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What could a Boolean category be?

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Abstract. In its most general meaning, a Boolean category should be to categories what a Boolean algebra is to posets. In a more specific meaning a Boolean category should provide the abstract algebraic structure underlying the proofs in Boolean Logic, in the same sense as a Cartesian closed category captures the proofs in intuitionistic logic and a *-autonomous category captures the proofs in linear logic. However, recent work has shown that there is no canonical axiomatisation of a Boolean category. In this talk I will sketch a series (with increasing strength) of possible such axiomatisations, all based on the notion of *-autonomous category. There will be some focus on the medial map, which has its origin in an inference rule in KS, a cut-free deductive system for Boolean logic in the calculus of structures.

1 Introduction

This work is mainly motivated by the question how to identify proofs in classical propositional logic. These are usually presented as syntactic objects within some deductive system (e.g., tableaux, sequent calculus, resolution, ...). Here we will take the point of view that these syntactic objects (also known as proof trees) should be considered as concrete representations of certain abstract proof objects, and that such an abstract proof object can be represented by a resolution proof tree and a sequent calculus proof tree, or even by several different sequent calculus proof trees.

Under this point of view the motivation for this work is to provide an abstract algebraic theory of proofs. Already Lambek [Lam68,Lam69] observed that such an algebraic treatment can be provided by category theory. For this, it is necessary to accept the following postulates about proofs:

- for every proof f of conclusion B from hypothesis A (denoted by $f\colon A \to B$) and every proof g of conclusion C from hypothesis B (denoted by $g\colon B \to C$) there is a uniquely defined composite proof $g \circ f$ of conclusion C from hypothesis A (denoted by $g \circ f\colon A \to C$),
- this composition of proofs is associative,
- for each formula A there is an identity proof $1_A\colon A \to A$ such that for $f\colon A \to B$ we have $f \circ 1_A = f = 1_B \circ f$.

Under these assumptions[1] the proofs are the arrows in a category whose objects are the formulas of the logic. What remains is to provide the right axioms for the "category of proofs".

It seems that finding these axioms is particularly difficult for the case of Boolean logic. For intuitionistic logic, Prawitz [Pra71] proposed the notion of *proof normalization* for identifying proofs. It was soon discovered that this notion of identity coincides with the notion of identity that results from the axioms of a Cartesian closed category (see, e.g., [LS86]). In fact, one can say that the proofs of intuitionistic logic are the arrows in the free (bi-)cartesian closed category generated by the set of propositional variables. An alternative way of representing the arrows in that category is via terms in the simply-typed λ-calculus: arrow composition is normalization of terms. This observation is well-known as Curry-Howard-correspondence [How80].

In the case of linear logic, the relation to *-autonomous categories [Bar79] was noticed immediately after its discovery [Laf88,See89]. In the sequent calculus linear logic proofs are identified when they can be transformed into each other via "trivial" rule permutations [Laf95]. For multiplicative linear logic this coincides with the proof identifications induced by the axioms of a *-autonomous category [Blu93,SL04]. Therefore, we can safely say that a proof in multiplicative linear logic is an arrow in the free *-autonomous category generated by the propositional variables [BCST96,LS04,Hug05].

But for classical logic no such well-accepted category of proofs exists. We can distinguish two main reasons. First, if we start from a Cartesian closed category and add an involutive negation[2], we get the collapse into a Boolean algebra, i.e., any two proofs $f, g\colon A \to B$ are identified. For every formula there would be at most one proof (see, e.g., [LS86] or the appendix of [Gir91] for details). Alternatively, starting from a *-autonomous category and adding natural transformations $A \to A \wedge A$ and $A \to \mathbf{t}$, i.e., the proofs for weakening and contraction, yields the same collapse.[3]

The second reason is that cut elimination in the sequent calculus for classical logic is not confluent. Since cut elimination is the usual way of composing proofs, this means that there is no canonical way of composing two proofs, let alone associativity of composition.

Consequently, for avoiding these two problems, we have to accept that (i) cartesian closed categories might not provide an abstract algebraic axiomatisation for proofs in classical logic, and that (ii) the sequent calculus is not the right framework for investigating the identity of proofs in in classical logic.

[1] It can (and should) be argued that these assumptions are already to strong for a reasonable theory of proofs. However, in this paper we follow the approach induced by these assumptions, and see where it will bring us.

[2] i.e., a natural isomorphism between A and the double-negation of A (in this paper denoted by $\bar{\bar{A}}$)

[3] Since we are dealing with Boolean logic, we will use the symbols $\wedge$ and $\mathbf{t}$ for the tensor operation (usually $\otimes$) and the unit (usually $\mathbf{1}$ or $\mathbf{I}$) in a *-autonomous category.

There have already been several accounts for a proof theory for classical logic based on the axioms of cartesian closed categories. The first were probably Parigot's $\lambda\mu$-calculus [Par92] and Girard's LC [Gir91]. The work on polarized proof nets by Laurent [Lau99,Lau03] shows that there is in fact not much difference between the two. Later, the category theoretical axiomatisations underlying this proof theory has been investigated and the close relationship to continuations [Thi97,SR98] has been established, culminating in Selinger's *control categories* [Sel01]. However, by sticking to the axioms of cartesian closed categories, one has to sacrifice the perfect symmetry of Boolean logic.

In this paper, we will go the opposite way. In the attempt of going from a Boolean algebra to a Boolean category we insist on keeping the symmetry between $\wedge$ and $\vee$. By doing this we have to leave the realm of cartesian closed categories. That this is very well possible has recently been shown by several authors [DP04,FP04c,LS05a]. However, the fact that all three proposals considerably differ from each other suggests that there might be no canonical way of giving a categorical axiomatisation for proofs in classical logic.

In this paper we will provide a series of possible such axiomatisations with increasing strength. They will all build on the structure of a *-autonomous category in which every object has a monoid (and a comonoid) structure. In this respect it will follow the work of [FP04c] and [LS05a], but will differ from [DP04].

The main proof theoretical inspiration for this work comes from system SKS [BT01], which is a deductive system for Boolean logic within the formalism of the calculus of structures [Gug02,GS01,BT01], making crucial use of the concept of deep inference.

