bullet\}, [\Pi\{C^\circ\}]\}}{\Gamma'\{\Lambda\{A \supset B^\bullet\}, [\Pi\{C^\circ\}]\}}$$

where $\Gamma'\{\ \}$, $\Lambda\{\ \}$, and $\Pi\{\ \}$ are output contexts. In particular, let $\Lambda\{\ \} = \Lambda_0, [\Lambda_1, [\ldots, [\Lambda_n, \{\ \}] \ldots]]$ and $\Pi\{\ \} = \Pi_1, [\Pi_2, [\ldots, [\Pi_m, \{\ \}] \ldots]]$. Now let $L_i = \mathit{fm}(\Lambda_i)$ for $i = 0 \ldots n$ and $P_j = \mathit{fm}(\Pi_j)$ for $j = 1 \ldots m$, and let

$$\begin{aligned}
L_X &= \mathit{fm}(\Lambda\{A^\circ\}) = L_0 \supset \Box(L_1 \supset \Box(L_2 \supset \Box(\cdots \supset \Box(L_n \supset A) \cdots)))\\
L_Y &= \mathit{fm}(\Lambda\{B^\bullet\}) = L_0 \wedge \Diamond(L_1 \wedge \Diamond(L_2 \wedge \Diamond(\cdots \wedge \Diamond(L_n \wedge B) \cdots)))\\
L_Z &= \mathit{fm}(\Lambda\{A \supset B^\bullet\}) = L_0 \wedge \Diamond(L_1 \wedge \Diamond(L_2 \wedge \Diamond(\cdots \wedge \Diamond(L_n \wedge (A \supset B)) \cdots)))\\
P_\emptyset &= \mathit{fm}([\Pi\{\emptyset\}]) = \Diamond(P_1 \wedge \Diamond(P_2 \wedge \Diamond(\cdots \wedge \Diamond(P_{m-1} \wedge \Diamond P_m) \cdots)))\\
P_C &= \mathit{fm}([\Pi\{C^\circ\}]) = \Box(P_1 \supset \Box(P_2 \supset \Box(\cdots \supset \Box(P_{m-1} \supset \Box(P_m \supset C)) \cdots)))
\end{aligned}$$

We are first going to show that $(L_X \wedge (L_Y \supset P_C)) \supset (L_Z \supset P_C)$ is X-valid. For this, it suffices to show that for every world w_0 of an arbitrary X-frame, if $w_0 \Vdash L_X$ and $w_0 \Vdash L_Y \supset P_C$ then $w_0 \Vdash L_Z \supset P_C$. So, assume that $w_0 \Vdash L_X$ and $w_0 \Vdash L_Y \supset P_C$. By definition, $w_0 \Vdash L_X$ means that

for all worlds $w_0', w_0'', w_1, w_1', w_1'', \ldots, w_n, w_n'$, if $w_j'' R w_{j+1}$ and $w_i \leq w_i' \leq w_i''$ and $w_i' \Vdash L_i$ then $w_n' \Vdash A$, (9)

and $w_0 \Vdash L_Y \supset P_C$ means that

for all worlds $\hat{w}_0$ with $w_0 \leq \hat{w}_0$, if there are worlds $\hat{w}_1, \ldots, \hat{w}_n$ with $\hat{w}_i R \hat{w}_{i+1}$ and $\hat{w}_i \Vdash L_i$ and $\hat{w}_n \Vdash B$ then $\hat{w}_0 \Vdash P_C$. (10)

We want to show $w_0 \Vdash L_Z \supset P_C$, which means that

for all worlds $\tilde{w}_0$ with $w_0 \leq \tilde{w}_0$, if there are worlds $\tilde{w}_1, \ldots, \tilde{w}_n$ with $\tilde{w}_i R \tilde{w}_{i+1}$ and $\tilde{w}_i \Vdash L_i$ and $\tilde{w}_n \Vdash A \supset B$ then $\tilde{w}_0 \Vdash P_C$. (11)

So, let us assume we have a chain $\tilde{w}_0 R \tilde{w}_1 R \ldots R \tilde{w}_n$ with $\tilde{w}_i \Vdash L_i$ and $\tilde{w}_n \Vdash A \supset B$. By (9), (F1), and monotonicity (Lemma 2.1), we can conclude that $\tilde{w}_n \Vdash A$. Therefore, we also get $\tilde{w}_n \Vdash B$. Thus, by (10), we get $\tilde{w}_0 \Vdash P_C$, as desired. In a similar way, one can show that $(P_\emptyset \supset P_C) \supset P_C$ is X-valid. Now note that

$$\big((P_\emptyset \supset P_C) \supset P_C\big) \wedge \big(L_X \wedge (L_Y \supset P_C) \supset (L_Z \supset P_C)\big) \supset \big((P_\emptyset \supset L_X) \wedge (L_Y \supset P_C) \supset (L_Z \supset P_C)\big)$$

$$
{\supset^\circ}\dfrac{{\supset^\circ}\dfrac{{\Box^\circ}\dfrac{{\Box^\bullet}\dfrac{{\Box^\bullet}\dfrac{{\supset^\bullet}\dfrac{\mathsf{id}\dfrac{}{\Box(A\supset B)^\bullet,\Box A^\bullet,[A\supset B^\bullet,A^\circ,A^\bullet]}\quad \mathsf{id}\dfrac{}{\Box(A\supset B)^\bullet,\Box A^\bullet,[B^\bullet,A^\bullet,B^\circ]}}{\Box(A\supset B)^\bullet,\Box A^\bullet,[A\supset B^\bullet,A^\bullet,B^\circ]}}{\Box(A\supset B)^\bullet,\Box A^\bullet,[A^\bullet,B^\circ]}}{\Box(A\supset B)^\bullet,\Box A^\bullet,[B^\circ]}}{\Box(A\supset B)^\bullet,\Box A^\bullet,\Box B^\circ}}{\Box(A\supset B)^\bullet,\Box A\supset\Box B^\circ}}{\Box(A\supset B)\supset(\Box A\supset\Box B)^\circ}
$$

