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On Proof Nets for Multiplicative Linear Logic with Units

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Abstract. In this paper we present a theory of proof nets for full multiplicative linear logic, including the two units. It naturally extends the well-known theory of unit-free multiplicative proof nets. A linking is no longer a set of axiom links but a tree in which the axiom links are subtrees. These trees will be identified according to an equivalence relation based on a simple form of graph rewriting. We show the standard results of sequentialization and strong normalization of cut elimination. Furthermore, the identifications enforced on proofs are such that the proof nets, as they are presented here, form the arrows of the free (symmetric) *-autonomous category.

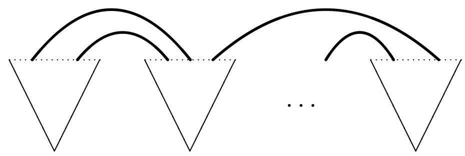
1 Introduction

For a long time formal logicians have been aware of the need to determine, given a formal system $\mathcal{S}$ and two proofs of a formula A in that system, when these two proofs are "the same" proof. As a matter of fact this was already a concern of Hilbert when he was preparing his famous lecture in 1900 [Thi03]. This problem has taken more importance during the last few years, because many logical systems permit a close correspondence between proofs and programs.

In a formalism like the sequent calculus (and to a lesser degree, natural deduction), it is oftentimes very easy to see that two derivations π_1 and π_2 should be identified because π_1 can be transformed in to π_2 by a sequence of rule permutations that are obviously trivial. It is less immediately clear *in general* what transformations can be effected on a proof without changing its essence. But here category theory is very helpful, providing criteria for the identification of proofs that are simple, general and unambiguous, if sometimes too strong [Gir91].

The advent of linear logic marked a significant advance in that quest. In particular the multiplicative fragment of linear logic comes equipped with an extremely successful theory of proof identification: not only do we know exactly when two sequent proofs should be identified (the allowed rule permutations are described in [Laf95]), but there is a class of simple formal objects that precisely represent these equivalence classes of sequent proofs. These objects are called proof nets, and they have a strong geometric character, corresponding to additional graph structure ("axiom links") on the syntactical forest of the sequent.

More precisely, given a sequent $\Gamma = A_1, \ldots, A_n$ and a proof π of that sequent, then the proof net that represents π is simply given by the syntactical forest of Γ decorated with additional edges (shown in thick lines) that represent the identity axioms that appeared in the proof:



Moreover proof nets are vindicated by category theory, since the category of two-formula sequents and proof nets is precisely the free $*$-autonomous category [Bar79] (without units) on the set of generating atomic formulas [Blu93]. As a matter of fact axiom links were already visible, under the name of *Kelly-Mac Lane graphs* in the early work [KL71] that tried to describe free autonomous categories; Girard's key insights [Gir87] here were in noticing that there was an inherent symmetry that could be formalized through a negation (thus the move from autonomous to $*$-autonomous), and that the addition of the axiom links to the sequent's syntactic forest were enough to completely characterize the proof.

The theory of proof nets has been extended to larger fragments of linear logic; when judged from the point of view of their ability to identify proofs that should be identified, these extensions can be shown to have varying degrees of success. One of these extensions, which complies particularly well with the categorical ideal, is the inclusion of additive connectives presented in [HvG03], in which the additives correspond exactly to categorical product and coproduct.

In this paper we give a theory of proof nets for the full multiplicative fragment. That is, our theory of proof nets includes the multiplicative units. We prove that it allows us to construct the free $*$-autonomous category with units on a given set of generating objects, thus getting full validation from the categorical imperative. There are only two other presentations for multiplicative units that we are aware of. In [KO99], the authors provide an internal language for autonomous and *-autonomous categories based on the $\lambda\mu$-calculus, and in [BCST96], a nonstandard version of two-sided proof nets for a weaker logic is developped from which the authors also claim to have constructed free $*$-autonomous categories. Our approach is different in the following way. By making full use of the symmetry given by the combination of an involutive negation and a one-sided sequent calculus, we get a notion of proof net which is considerably simpler than the one provided in [BCST96].

The main problem

We assume that the reader is familiar with the sequent calculus for classical multiplicative linear logic.

The theory of $*$-autonomous categories tells us that whenever a proof contains a rule instance r which appears after a $\bot$-introduction rule and which does

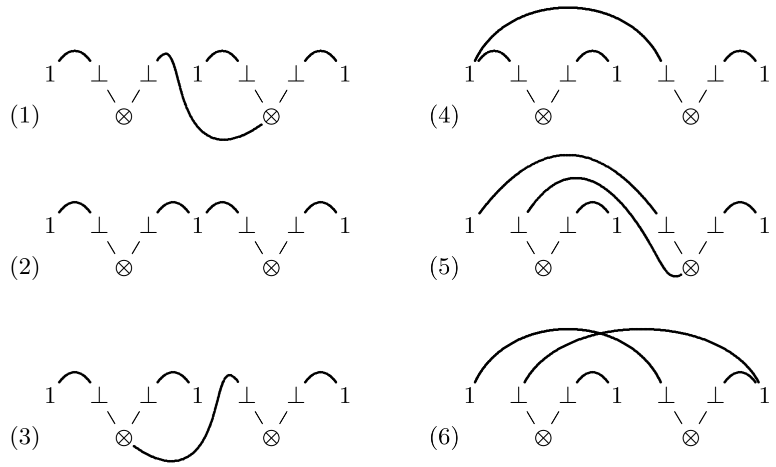


Fig. 1. Different representations of the same proof

not introduce a connective under that $\bot$, then r can be pushed above that $\bot$-introduction without changing the proof:

$$\mathsf{r}\,\frac{\bot\,\dfrac{\mathit{\Gamma}}{\bot, \mathit{\Gamma}} \qquad \cdots}{\bot, \mathit{\Gamma}'} \qquad \longleftrightarrow \qquad \bot\,\frac{\mathsf{r}\,\dfrac{\mathit{\Gamma} \qquad \cdots}{\mathit{\Gamma}'}}{\bot, \mathit{\Gamma}'}$$

This seemingly trivial permutation actually has deep consequences. Supposing that rule r was a $\otimes$-introduction, there is now a choice of two branches on which to do the $\bot$-introduction.

$$\otimes\,\frac{\bot\,\dfrac{\mathit{\Gamma}, A}{\bot, \mathit{\Gamma}, A} \quad B, \mathit{\Delta}}{\bot, \mathit{\Gamma}, A \otimes B, \mathit{\Delta}} \quad \longleftrightarrow \quad \bot\,\frac{\otimes\,\dfrac{\mathit{\Gamma}, A \quad B, \mathit{\Delta}}{\mathit{\Gamma}, A \otimes B, \mathit{\Delta}}}{\bot, \mathit{\Gamma}, A \otimes B, \mathit{\Delta}} \quad \longleftrightarrow \quad \otimes\,\frac{\mathit{\Gamma}, A \quad \bot\,\dfrac{B, \mathit{\Delta}}{\bot, B, \mathit{\Delta}}}{\bot, \mathit{\Gamma}, A \otimes B, \mathit{\Delta}}$$

Ordinary proof nets for multiplicative linear logic are characterized by the presence of *links*, which connect the atoms of the syntactical forest of the sequent. When extending them to multiplicative units, the first impulse is probably to try to attach the $\bot$s that are present on the sequent forest on other atomic formulas. This is what is done in [BCST96] and corresponds, in the sequent calculus, to doing the $\bot$-introductions as early as possible, that is, as high up on the sequent tree as can be done. The paragraph above shows that an arbitrary choice has to be made because of tensor introductions: in a $\otimes$-intro one branch of the sequent proof tree or the other has to be chosen for doing the $\bot$-intro. In such a situation correct identification of proofs can only be achieved by considering equivalences classes of graphs, and the theory of proof nets involves an equivalence relation on a set of "correct" graphs.

Another possibility is to attach these $\bot$s "as low as possible" on the forest, corresponding to the idea that in the sequent calculus deduction the $\bot$-intro would be done as late as possible, for example just before the $\bot$ instance gets a connective introduced under it. One way of implementing this is linking the $\bot$ instance to the last connective that was introduced above it. This is not the only way of doing things, for example we could imagine links that attach that $\bot$ instance to several subformulas of the sequent forest, corresponding to the several conclusions of the sequent that existed above the $\bot$-introduction. But whatever way we choose to "normalize" proofs, we claim that if the conventional notion of "link" is used for $\bot$s (i.e., if we consider a proof π on the sequent Γ as the sequent forest of Γ decorated with special edges that encode information about the essence of π) we still need to use equivalence classes of such graphs, and there is no hope of having a normal form in that universe of enriched sequent graphs. For instance, the six graphs in Figure 1 are easily seen to represent equivalent proofs, because going from an odd-numbered example to its successor is just sliding a $\bot$-intro up in one of the $\otimes$-intro branches, and going from an even-numbered example to its successor is just doing the reverse transformation. But notice that examples (3) and (5) are *distinct but isomorphic* graphs, since one can be exactly superposed on the other *by only using the Exchange rule.* Thus it is impossible, given the information at our disposal, to choose one instead of the other to represent the abstract proof they both denote. The only way this could be done would be by using arbitrary extra information, like the order of the formulas in the sequent, a strategy that only replaces the overdeterminism of the sequent calculus by another kind of overdeterminism.

The same can be said of Examples (2) and (6), which are also isomorphic modulo Exchange. But notice that these two comply to the "as early as possible" strategy, while the previous two were of the "as late as possible" kind. So for neither strategy can there be a hope a graphical normal form. The interested reader can verify that the six examples above are part of a "ring" of 24 graphs that are all equivalent from the point of view of category theory.

Thus there is one aspect of our work that does not differ from [BCST96], which is our presentation of abstract proofs as equivalence classes of graphs. But some related aspects are significantly different:

- The graphs that belong to our equivalence classes are *standard multiplicative proof nets,* where the usual notions, like correctness criteria and the empire of a tensor branch, will apply. It is just that some $\parr$ and $\otimes$ links are used in a particular fashion to deal with the units. (The readers can choose their favorite correctness criterion since they are all equivalent; in this paper we will use the one of [DR89] because of its popularity.)
- The equivalence relation we will present is based on a very simple set of rewriting rules on proof graphs. As a matter of fact, there is only *one* non-trivial rule, since the other rules have to do with commutativity and associativity of the connectives and can be dispensed with if we use, for example, n-ary connectives.