It is a trivial but important observation that deep inference allows to establish the relationship between proof theory and algebra in a much cleaner way than this is possible with shallow inference formalisms like the sequent calculus. The reason is that from a derivation in a deep inference formalism one can directly "read off the morphisms". Take for example the following derivation in SKS:

$$\mathsf{m}\,\frac{\mathsf{r}\,\dfrac{(A' \wedge B) \vee (C \wedge D)}{(A \wedge B) \vee (C \wedge D)}}{(A \vee C) \wedge (B \vee D)} \tag{1}$$

where A, A', B, C, and D are arbitrary formulas, m is the medial rule and r is any inference rule taking A' to A. In category theoretical language this would be written as a composition of maps:

$$(A' \wedge B) \vee (C \wedge D) \xrightarrow{(\mathsf{r} \wedge B) \vee (C \wedge D)} (A \wedge B) \vee (C \wedge D) \xrightarrow{\mathsf{m}_{A,B,C,D}} (A \vee C) \wedge (B \vee D)$$

where $\mathsf{m}_{A,B,C,D} \colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D)$ is called the *medial map*, and $\mathsf{r} \colon A' \to A$ is the map corresponing to the rule r.

This means that although we use in the following the language of category theory, the seasoned proof theorist might find it easier to understand if he substitutes everywhere *"object"* by *"formula"* and *"map"/"morphism"/"arrow"* by *"proof"*. Every commuting diagram in the paper is nothing but an equation between proofs written as derivations in a deep inference system.

This paper is an extended abstract of [Str05c] which has been written at the same time as Lamarche's [Lam05]. Some of the basic axioms are necessarily in common in both papers. But while [Lam05] is mainly concerned with the construction of concrete models of proofs in classical logic, we are mainly concerned with syntax.

Detailed proofs of all the claims made here can be found in [Str05c].

2 Some axioms for Boolean categories

In its most general sense, a Boolean category should be for categories, what a Boolean algebra is for posets. This leads to the following definition:

2.1 Definition We say a category $\mathscr{C}$ is a $\mathsf{B0}$-*category* if there is a Boolean algebra $\mathscr{B}$ and a mapping $F\colon \mathscr{C} \to \mathscr{B}$ from objects of $\mathscr{C}$ to elements of $\mathscr{B}$, such that for all objects A and B in $\mathscr{C}$, we have $F(A) \leq F(B)$ in $\mathscr{B}$ if and only if there is an arrow $f\colon A \to B$ in $\mathscr{C}$.

In other words, a $\mathsf{B0}$-category is a category whose image under the forgetful functor from the category of categories to the category of posets is a Boolean algebra. This definition is neither enlightening nor useful. It is necessary to add some additional structure in order to obtain a "nicely behaved" theory of Boolean categories. Before, let us make the following (trivial) observation.

2.2 Observation In a $\mathsf{B0}$-category, we can for any pair of objects A and B, provide objects $A \wedge B$ and $A \vee B$ and $\bar{A}$, and there are objects t and f, such that there are maps

$$
\begin{array}{cc}
\hat{\alpha}_{A,B,C}\colon A \wedge (B \wedge C) \to (A \wedge B) \wedge C & \check{\alpha}_{A,B,C}\colon A \vee (B \vee C) \to (A \vee B) \vee C \\
\hat{\sigma}_{A,B}\colon A \wedge B \to B \wedge A & \check{\sigma}_{A,B}\colon A \vee B \to B \vee A \\
\hat{\varrho}_{A}\colon A \wedge \mathsf{t} \to A & \check{\varrho}_{A}\colon A \vee \mathsf{f} \to A \\
\hat{\lambda}_{A}\colon \mathsf{t} \wedge A \to A & \check{\lambda}_{A}\colon \mathsf{f} \vee A \to A \\
\hat{\imath}_{A}\colon A \wedge \bar{A} \to \mathsf{f} & \check{\imath}_{A}\colon \mathsf{t} \to \bar{A} \vee A
\end{array}
\tag{2}
$$

$$
\mathsf{s}_{A,B,C}\colon (A \vee B) \wedge C \to A \vee (B \wedge C)
$$

$$
\mathsf{m}_{A,B,C,D}\colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D)
$$

$$
\begin{array}{cc}
\Delta_{A}\colon A \to A \wedge A & \nabla_{A}\colon A \vee A \to A \\
\Pi^{A}\colon A \to \mathsf{t} & \amalg^{A}\colon \mathsf{f} \to A
\end{array}
$$

for all objects A, B, and C. This can easily be shown by verifying that all of them correspond to valid implications in Boolean logic. Conversely, a category in which every arrow can be given as a composite of the ones given above by using only the operations of $\wedge$, $\vee$, and the usual arrow composition, is a $\mathsf{B0}$-category. This is a consequence of the completeness of system SKS [BT01], which is a deep inference deductive system for Boolean logic incorporating the maps in (2) as inference rules.

Let us stress the fact that in a plain $\mathsf{B0}$-category there is no relation between the maps listed in (2). In particular, there is no functoriality of $\vee$ and $\wedge$, no naturality of $\hat{\alpha}$, $\hat{\sigma}$, $\ldots$, and no deMorgan duality. Adding this structure means exactly adding the structure of a *-autonomous category [Bar79].[4]

This in particular means that $\wedge$, $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\varrho}$, and $\hat{\lambda}$, obey the axioms of a symmetric monoidal category. We will not repeat here all the coherence diagrams. The only important fact to know is the coherence theorem [Mac63], which says that every diagram containing only maps composed of $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\varrho}$, and $\hat{\lambda}$, via $\wedge$ and $\circ$ must commute (for details, see [Mac71]). Dually, we have a second symmetric monoidal structure, implying that all diagrams built with $\check{\alpha}$, $\check{\sigma}$, $\check{\varrho}$, and $\check{\lambda}$, via $\vee$ and $\circ$ do commute.

As a consequence of the coherence theorem, we can omit certain parentheses to ease the reading. For example, we will write $A \wedge B \wedge C \wedge D$ for $(A \wedge B) \wedge (C \wedge D)$ as well as for $A \wedge ((B \wedge C) \wedge D)$. This can be done because there is a uniquely defined "coherence isomorphism" between any two of these objects.