$$
{\supset^\circ}\dfrac{{\supset^\circ}\dfrac{{\Diamond^\bullet}\dfrac{{\Diamond^\circ}\dfrac{{\Box^\bullet}\dfrac{{\supset^\bullet}\dfrac{\mathsf{id}\dfrac{}{\Box(A\supset B)^\bullet,[A\supset B^\bullet,A^\circ,A^\bullet]}\quad \mathsf{id}\dfrac{}{\Box(A\supset B)^\bullet,[B^\bullet,A^\bullet,B^\circ]}}{\Box(A\supset B)^\bullet,[A\supset B^\bullet,A^\bullet,B^\circ]}}{\Box(A\supset B)^\bullet,[A^\bullet,B^\circ]}}{\Box(A\supset B)^\bullet,[A^\bullet],\Diamond B^\circ}}{\Box(A\supset B)^\bullet,\Diamond A^\bullet,\Diamond B^\circ}}{\Box(A\supset B)^\bullet,\Diamond A\supset\Diamond B^\circ}}{\Box(A\supset B)\supset(\Diamond A\supset\Diamond B)^\circ}
\qquad
{\supset^\circ}\dfrac{{\Diamond^\bullet}\dfrac{{\bot^\bullet}\dfrac{}{[\bot^\bullet],\bot^\circ}}{\Diamond\bot^\bullet,\bot^\circ}}{\Diamond\bot\supset\bot^\circ}
$$

$$
{\supset^\circ}\dfrac{{\Diamond^\bullet}\dfrac{{\vee^\bullet}\dfrac{{\vee^\circ}\dfrac{{\Diamond^\circ}\dfrac{\mathsf{id}\dfrac{}{[A^\bullet,A^\circ]}}{[A^\bullet],\Diamond A^\circ}}{[A^\bullet],\Diamond A\vee\Diamond B^\circ}\quad {\vee^\circ}\dfrac{{\Diamond^\circ}\dfrac{\mathsf{id}\dfrac{}{[B^\bullet,B^\circ]}}{[B^\bullet],\Diamond B^\circ}}{[B^\bullet],\Diamond A\vee\Diamond B^\circ}}{[A\vee B^\bullet],\Diamond A\vee\Diamond B^\circ}}{\Diamond(A\vee B)^\bullet,\Diamond A\vee\Diamond B^\circ}}{\Diamond(A\vee B)\supset(\Diamond A\vee\Diamond B)^\circ}
\qquad
{\supset^\circ}\dfrac{{\Box^\circ}\dfrac{{\supset^\circ}\dfrac{{\supset^\bullet}\dfrac{{\Diamond^\circ}\dfrac{\mathsf{id}\dfrac{}{\Diamond A\supset\Box B^\bullet,[A^\circ,A^\bullet]}}{\Diamond A\supset\Box B^\bullet,\Diamond A^\circ,[A^\bullet]}\quad {\Box^\bullet}\dfrac{\mathsf{id}\dfrac{}{\Box B^\bullet,[B^\bullet,A^\bullet,B^\circ]}}{\Box B^\bullet,[A^\bullet,B^\circ]}}{\Diamond A\supset\Box B^\bullet,[A^\bullet,B^\circ]}}{\Diamond A\supset\Box B^\bullet,[A\supset B^\circ]}}{\Diamond A\supset\Box B^\bullet,\Box(A\supset B)^\circ}}{(\Diamond A\supset\Box B)\supset\Box(A\supset B)^\circ}
$$

Fig. 6. Proofs of $\mathsf{k}_1,\ldots,\mathsf{k}_5$ in NIK

is a valid intuitionistic formula (for arbitrary $P_\emptyset, P_C, L_X, L_Y, L_Z$). Thus, we can conclude that $(P_\emptyset \supset L_X) \wedge (L_Y \supset P_C) \supset (L_Z \supset P_C)$ is X-valid, and we can apply Lemma 4.10. □

Now we can put everything together to prove Theorem 4.1.

Proof (of Theorem 4.1) Point (i) is just Lemmas 4.5, 4.6, 4.12, and 4.8. Point (ii) follows immediately from (i) using induction on the size of the derivation. □

5 Completeness

For simplifying the presentation, we show completeness with respect to the Hilbert system.

Theorem 5.1 *Let* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$. *Then every theorem of the logic* $\mathsf{IK}+\mathsf{X}$ *is provable in* $\mathsf{NIK}+\mathsf{X}^\bullet+\mathsf{X}^\circ+\mathsf{cut}$.

Proof Clearly, all axioms of propositional intuitionistic logic are provable in NIK. The axioms $\mathsf{k}_1,\ldots,\mathsf{k}_5$ are provable in NIK, as shown in Figure 6. Furthermore, each axiom $\mathsf{x} \in \mathsf{X}$ is provable in $\mathsf{NIK} + \mathsf{x}^\bullet + \mathsf{x}^\circ$. This is left to the reader, as these proofs are very similar to the classical setting [3]. Finally, the rules mp and nec, shown in (3), can be simulated by the rules cut and $\mathsf{nec}^{[]}$, shown in (7). Then, the $\mathsf{nec}^{[]}$-rule is admissible, which can be seen by a straightforward induction on the size of the proof. □

In the next section we show cut elimination for $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ$, yielding completeness for the cut-free system. However, it turns out that this system is not for every X complete. As observed by Brünnler, in the classical case X needs to be 45-closed [3]. In the intuitionistic case, X needs to be t45-closed:

Definition 5.2 Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. We say that X is *45-closed* if the following two conditions are fulfilled:
- if $\mathsf{4}$ is derivable in $\mathsf{IK} + \mathsf{X}$ then $\mathsf{4} \in \mathsf{X}$, and
- if $\mathsf{5}$ is derivable in $\mathsf{IK} + \mathsf{X}$ then $\mathsf{5} \in \mathsf{X}$.

We say that X is *t45-closed* if additionally the following condition holds:
- if t is derivable in $\mathsf{IK} + \mathsf{X}$ then $\mathsf{t} \in \mathsf{X}$.