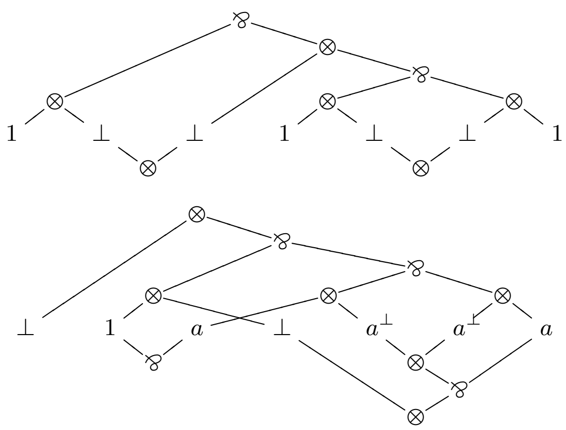


Fig. 2. Two examples of proof graphs

2 Cut free proof nets for MLL

Let $\mathcal{A} = \{a, b, \dots\}$ be an arbitrary set of atoms, and let $\mathcal{A}^\perp = \{a^\perp, b^\perp, \dots\}$. The set of MLL *formulas* is defined as follows:

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{A}^\perp \mid 1 \mid \perp \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \mathbin{⅋} \mathcal{F} \quad .$$

Additionally, we will define the set of MLL *linkings* (which can be seen as a special kind of formulas) as follows:

$$\mathcal{L} ::= 1 \mid a \otimes a^\perp \mid a^\perp \otimes a \mid \perp \otimes \mathcal{L} \mid \mathcal{L} \otimes \perp \mid \mathcal{L} \mathbin{⅋} \mathcal{L} \quad .$$

Here, a stands for any element of $\mathcal{A}$. We will use A, B, ... to denote formulas, and P, Q, ... to denote linkings. *Sequents* (denoted by Γ, Δ, ...) are finite lists of formulas (separated by comma).

In the following, we will consider formulas and linkings always as binary trees (and sequents as forests), whose leaves are decorated by elements of $\mathcal{A} \cup \mathcal{A}^\perp \cup \{1, \perp\}$, and whose inner nodes are decorated by ⅋ or $\otimes$. We can also think of the nodes being decorated by the whole subformula above that node.

2.1 Definition A *pre-proof graph* is a graph consisting of a linking P and a sequent Γ, which both share the same set of leaves. It will be denoted as $P \triangleright \Gamma$.

Following the tradition, we will draw these graphs such that the roots of the formula trees are at the bottom, the root of the linking tree is at the top, and the leaves are in between. Figure 2 shows two examples. The first of them corresponds to the first graph in Figure 1. In a more compact notation we will write them as

$$\begin{array}{c}(1_1 \otimes \perp_2) \mathbin{⅋} (\perp_3 \otimes ((1_4 \otimes \perp_5) \mathbin{⅋} (\perp_6 \otimes 1_7))) \\ \triangledown \\ 1_1, \perp_2 \otimes \perp_3, 1_4, \perp_5 \otimes \perp_6, 1_7\end{array}$$

and

$$\begin{array}{c}\bot_1 \otimes ((1_2 \otimes \bot_4) \bindnasrepma ((a_3 \otimes a_5^\bot) \bindnasrepma (a_6^\bot \otimes a_7)))\\ \triangledown \\ \bot_1, 1_2 \bindnasrepma a_3, \bot_4 \otimes ((a_5^\bot \otimes a_6^\bot) \bindnasrepma a_7)\end{array} \quad .$$

Here, the indices are used to show how the leaves of the linking and the leaves of the sequent are identified. In this way we will, throughout this paper, use indices on atoms to distinguish between different occurrences of the same atom (i.e. a_3 and a_7 do not denote different atoms). In the same way, indices on the units 1 and $\bot$ are used to distinguish different occurrences.

2.2 Definition A *switching* of a pre-proof graph $P \rhd \Gamma$ is a graph G that is obtained from $P \rhd \Gamma$ by omitting for each $\bindnasrepma$-node one of the two edges that connect the node to its children. [DR89]

2.3 Definition A pre-proof graph $P \rhd \Gamma$ is called *correct* if all its switchings are connected and acyclic. A *proof graph* is a correct pre-proof graph.

The examples in Figure 2 are proof graphs.

Let $P \rhd \Gamma$ be a pre-proof graph where one $\bot$ is selected. Let it be indexed as $\bot_i$. Now, let G be a switching of $P \rhd \Gamma$, and let G' be the graph obtained from G by removing the edge between $\bot_i$ and its parent in P (which is always a $\otimes$). Then G' is called an *extended switching* of $P \rhd \Gamma$ with respect to $\bot_i$. Observe that, if $P \rhd \Gamma$ is correct, then each extended switching is disconnected and consists of two connected components.

We will use the notation $P\{Q\} \rhd \Gamma$ to distinguish the subtree Q of the linking tree of the graph. Then $P\{\ \}$ is the context of Q.

2.4 Equivalence on pre-proof graphs On the set of pre-proof graphs we will define the relation $\sim$ to be the smallest equivalence relation satisfying

$$\begin{array}{rcl}P\{Q \bindnasrepma R\} \rhd \Gamma & \sim & P\{R \bindnasrepma Q\} \rhd \Gamma\\ P\{(Q \bindnasrepma R) \bindnasrepma S\} \rhd \Gamma & \sim & P\{Q \bindnasrepma (R \bindnasrepma S)\} \rhd \Gamma\\ P\{Q \otimes R\} \rhd \Gamma & \sim & P\{R \otimes Q\} \rhd \Gamma\\ P\{\bot_i \otimes (Q \otimes \bot_j)\} \rhd \Gamma & \sim & P\{(\bot_i \otimes Q) \otimes \bot_j)\} \rhd \Gamma\\ P\{Q \bindnasrepma (R \otimes \bot_i)\} \rhd \Gamma & \overset{(*)}{\sim} & P\{(Q \bindnasrepma R) \otimes \bot_i)\} \rhd \Gamma\end{array} \quad ,$$

where the last equation only holds if the following side condition is fulfilled:

($*$) In each extended switching of $P\{Q \bindnasrepma (R \otimes \bot_i)\} \rhd \Gamma$ with respect to $\bot_i$ no node of the subtree Q is connected to $\bot_i$.

The following proof graph is equivalent to the second one in Figure 2:

$$\begin{array}{c}(((\bot_1 \otimes 1_2) \otimes \bot_4) \bindnasrepma ((a_3 \otimes a_5^\bot)) \bindnasrepma (a_6^\bot \otimes a_7)\\ \triangledown \\ \bot_1, 1_2 \bindnasrepma a_3, \bot_4 \otimes ((a_5^\bot \otimes a_6^\bot) \bindnasrepma a_7)\end{array} \quad .$$

2.5 Definition A *pre-proof net*[1] is an equivalence class $[P \rhd \Gamma]_\sim$. A pre-proof net is *correct* if one of its elements is correct. In this case it is called a *proof net*.

[1] What we call *pre-proof net* is in the literature often called *proof structure*.

$$
\mathsf{id}\,\frac{}{a \otimes a^\perp \rhd a, a^\perp} \qquad \mathsf{ex}\,\frac{P \rhd \Gamma, A, B, \Delta}{P \rhd \Gamma, B, A, \Delta}
$$

$$
1\,\frac{}{1 \rhd 1} \qquad \bot\,\frac{P \rhd \Gamma}{\bot \otimes P \rhd \bot, \Gamma}
$$

$$
\invamp\,\frac{P \rhd A, B, \Gamma}{P \rhd A \invamp B, \Gamma} \qquad \otimes\,\frac{P \rhd \Gamma, A \qquad Q \rhd B, \Delta}{P \invamp Q \rhd \Gamma, A \otimes B, \Delta}
$$

Fig. 3. Translation of cut free sequent calculus proofs into pre-proof graphs

In the following, we will for a given proof graph $P \rhd \Gamma$ write $[P \rhd \Gamma]$ to denote the proof net formed by its equivalence class (i.e. we will omit the $\sim$ subscript).

2.6 Lemma *If $P \rhd \Gamma$ is correct and $P \rhd \Gamma \sim P' \rhd \Gamma$, then $P' \rhd \Gamma$ is also correct.*

Proof: That the first four equations preserve correctness is obvious. If in the last equation there is a switching that disconnects one side, then it also disconnects the other. For acyclicity, we have to check whether there is a switching that produces a cycle on the right-hand side of the equation and not on the left-hand side. This is only possible if the cycle contains some nodes of Q and the $\bot_i$. But this case is ruled out by the side condition $(*)$. □

Lemma 2.6 ensures that the notion of proof net is well-defined, in the sense that all its members are proof graphs, i.e. correct.

3 Sequentialization

Figure 3 shows how cut free sequent proofs of MLL can be inductively translated into pre-proof graphs.

We will call a pre-proof net *sequentializable* if one of its representatives can be obtained from a sequent calculus proof via this translation.

3.1 Theorem *A pre-proof net is sequentializable iff it is a proof net.*

For the proof we will need the observation that any proof graph is an ordinary unit-free proof net, and the well-known fact that there is always a splitting tensor in such a net.

3.2 Observation Every proof graph $P \rhd \Gamma$ is an ordinary unit-free proof net in the style of [DR89]. To make this precise, define for the linking P the *linking formula* $P^\star$ inductively as follows:

$$
\begin{array}{rcl}
a^{\perp\star} = a & 1^\star = \bot & (A \otimes B)^\star = A^\star \otimes B^\star \\
a^\star = a^\perp & \bot^\star = 1 & (A \invamp B)^\star = A^\star \invamp B^\star
\end{array} .
$$

In other words, $P^\star$ is obtained from P by replacing each leaf by its dual and by leaving all inner nodes unchanged. We now connect the leaves of $P^\star$ and Γ by ordinary axiom links according to the leaf identification in $P \rhd \Gamma$. If we forget the fact that $\bot$ and 1 are the units and think of them as ordinary dual atoms, then we have an ordinary unit-free proof net[2].