For obtaining the duality between $\wedge$ and $\vee$, we need to have a a contravariant functor $\overline{(-)} : \mathscr{C} \to \mathscr{C}$, such that $\overline{\overline{(-)}} : \mathscr{C} \to \mathscr{C}$ is a natural isomorphism and such that for any three objects A, B, C there is a natural bijection

$$\mathrm{Hom}_{\mathscr{C}}(A \wedge B, C) \quad \cong \quad \mathrm{Hom}_{\mathscr{C}}(A, \bar{B} \vee C) \quad . \qquad (\star)$$

We also define $A \vee B = \overline{\bar{B} \wedge \bar{A}}$ and $\mathsf{f} = \bar{\mathsf{t}}$.[5]

This is all what is needed for a *-autonomous category. Here some important properties: Via the bijection $(\star)$ we can assign to every map $f \colon A \to B \vee C$ a map $g \colon A \wedge \bar{B} \to C$, and vice versa. We say that f and g are *transposes* of each other if they determine each other via $(\star)$. We will use the term "transpose" in a very general sense: given objects A, B, C, D, E such that $D \cong A \wedge B$ and $E \cong \bar{B} \vee C$, then any $f \colon D \to C$ uniquely determines a $g \colon A \to E$, and vice versa. Also in that general case we will say that f and g are transposes of each other. For example, $\hat{\lambda}_A \colon \mathsf{t} \wedge A \to A$ and $\check{\varrho}_A \colon A \to A \vee \mathsf{f}$ are transposes of each other, and another way of transposing them yields the maps

$$\check{\imath}_A \colon \mathsf{t} \to \bar{A} \vee A \qquad \text{and} \qquad \hat{\imath}_A \colon A \wedge \bar{A} \to \mathsf{f} \quad .$$

Let us now transpose the identity $1_{B \vee C} \colon B \vee C \to B \vee C$. This yields the *evaluation map* $\mathsf{eval} \colon (B \vee C) \wedge \bar{C} \to B$. Taking the $\wedge$ of this with $1_A \colon A \to A$ and transposing back determines a map $\mathsf{s}_{A,B,C} \colon A \wedge (B \vee C) \to (A \wedge B) \vee C$ that is

[4] Since we are working in classical logic, we will here use the symbols $\wedge, \vee, \mathbf{t}, \mathbf{f}$ for the usual $\otimes, \invamp, 1, \bot$.

[5] Although we live in the commutative world, we invert the order of the arguments when taking the negation.

natural in all three arguments, and that we call the *switch map* [Gug02,BT01][6]. In a similar fashion we obtain maps $(A \vee B) \wedge C \to A \vee (B \wedge C)$ and $A \wedge (B \vee C) \to B \vee (A \wedge C)$ and $(A \vee B) \wedge C \to (A \wedge C) \vee B$. Alternatively these maps can be obtained from s by composing with $\hat{\sigma}$ and $\check{\sigma}$. For this reason we will use the term "switch" for all of them, and denote them by $\mathsf{s}_{A,B,C}$ if it is clear from context which one is meant, as for example in

$$\begin{array}{ccc}
(A \vee B) \wedge (C \vee D) & \xrightarrow{\mathsf{s}_{A,B,C\vee D}} & A \vee (B \wedge (C \vee D)) \\
\Big\downarrow {\scriptstyle \mathsf{s}_{A\vee B,C,D}} & & \Big\downarrow {\scriptstyle A \vee \mathsf{s}_{B,C,D}} \\
((A \vee B) \wedge C) \vee D & \xrightarrow[\mathsf{s}_{A,B,C} \vee D]{} & A \vee (B \wedge C) \vee D
\end{array} \tag{3}$$

which always commutes in a *-autonomous category. Sometimes we will denote the map defined by the diagonal of (3) by $\hat{\mathsf{t}}_{A,B,C,D} \colon (A \vee B) \wedge (C \vee D) \to A \vee (B \wedge C) \vee D$, called the *tensor map*[7] and its dual by $\check{\mathsf{t}}_{A,B,C,D} \colon A \wedge (B \vee C) \wedge D \to (A \wedge B) \vee (C \wedge D)$, called the *cotensor map*.

Now, let us recall some well-known facts about mix.

2.3 Theorem *Let $e \colon \mathsf{f} \to \mathsf{t}$ be a map in a *-autonomous category. Then*

$$\begin{array}{ccc}
\mathsf{f} \wedge \mathsf{f} & \xrightarrow{e \wedge \mathsf{f}} & \mathsf{t} \wedge \mathsf{f} \\
\Big\downarrow {\scriptstyle \mathsf{f} \wedge e} & & \Big\downarrow {\scriptstyle \hat{\lambda}_{\mathsf{f}}} \\
\mathsf{f} \wedge \mathsf{t} & \xrightarrow[\hat{\varrho}_{\mathsf{f}}]{} & \mathsf{f}
\end{array} \tag{4}$$

if and only if

$$\begin{array}{ccccc}
A \wedge B & \xrightarrow{A \wedge \check{\lambda}_B^{-1}} & A \wedge (\mathsf{f} \vee B) & \xrightarrow{\mathsf{s}_{A,\mathsf{f},B}} & (A \wedge \mathsf{f}) \vee B \\
\Big\downarrow {\scriptstyle \check{\varrho}_A^{-1} \wedge B} & & & & \Big\downarrow {\scriptstyle (A \wedge e) \vee B} \\
(A \vee \mathsf{f}) \wedge B & & & & (A \wedge \mathsf{t}) \vee B \\
\Big\downarrow {\scriptstyle \mathsf{s}_{A,\mathsf{f},B}} & & & & \Big\downarrow {\scriptstyle \hat{\varrho}_A \vee B} \\
A \vee (\mathsf{f} \wedge B) & \xrightarrow[A \vee (e \wedge B)]{} & A \vee (\mathsf{t} \wedge B) & \xrightarrow[A \vee \hat{\lambda}_B]{} & A \vee B
\end{array} \tag{5}$$

for all objects A and B.

In fact, in a *-autonomous category every map $e \colon \mathsf{f} \to \mathsf{t}$ obeying (4) uniquely determines a map $\mathsf{mix}_{A,B} \colon A \wedge B \to A \vee B$, defined by the diagonal of (5), which is natural in A and B. It can be shown that this *mix map* goes well with the twist, associativity, and switch maps:

[6] To category theorists it is probably better known under the names *weak distributivity* [HdP93,CS97] or *linear distributivity*. However, strictly speaking, it is not a form of distributivity. An alternative is the name *dissociativity* [DP04].

[7] This map describes precisely the tensor rule in the sequent system for linear logic.