This is needed, because, for example, the formula $\Box A \supset \Box\Box A$ holds in any $\{\mathsf{t}, \mathsf{5}\}$-frame, but for proving it *without cut*, one would need the rules $\mathsf{4}^\bullet$ and $\mathsf{4}^\circ$. The cut elimination result of the next section will entail the following theorem:

Theorem 5.3 (Completeness) *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ be t45-closed. Then every theorem of the logic $\mathsf{IK} + \mathsf{X}$ is provable in $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ$.*

6 Cut Elimination

We define the *depth* of a formula A, denoted by $depth(A)$, inductively as follows:

$$depth(a) = depth(\bot) = 1$$
$$depth(\Box A) = depth(\Diamond A) = depth(A) + 1$$
$$depth(A \wedge B) = depth(A \vee B) = depth(A \supset B) = \max(depth(A), depth(B)) + 1$$

Definition 6.1 Given an instance of cut (as shown in (7)), its *cut formula* is A, and its *cut rank* is $depth(A)$. The *cut rank* of a derivation $\mathcal{D}$, denoted by $rank(\mathcal{D})$, is the maximum of the cut ranks of the cut instances of $\mathcal{D}$. Thus, a derivation with cut rank 0 is cut-free. For $r > 0$, we define the rule cut_r as cut whose cut rank is $\leq r$. As usual, the *height* of a derivation $\mathcal{D}$, denoted by $|\mathcal{D}|$, is defined to be the length of the maximal branch in the derivation tree.

Definition 6.2 We say that a rule r with one premise is *height* (respectively *cut rank*) *preserving admissible* in a system S, if for each derivation $\mathcal{D}$ in S of r's premise there is a derivation $\mathcal{D}'$ of r's conclusion in S, such that $|\mathcal{D}'| \leq |\mathcal{D}|$ (respectively $rank(\mathcal{D}') \leq rank(\mathcal{D})$). Similarly, a rule r is *height* (respectively *cut rank*) *preserving invertible* in a system S, if for every derivation of the conclusion of r there are derivations for each of r's premises with at most the same height (respectively at most the same rank).

$$\mathsf{d}^{[]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \quad \mathsf{t}^{[]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \quad \mathsf{b}^{[]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{[\Sigma],\Delta\}} \quad \mathsf{4}^{[]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}} \quad \mathsf{5}^{[]}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}$$

(where $depth(\Gamma\{\ \}\{\emptyset\}) > 0$)

Fig. 7. Structural rules for the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$

Figure 7 shows for each axiom in $\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ a corresponding structural rule. They will occur during the cut elimination process. Note that these rules are exactly the same as in the classical case [4]. These rules are admissible for the corresponding system, provided it is t45-closed. This lemma is the only place in the cut elimination proof, where this property is needed. As in the classical case [3], the $\mathsf{d}^{[]}$-rule needs special treatment.

Lemma 6.3 *(i) Let* $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *be 45-closed, and let* $\mathsf{r} \in \mathsf{X}^{[]}$*. Then the rule* r *is cut-rank preserving admissible for* $\mathsf{NIK} \cup \mathsf{X}^\bullet \cup \mathsf{X}^\circ \cup \{\mathsf{cut}\}$ *as well as for* $\mathsf{NIK} \cup \mathsf{X}^\bullet \cup \mathsf{X}^\circ \cup \{\mathsf{cut},\mathsf{d}^{[]}\}$.
(ii) Let $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *be t45-closed with* $\mathsf{d} \in \mathsf{X}$*. Then the rule* $\mathsf{d}^{[]}$ *is admissible for* $\mathsf{NIK} \cup \mathsf{X}^\bullet \cup \mathsf{X}^\circ$*.*

Proof The proof for (i) is similar to the one in [3]. But in the case analysis every case appears twice, once for the $\mathsf{x}^\bullet$ and once for the x° rule. For (ii), the proof is also almost the same as in [3], except that the rule t° can be introduced when $\{\mathsf{d},\mathsf{b},\mathsf{4}\} \subseteq \mathsf{X}$, because there is no contraction available for output formulas. □

Lemma 6.4 *Let* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *and either* $\mathsf{Z} = \mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ + \mathsf{cut}$ *or* $\mathsf{Z} = \mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ + \mathsf{d}^{[]} + \mathsf{cut}$.

- *(i) The rules* $\mathsf{nec}^{[]}$*,* w*,* c*,* $\mathsf{m}^{[]}$ *are height and cut rank preserving admissible for* Z*.*
- *(ii) All rules* $\mathsf{r}^\bullet$ *(except* $\bot^\bullet$ *and* $\supset^\bullet$*) in* Z *are height and cut rank preserving invertible.*

Proof For m, we can proceed by a straightforward induction on the height of the derivation. For all other rules, this proof is exactly the same as in [3]. □

When we eliminate the cut rule from a proof, we will at some point rely on local transformations that reduce the cut rank. However when the cut meets the rules $\mathsf{4}^\bullet, \mathsf{4}^\circ$ or $\mathsf{5}^\bullet, \mathsf{5}^\circ$ while moving upwards, its rank does not decrease. For this reason, we use the Y-cut-rules [3], defined below for $\mathsf{Y} \subseteq \{\mathsf{4},\mathsf{5}\}$:

$$\Diamond\mathsf{Y\text{-}cut}\ \frac{\Gamma^{\downarrow}\{\emptyset\}\{\Diamond A^\circ\} \qquad \Gamma\{\Diamond A^\bullet\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\emptyset\}} \qquad \Box\mathsf{Y\text{-}cut}\ \frac{\Gamma^{\downarrow}\{\Box A^\circ\}\{\emptyset\}^n \qquad \Gamma\{\Box A^\bullet\}\{\Box A^\bullet\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

where for $\Diamond\mathsf{Y\text{-}cut}$ there must be a derivation from $\Gamma^{\downarrow}\{\emptyset\}\{\Diamond A^\circ\}$ to $\Gamma^{\downarrow}\{\Diamond A^\circ\}\{\emptyset\}$ in Y°, and for $\Box\mathsf{Y\text{-}cut}$ there must be a derivation from $\Gamma\{\Box A^\bullet\}\{\Box A^\bullet\}^n$ to $\Gamma\{\Box A^\bullet\}\{\emptyset\}^n$ in $\mathsf{Y}^\bullet$. Here, we use the notation $\{\Delta\}^n$ as abbreviation for n holes that are all filled with the same Δ. For $r \geq 0$, the rules $\Diamond\mathsf{Y\text{-}cut}_r$ and $\Box\mathsf{Y\text{-}cut}_r$ are defined analogous to cut_r.