3.3 Lemma *If in a unit-free proof net all roots are $\otimes$-nodes, then one of them is splitting, i.e. by removing it the net becomes disconnected.* [Gir87]

Proof of Theorem 3.1 (Sketch): It is easy to see that the rules 1 and id give proof graphs and that the rules $\bot$, $⅋$, and $\otimes$ preserve the correctness. Therefore every sequentializable pre-proof net is correct.

For the other direction pick one representative $P \rhd \Gamma$ of the proof net and proceed by induction on the sum of the number of $\otimes$-nodes in the graph and the number of $⅋$-nodes in Γ. We now interpret $P \rhd \Gamma$ as an ordinary unit-free proof net (according to Observation 3.2), and remove all $⅋$-roots (for those inside Γ apply the $⅋$ rule and proceed by induction hypothesis). Then apply Lemma 3.3. If the splitting $\otimes$ is inside Γ, we can apply the $\otimes$-rule and proceed by induction hypothesis; if it is inside P, it must come from an axiom link or a bottom link. In both cases we can obtain two smaller proof graphs, to which we can apply the induction hypothesis to get two sequent proofs, which can can be composed by plugging one into a leaf of the other. □

4 Cut and cut elimination

A *cut* is a formula $A \varobar A^\bot$, where $\varobar$ is called the *cut connective*, and where the function $(-)^\bot$ is defined on formulas as follows:

$$\begin{array}{lll} a^{\bot\bot} = a & 1^\bot = \bot & (A \otimes B)^\bot = A^\bot ⅋ B^\bot \\ a^\bot = a^\bot & \bot^\bot = 1 & (A ⅋ B)^\bot = A^\bot \otimes B^\bot \end{array} .$$

A *sequent with cuts* is a sequent where some of the formulas are cuts. But cuts are not allowed to occur inside formulas, i.e. all $\varobar$-nodes are roots. A *pre-proof graph with cuts* is a pre-proof graph $P \rhd \Gamma$, where Γ may contain cuts. The $\varobar$-nodes have the same geometric behavior as the $\otimes$-nodes. Therefore the correctness criterion stays literally the same, and we can define *proof graphs with cuts* and *proof nets with cuts* accordingly. In the translation from sequent proofs containing the cut rule into pre-proof graphs with cuts, the cut is treated as follows:

$$\mathsf{cut}\,\frac{\Gamma, A \quad A^\bot, \Delta}{\Gamma, \Delta} \quad\leadsto\quad \mathsf{cut}\,\frac{P \rhd \Gamma, A \quad Q \rhd A^\bot, \Delta}{P ⅋ Q \rhd \Gamma, A \varobar A^\bot, \Delta} .$$

Since the $\varobar$ behaves in the same way as the $\otimes$, we immediately have the generalization of the sequentialization:

[2] If Γ consists of only one formula, then we have an object which is in [BC99] called a *bipartite proof net*. In fact, two proof graphs (in our sense) are equivalent if and only if the two linkings (seen as formulas) are isomorphic (in the sense of [BC99]).

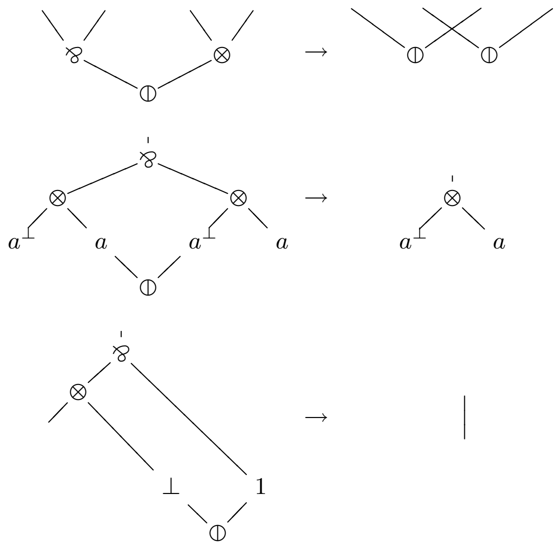


Fig. 4. Cut elimination reduction steps

4.1 Theorem *A pre-proof net with cuts is sequentializable if and only if it is correct, i.e. it is a proof net with cuts.*

On the set of cut pre-proof graphs we can define the cut reduction relation $\rightarrow$ as follows:

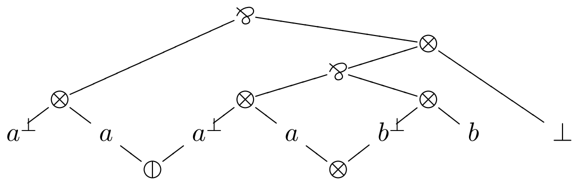
$$
\begin{array}{ccc}
\begin{array}{c} P \\ \triangledown \\ (A \parr B) \obar (A^{\perp} \otimes B^{\perp}), \Gamma \end{array} & \rightarrow & \begin{array}{c} P \\ \triangledown \\ A \obar A^{\perp}, B \obar B^{\perp}, \Gamma \end{array} \\[2ex]
\begin{array}{c} P\{(a_h^{\perp} \otimes a_i) \parr (a_j^{\perp} \otimes a_k)\} \\ \triangledown \\ a_i \obar a_j^{\perp}, \Gamma \end{array} & \rightarrow & \begin{array}{c} P\{a_h^{\perp} \otimes a_k\} \\ \triangledown \\ \Gamma \end{array} \\[2ex]
\begin{array}{c} P\{(Q \otimes \perp_i) \parr 1_j\} \\ \triangledown \\ \perp_i \obar 1_j, \Gamma \end{array} & \rightarrow & \begin{array}{c} P\{Q\} \\ \triangledown \\ \Gamma \end{array}
\end{array}
$$

These reduction steps are shown in graphical notation in Figure 4.

4.2 Lemma *If $P \triangleright \Gamma$ is correct and $P \triangleright \Gamma \rightarrow P' \triangleright \Gamma'$, then $P' \triangleright \Gamma'$ is also correct.*

Proof: It is impossible that a cut reduction step introduces a cycle in a switching or disconnects it. □

Observe that it can happen that in a proof graph no reduction is possible, although there are cuts present in the sequent. For example, in



the cut cannot be reduced.

In a given proof graph $P \rhd \Gamma$, a $\oplus$-node that can be reduced will be called *ready*. Obviously, a cut on a $\otimes$-$\parr$-pair is always ready, but for a cut on atoms or units this is not necessarily the case, as the example above shows. However, we have the following

4.3 Theorem *Given a proof graph $P \rhd \Gamma$ and a $\oplus$-node in Γ, there is an equivalent proof graph $P' \rhd \Gamma$, in which that $\oplus$-node is ready, i.e. can be reduced.*

This is an immediate consequence of the following two lemmas.

4.4 Lemma *For every proof graph $P \rhd a_i \oplus a_j^\perp, \Gamma$ that contains an atomic cut, there is an equivalent proof graph $P'\{(a_h^\perp \otimes a_i) \parr (a_j^\perp \otimes a_k)\} \rhd a_i \oplus a_j^\perp, \Gamma$.*

4.5 Lemma *For every proof graph $P \rhd \perp_i \oplus 1_j^\perp, \Gamma$ that contains a cut on the units, there is an equivalent proof graph $P'\{(Q \otimes \perp_i) \parr 1_j\} \rhd \perp_i \oplus 1_j, \Gamma$.*

For proving them, we will use the following three lemmas.

4.6 Lemma *Let $P\{(\perp_k \otimes R\{x_i\}) \parr (S\{x_j^\perp\} \otimes \perp_h)\} \rhd x_i \oplus x_j^\perp, \Gamma$ be a proof graph, where x is an arbitrary atom or a unit, and $x^\perp$ its dual.*
Then at least one of $P\{\perp_k \otimes (R\{x_i\} \parr (S\{x_j^\perp\} \otimes \perp_h))\} \rhd x_i \oplus x_j^\perp, \Gamma$ and $P\{((\perp_k \otimes R\{x_i\}) \parr S\{x_j^\perp\}) \otimes \perp_h\} \rhd x_i \oplus x_j^\perp, \Gamma$ is equivalent to it.

4.7 Lemma *Let $P\{(\perp_k \otimes R\{x_i\}) \parr (x_j^\perp \otimes Q)\} \rhd x_i \oplus x_j^\perp, \Gamma$ be a proof graph, where x is an arbitrary atom or a unit, and $x^\perp$ its dual.*
Then $P\{\perp_k \otimes (R\{x_i\} \parr (x_j^\perp \otimes Q))\} \rhd x_i \oplus x_j^\perp, \Gamma$ is equivalent to it.

4.8 Lemma *Let $P\{(\perp_k \otimes R\{x_i\}) \parr x_j^\perp\} \rhd x_i \oplus x_j^\perp, \Gamma$ be a proof graph, where x is an arbitrary atom or a unit, and $x^\perp$ its dual.*
Then $P\{\perp_k \otimes (R\{x_i\} \parr x_j^\perp)\} \rhd x_i \oplus x_j^\perp, \Gamma$ is equivalent to it.

Proof of Lemma 4.4 (Sketch): By the definition of proof graph and the correctness, the linking P must be of the shape $P''\{R\{a_h^\perp \otimes a_i\} \parr S\{a_j^\perp \otimes a_k\}\}$ for some contexts $P''\{\ \}$, $R\{\ \}$ and $S\{\ \}$. The contexts $R\{\ \}$ and $S\{\ \}$ can be reduced to $\{\ \}$ by applying Lemma 4.6 and Lemma 4.7.

Proof of Lemma 4.5 (Sketch): Similar to Lemma 4.4, but this time Lemma 4.8 is also used.