2.4 Proposition *The map* $\mathsf{mix}_{A,B} \colon A \wedge B \to A \vee B$ *obtained from* (5) *is natural in both arguments and obeys the equations*

$$\begin{array}{ccc} A \wedge B & \xrightarrow{\mathsf{mix}_{A,B}} & A \vee B \\ {\scriptstyle \hat{\sigma}_{A,B}} \downarrow & & \downarrow {\scriptstyle \check{\sigma}_{A,B}} \\ B \wedge A & \xrightarrow[\mathsf{mix}_{B,A}]{} & B \vee A \end{array} \qquad (\mathsf{mix}\text{-}\hat{\sigma})$$

and

$$\begin{array}{ccccc} A \wedge (B \wedge C) & \xrightarrow{A \wedge \mathsf{mix}_{B,C}} & A \wedge (B \vee C) & \xrightarrow{\mathsf{mix}_{A,B \vee C}} & A \vee (B \vee C) \\ {\scriptstyle \hat{\alpha}_{A,B,C}} \downarrow & & \downarrow {\scriptstyle \mathsf{s}_{A,B,C}} & & \downarrow {\scriptstyle \check{\alpha}_{A,B,C}} \\ (A \wedge B) \wedge C & \xrightarrow[\mathsf{mix}_{A \wedge B,C}]{} & (A \wedge B) \vee C & \xrightarrow[\mathsf{mix}_{A,B} \vee C]{} & (A \vee B) \vee C \end{array} \qquad (\mathsf{mix}\text{-}\hat{\alpha})$$

2.5 Corollary *In a *-autonomous category there is a one-to-one correspondence between the maps* $e \colon \mathsf{f} \to \mathsf{t}$ *obeying* (4) *and the natural transformations* $\mathsf{mix}_{A,B} \colon A \wedge B \to A \vee B$ *obeying* (mix-$\hat{\sigma}$) *and* (mix-$\hat{\alpha}$).

Note that a *-autonomous category can have many different maps $e \colon \mathsf{f} \to \mathsf{t}$ with the property of Theorem 2.3, each of them defining its own natural mix obeying (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$).

The structure investigated so far is exactly the same as for proofs in linear logic (with or without mix). For classical logic, we need to provide algebraic structure for the maps $\nabla_A \colon A \vee A \to A$ and $\amalg^A \colon \mathsf{f} \to A$, as well as $\Delta_A \colon A \to A \wedge A$ and $\Pi^A \colon A \to \mathsf{t}$, which are listed in (2). This is done via monoids and comonoids, in other words, ∇_A and Δ_A are asked to (co-)associative and (co-)commutative, and $\amalg^A$ and Π^A are their (co-)units.

2.6 Definition A $\mathsf{B1}$-*category* is a *-autonomous $\mathsf{B0}$-category in which every object A is equipped with a commutative $\vee$-monoid structure $(\nabla_A, \amalg^A)$ and a cocommutative $\wedge$-comonoid structure (Δ_A, Π^A), such that

$$\overline{\nabla_A} = \Delta_{\bar{A}} \qquad \text{and} \qquad \overline{\amalg^A} = \Pi^{\bar{A}} \quad .$$

Let $f \colon A \to B$ be a map in a $\mathsf{B1}$-category. We say that

- f *preserves the* $\vee$*-multiplication* if $\nabla_B \circ (f \vee f) = f \circ \nabla_A$,
- f *preserves the* $\vee$*-unit* if $\amalg^B = f \circ \amalg^A$,
- f *preserves the* $\wedge$*-comultiplication* if $\Delta_B \circ f = (f \wedge f) \circ \Delta_A$,
- f *preserves the* $\wedge$*-counit* if $\Pi^B \circ f = \Pi^A$,
- f is a $\vee$-*monoid morphism* if it preserves the $\vee$-multiplication and the $\vee$-unit,
- f is a $\wedge$-*comonoid morphism* if it preserves the $\wedge$-comultiplication and the $\wedge$-counit.

In a $\mathsf{B1}$-category we have two canonical maps $\mathsf{f} \to \mathsf{t}$, namely $\amalg^{\mathbf{f}}$ and $\amalg^{\mathbf{t}}$. Because of the $\wedge$-comonoid structure on f and the $\vee$-monoid structure on t, we have

$$
\begin{array}{ccccc}
\mathsf{f} \vee \mathsf{t} & \xrightarrow{\amalg^{\mathbf{t}} \vee \mathbf{t}} & \mathsf{t} \vee \mathsf{t} & \xleftarrow{\mathbf{t} \vee \amalg^{\mathbf{t}}} & \mathsf{t} \vee \mathsf{f} \\
& {\scriptstyle \check{\lambda}_{\mathbf{t}}} \searrow & \downarrow {\scriptstyle \nabla_{\mathbf{t}}} & \swarrow {\scriptstyle \check{\varrho}_{\mathbf{t}}} & \\
& & \mathsf{t} & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
\mathsf{t} \wedge \mathsf{f} & \xleftarrow{\amalg^{\mathbf{f}} \wedge \mathbf{f}} & \mathsf{f} \wedge \mathsf{f} & \xrightarrow{\mathbf{f} \wedge \amalg^{\mathbf{f}}} & \mathsf{f} \wedge \mathsf{t} \\
& {\scriptstyle \hat{\lambda}_{\mathbf{f}}^{-1}} \nwarrow & \uparrow {\scriptstyle \Delta_{\mathbf{f}}} & \nearrow {\scriptstyle \hat{\varrho}_{\mathbf{f}}^{-1}} & \\
& & \mathsf{f} & &
\end{array}
$$

(which even hold if the (co)monoids are not (co)commutative.) Since $\check{\lambda}_{\mathbf{t}}$, $\check{\varrho}_{\mathbf{t}}$, $\hat{\lambda}_{\mathbf{f}}$, and $\hat{\varrho}_{\mathbf{f}}$ are isomorphisms, we immediately can conclude that the following two diagrams commute (cf. [FP04a]):

$$
\begin{array}{ccc}
\mathsf{t} & \xrightarrow{\check{\lambda}_{\mathbf{t}}^{-1}} & \mathsf{f} \vee \mathsf{t} \\
{\scriptstyle \check{\varrho}_{\mathbf{t}}^{-1}} \downarrow & & \downarrow {\scriptstyle \amalg^{\mathbf{t}} \vee \mathbf{t}} \\
\mathsf{t} \vee \mathsf{f} & \xrightarrow[\mathbf{t} \vee \amalg^{\mathbf{t}}]{} & \mathsf{t} \vee \mathsf{t}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathsf{f} \wedge \mathsf{f} & \xrightarrow{\mathbf{f} \wedge \amalg^{\mathbf{f}}} & \mathsf{f} \wedge \mathsf{t} \\
{\scriptstyle \amalg^{\mathbf{f}} \wedge \mathbf{f}} \downarrow & & \downarrow {\scriptstyle \hat{\varrho}_{\mathbf{f}}} \\
\mathsf{t} \wedge \mathsf{f} & \xrightarrow[\hat{\lambda}_{\mathbf{f}}]{} & \mathsf{f}
\end{array}
$$