Observation 6.5 If $\mathsf{Y} = \emptyset$ then $\Gamma\{\ \}\{\ \} = \Gamma'\{\{\ \},\{\ \}\}$, for some input context $\Gamma'\{\ \}$, and both $\Diamond\mathsf{Y}$-cut and $\Box\mathsf{Y}$-cut are just ordinary cuts. If $\mathsf{Y} = \{4\}$ then in $\Diamond\mathsf{Y}$-cut we have $\Gamma\{\ \}\{\ \} = \Gamma'\{\{\ \},\Gamma''\{\ \}\}$ for some input contexts $\Gamma'\{\ \}$ and $\Gamma''\{\ \}$, and in $\Box\mathsf{Y}$-cut we have $\Gamma\{\ \}\{\ \}^n = \Gamma'\{\{\ \},\Gamma''\{\ \}^n\}$. If $\mathsf{Y} = \{5\}$ then the first hole must be "inside a box", i.e., in $\Diamond\mathsf{Y}$-cut we have $\mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$ and in $\Box\mathsf{Y}$-cut we have $\mathit{depth}(\Gamma\{\ \}\{\emptyset\}^n) > 0$. If $\mathsf{Y} = \{4, 5\}$ then there is no restriction on the context.

Lemma 6.6 *Let* $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, 4, 5\}$ *be 45-closed, let* $\mathsf{Y} \subseteq \{4, 5\} \cap \mathsf{X}$*, let either* $\mathsf{Z} = \mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ$ *or* $\mathsf{Z} = \mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ + \mathsf{d}^{[]}$*, and let* $r, n \geq 0$.

(i) If there is a proof of shape

$$\mathsf{cut}_{r+1}\,\frac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{A^\circ\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}}$$

with $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *in* $\mathsf{Z} + \mathsf{cut}_r$*, then there is a proof of* $\Gamma\{\emptyset\}$ *in* $\mathsf{Z} + \mathsf{cut}_r$.

(ii) If there is a proof of shape

$$\Diamond\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\,\frac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{\Diamond A^\circ\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond A^\bullet\}\{\emptyset\}}}{\Gamma\{\emptyset\}\{\emptyset\}}$$

with $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *in* $\mathsf{Z} + \mathsf{cut}_r$*, then there is a proof of* $\Gamma\{\emptyset\}\{\emptyset\}$ *in* $\mathsf{Z} + \mathsf{cut}_r$.

(iii) If there is a proof of shape

$$\Box\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\,\frac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\Box A^\circ\}\{\emptyset\}^n} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Box A^\bullet\}\{\Box A^\bullet\}^n}}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

with $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *in* $\mathsf{Z} + \mathsf{cut}_r$*, then there is a proof of* $\Gamma\{\emptyset\}\{\emptyset\}^n$ *in* $\mathsf{Z} + \mathsf{cut}_r$.

Proof (Sketch) This is proved for all three points simultaneously by induction on $|\mathcal{D}_1| + |\mathcal{D}_2|$. If one of $\mathcal{D}_1$ or $\mathcal{D}_2$ is an axiom, the cut disappears. One case is shown below

$$\mathsf{cut}_1\,\frac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\bot^\circ\}} \qquad \bot^\bullet\,\frac{}{\Gamma\{\bot^\bullet\}}}{\Gamma\{\emptyset\}} \quad \rightsquigarrow \quad \overset{\mathcal{D}_1'}{\Gamma\{\emptyset\}}$$

where $\mathcal{D}_1'$ is obtained from $\mathcal{D}_1$ by removing the $\bot^\circ$ in every line and keeping the output formula of $\Gamma\{\emptyset\}$ instead. This is possible because there is no rule for $\bot^\circ$. The other axiomatic cases are more standard. If in one of $\mathcal{D}_1$ or $\mathcal{D}_2$ the bottommost rule does not work on the cut formula, we have one of the commutative cases, which are very similar to the standard sequent calculus and make crucial use of the invertability of the $\mathsf{r}^\bullet$-rules. Finally, we have the so called key cases. We show the case involving $\Box\mathsf{Y}\text{-}\mathsf{cut}$ and $\mathsf{b}^\bullet$, in which the derivation

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^\circ]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\Box A^\circ\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}\,\Box^\circ \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{\Box A^\bullet\}^{n}\{A^\bullet,[\Box A^\bullet,\Delta]\}}}{\Gamma\{\Box A^\bullet\}^{n}\{[\Box A^\bullet,\Delta]\}}\,\mathsf{b}^\bullet}{\Gamma\{\emptyset\}^{n}\{[\Delta]\}}\,\Box\mathsf{Y}\text{-}\mathsf{cut}_{r+1}$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^\circ]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\emptyset\}^{n}\{[[A^\circ],\Delta^{\downarrow}]\}}\,\mathsf{Y}^{[]}}{\Gamma^{\downarrow}\{\emptyset\}^{n}\{A^\circ,[\Delta^{\downarrow}]\}}\,\mathsf{b}^{[]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^\circ]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{[A^\circ]\}\{\emptyset\}^{n-1}\{A^\bullet,[\Delta^{\downarrow}]\}}\,\mathsf{w}}{\Gamma^{\downarrow}\{\Box A^\circ\}\{\emptyset\}^{n-1}\{A^\bullet,[\Delta^{\downarrow}]\}}\,\Box^\circ \quad \dfrac{\mathcal{D}_2}{\Gamma\{\Box A^\bullet\}^{n}\{A^\bullet,[\Box A^\bullet,\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{A^\bullet,[\Delta]\}}\,\Box\mathsf{Y}\text{-}\mathsf{cut}_{r+1}}{\Gamma\{\emptyset\}^{n}\{[\Delta]\}}\,\mathsf{cut}_r$$

where $\mathsf{Y}^{[]}$ stands for a derivation consisting of $\mathsf{4}^{[]}$ and $\mathsf{5}^{[]}$, depending on the chosen Y. Then, on the left branch, we use cut rank preserving admissibility of the $\mathsf{b}^{[]}$-, $\mathsf{4}^{[]}$-, and $\mathsf{5}^{[]}$-rules. On the right branch, we use cut rank and height preserving admissibility of weakening together with the induction hypothesis. The other cases are similar. ◻

Theorem 6.7 *Let* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *be t45-closed. If a sequent* Γ *is provable in* $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ + \mathsf{cut}$ *then it is also provable in* $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ$.