Let us now extend the relation $\to$ to proof nets as follows: $[P \rhd \Gamma] \to [Q \rhd \Delta]$ if an only if there are proof graphs $P' \rhd \Gamma$ and $Q' \rhd \Delta$ such that

$$P \rhd \Gamma \sim P' \rhd \Gamma \to Q' \rhd \Delta \sim Q \rhd \Delta \quad .$$

4.9 Lemma *Let $P \rhd \Gamma \sim P' \rhd \Gamma$, and let $P \rhd \Gamma \to Q \rhd \Delta$ and $P' \rhd \Gamma \to Q' \rhd \Delta$, i.e. in both reductions the same cut is reduced. Then we have $Q \rhd \Delta \sim Q' \rhd \Delta$.*

4.10 Lemma *There is no infinite sequence*

$$[P \rhd \Gamma] \to [P' \rhd \Gamma'] \to [P'' \rhd \Gamma''] \to \cdots$$

Proof: In each reduction step the size of the sequent (i.e. the number of $⅋$, $\otimes$ and $⦶$-nodes) is reduced. □

4.11 Lemma *If $Q \rhd \Delta \leftarrow P \rhd \Gamma \to R \rhd \Sigma$, then either $Q \rhd \Delta = R \rhd \Sigma$, or there is a proof graph $S \rhd \Phi$ such that $Q \rhd \Delta \to S \rhd \Phi \leftarrow R \rhd \Sigma$.*

4.12 Lemma *If $[Q \rhd \Delta] \leftarrow [P \rhd \Gamma] \to [R \rhd \Sigma]$, then either $[Q \rhd \Delta] = [R \rhd \Sigma]$, or there is a proof net $[S \rhd \Phi]$ such that $[Q \rhd \Delta] \to [S \rhd \Phi] \leftarrow [R \rhd \Sigma]$.*

Proof (Sketch): Let $⦶_1$ denote the cut that is reduced in Γ to obtain Δ and $⦶_2$ the one that is reduced to obtain Σ. The basic idea is to apply Theorem 4.3 in order to make both cuts ready at the same time and then apply Lemma 4.11 and Lemma 4.9. There is essentially only one case in which it is not possible to make both cuts ready at the same time, namely, when they use the same axiom link. In other words, $P \rhd \Gamma$ is of the following shape:

$$\begin{array}{c} P'\{(P''\{a_h^\perp \otimes a_i\} ⅋ P'''\{a_j^\perp \otimes a_k\}) ⅋ P''''\{a_l^\perp \otimes a_m\}\} \\ \triangledown \\ a_i ⦶_1 a_j^\perp, a_k ⦶_2 a_l^\perp, \Phi \end{array}$$

But whatever order of reduction is used, in both cases we get something of the shape $S'\{a_h^\perp \otimes a_m\} \rhd \Phi$. □

4.13 Theorem *The cut elimination reduction $\to$ on proof nets is strongly normalizing. The normal forms are cut free proof nets.*

Proof: Termination is provided by Lemma 4.10 and confluence follows from Lemma 4.12. That the normal form is cut free is ensured by Theorem 4.3. □

5 *-Autonomy

For any formula A, we can provide an identity proof net $\mathrm{id}_A = [I_A \rhd A^\perp, A]$, where I_A is called the *identity linking* which is defined inductively on A as follows:

$$\begin{array}{lll} I_a & = I_{a^\perp} & = a \otimes a^\perp \\ I_\perp & = I_1 & = \perp \otimes 1 \\ I_{A ⅋ B} & = I_{A \otimes B} & = I_A ⅋ I_B \end{array}$$

Observe that we can have that $I_A = I_{A^\perp}$ because changing the order of the arguments of a $\otimes$ or $⅋$ in the linking of a proof graph does not change the proof net (see 2.4).

Furthermore, for any two proof nets $f = [P \rhd A^\perp, B]$ and $g = [Q \rhd B^\perp, C]$, we can define their composition $g \circ f$ to be the result of the cut elimination procedure to $[P ⅋ Q \rhd A^\perp, B ⦶ B^\perp, C]$. That this is well-defined and associative follows immediately from the strong normalization of cut elimination. We also have that $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

This gives rise to a category $\mathbf{PN}(\mathcal{A})$ whose objects are the MLL formulas built over $\mathcal{A} \cup \mathcal{A}^\perp \cup \{\perp, 1\}$, and whose arrows are the proof nets. More precisely, the arrows between two objects A and B are the (cut-free) proof nets $[P \rhd A^\perp, B]$. The operation $\otimes$ on formulas can be extended to a bifunctor $\otimes : \mathbf{PN}(\mathcal{A}) \times \mathbf{PN}(\mathcal{A}) \to \mathbf{PN}(\mathcal{A})$ by defining for two arrows $f = [P \rhd A^\perp, B]$ and $g = [Q \rhd C^\perp, D]$ the arrow $f \otimes g = [P \parr Q \rhd A^\perp \parr C^\perp, B \otimes D]$. It can easily be seen that this bifunctor makes our category symmetric monoidal (with unit 1): The basic natural isomorphisms demanded by the definition (associativity, right unit, left unit, symmetry) are

$$\begin{aligned}
\alpha_{A,B,C} &= [I_A \parr I_B \parr I_C \rhd A^\perp \parr (B^\perp \parr C^\perp), (A \otimes B) \otimes C] \\
\rho_A &= [\perp \otimes I_A \rhd A^\perp \parr \perp, A] \\
\lambda_A &= [\perp \otimes I_A \rhd \perp \parr A^\perp, A] \\
\sigma_{A,B} &= [I_A \parr I_B \rhd A^\perp \parr B^\perp, B \otimes A]
\end{aligned}$$

It is easy to check these are indeed proof nets, that α, ρ, λ, and σ are natural isomorphisms for all formulas A, B, and C, and that the corresponding diagrams (see [BW99]) commute.

Furthermore, we can exhibit the (contravariant) duality functor $(-)^\perp$ whose object function has already been defined. For an arrow $f = [P \rhd A^\perp, B] : A \to B$ let $f^\perp = [P \rhd B, A^\perp] : B^\perp \to A^\perp$. This determines a symmetric *-autonomous category structure [Bar79,BW99]. In particular, we define the bifunctor $- \parr -$ as $A \parr B = (A^\perp \otimes B^\perp)^\perp$ and its unit object as $\perp = 1^\perp$. The last thing to check is that we have the natural bijection

$$\begin{aligned}
\mathrm{Hom}(A \otimes B, C) &\cong \mathrm{Hom}(A, B^\perp \parr C) \\
[P \rhd A^\perp \parr B^\perp, C] &\mapsto [P \rhd A^\perp, B^\perp \parr C] \quad .
\end{aligned}$$

6 The free *-autonomous category

In this section we will show that the category of proof nets is the free symmetric *-autonomous category. Let $\mathcal{A}$ be a set and let $\eta_\mathcal{A} : \mathcal{A} \to \mathrm{Obj}(\mathbf{PN}(\mathcal{A}))$ be the function that maps every element of $\mathcal{A}$ to itself seen as atomic formula. To say that $\mathbf{PN}(\mathcal{A})$ is the *free *-autonomous category generated by* $\mathcal{A}$ amounts to saying that

6.1 Theorem *For any *-autonomous category*[3] $(\mathcal{C}, \otimes, 1_\mathcal{C}, (-)^\perp)$ *and any map* $G^\circ : \mathcal{A} \to \mathrm{Obj}(\mathcal{C})$*, there is a unique functor* $G : \mathbf{PN}(\mathcal{A}) \to \mathcal{C}$*, preserving the *-autonomous structure, such that* $G^\circ = \mathrm{Obj}(G) \circ \eta_\mathcal{A}$*, where* $\mathrm{Obj}(G)$ *is the restriction of* G *on objects.*

The remainder of this section is devoted to a sketch the proof of this theorem. For this we will introduce the following notation.

[3] For simplicity we assume that for every object C of $\mathcal{C}$ we have $C^{\perp\perp} = C$. This can be relaxed to a natural isomorphism by standard trickery.

Let I be an index set. A *bracketing* of I is given by a total order $I = \{i_1, \ldots, i_k\}$ and a binary tree with k leaves indexed by I that respects the order. We will denote bracketings of I also by I. The whole point of this is, given an I-indexed family $(C_{i_1})_{i \in I}$ of objects of $\mathcal{C}$, that we can use the notation $\bigotimes_I\{C_{i_1}, \ldots, C_{i_k}\}$ to denote the object of $\mathcal{C}$ that is obtained by applying the functor $- \otimes -$ according to the bracketing I. By a standard theorem of symmetric monoidal categories, any two objects obtained from different bracketings have a unique "coherence" isomorphism between them. Similarly, $\bigparr_I\{C_{i_1}, \ldots, C_{i_k}\}$ is defined. For empty I, let $\bigotimes_\emptyset \emptyset = 1_\mathcal{C}$ and $\bigparr_\emptyset \emptyset = \bot_\mathcal{C} = 1_\mathcal{C}^\bot$. The purpose of this notation is to state the following property of *-autonomous categories.

6.2 Proposition *Let $\mathcal{C}$ be a *-autonomous category, and let $C_1, \ldots, C_n$ be objects of $\mathcal{C}$. Let $I, J \subseteq \{1, \ldots, n\}$, and let $\complement I = \{1, \ldots, n\} \setminus I$ and $\complement J = \{1, \ldots, n\} \setminus J$ be their complements. Then for all bracketings of $I, J, \complement I, \complement J$, we have a natural bijection between* $\mathrm{Hom}_\mathcal{C}\left(\bigotimes_I\{C_i^\bot \mid i \in I\}, \bigparr_{\complement I}\{C_i \mid i \in \complement I\}\right)$ *and* $\mathrm{Hom}_\mathcal{C}\left(\bigotimes_J\{C_j^\bot \mid j \in J\}, \bigparr_{\complement J}\{C_j \mid j \in \complement J\}\right)$.