This gives us two different mix maps $A \wedge B \to A \vee B$, and motivates the following definition:

2.7 Definition A $\mathsf{B1}$-category is called *single-mixed* if $\amalg^{\mathbf{f}} = \amalg^{\mathbf{t}}$.

In a single-mixed $\mathsf{B1}$-category we have, as the name says, a single canonical mix map $\mathsf{mix}_{A,B} \colon A \wedge B \to A \vee B$ obeying (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$). The naturality of mix, i.e., the commutativity of

$$
\begin{array}{ccc}
A \wedge B & \xrightarrow{\mathsf{mix}_{A,B}} & A \vee B \\
{\scriptstyle f \wedge g} \downarrow & & \downarrow {\scriptstyle f \vee g} \\
C \wedge D & \xrightarrow[\mathsf{mix}_{C,D}]{} & C \vee D
\end{array}
\tag{6}
$$

for all maps $f \colon A \to C$ and $g \colon B \to D$, uniquely determines a map $f \between g \colon A \wedge B \to C \vee D$. Then, for every $f, g \colon A \to B$ we can define

$$
f + g = \nabla_B \circ (f \between g) \circ \Delta_A \colon A \to B \quad .
$$

It follows from (co)-associativity and (co)-commutativity of Δ and ∇, along with naturality of mix, that the operation $+$ on maps is associative and commutative. This gives us for $\mathrm{Hom}(A, B)$ a commutative semigroup structure.

Note that in general the semigroup structure on the Hom-sets is not an enrichment, e.g., $(f + g)h$ is in general not the same as $fh + gh$.

2.8 Definition Let $\mathscr{C}$ be a single-mixed $\mathsf{B1}$-category. Then $\mathscr{C}$ is called *idempotent* if for every A and B, the semigroup on $\mathrm{Hom}(A, B)$ is idempotent, i.e., for every $f \colon A \to B$ we have $f + f = f$.

In an idempotent $\mathsf{B1}$-category the semigroup structure on $\mathrm{Hom}(A, B)$ is in fact a sup-semilattice structure, given by $f \leq g$ iff $f + g = g$.

Let us now move to the next level:

2.9 Definition A B2-*category* is a B1-category which obeys equations

$$\Pi^{\mathsf{t}} = 1_{\mathsf{t}} \colon \mathsf{t} \to \mathsf{t} \tag{B2a}$$

and

$$\begin{array}{ccc} & A \wedge B & \\ {}^{\Delta_A \wedge \Delta_B}\swarrow & & \searrow^{\Delta_{A\wedge B}} \\ A \wedge A \wedge B \wedge B & \xrightarrow[A\wedge\hat{\sigma}_{A,B}\wedge B]{} & A \wedge B \wedge A \wedge B \end{array} \tag{B2c}$$

for all objects A and B.

One can easily show that in a B2-category also

$$\begin{array}{ccc} & A \wedge B & \\ {}^{\Pi^A \wedge \Pi^B}\swarrow & & \searrow^{\Pi^{A\wedge B}} \\ \mathsf{t} \wedge \mathsf{t} & \xrightarrow[\hat{\varrho}_{\mathsf{t}}]{} & \mathsf{t} \end{array} \tag{B2b}$$

does hold. Furthermore, we have

$$1_{\mathsf{t}} + 1_{\mathsf{t}} = 1_{\mathsf{t}} \qquad \text{and} \qquad 1_{\mathsf{f}} + 1_{\mathsf{f}} = 1_{\mathsf{f}} \tag{7}$$

This is a consequence of having proper units. In the case of weak units (see [LS05b,LS05a]), the equations (7) do not hold. The following theorem summarizes the properties of B2-categories.

2.10 Theorem *In a* B2*-category, the maps* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$, Δ_A, Π^A, $\Pi^B_{A\square}$, *and* $\Pi^A_{\square B}$, *all are* $\wedge$*-comonoid morphisms, and the* $\wedge$*-comonoid morphisms are closed under* $\wedge$*. Dually, the maps* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, ∇_A, $\amalg^A$, $\amalg^B_{A\square}$, *and* $\amalg^A_{\square B}$, *all are* $\vee$*-monoid morphisms, and the* $\vee$*-monoid morphisms are closed under* $\vee$*.*

As observed before, if a B1-category is single-mixed then $\mathrm{Hom}(A, B)$ carries a semigroup structure. If we additionally have the structure of a B2-category, then the bijection ($\star$) preserves this semigroup structure:

2.11 Proposition *In a single-mixed* B2*-category the bijection* ($\star$) *is a semigroup isomorphism.*

3 Order enrichment

In [FP04c], Führman and Pym equipped B2-categories with an order enrichment, such that the proof identifications induced by the axioms are exactly the same as the proof identifications made by Gentzen's sequent calculus LK [Gen34], modulo "trivial rule permutations" (see [Laf95,Rob03]), and such that $f \square g$ if g is obtained from f via cut elimination (which is not confluent in LK).

3.1 Definition A B2-category is called an LK-*category* if for every A, B, the set $\mathrm{Hom}(A,B)$ is equipped with a partial order structure $\square$ such that

(i) the arrow composition $\circ$, as well as the bifunctors $\wedge$ and $\vee$ are monotonic in both arguments,
(ii) for every map $f\colon A \to B$ we have

$$\Pi^B \circ f \;\square\; \Pi^A \qquad \text{(LK-}\Pi\text{)}$$

$$\Delta_B \circ f \;\square\; (f \wedge f) \circ \Delta_A \qquad \text{(LK-}\Delta\text{)}$$

(iii) and the bijection $(\star)$ is an order isomorphism for $\square$.

In [FP04c,FP04b] Führmann and Pym use the term "classical category". We use here the term LK-categories because—as worked out in detail in [FP04c]—they provide a category theoretical axiomatisation of sequent calculus proofs in Gentzens's system LK [Gen34]. However, it should be clear that LK-categories are only one particular example of a wide range of possible category theoretical axiomatisations of proofs in classical logic.

3.2 Remark In [FP04c], Führmann and Pym give a different definition for LK-categories. Since they start from a weakly distributive category [CS97] instead of a *-autonomous one, they do not have immediate access to transposition. For this reason, they have to give a larger set of inequalities, defining the order $\square$. But one can easily show that both definitions are equivalent.

The following theorem states the main properties of LK-categories. It has first been observed and proved by Führmann and Pym in [FP04a].