Proof If $\mathsf{d} \notin \mathsf{X}$ the result follows from Lemma 6.6 by a straightforward induction on the cut rank of the derivation. If $\mathsf{d} \in \mathsf{X}$, we first replace all instances of $\mathsf{d}^\bullet$ by $\Box^\bullet$ and $\mathsf{d}^{[]}$, and all instances of d° by $\Diamond^\circ$ and $\mathsf{d}^{[]}$. Then we proceed as before, and finally we apply Lemma 6.3.(ii) to remove $\mathsf{d}^{[]}$. ◻

Finally, we can drop the t45-closed condition and obtain full modularity by also allowing the structural rules of Figure 7 in the system:

Theorem 6.8 *Let* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$. *If a sequent* Γ *is provable in* $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ + \mathsf{cut}$ *then it is also provable in* $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ + \mathsf{X}^{[]}$.

Proof (Sketch) We first transform a proof in $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ + \mathsf{cut}$ into one in $\mathsf{NIK} + \mathsf{Y}^\bullet + \mathsf{Y}^\circ$ by Theorem 6.7, where Y is the t45-closure of X. Trivially, this is also a proof in $\mathsf{NIK} + \mathsf{Y}^\bullet + \mathsf{Y}^\circ + \mathsf{X}^{[]}$. This is then transformed into a proof in $\mathsf{NIK} + \mathsf{X}^\bullet + \mathsf{X}^\circ + \mathsf{X}^{[]}$ by showing admissibility of the superfluous rules. ◻

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A Cut reduction cases

This appendix contains a list of the cut reduction cases. It is not part of the published paper in the proceedings of FoSSaCS 2013.

A.1 Axiomatic cases

$$\mathsf{cut}_1\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\bot^\circ\}} \qquad \bot^\bullet\,\dfrac{}{\Gamma\{\bot^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\bot^\bullet)}{\leadsto} \quad \overset{\mathcal{D}_1'}{\Gamma\{\emptyset\}}$$

$$\mathsf{cut}_1\ \dfrac{\mathsf{id}\,\dfrac{}{\Gamma^{\downarrow}\{a^\bullet, a^\circ\}} \qquad \overset{\mathcal{D}_1}{\Gamma\{a^\bullet, a^\bullet\}}}{\Gamma\{a^\bullet\}} \quad \overset{(a^\bullet)}{\leadsto} \quad \overset{\mathcal{D}_1'}{\Gamma\{a^\bullet\}}$$

$$\mathsf{cut}_1\ \dfrac{\overset{\mathcal{D}_1}{\Gamma\{a^\circ\}} \qquad \mathsf{id}\,\dfrac{}{\Gamma\{a^\bullet, a^\circ\}}}{\Gamma\{a^\circ\}} \quad \overset{(a^\circ)}{\leadsto} \quad \overset{\mathcal{D}_1}{\Gamma\{a^\circ\}}$$

$$\mathsf{cut}_{r+1}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma\{a^\bullet\}\{A^\circ\}} \qquad \mathsf{id}\,\dfrac{}{\Gamma\{a^\bullet, a^\circ\}\{A^\bullet\}}}{\Gamma\{a^\bullet, a^\circ\}\{\emptyset\}} \quad \overset{(\mathsf{id}_1)}{\leadsto} \quad \mathsf{id}\,\dfrac{}{\Gamma\{a^\bullet, a^\circ\}\{\emptyset\}}$$

In the $\bot^\bullet$-reduction, $\mathcal{D}_1'$ is obtained from $\mathcal{D}_1$ by removing the $\bot^\circ$ in every line and keeping the output formula of $\Gamma\{\emptyset\}$ instead. This is possible because there is no rule for $\bot^\circ$. In the $a^\bullet$-reduction we use the cut-rank preserving admissibility of contraction. In the a°-reduction, note that here $\Gamma^{\downarrow}\{a^\circ\} = \Gamma\{a^\circ\}$. For the last reduction, there are three more cases that are analogous and that are not shown.

A.2 Commutative cases

$$\mathsf{cut}_r \dfrac{\mathsf{r}^\bullet \dfrac{\overset{\mathcal{D}_1}{\Gamma_1^\downarrow\{A^\circ\}}}{\Gamma^\downarrow\{A^\circ\}} \quad \overset{\mathcal{D}_2}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\mathsf{r}^\bullet)}{\rightsquigarrow} \quad \mathsf{r}^\bullet \dfrac{\mathsf{cut}_r \dfrac{\overset{\mathcal{D}_1}{\Gamma_1^\downarrow\{A^\circ\}} \quad \overset{\mathcal{D}_2'}{\Gamma_1\{A^\bullet\}}}{\Gamma_1\{\emptyset\}}}{\Gamma\{\emptyset\}}$$

$$\mathsf{cut}_r \dfrac{\overset{\mathcal{D}_1}{\Gamma^\downarrow\{A^\circ\}} \quad \mathsf{r}^\circ \dfrac{\overset{\mathcal{D}_2}{\Gamma_1\{A^\bullet\}}}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\mathsf{r}^\circ)}{\rightsquigarrow} \quad \mathsf{r}^\circ \dfrac{\mathsf{cut}_r \dfrac{\overset{\mathcal{D}_1'}{\Gamma_1^\downarrow\{A^\circ\}} \quad \overset{\mathcal{D}_2}{\Gamma_1\{A^\bullet\}}}{\Gamma_1\{\emptyset\}}}{\Gamma\{\emptyset\}}$$