Proof: The proof is done by repeatedly applying the associativity and commutativity of the two functors $- \otimes -$ and $- \parr -$, the natural isomorphisms for the units, and the natural bijection $\mathrm{Hom}_\mathcal{C}(A \otimes B^\bot, C) \cong \mathrm{Hom}_\mathcal{C}(A, B \parr C)$, which is imposed by the *-autonomous structure. ☐

Let now the *-autonomous category $\mathcal{C}$ and the embedding $G^\circ : \mathcal{A} \to \mathrm{Obj}(\mathcal{C})$ be given. We will exhibit the functor $G : \mathbf{PN}(\mathcal{A}) \to \mathcal{C}$ which has the desired properties. On the objects, this functor is uniquely determined as follows:

$$\begin{array}{ccc} G(a) = G^\circ(a) & G(\bot) = \bot_\mathcal{C} & G(A \parr B) = G(A) \parr G(B) \\ G(a^\bot) = G^\circ(a)^\bot & G(1) = 1_\mathcal{C} & G(A \otimes B) = G(A) \otimes G(B) \end{array}$$

There is no other choice since the objects $1_\mathcal{C}$ and $\bot_\mathcal{C}$ in $\mathcal{C}$, as well as the functors $(-)^\bot$, $- \otimes -$, and $- \parr -$ are uniquely determined by the *-autonomous structure on $\mathcal{C}$.

For defining G on the morphisms, the situation is not as simple. We will first ignore the fact that the units are units and interpret proof graph (with cuts) $P \rhd \Gamma$ as ordinary unit-free proof net with conclusions $A_0, \ldots, A_n, B_1 \oplus B_1^\bot, \ldots, B_m \oplus B_m^\bot$, where $A_0 = P^\star$ (see Observation 3.2), $A_1, \ldots, A_n$ are the formulas in Γ that are not cuts, and $B_1 \oplus B_1^\bot, \ldots, B_m \oplus B_m^\bot$ are the cuts in Γ. To each such object we will uniquely assign a family of morphisms

$$\bigotimes_I\{G(A_i)^\bot \mid i \in I\} \to \bigparr_{\complement I}\{G(A_i) \mid i \in \complement I\}$$

indexed by the bracketings on the subsets $I \subseteq \{0, \ldots, n\}$ and their complements. Proposition 6.2 ensures that each member of such a family of morphisms determines the others uniquely. The construction is done by induction on the size of the proof graph and by employing Lemma 3.3. (In fact, it is quite similar to the sequentialization.)

Observe that this construction gives us in particular for each proof graph $P \rhd A^\bot, B$ a unique arrow $\psi_{P \rhd A^\bot, B} : G(P^\star)^\bot \to G(A^\bot) \parr G(B)$. Furthermore, observe that for every linking P, the object $G(P^\star)^\bot$ in $\mathcal{C}$ is isomorphic to $\bigotimes_{a \otimes a^\bot}\{G(a) \parr G(a)^\bot\}$, where $a \otimes a^\bot$ ranges over the axiom links in

P. This means that the *-autonomous structure on $\mathcal{C}$ uniquely determines a morphism $\phi_P : 1_{\mathcal{C}} \to G(P^\star)^\perp$. This can be composed with $\psi_{P\triangleright A^\perp,B}$ to get $\xi_{[P\triangleright A^\perp,B]} : 1_{\mathcal{C}} \to G(A^\perp) \parr G(B)$. That this is well-defined, is ensured by the following lemma (in which we no longer ignore the fact that the units are units).

6.3 Lemma *If $Q \triangleright A^\perp, B \sim P \triangleright A^\perp, B$, then $\xi_{[P\triangleright A^\perp,B]} = \xi_{[Q\triangleright A^\perp,B]}$.*

Consequently, to each proof net $f = [P \triangleright A^\perp, B]$, we can uniquely assign the arrow $G(f) : G(A) \to G(B)$ that is determined by $\xi_{[P\triangleright A^\perp,B]}$ via Proposition 6.2.

It remains to show that $G : \mathbf{PN}(\mathcal{A}) \to \mathcal{C}$ is indeed a functor (i.e. identities and composition are preserved). That for each formula A, the proof $[I_A \triangleright A^\perp, A]$ is mapped to identity $\mathrm{id} : G(A) \to G(A)$ is an easy induction on the structure of A and left to the reader. The preservation of composition is ensured by the following lemma.

6.4 Lemma *Let $T \triangleright \Gamma \to S \triangleright \Delta$, i.e. the proof graph $S \triangleright \Delta$ is obtained from $T \triangleright \Gamma$ by applying a single cut reduction step. Then $\xi_{[T\triangleright\Gamma]}$ and $\xi_{[S\triangleright\Delta]}$ denote the same morphism $1_{\mathcal{C}} \to \parr\{G(A_1), \ldots, G(A_n)\}$, where $A_1, \ldots, A_n$ are the formulas in Γ (resp. Δ) that are not cuts.*

It might be worth mentioning, that Theorem 6.1 provides a decision procedure for the equality of morphisms in the free symmetric *-autonomous category, which is in our opinion simpler than the ones provided in [BCST96] and [KO99].

7 Conclusion

We think we made a convincing case for the the cleanest approach yet to proof nets with the multiplicative units. There is always the possibility that another "ideology" than category theory will arise and will tell us to identify sequent proofs in a different way, perhaps collapsing fewer proofs, and help us construct more rigid proof objects. But we doubt very much that such a thing exists, given that the permutation rules that category theory imposes on the sequent calculus are so natural and so hard to weaken.

There are some issues that are left open and that we want to explore in the future:

- The relation with the new proof formalism called the calculus of structures [GS01,BT01]. We should mention that the idea behind our approach originates from the new viewpoints that are given by the calculus of structures.
- The addition of additives to our theory. This should not be very hard, given the work done in [HvG03]. The true challenge is to include also the additive units.
- The development of a theory of proof nets for classical logic. The problem is finding the right extension of the axioms of a *-autonomous category, such that on the one hand classical proofs are identified in a natural way, and on the other hand there is no collapse to a boolean algebra.
- The search for meaningful invariants. It is very probable that the equivalence classes of graphs we define have a geometric meaning, and can be related

to more abstract invariants like those given by homological algebra. We are convinced that the work in in [Mét94] is only the tip of the iceberg.

References

[Bar79] Michael Barr. **-Autonomous Categories*, volume 752 of *Lecture Notes in Mathematics*. Springer-Verlag, 1979.

[BC99] Vincent Balat and Roberto Di Cosmo. A linear logical view of linear type isomorphisms. In *Computer Science Logic, CSL 1999*, volume 1683 of *LNCS*, pages 250–265. Springer-Verlag, 1999.

[BCST96] Richard Blute, Robin Cockett, Robert Seely, and T. Trimble. Natural deduction and coherence for weakly distributive categories. *Journal of Pure and Applied Algebra*, 113:229–296, 1996.

[Blu93] Richard Blute. Linear logic, coherence and dinaturality. *Theoretical Computer Science*, 115:3–41, 1993.

[BT01] Kai Brünnler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347–361. Springer-Verlag, 2001.

[BW99] Michael Barr and Charles Wells. *Category Theory for Computing Science*. Les Publications CRM, Montréal, third edition, 1999.

[DR89] Vincent Danos and Laurent Regnier. The structure of multiplicatives. *Annals of Mathematical Logic*, 28:181–203, 1989.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Gir91] J.-Y. Girard. A new constructive logic: Classical logic. *Math. Stuct. in Comp. Sci*, 1:255–296, 1991.

[GS01] Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In Laurent Fribourg, editor, *Computer Science Logic, CSL 2001*, volume 2142 of *LNCS*, pages 54–68. Springer-Verlag, 2001.

[HvG03] Dominic Hughes and Rob van Glabbeek. Proof nets for unit-free multiplicative-additive linear logic. In *18'th IEEE Symposium on Logic in Computer Science (LICS 2003)*, 2003.

[KL71] G.M. Kelly and S. Mac Lane. Coherence in closed categories. *Journal of Pure and Applied Algebra*, 1:97–140, 1971.

[KO99] Thong-Wei Koh and Chih-Hao Luke Ong. Internal languages for autonomous and *-autonomous categories. In Martin Hofmann, Giuseppe Rosolini, and Dusko Pavlovic, editors, *Proceedings of the 8th Conference on Category Theory and Computer Science, Edinburgh, September 1999*, volume 29 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 1999.

[Laf95] Y. Lafont. From proof nets to interaction nets. In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, volume 222 of *London Mathematical Society Lecture Notes*, pages 225–247. Cambridge University Press, 1995.

[Mét94] F. Métayer. Homology of proof nets. *Archive of Mathematical Logic*, 33:169–188, 1994.

[Thi03] Rüdiger Thiele. Hilbert's twenty-fourth problem. *American Mathematical Monthly*, 110:1–24, 2003.

A Appendix — Full proofs

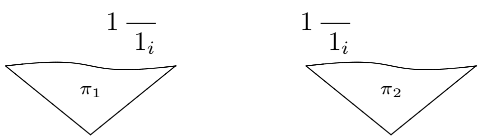
In this appendix we provide all proofs that have been left out in the main text. We also show detailed versions of those proofs that have only been sketched.

A.1 Sequentialization

Proof of Theorem 3.1: It is easy to see that the rules 1 and id give proof graphs and that the rules $\bot$, $\parr$, and $\otimes$ preserve the correctness. Therefore every sequentializable pre-proof net is correct.