3.3 Theorem *Every* LK*-category is single-mixed and idempotent. Furthermore, for all maps $f, g\colon A \to B$, we have $f \leq g$ iff $g \;\square\; f$.*

4 The medial map and the nullary medial map

That LK-categories are idempotent means that they are already at the degenerate end of the spectrum of "Boolean categories". On the other hand, B2-categories have (apart from Theorem 2.10) very little structure. The question that arises now is therefore, how we can add additional structure to B2-categories without getting too much collapse. In particular, can we extend the structure such that all the maps mentioned in Therorem 2.10 become $\vee$-monoid morphisms *and* $\wedge$-comonoid morphisms? This is where medial enters the scene.

4.1 Definition We say, a B2-category $\mathscr{C}$ *has medial* if for all objects A, B, C, and D there is a map $\mathsf{m}_{A,B,C,D}\colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D)$ with the following properties:

- it is natural in A, B, C and D,
- it is self-dual, and

– it obeys the equation

$$
\begin{array}{ccc}
 & A \vee B & \\
 {\scriptstyle \Delta_A \vee \Delta_B} \swarrow & & \searrow {\scriptstyle \Delta_{A\vee B}} \\
(A \wedge A) \vee (B \wedge B) & \xrightarrow[\mathsf{m}_{A,A,B,B}]{} & (A \vee B) \wedge (A \vee B)
\end{array}
\tag{B3c}
$$

for all objects A and B.

Before we state main properties of medial, let us introduce the following notation:

$$
\langle f, g\rangle = (f \wedge g) \circ \Delta_A \colon A \to C \wedge D \quad \text{and} \quad [f, h] = \nabla_C \circ (f \vee h) \colon A \vee B \to C \tag{8}
$$

where $f\colon A \to C$ and $g\colon A \to D$ and $h\colon B \to C$ are arbitrary maps. Another helpful notation (cf. [LS05a]) is the following:

$$
\begin{array}{ll}
\Pi^B_{A\square} = \hat{\varrho}_A \circ (A \wedge \Pi^B)\colon A \wedge B \to A & \Pi^A_{\square B} = \hat{\lambda}_B \circ (\Pi^A \wedge B)\colon A \wedge B \to B \\
\amalg^B_{A\square} = (A \vee \amalg^B) \circ \check{\varrho}^{-1}_A \colon A \to A \vee B & \amalg^A_{\square B} = (\amalg^A \vee B) \circ \check{\lambda}^{-1}_B \colon B \to A \vee B
\end{array}
$$

Note that

$$
\nabla_A \circ \amalg^A_{\square A} = 1_A = \nabla_A \circ \amalg^A_{A\square} \qquad \text{and} \qquad \Pi^A_{\square A} \circ \Delta_A = 1_A = \Pi^A_{A\square} \circ \Delta_A
$$

4.2 Theorem *Let $\mathscr{C}$ be a* B2*-category that has medial. Then*

(i) *The maps that preserve the $\wedge$-comultiplication are closed under $\vee$, and dually, the maps that preserve the $\vee$-multiplication are closed under $\wedge$.*

(ii) *For all maps $A \xrightarrow{f} C$, $A \xrightarrow{g} D$, $B \xrightarrow{h} C$, and $B \xrightarrow{k} D$, we have that*

$$
[\langle f, g\rangle, \langle h, k\rangle] = \langle [f, h], [g, k]\rangle \colon A \vee B \to C \wedge D \quad .
$$

(iii) *For all objects A, B, C, and D, the following diagram commutes:*

$$
\begin{array}{ccc}
 & ((A \wedge B) \vee (C \wedge D)) \wedge ((A \wedge B) \vee (C \wedge D)) & \\
 {\scriptstyle \Delta_{(A\wedge B)\vee(C\wedge D)}} \nearrow & & \searrow {\scriptstyle (\Pi^B_{A\square} \vee \Pi^D_{C\square}) \wedge (\Pi^A_{\square B} \vee \Pi^C_{\square D})} \\
(A \wedge B) \vee (C \wedge D) & & (A \vee C) \wedge (B \vee D) \\
 {\scriptstyle (\amalg^C_{A\square} \wedge \amalg^D_{B\square}) \vee (\amalg^A_{\square C} \wedge \amalg^B_{\square D})} \searrow & & \nearrow {\scriptstyle \nabla_{(A\vee C)\wedge(B\vee D)}} \\
 & ((A \vee C) \wedge (B \vee D)) \vee ((A \vee C) \wedge (B \vee D)) &
\end{array}
$$

and the horizontal diagonal is equal to $\mathsf{m}_{A,B,C,D}$.

4.3 Definition We say, a B2-category $\mathscr{C}$ *has nullary medial* if there is a map $\check{\mathsf{nm}}\colon \mathbf{t}\vee\mathbf{t}\to\mathbf{t}$ (called the *nullary medial map*) such that for all objects A, B, the following holds:

$$\begin{array}{ccc} & A\vee B & \\ {\scriptstyle \Pi^A\vee\Pi^B}\nearrow & & \searrow{\scriptstyle \Pi^{A\vee B}} \\ \mathbf{t}\vee\mathbf{t} & \xrightarrow[\check{\mathsf{nm}}]{} & \mathbf{t} \end{array} \tag{B3b}$$

Clearly, if a a B2-category has nullary medial, then $\check{\mathsf{nm}} = \Pi^{\mathbf{t}\vee\mathbf{t}}$. This can be seen by plugging in $\mathbf{t}$ for A and B in (B3b). By duality, $\Pi^{\mathbf{f}\wedge\mathbf{f}} = \hat{\mathsf{nm}}\colon \mathbf{f}\to\mathbf{f}\wedge\mathbf{f}$ (the *nullary comedial map*) obeys the dual of (B3b).

4.4 Proposition *Every* B2*-category with medial and nullary medial obeys*

$$\begin{array}{ccc} (A\wedge\mathbf{t})\vee(B\wedge\mathbf{t}) & \xrightarrow{\mathsf{m}_{A,\mathbf{t},B,\mathbf{t}}} & (A\vee B)\wedge(\mathbf{t}\vee\mathbf{t}) \\ {\scriptstyle \hat{\varrho}_A\vee\hat{\varrho}_B}\downarrow & & \downarrow{\scriptstyle (A\vee B)\wedge\check{\mathsf{nm}}} \\ A\vee B & \xrightarrow[\hat{\varrho}^{-1}_{A\vee B}]{} & (A\vee B)\wedge\mathbf{t} \end{array} \tag{m-$\hat{\varrho}$}$$

4.5 Definition A B3-*category* is a B2-category that obeys

$$\Pi^{\mathbf{t}\vee\mathbf{t}} = \check{\mathsf{nm}} = \nabla_{\mathbf{t}}\colon \mathbf{t}\vee\mathbf{t}\to\mathbf{t} \tag{B3a}$$

and has medial and nullary medial.