$$\mathsf{cut}_r \dfrac{\supset^\bullet \dfrac{\overset{\mathcal{D}_1}{\Gamma^\downarrow\{B \supset C^\bullet, B^\circ\}\{\emptyset\}} \quad \overset{\mathcal{D}_2}{\Gamma^\downarrow\{C^\bullet\}\{A^\circ\}}}{\Gamma^\downarrow\{B \supset C^\bullet\}\{A^\circ\}} \quad \overset{\mathcal{D}_3}{\Gamma\{B \supset C^\bullet\}\{A^\bullet\}}}{\Gamma\{B \supset C^\bullet\}\{\emptyset\}} \quad \overset{(\supset_1^\bullet)}{\rightsquigarrow}$$

$$\supset^\bullet \dfrac{\overset{\mathcal{D}_1}{\Gamma^\downarrow\{B \supset C^\bullet, B^\circ\}\{\emptyset\}} \quad \mathsf{cut}_r \dfrac{\overset{\mathcal{D}_2}{\Gamma^\downarrow\{C^\bullet\}\{A^\circ\}} \quad \overset{\mathcal{D}_3'}{\Gamma\{C^\bullet\}\{A^\bullet\}}}{\Gamma\{C^\bullet\}\{\emptyset\}}}{\Gamma\{B \supset C^\bullet\}\{\emptyset\}}$$

$$\mathsf{cut}_r \dfrac{\overset{\mathcal{D}_1}{\Gamma^\downarrow\{B \supset C^\bullet\}\{A^\circ\}} \quad \supset^\bullet \dfrac{\overset{\mathcal{D}_2}{\Gamma^\downarrow\{B \supset C^\bullet, B^\circ\}\{A^\bullet\}} \quad \overset{\mathcal{D}_3}{\Gamma\{C^\bullet\}\{A^\bullet\}}}{\Gamma\{B \supset C^\bullet\}\{A^\bullet\}}}{\Gamma\{B \supset C^\bullet\}\{\emptyset\}} \quad \overset{(\supset_2^\bullet)}{\rightsquigarrow}$$

$$\supset^\bullet \dfrac{\mathsf{cut}_r \dfrac{\overset{\mathcal{D}_1}{\Gamma^\downarrow\{B \supset C^\bullet\}\{A^\circ\}} \quad \overset{\mathcal{D}_2}{\Gamma^\downarrow\{B \supset C^\bullet, B^\circ\}\{A^\bullet\}}}{\Gamma^\downarrow\{B \supset C^\bullet, B^\circ\}\{\emptyset\}} \quad \mathsf{cut}_r \dfrac{\overset{\mathcal{D}_1'}{\Gamma^\downarrow\{C^\bullet\}\{A^\circ\}} \quad \overset{\mathcal{D}_3}{\Gamma\{C^\bullet\}\{A^\bullet\}}}{\Gamma\{C^\bullet\}\{\emptyset\}}}{\Gamma\{B \supset C^\bullet\}\{\emptyset\}}$$

In the $\mathsf{r}^\bullet$-reduction, $\mathcal{D}_2'$ is obtained from $\mathcal{D}_2$ by height and cut rank preserving invertability of the $\mathsf{r}^\bullet$-rules. In the r°-reduction, $\mathcal{D}_1'$ is in almost all cases identical to $\mathcal{D}_1$, except for $\supset^\circ$, where we need height and cut rank preserving admissibility of w. The cases for the branching rules $\vee^\bullet$ and $\wedge^\circ$ are similar. In the cases for $\supset^\bullet$, the derivations $\mathcal{D}_1'$ and $\mathcal{D}_3'$ are obtained from $\mathcal{D}_1$ and $\mathcal{D}_3$, respectively, by observing that whenever there is a proof of $\Gamma\{B \supset C^\bullet\}$ then there is one of $\Gamma\{C^\bullet\}$ having at most the same cut rank and height. This can be shown in the same way as the height and cut rank preserving invertability of the other $\mathsf{r}^\bullet$-rules.

A.3 Key cases for non-modal formulas

$$\mathsf{cut}_{r+1}\dfrac{\vee^\circ\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{A^\circ\}}}{\Gamma^{\downarrow}\{A\vee B^\circ\}} \quad \vee^\bullet\dfrac{\overset{\mathcal{D}_2}{\Gamma\{A^\bullet\}} \quad \overset{\mathcal{D}_3}{\Gamma\{B^\bullet\}}}{\Gamma\{A\vee B^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\vee)}{\rightsquigarrow} \quad \mathsf{cut}_r\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{A^\circ\}} \quad \overset{\mathcal{D}_2}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}}$$

$$\mathsf{cut}_{r+1}\dfrac{\wedge^\circ\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{A^\circ\}} \quad \overset{\mathcal{D}_2}{\Gamma^{\downarrow}\{B^\circ\}}}{\Gamma^{\downarrow}\{A\wedge B^\circ\}} \quad \wedge^\bullet\dfrac{\overset{\mathcal{D}_3}{\Gamma\{A^\bullet, B^\bullet\}}}{\Gamma\{A\wedge B^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\wedge)}{\rightsquigarrow} \quad \mathsf{cut}_r\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{A^\circ\}} \quad \mathsf{cut}_r\dfrac{\overset{\mathcal{D}_2'}{\Gamma^{\downarrow}\{A^\bullet, B^\circ\}} \quad \overset{\mathcal{D}_3}{\Gamma\{A^\bullet, B^\bullet\}}}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}}$$

$$\mathsf{cut}_{r+1}\dfrac{\supset^\circ\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{A^\bullet, B^\circ\}}}{\Gamma^{\downarrow}\{A\supset B^\circ\}} \quad \supset^\bullet\dfrac{\overset{\mathcal{D}_2}{\Gamma^{\downarrow}\{A\supset B^\bullet, A^\circ\}} \quad \overset{\mathcal{D}_3}{\Gamma\{B^\bullet\}}}{\Gamma\{A\supset B^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\supset)}{\rightsquigarrow} \quad \mathsf{cut}_r\dfrac{\mathsf{cut}_{r+1}\dfrac{\overset{\mathcal{D}_2}{\Gamma^{\downarrow}\{A\supset B^\bullet, A^\circ\}} \quad \supset^\circ\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{A^\bullet, B^\circ\}}}{\Gamma^{\downarrow}\{A\supset B^\circ\}}}{\Gamma^{\downarrow}\{A^\circ\}} \quad \mathsf{cut}_r\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{A^\bullet, B^\circ\}} \quad \overset{\mathcal{D}_3}{\Gamma\{B^\bullet\}}}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}}$$