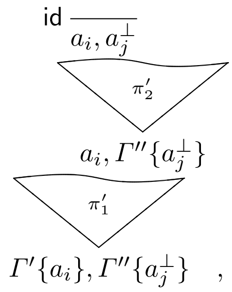
For the other direction pick one representative $P \rhd \Gamma$ of the proof net and proceed by induction on the sum of the number of $\otimes$-nodes in the graph and the number of $\parr$-nodes in Γ. In other words, the number of $\parr$-nodes in P is not relevant. (We will end up by exhibiting a sequentialization of an equivalent proof graph $Q \rhd \Gamma$, obtained from $P \rhd \Gamma$ by only applying associativity and commutativity of $\parr$)

The base case is trivial (the graph consists of a single node which is labelled by 1). For the inductive case look at the root nodes in Γ. If one of them is a $\parr$, we can remove it by applying the $\parr$-rule and proceed by induction hypothesis. If all roots in Γ are $\otimes$ nodes, we interpret $P \rhd \Gamma$ as an ordinary unit-free proof net (according to Observation 3.2), which remains correct if we remove in P all $\parr$-nodes that do not have a $\otimes$-node as ancestor. Now all roots are $\otimes$-nodes and one of them is splitting (by Lemma 3.3). If it belongs to Γ, we restore the $\parr$-structure under the $\otimes$-nodes in P such that each of the two subtrees of the root-$\parr$ covers one of the two components in which the graph is divided by removing the splitting $\otimes$. We can now apply the $\otimes$-rule and proceed by induction hypothesis on the two premises. If the splitting $\otimes$ belongs to P, there are two possibilities. Either it comes from an axiom link (i.e. both children are dual atoms), or it comes from a bottom link (i.e. one child is a $\bot$). In the first case, we have that the graph is of the shape $P', a_i \otimes a_j^\bot, P'' \rhd \Gamma'\{a_i\}, \Gamma''\{a_j^\bot\}$, where the linking is written as sequent because the $\parr$-roots are removed, and where $\Gamma'\{a_i\}$ denotes a sequent where one formula contains the atom a, indexed as a_i, such that P', a_i and $\Gamma'\{a_i\}$ share the same atoms and units, as well as $a_j^\bot, P''$ and $\Gamma''\{a_j^\bot\}$. If we replace a_i by 1_i and $a_j^\bot$ by 1_j, we obtain two proof graphs $P', 1_i \rhd \Gamma'\{1_i\}$ and $1_j, P'' \rhd \Gamma''\{1_j\}$ of strictly smaller size. Therefore, by induction hypothesis, we have two sequent proofs



$\Gamma'\{1_i\}$ and $\Gamma''\{1_j\}$.

From π_1 and π_2 we can construct the following sequent proof:



,

where π_1' is obtained from π_1 by replacing 1_i everywhere by a_i and by adding $\Gamma''\{a_j^{\perp}\}$ everywhere to the sequent that contains the a_i. Similarly π_2' is obtained from π_2. It is easy to see that this proof translates into $[P \rhd \Gamma]$. The case where the splitting tensor in P has a $\perp$ as child is similar and left to the reader. □

A.2 Cut elimination

Proof of Lemma 4.4: By the definition of proof graph, the linking P must contain two subtrees $a_h^{\perp} \otimes a_i$ and $a_j^{\perp} \otimes a_k$. By the correctness criterion, they must be in a $⅋$-relation, i.e. $P = P''\{R\{a_h^{\perp} \otimes a_i\} ⅋ S\{a_j^{\perp} \otimes a_k\}\}$ for some contexts $P''\{\ \}$, $R\{\ \}$ and $S\{\ \}$. We will proceed by induction on the size of $R\{\ \}$ and $S\{\ \}$, i.e. the sum of the number of $⅋$- and $\otimes$-nodes in them. We have th following cases:

- Both are empty. In this case we are done.
- $R\{\ \}$ has a $\otimes$ as root and $S\{\ \}$ is empty. In this case $R\{\ \} = \perp \otimes R'\{\ \}$ for some $R'\{\ \}$, and we can apply Lemma 4.7 with $Q = a_k$.
- $R\{\ \}$ is empty and $S\{\ \}$ has a $\otimes$ as root. This case is symmetric to the previous one, and we can apply Lemma 4.7 with $Q = a_h^{\perp}$.
- Both $R\{\ \}$ and $S\{\ \}$ have a $\otimes$ as root. In this case, we can apply Lemma 4.6, and proceed by induction hypothesis.
- One of $R\{\ \}$ and $S\{\ \}$ has a $⅋$ as root. In this case we apply the associativity of the $⅋$ (which is not subject to a side condition), and proceed by induction hypothesis. □

Proof of Lemma 4.5: This proof is very similar to the previous one. Because of the correctness, the proof graph $P \rhd \Gamma$ is of the shape $P''\{R\{Q \otimes \perp_i\} ⅋ S\{1_j\}\} \rhd \Gamma$. Again, we will proceed by induction on the size of $R\{\ \}$ and $S\{\ \}$. The cases are similar to the previous proof. The only difference is:

- $R\{\ \}$ has a $\otimes$ as root and $S\{\ \}$ is empty. In this case we apply Lemma 4.8 (instead of Lemma 4.7). □

Proof of Lemma 4.6: By way of contradiction, assume that both are not equivalent to the original proof graph. This means that in both cases the side condition ($*$) of 2.4 is not fulfilled, which means that we have in the original proof graph

– an extended switching wrt. $\bot_k$ such that one node of $S\{x_j^\bot\} \otimes \bot_h$ is connected to it, and

– an extended switching wrt. $\bot_h$ such that one node of $\bot_k \otimes R\{x_i\}$ is connected to it.

If the two paths do not have a common node, then we can construct a switching in which both are present. But this switching contains a cycle in which the two paths are connected by the $\otimes$-roots of $\bot_k \otimes R\{x_i\}$ and $S\{x_j^\bot\} \otimes \bot_h$, contradicting the assumption of correctness. If the two paths have at least one common node, we can also construct a switching with a cycle as follows. Starting from $\bot_k$, we take the first path until the first intersection node with the second path. Then the next node of the first path does also belong to the second one (otherwise we would have a graph node with four edges attached to it). We follow the second path in the direction determined by the two nodes. There are now three possibilities:

1. We reach a leaf in $R\{x_i\}$, which yields a cycle immediately.
2. We reach $\bot_h$. In this case we can construct a cycle by using the $\oplus$-node between x_i and $x_j^\bot$.
3. We reach a node in $S\{x_j^\bot\}$. In this case there is a part of the second path connecting $\bot_h$ with some node in $S\{x_j^\bot\}$, which yields a cycle. □

Proof of Lemma 4.7: By way of contradiction, assume this is not the case, i.e. the side condition ($*$) of 2.4 is not fulfilled, which means that we have in the original proof graph an extended switching wrt. $\bot_k$ such that a node of $x_j^\bot \otimes Q$ is connected to it. If this path goes through $R\{x_i\}$, we have a cycle immediately. If this is not the case, it has to enter $x_j^\bot \otimes Q$ either from the $\invamp$-node above or through a leaf of Q. In both cases we can extend the path through $x_j^\bot$ and the $\oplus$-node to x_i and the root of $R\{x_i\}$, which yields a cycle. □

Proof of Lemma 4.8: Similar to Lemma 4.7. □

Proof of Lemma 4.9: Since $P \rhd \Gamma \sim P' \rhd \Gamma$, we have

$$P \rhd \Gamma = P_0 \rhd \Gamma \sim P_1 \rhd \Gamma \sim \cdots \sim P_n \rhd \Gamma = P' \rhd \Gamma \ ,$$

for some linkings $P_0, P_1, \ldots, P_n$, where for each $i = 1, \ldots, n$ the equivalence $P_{i-1} \rhd \Gamma \sim P_i \rhd \Gamma$ is a direct application of the equations in 2.4. We can now distinguish three cases.

First, the reduced cut is on binary connectives. Then in each of the proof graphs $P_i \rhd \Gamma$ the cut is ready and we have $Q \rhd \Delta = Q_0 \rhd \Delta \sim \cdots \sim Q_n \rhd \Delta = Q' \rhd \Delta$, where each $Q_i \rhd \Delta$ is obtained from reducing the cut in $P_i \rhd \Gamma$.

In the second case the reduced cut is an atomic one, say $a_i \varotimes a_j^\perp$. Here it might happen that in some of the $P_i \rhd \Gamma$ the cut is not ready because of unnecessary applications of associativity. But it is easy to see that there is a transformation from $P \rhd \Gamma$ to $P' \rhd \Gamma$ in which the readiness of the cut is not destroyed, i.e. the sublinking $(a_h^\perp \otimes a_i) \bindnasrepma (a_j^\perp \otimes a_k)$ of P and P' is not touched. We can therefore proceed as in the first case.

The most difficult case occurs if the cut is on the units, say $\perp_i \varotimes 1_j$. Although $P = R\{(S \otimes \perp_i) \bindnasrepma 1_j\}$ and $P' = R'\{(S' \otimes \perp_i) \bindnasrepma 1_j\}$, the sublinking $(- \otimes \perp_i) \bindnasrepma 1_j$ might be destroyed in the transformation because other subtrees might leave or enter the scope of the $\perp_i$, and can therefore occur "between" $\perp_i$ and 1_j. However, in the reduction $\perp_i$ and 1_j disappear. Hence this intermediate steps become vacuous. We can therefore proceed similarly to the other two cases. □

Proof of Lemma 4.11: If $Q \rhd \Delta$ and $R \rhd \Sigma$ are obtained from $P \rhd \Gamma$ by reducing the same $\varotimes$-node in Γ then they must be equal. If different $\varotimes$-nodes have been reduced in the two reductions, then both $\varotimes$-nodes must have been ready in $P \rhd \Gamma$. But reducing one of the two $\varotimes$-nodes does not destroy the readiness of the other, which can therefore be reduced afterwards. Since the redexes of the reductions do not overlap, the result $S \rhd \Phi$ is independent of the order of the two reductions. □

Proof of Lemma 4.12: Let $\varotimes_1$ denote the cut that is reduced in Γ to obtain Δ and $\varotimes_2$ the one that is reduced to obtain Σ. If they are identical, we immediately have $[Q \rhd \Delta] = [R \rhd \Sigma]$. If not, we distinguish the following cases:

- One of the two cuts is on binary connectives, i.e. it is ready in each representation of the proof net. We can therefore choose a representation in which the other cut is also ready and apply Lemma 4.11 and Lemma 4.9.
- One of the two cuts is on units, say $\varotimes_1$. Then we can first make $\varotimes_2$ ready by applying Lemma 4.4 or Lemma 4.5. Then we apply Lemma 4.5 to also make $\varotimes_1$ ready. By this, the readiness of $\varotimes_2$ is not touched. We can therefore obtain a presentation of $[P \rhd \Gamma]$ in which both cuts are ready, and proceed as in the previous case.
- Both cuts are atomic, but are not directly connected to each other via an axiom link. Then we can proceed as in the previous case to obtain a presentation of $[P \rhd \Gamma]$ in which both cuts are ready.
- Both cuts are atomic and share a common axiom link. In other words, $P \rhd \Gamma$ is of the following shape:
$$\begin{array}{c} P'\{(P''\{a_h^\perp \otimes a_i\} \bindnasrepma P'''\{a_j^\perp \otimes a_k\}) \bindnasrepma P''''\{a_l^\perp \otimes a_m\}\} \\ \triangledown \\ a_i \varotimes_1 a_j^\perp, a_k \varotimes_2 a_l^\perp, \Phi \end{array}$$
In this case it is not possible to make both cuts ready at the same time. But we can transform the above graph into
$$\begin{array}{c} S'\{((a_h^\perp \otimes a_i) \bindnasrepma (a_j^\perp \otimes a_k)) \bindnasrepma (a_l^\perp \otimes a_m)\} \\ \triangledown \\ a_i \varotimes_1 a_j^\perp, a_k \varotimes_2 a_l^\perp, \Phi \end{array} \quad ,$$

as well as into

$$\begin{array}{c} S'\{(a_h^\perp \otimes a_i) \parr ((a_j^\perp \otimes a_k) \parr (a_l^\perp \otimes a_m))\} \\ \triangledown \\ a_i \obar_1 a_j^\perp, a_k \obar_2 a_l^\perp, \Phi \quad . \end{array}$$

In the first case $\obar_1$ is ready and in the second $\obar_2$. In both cases, after the reduction of one cut, the other becomes ready. After the second reduction, the result is in both cases $S'\{a_h^\perp \otimes a_m\} \rhd \Phi$. □

A.3 The free *-autonomous category

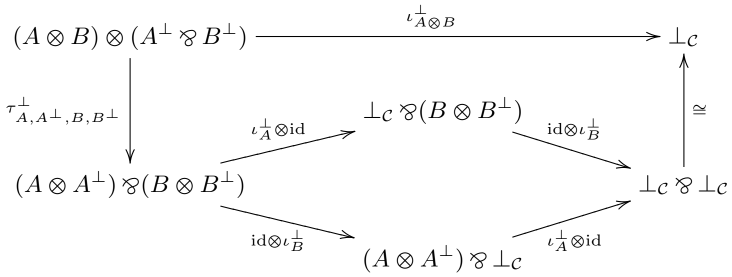
Let us now complete the proof of Theorem 6.1. Before we start, let us recall some some well-known facts about symmetric *-autonomous categories, that we will heavily use.

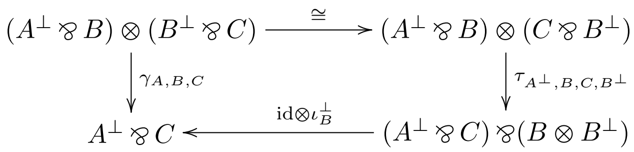
A.1 Observation Let $\mathcal{C}$ be a symmetric *-autonomous category. We will make heavy use of the adjunction between the two functors $A \otimes -$ and $(-)^\perp \parr C$ for any two objects A and C (i.e. the arrows $A \otimes B \to C$ are in one-to-one correspondence to the arrows $A \to B^\perp \parr C$). We have (besides the natural isomorphisms $\alpha, \sigma, \rho, \lambda$; see Section 5) for each object A in $\mathcal{C}$, canonical morphism $\iota_A : 1_{\mathcal{C}} \to A^\perp \parr A$ and $\iota_A^\perp : A \otimes A^\perp \to \perp_{\mathcal{C}}$, which correspond via the adjunction to the identity $\mathrm{id}_A : A \to A$. Further, for any two objects A and B, there is the *evaluation map* $\epsilon_{A,B} : A \otimes (A^\perp \parr B) \to B$ that corresponds under the adjunction to $\mathrm{id}_{A^\perp \parr B}$, and which is natural in A and B. From this, we can obtain for any three objects A, B, and C, the *internalized composition* arrow $\gamma_{A,B,C} : (A^\perp \parr B) \otimes (B^\perp \parr C) \to A^\perp \parr C$, which corresponds under the adjunction to

$$A \otimes (A^\perp \parr B) \otimes (B^\perp \parr C) \xrightarrow{\epsilon_{A,B} \otimes \mathrm{id}} B \otimes (B^\perp \parr C) \xrightarrow{\epsilon_{B,C}} C$$

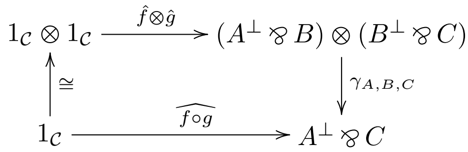
and is also natural in all three variables. Besides that, for any four objects A, B, C, and D, we have the *internalized tensor* $\tau_{A,B,C,D} : (A \parr B) \otimes (C \parr D) \to (A \parr C) \parr (B \otimes D)$, which is natural in all four variables, and which corresponds under the adjunction to $\epsilon_{A^\perp,B} \otimes \epsilon_{C^\perp,D} : A^\perp \otimes C^\perp \otimes (A \parr B) \otimes (C \parr D) \to B \otimes D$. By duality, we also have $\tau^\perp_{A,B,C,D} : (A \otimes C) \otimes (B \parr D) \to (A \otimes B) \parr (C \otimes D)$, natural in all four variables. In particular, we have that the following diagrams commute because of naturality.

$$\begin{array}{ccc}
((X \parr A) \otimes ((Y \parr C) \parr B)) \otimes (Z \parr D) & \xrightarrow{\cong} & (X \parr A) \otimes (((Y \parr B) \parr C) \otimes (Z \parr D)) \\
\downarrow{\scriptstyle \tau \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \otimes \tau} \\
((X \parr (Y \parr C)) \parr (A \otimes B)) \otimes (Z \parr D) & & (X \parr A) \otimes (((Y \parr B) \parr Z) \parr (C \otimes D)) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
(((X \parr Y) \parr (A \otimes B)) \parr C) \otimes (Z \parr D) & & (X \parr A) \otimes (((Y \parr Z) \parr (C \otimes D)) \parr B) \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\
(((X \parr Y) \parr (A \otimes B)) \parr Z) \parr (C \otimes D) & \xrightarrow{\cong} & (X \parr ((Y \parr Z) \parr (C \otimes D))) \parr (A \otimes B)
\end{array}$$





A.2 Observation The internalized composition arrow γ has the following property. Let $f : A \to B$ and $g : B \to C$ be any arrows in a *-autonomous category, and let $\hat{f} : 1_{\mathcal{C}} \to A^{\perp} \parr B$ and $\hat{g} : 1_{\mathcal{C}} \to B^{\perp} \parr C$ be the arrows corresponding to them under the adjunction. Then we have that



commutes, which is the reason why γ is called the internalized composition.

Now we have all necessary ingredients for completing the proof of Theorem 6.1. We will start by showing how we uniquely construct for each (cut free) proof graph $P \rhd \Gamma$ a family of morphisms

$$\bigotimes_I \{G(A_i)^{\perp} \mid i \in I\} \to \bigparr_{\complement I} \{G(A_i) \mid i \in \complement I\}$$

indexed by the bracketings on the subsets $I \subseteq \{0, \ldots, n\}$ and their complements, where $A_1, \ldots, A_n$ are the formulas in Γ, and $A_0 = P^{\star}$, i.e. $P \rhd \Gamma$ is seen as ordinary unit-free proof net with conclusions $A_0, \ldots, A_n$ (see Observation 3.2).

Proposition 6.2 ensures that each member of such a family of morphisms determines the others uniquely. We will call such a family an *equivariant family*, and a member of it a *representative*. We will proceed by induction on the size of the proof graph (i.e. the sum of the numbers of $\otimes$ and $\parr$-nodes). We will again make crucial use Lemma 3.3, i.e. the existence of a splitting tensor.

- If the graph contains no $\otimes$ or $\parr$-nodes, then it is a single axiom link with conclusion $a, a^{\perp}$. In this case our unique equivariant family is determined by the identity $\mathrm{id} : G(a) \to G(a)$.

- If one of the root nodes in the graph is a $\parr$, i.e. $A_j = A'_j \parr A''_j$ for some $j \in \{0, \ldots, n\}$, then we have by induction hypothesis the unique equivariant family with representative

$$\bigotimes\{G(A_i)^\perp \mid i \in \{0,\dots,n\} \setminus \{j\}\} \to G(A'_j) \parr G(A''_j)$$

from which we get immediately

$$\bigotimes\{G(A_i)^\perp \mid i \in \{0,\dots,n\} \setminus \{j\}\} \to G(A_j)$$

because $G(A'_j) \parr G(A''_j) = G(A_j)$. Because of the associativity of the functor $- \parr -$, this is independent from the choice of the $\parr$-root, in case there is more than one.

– If all roots are $\otimes$, then one of them is splitting, say $A_j = A'_j \otimes A''_j$. Removing the splitting $\otimes$ splits the graph into two smaller graphs, which are both correct. Without loss of generality, one contains the formulas $A_0, \dots, A_{j-1}, A'_j$ and the other contains $A''_j, A_{j+1}, \dots, A_n$ (i.e. we might have to choose a different ordering of the A_i). By induction hypothesis, we have two uniquely determined equivariant families, with representatives

$$\bigotimes\{G(A_0)^\perp, \dots, G(A_{j-1})^\perp\} \to G(A'_j) \quad \text{and}$$
$$\bigotimes\{G(A_{j+1})^\perp, \dots, G(A_n)^\perp\} \to G(A''_j) \quad ,$$

from which we get

$$\bigotimes\{G(A_i)^\perp | i \in \{0,\dots,n\}, i \neq j\} \to G(A_j)$$

by applying the functor $- \otimes -$ and the fact that $G(A_j) = G(A'_j) \otimes G(A''_j)$. This is uniquely determined by the *-autonomous structure on $\mathcal{C}$. Observe that this is independent from choice of the splitting $\otimes$ (in case there is more than one) because of the first diagram in Observation A.1.