The following surprising fact has first been observed by Lamarche [Lam05].

4.6 Theorem *In a* B3*-category we have* $\Pi^{\mathbf{f}} = \Pi^{\mathbf{t}}$*, i.e., every* B3*-category is single-mixed.*

Other important properties of B3-categories are the following.

4.7 Theorem *In a* B3*-category, the* $\vee$*-monoid morphisms and the* $\wedge$*-comonoid morphisms are closed under* $\wedge$ *and* $\vee$*. Furthermore, the maps* $\mathsf{m}_{A,B,C,D}$ *and* $\check{\mathsf{nm}}$ *and* $\hat{\mathsf{nm}}$ *are* $\vee$*-monoid morphisms and* $\wedge$*-comonoid morphisms.*

4.8 Proposition *In a* B3*-category the maps* $\check{\alpha}_{A,B,C}$*,* $\check{\sigma}_{A,B}$*,* $\check{\lambda}_A$*, and* $\check{\varrho}_A$ *preserve the* $\wedge$*-counit, and dually, the maps* $\hat{\alpha}_{A,B,C}$*,* $\hat{\sigma}_{A,B}$*,* $\hat{\lambda}_A$*, and* $\hat{\varrho}_A$ *all preserve the* $\vee$*-unit.*

4.9 Proposition *A* B3*-category obeys the equation*

$$\begin{array}{ccc} (A\wedge B)\vee(C\wedge D) & \xrightarrow{\hat{\sigma}_{A,B}\vee\hat{\sigma}_{C,D}} & (B\wedge A)\vee(D\wedge C) \\ {\scriptstyle \mathsf{m}_{A,B,C,D}}\downarrow & & \downarrow{\scriptstyle \mathsf{m}_{B,A,D,C}} \\ (A\vee C)\wedge(B\vee D) & \xrightarrow[\hat{\sigma}_{A\vee C,B\vee D}]{} & (B\vee D)\wedge(A\vee C) \end{array} \tag{m-$\hat{\sigma}$}$$

if and only if $\hat{\sigma}_{A,B}\colon A\wedge B\to B\wedge A$ *preserves the* $\vee$*-multiplication.*

4.10 Proposition *A* B3*-category obeys the equation*

$$\begin{array}{ccc}
(A \wedge (B \wedge C)) \vee (D \wedge (E \wedge F)) & \xrightarrow{\hat{\alpha}_{A,B,C} \vee \hat{\alpha}_{D,E,F}} & ((A \wedge B) \wedge C) \vee ((D \wedge E) \wedge F) \\
\downarrow {\scriptstyle \mathsf{m}_{A,B\wedge C,D,E\wedge F}} & & \downarrow {\scriptstyle \mathsf{m}_{A\wedge B,C,D\wedge E,F}} \\
(A \vee D) \wedge ((B \wedge C) \vee (E \wedge F)) & & ((A \wedge B) \vee (D \wedge E)) \wedge (C \vee F) \\
\downarrow {\scriptstyle (A\vee D)\wedge \mathsf{m}_{B,C,E,F}} & & \downarrow {\scriptstyle \mathsf{m}_{A,B,D,E}\wedge (C\vee F)} \\
(A \vee D) \wedge ((B \vee E) \wedge (C \vee F)) & \xrightarrow[\hat{\alpha}_{A\vee D,B\vee E,C\vee F}]{} & ((A \vee D) \wedge (B \vee E)) \wedge (C \vee F)
\end{array} \qquad (\mathsf{m}\text{-}\hat{\alpha})$$

if and only if $\hat{\alpha}_{A,B,C} \colon A \wedge (B \wedge C) \to (A \wedge B) \wedge C$ *preserves the* $\vee$*-multiplication.*

4.11 Proposition *A* B3*-category obeying* (m-$\hat{\sigma}$) *and* (m-$\hat{\alpha}$) *does also obey the equation*

$$\begin{array}{ccc}
((A \wedge B) \vee (C \wedge D)) \wedge E & \xrightarrow{\mathsf{s}_{A\wedge B,C\wedge D,E}} & (A \wedge B) \vee (C \wedge D \wedge E) \\
\downarrow {\scriptstyle \mathsf{m}_{A,B,C,D}\wedge E} & & \downarrow {\scriptstyle \mathsf{m}_{A,B,C,D\wedge E}} \\
(A \vee C) \wedge (B \vee D) \wedge E & \xrightarrow[(A\vee C)\wedge \mathsf{s}_{B,D,E}]{} & (A \vee C) \wedge (B \vee (D \wedge E))
\end{array} \qquad (\mathsf{m}\text{-}\mathsf{s})$$

if and only if $\mathsf{s}_{A,B,C} \colon A \wedge (B \vee C) \to (A \wedge B) \vee C$ *preserves the* $\wedge$*-comultiplication.*

4.12 Definition A B4*-category* is a B3-category that obeys the equations (m-$\hat{\sigma}$), (m-$\hat{\alpha}$), and (m-s).

4.13 Remark Equivalently, one can define a B4-category as a B3-category in which $\hat{\sigma}$, $\hat{\alpha}$, and s preserve the $\vee$-multiplication. We have chosen the form of Definition (4.12) to show the resemblance to the work [Lam05] where the equations (m-$\hat{\sigma}$), (m-$\hat{\alpha}$), and (m-s) are also considered as primitives.

4.14 Theorem *In a* B4*-category, the maps* $\hat{\alpha}_{A,B,C}$*,* $\hat{\sigma}_{A,B}$*,* $\hat{\varrho}_A$*,* $\hat{\lambda}_A$ *and* $\check{\alpha}_{A,B,C}$*,* $\check{\sigma}_{A,B}$*,* $\check{\varrho}_A$*,* $\check{\lambda}_A$*, as well as* $\mathsf{s}_{A,B,C}$ *and* $\mathsf{mix}_{A,B}$ *are all are* $\vee$*-monoid morphisms and* $\wedge$*-comonoid morphisms.*

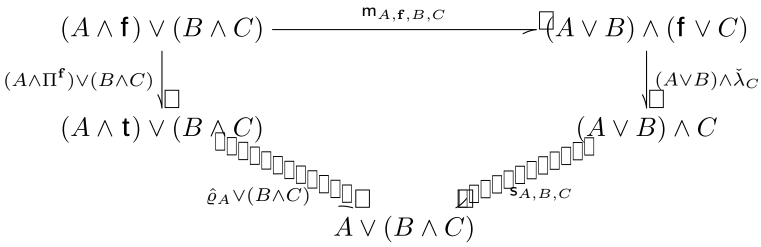
It has first been observed by Lamarche in [Lam05] that the equation (m-mix) (see below) is a consequence of the equations (m-$\hat{\alpha}$), (m-$\hat{\sigma}$), and (m-s).