For $\vee$, there is a similar case where $\vee^\circ$ chooses B. For $\wedge$, $\mathcal{D}_2'$ is obtained from $\mathcal{D}_2$ by depth-preserving admissibility of w. For $\supset$, we can apply the induction hypothesis to the left branch to reduce the cut_{r+1}.

A.4 Key cases for $\Diamond$-formulas

$$
\diamond\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\Diamond^{\circ}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{[A^{\circ},\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\emptyset\}\{\Diamond A^{\circ},[\Delta^{\downarrow}]\}}\qquad \Diamond^{\bullet}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{[A^{\bullet}]\}\{[\Delta]\}}}{\Gamma\{\Diamond A^{\bullet}\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{[\Delta]\}}
\;\overset{(\Diamond)}{\leadsto}\;
\mathsf{cut}_r\;\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{[A^{\circ},\Delta^{\downarrow}]\}}\qquad \mathsf{m}^{[]}\,\dfrac{\mathsf{Y}^{[]}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{[A^{\bullet}]\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{[A^{\bullet}],[\Delta]\}}}{\Gamma\{\emptyset\}\{[A^{\bullet},\Delta]\}}}{\Gamma\{\emptyset\}\{[\Delta]\}}
$$

$$
\diamond\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\mathsf{t}^{\circ}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{A^{\circ}\}}}{\Gamma^{\downarrow}\{\emptyset\}\{\Diamond A^{\circ}\}}\qquad \Diamond^{\bullet}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{[A^{\bullet}]\}\{\emptyset\}}}{\Gamma\{\Diamond A^{\bullet}\}\{\emptyset\}}}{\Gamma\{\emptyset\}\{\emptyset\}}
\;\overset{(\Diamond_{\mathsf{t}})}{\leadsto}\;
\mathsf{cut}_r\;\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{A^{\circ}\}}\qquad \mathsf{t}^{[]}\,\dfrac{\mathsf{Y}^{[]}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{[A^{\bullet}]\}\{\emptyset\}}}{\Gamma\{\emptyset\}\{[A^{\bullet}]\}}}{\Gamma\{\emptyset\}\{A^{\bullet}\}}}{\Gamma\{\emptyset\}\{\emptyset\}}
$$

$$
\diamond\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\mathsf{b}^{\circ}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{A^{\circ},[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\emptyset\}\{[\Diamond A^{\circ},\Delta^{\downarrow}]\}}\qquad \Diamond^{\bullet}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{[A^{\bullet}]\}\{[\Delta]\}}}{\Gamma\{\Diamond A^{\bullet}\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{[\Delta]\}}
\;\overset{(\Diamond_{\mathsf{b}})}{\leadsto}\;
\mathsf{cut}_r\;\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{A^{\circ},[\Delta^{\downarrow}]\}}\qquad \mathsf{b}^{[]}\,\dfrac{\mathsf{Y}^{[]}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{[A^{\bullet}]\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{[[A^{\bullet}],\Delta]\}}}{\Gamma\{\emptyset\}\{A^{\bullet},[\Delta]\}}}{\Gamma\{\emptyset\}\{[\Delta]\}}
$$

$$
\diamond\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\mathsf{4}^{\circ}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{[\Diamond A^{\circ},\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\emptyset\}\{\Diamond A^{\circ},[\Delta^{\downarrow}]\}}\qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond A^{\bullet}\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{[\Delta]\}}
\;\overset{(\Diamond_4)}{\leadsto}\;
\diamond\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{[\Diamond A^{\circ},\Delta^{\downarrow}]\}}\qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond A^{\bullet}\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{[\Delta]\}}
$$

$$
\diamond\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\mathsf{5}^{\circ}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{\emptyset\}\{\Diamond A^{\circ}\}}}{\Gamma^{\downarrow}\{\emptyset\}\{\Diamond A^{\circ}\}\{\emptyset\}}\qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond A^{\bullet}\}\{\emptyset\}\{\emptyset\}}}{\Gamma\{\emptyset\}\{\emptyset\}\{\emptyset\}}
\;\overset{(\Diamond_5)}{\leadsto}\;
\diamond\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\emptyset\}\{\emptyset\}\{\Diamond A^{\circ}\}}\qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond A^{\bullet}\}\{\emptyset\}\{\emptyset\}}}{\Gamma\{\emptyset\}\{\emptyset\}\{\emptyset\}}
$$

In the $\Diamond$-, $\Diamond_{\mathsf{t}}$-, and $\Diamond_{\mathsf{b}}$-reductions, we use cut rank preserving admissibility of the $\mathsf{Y}^{[]}$-, $\mathsf{m}^{[]}$-, $\mathsf{t}^{[]}$-, and $\mathsf{b}^{[]}$-rules. In the $\Diamond_4$- and $\Diamond_5$-reductions, we just apply the induction hypothesis.