Now we will extend this construction to proof graphs with cuts. More precisely, to every proof graph $P \rhd \Gamma$, where $\Gamma = A_1, \dots, A_n, B_1 \oplus B_1^\perp, \dots, B_m \oplus B_m^\perp$, we assign a unique equivariant family of morphisms

$$\bigotimes_I\{G(A_i)^\perp \mid i \in I\} \to \parr_{\complement I}\{G(A_i) \mid i \in \complement I\}$$

indexed by the bracketings on the subsets $I \subseteq \{0,\dots,n\}$ and their complements, where $A_0 = P^\star$, an none of $A_1, \dots, A_n$ is a cut. For this, we will first interpret each cut $B_i \oplus B_i^\perp$ as $\otimes$-formula $B_i \otimes B_i^\perp$, and apply the previous construction to $P \rhd A_1, \dots, A_n, B_1 \otimes B_1^\perp, \dots, B_m \otimes B_m^\perp$, which yields in particular

$$\bigotimes\{G(A_i)^\perp | i \in \{0,\dots,n\}\} \to \parr\{G(B_j) \otimes G(B_j)^\perp | j \in \{1,\dots,m\}\} \quad .$$

By the *-autonomous structure on $\mathcal{C}$ we have for each $j \in \{1,\dots,m\}$, the morphism $G(B_j) \otimes G(B_j)^\perp \to \bot_\mathcal{C}$, which is determined by the identity on $G(B_j)$ and Proposition 6.2. From these, we get

$$\zeta_\Gamma : \parr\{G(B_j) \otimes G(B_j)^\perp | j \in \{1,\dots,m\}\} \to \bot_\mathcal{C}$$

by applying the functor $- \parr -$ according to the desired bracketing. By composition we get

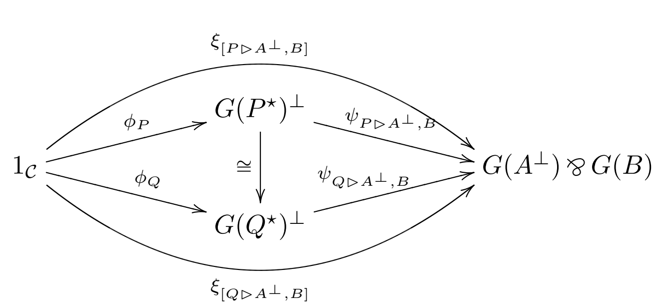
$$\bigotimes\{G(A_i)^\perp | i \in \{0,\dots,n\}\} \to \bot_\mathcal{C} \quad ,$$

as desired. The uniqueness of this follows from the lower diamond in the second diagram of Observation A.1.

A.3 Observation Let $P \rhd A_1, \dots, A_n$ be a cut free proof graph with k leaves. Then $G(P^\star)^\perp$ as well as $\bigparr\{G(A_1)\dots, G(A_n)\}$ can be seen as functors $\mathcal{C}^k \to \mathcal{C}$ in k arguments which are filled by elements of the image of $G^\circ : \mathcal{A} \to \mathrm{Obj}(\mathcal{C})$ or the unit objects $1_\mathcal{C}$ and $\perp_\mathcal{C}$. The arrow $\psi_{P \rhd A_1,\dots,A_n} : G(P^\star)^\perp \to \bigparr\{G(A_1)\dots, G(A_n)\}$, determined by the construction of the equivariant families, is natural in all k arguments.

Let us now proceed by proving Lemma 6.3.

Proof of Lemma 6.3: Observe that for every linking P, the object $G(P^\star)^\perp$ in $\mathcal{C}$ is isomorphic to $\bigotimes_{a\otimes a^\perp}\{G(a) \parr G(a)^\perp\}$, where $a \otimes a^\perp$ ranges over the axiom links in P. This means that the *-autonomous structure on $\mathcal{C}$ uniquely determines a morphism $\phi_P : 1_\mathcal{C} \to G(P^\star)^\perp$. If we have $Q \rhd A^\perp, B \sim P \rhd A^\perp, B$, then $G(Q^\star)^\perp \cong G(P^\star)^\perp$ by the *-autonomy of $\mathcal{C}$ and the fact that the equivalence rules do not disturb the ordinary axiom links. Hence the diagram

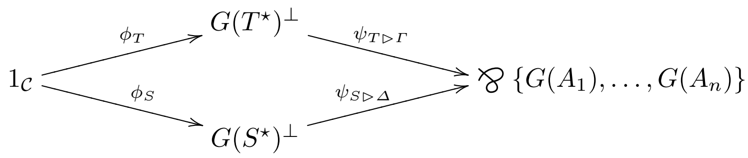


commutes.

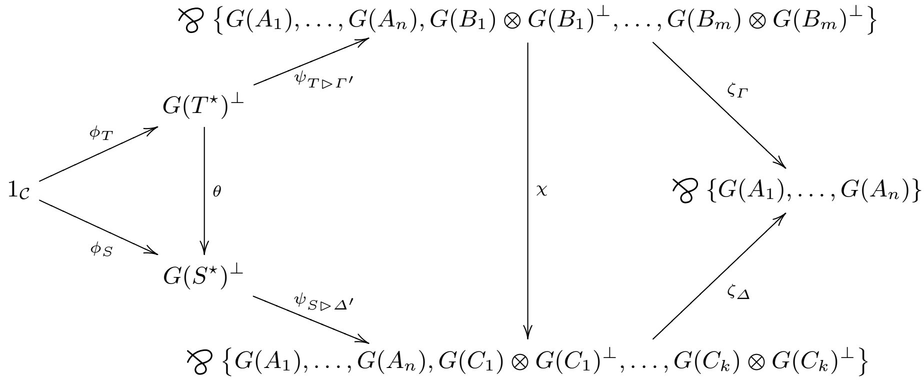
Therefore, we can to each proof net $f = [P \rhd A^\perp, B]$ uniquely assign the arrow $G(f) : G(A) \to G(B)$ that is determined by $\xi_{[P\rhd A^\perp,B]}$ via Proposition 6.2.

It remains to show that $G : \mathbf{PN}(\mathcal{A}) \to \mathcal{C}$ is indeed a functor (i.e. identities and composition are preserved). That for each formula A, the proof $[I_A \rhd A^\perp, A]$ is mapped to identity $\mathrm{id} : G(A) \to G(A)$ is an easy induction on the structure of A and left to the reader. The crucial part is to show that for two given proof nets $f = [P \rhd A^\perp, B]$ and $g = [Q \rhd B^\perp, C]$, the composition $G(g) \circ G(f)$ yields the same arrow in $\mathcal{C}$, as $G(g \circ f)$, where $g \circ f = [R \rhd A^\perp, C]$ is the proof net that is obtained by eliminating the cut in $[P \parr Q \rhd A^\perp, B \oplus B^\perp, C]$. For the proof graph $P \parr Q \rhd A^\perp, B \oplus B^\perp, C$ we get by the construction of the equivariant families a unique morphism $\psi_{P\parr Q\rhd A^\perp,B\oplus B^\perp,C} : G((P \parr Q)^\star)^\perp \to G(A^\perp) \parr G(C)$ which we can compose with $\phi_{P\parr Q} : 1_\mathcal{C} \to G((P \parr Q)^\star)^\perp$ to get $\xi_{[P\parr Q\rhd A^\perp,B\oplus B^\perp,C]} : 1_\mathcal{C} \to G(A^\perp) \parr G(C)$. It follows from the *-autonomous structure on $\mathcal{C}$ (Observation A.2), that this corresponds (via Proposition 6.2) to $G(g) \circ G(f)$. It remains to show that $\xi_{[P\parr Q\rhd A^\perp,B\oplus B^\perp,C]}$ and $\xi_{[R\rhd A^\perp,C]}$ (which corresponds via Proposition 6.2 to $G(g \circ f)$) are the same morphism $1_\mathcal{C} \to G(A)^\perp \parr G(C)$ in $\mathcal{C}$. This is done by induction on the length of the cut elimination reduction by employing Lemma 6.4.

Proof of Lemma 6.4: We have to show that the following diagram commutes:



This is done by showing that the following diagram commutes:



Here $B_1 \oplus B_1^\perp, \dots, B_m \oplus B_m^\perp$ are the cuts in Γ and $C_1 \oplus C_1^\perp, \dots, C_k \oplus C_k^\perp$ are the cuts in Δ. The sequents Γ' and Δ' are obtained from Γ and Δ by replacing these cuts by tensor fomulas, as it is done in the construction of the equivariant families. For θ and χ there are three cases to consider:

- The reduced cut is on binary connectives. In this case we have $k = m + 1$, $T = S$, and (without loss of generality) $B_m = C_{k-1} \otimes C_k$, and $C_i = B_i$ for all $i \in \{1, \dots, m-1\}$. We have that θ is the identity, and χ is determined by the canonical arrow $\tau^\perp_{G(C_{k-1}), G(C_{k-1})^\perp, G(C_k), G(C_k)^\perp}$ (see Observation A.1).
- The reduced cut is on atoms. In this case we have $k = m - 1$ and (without loss of generality) $B_m = a$, and $C_i = B_i$ for all $i \in \{1, \dots, m-1\}$. We have that χ is determined by $\iota^\perp_{G(a)} : G(a) \otimes G(a)^\perp \to \perp_{\mathcal{C}}$, and θ by the internalized composition $\gamma_{G(a), G(a), G(a)}$.
- The reduced cut is on units. In this case we have $k = m - 1$ and (without loss of generality) $B_m = \perp$, and $C_i = B_i$ for all $i \in \{1, \dots, m-1\}$. Observe, that in this case $G(T^\star)^\perp$ and $G(S^\star)^\perp$ are isomorphic. Furthermore, the arrow $\iota^\perp_{\perp_{\mathcal{C}}} : G(\perp) \otimes G(1) \to \perp_{\mathcal{C}}$, used in the construction of ζ_Γ is an isomorphism. Therefore θ and χ are both isomorphisms.

In all three cases we have that the leftmost triangle commutes because of the uniqueness of ϕ_S, the rightmost triangle commutes because of the uniqueness of ζ_Γ (and the second diagram in Observation A.1), and the inner square commutes because of the uniqueness of the construction of the equivariant families, the naturality of ψ (see Observation A.3), and the third diagram of Observation A.1 (in the second case only). □