A.5 Key cases for $\Box$-formulas

$$\mathsf{\Box Y\text{-}cut}_{r+1}\ \dfrac{\Box^{\circ}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}} \qquad \Box^{\bullet}\ \dfrac{\overset{\mathcal{D}_2}{\Gamma\{\Box A^{\bullet}\}^{n}\{\Box A^{\bullet},[A^{\bullet},\Delta]\}}}{\Gamma\{\Box A^{\bullet}\}^{n}\{\Box A^{\bullet},[\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{[\Delta]\}} \quad \overset{(\Box)}{\rightsquigarrow}$$

$$\mathsf{cut}_r\ \dfrac{\mathsf{m}^{[]}\ \dfrac{\mathsf{Y}^{[]}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\emptyset\}^{n}\{[A^{\circ}],[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\emptyset\}^{n}\{[A^{\circ},\Delta^{\downarrow}]\}} \qquad \mathsf{\Box Y\text{-}cut}_{r+1}\ \dfrac{\Box^{\circ}\ \dfrac{\mathsf{w}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{[A^{\bullet},\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n-1}\{[A^{\bullet},\Delta^{\downarrow}]\}} \quad \overset{\mathcal{D}_2}{\Gamma\{\Box A^{\bullet}\}^{n}\{\Box A^{\bullet},[A^{\bullet},\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{[A^{\bullet},\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{[\Delta]\}}$$

$$\mathsf{\Box Y\text{-}cut}_{r+1}\ \dfrac{\Box^{\circ}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n}}}{\Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n}} \qquad \mathsf{t}^{\bullet}\ \dfrac{\overset{\mathcal{D}_2}{\Gamma\{\Box A^{\bullet}\}^{n}\{A^{\bullet}\}}}{\Gamma\{\Box A^{\bullet}\}^{n+1}}}{\Gamma\{\emptyset\}^{n+1}} \quad \overset{(\Box_{\mathsf{t}})}{\rightsquigarrow}$$

$$\mathsf{cut}_r\ \dfrac{\mathsf{t}^{[]}\ \dfrac{\mathsf{Y}^{[]}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n}}}{\Gamma^{\downarrow}\{\emptyset\}^{n}\{[A^{\circ}]\}}}{\Gamma^{\downarrow}\{\emptyset\}^{n}\{A^{\circ}\}} \qquad \mathsf{\Box Y\text{-}cut}_{r+1}\ \dfrac{\Box^{\circ}\ \dfrac{\mathsf{w}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n}}}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{A^{\bullet}\}}}{\Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n-1}\{A^{\bullet}\}} \quad \overset{\mathcal{D}_2}{\Gamma\{\Box A^{\bullet}\}^{n}\{A^{\bullet}\}}}{\Gamma\{\emptyset\}^{n}\{A^{\bullet}\}}}{\Gamma\{\emptyset\}^{n+1}}$$

$$\mathsf{\Box Y\text{-}cut}_{r+1}\ \dfrac{\Box^{\circ}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}} \qquad \mathsf{b}^{\bullet}\ \dfrac{\overset{\mathcal{D}_2}{\Gamma\{\Box A^{\bullet}\}^{n}\{A^{\bullet},[\Box A^{\bullet},\Delta]\}}}{\Gamma\{\Box A^{\bullet}\}^{n}\{[\Box A^{\bullet},\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{[\Delta]\}} \quad \overset{(\Box_{\mathsf{b}})}{\rightsquigarrow}$$

$$\mathsf{cut}_r\ \dfrac{\mathsf{b}^{[]}\ \dfrac{\mathsf{Y}^{[]}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\emptyset\}^{n}\{[[A^{\circ}],\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\emptyset\}^{n}\{A^{\circ},[\Delta^{\downarrow}]\}} \qquad \mathsf{\Box Y\text{-}cut}_{r+1}\ \dfrac{\Box^{\circ}\ \dfrac{\mathsf{w}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{[A^{\circ}]\}\{\emptyset\}^{n-1}\{A^{\bullet},[\Delta^{\downarrow}]\}}}{\Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n-1}\{A^{\bullet},[\Delta^{\downarrow}]\}} \quad \overset{\mathcal{D}_2}{\Gamma\{\Box A^{\bullet}\}^{n}\{A^{\bullet},[\Box A^{\bullet},\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{A^{\bullet},[\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{[\Delta]\}}$$

$$\mathsf{\Box Y\text{-}cut}_{r+1}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}} \qquad \mathsf{4}^{\bullet}\ \dfrac{\overset{\mathcal{D}_2}{\Gamma\{\Box A^{\bullet}\}^{n}\{\Box A^{\bullet},[\Box A^{\bullet},\Delta]\}}}{\Gamma\{\Box A^{\bullet}\}^{n}\{\Box A^{\bullet},[\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{[\Delta]\}} \quad \overset{(\Box_{4})}{\rightsquigarrow}$$

$$\mathsf{\Box Y\text{-}cut}_{r+1}\ \dfrac{\overset{\mathcal{D}_1}{\Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n-1}\{[\Delta^{\downarrow}]\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Box A^{\bullet}\}^{n}\{\Box A^{\bullet},[\Box A^{\bullet},\Delta]\}}}{\Gamma\{\emptyset\}^{n}\{[\Delta]\}}$$

$$
\Box\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n+1}\end{array} \qquad \mathsf{5}^{\bullet}\;\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma\{\Box A^{\bullet}\}^{n+1}\{\Box A^{\bullet}\}\end{array}}{\Gamma\{\Box A^{\bullet}\}^{n+1}\{\emptyset\}}}{\Gamma\{\emptyset\}^{n+2}} \quad \overset{(\Box_5)}{\rightsquigarrow}
$$

$$
\Box\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\;\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^{\downarrow}\{\Box A^{\circ}\}\{\emptyset\}^{n+1}\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\Box A^{\bullet}\}^{n+1}\{\Box A^{\bullet}\}\end{array}}{\Gamma\{\emptyset\}^{n+2}}
$$

In the $\Box$-, $\Box_{\mathsf{t}}$-, and $\Box_{\mathsf{b}}$-reductions, we use cut rank preserving admissibility of the $\mathsf{Y}^{[]}$-, $\mathsf{m}^{[]}$-, $\mathsf{t}^{[]}$-, and $\mathsf{b}^{[]}$-rules on the left branch, and cut rank and height preserving admissibility of weakening together with the induction hypothesis on the right branch. In the $\Diamond_{\mathsf{4}}$- and $\Diamond_{\mathsf{5}}$-reductions, we just apply the induction hypothesis.