4.15 Corollary *In a* B4*-category, the diagram*

$$\begin{array}{ccc}
A \wedge B \wedge C \wedge D & \xrightarrow{A\wedge \hat{\sigma}_{B,C}\wedge D} & A \wedge C \wedge B \wedge D \\
\downarrow {\scriptstyle \mathsf{mix}_{A\wedge B,C\wedge D}} & & \downarrow {\scriptstyle \mathsf{mix}_{A,C}\wedge \mathsf{mix}_{B,D}} \\
(A \wedge B) \vee (C \wedge D) & \xrightarrow[\mathsf{m}_{A,B,C,D}]{} & (A \vee C) \wedge (B \vee D)
\end{array} \qquad (\mathsf{m}\text{-}\mathsf{mix})$$

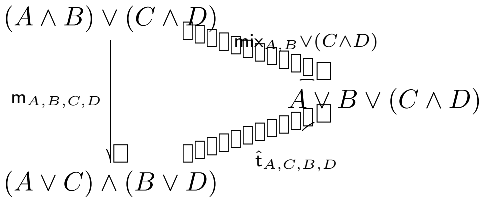
commutes.

Obviously one can come up with more diagrams like (m-mix) or (m-$\hat{\varrho}$) and ask whether they commute, for example the following due to McKinley [McK05]:



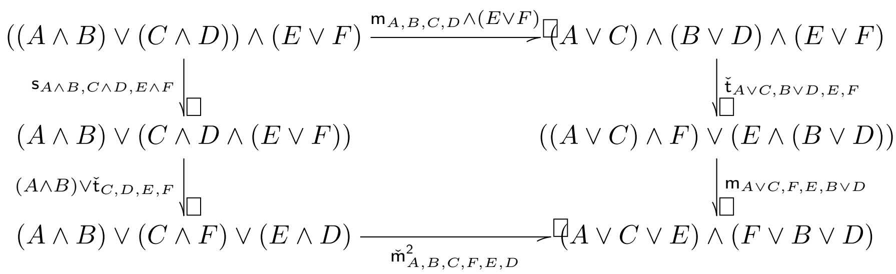
(9)

which is equivalent to



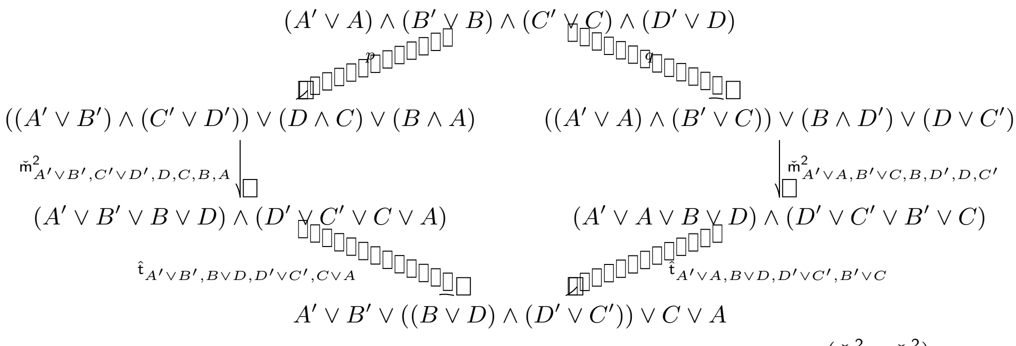
(mix-m-$\hat{\mathsf{t}}$)

Here are two more example that do not contain the units:



(m-$\check{\mathsf{t}}$-s)

and



($\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$)

where p and q are the canonical maps (composed of several switches, twists, and associativity) that are determined by the *-autonomous structure, and $\check{\mathsf{m}}^2$ is the dual of the diagonal of (m-$\hat{\alpha}$).

One usually speaks of "coherence" [Mac71] if all diagrams of this kind commute. Very often a "coherence theorem" is based on so-called "coherence graphs" [KM71,DP04]. In certain cases (see, e.g., [Str05a]) the notion of coherence graph is too restricted. For this reason, in [LS05a], the notion of "graphicality" is introduced.

In a graphical B4-category the equations (mix-m-$\hat{\mathsf{t}}$), (m-$\check{\mathsf{t}}$-s), and ($\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$) all hold. However, at the current state of the art it is not known whether they hold in every B4-category.[8]

4.16 Definition A $\mathsf{B4}'$*-category* is a B4-category that obeys equations (mix-m-$\hat{\mathsf{t}}$), (m-$\check{\mathsf{t}}$-s), and ($\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$) for all objects.

The motivation for this definition is that the equations (mix-m-$\hat{\mathsf{t}}$), (m-$\check{\mathsf{t}}$-s), and ($\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$) are exactly the ones that are needed to establish Theorem 5.6 in the next section.

5 Beyond medial

The definition of monoidal categories settles how the maps $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, and $\hat{\lambda}_A$ behave with respect to each other, and how the maps $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, and $\check{\lambda}_A$ behave with respect to each other. The notion of *-autonomous category then settles via the bijection ($\star$) how the two monoidal structures interact. Then, the structure of a B1-category adds $\vee$-monoids and $\wedge$-comonoids, and the structure of B2-categories allows the $\vee$-monoidal structure to go well with the $\vee$-monoids and the $\wedge$-monoidal structure to go well with the $\wedge$-comonoids. Finally, the structure of B4-categories ensures that *both* monoidal structures go well with the $\vee$-monoids *and* the $\wedge$-comonoids.

However, what has been neglected so far is how the $\vee$-monoids and the $\wedge$-comonoids go along with each other. Recall that in any B2-category the maps ∇ and $\amalg$ preserve the $\vee$-monoid structure and the maps Δ and Π preserve the $\wedge$-comonoid structure (Theorem 4.14).

5.1 Combatibility of $\vee$-monoids and $\wedge$-comonoids: We have the following possibilities:

(i) The maps Π and $\amalg$ both preserve the $\vee$-unit and the $\wedge$-counit.
(ii) The maps Π and $\amalg$ both preserve the $\vee$-multiplication and the $\wedge$-comultiplication.
(iii) The maps Δ and ∇ both preserve the $\vee$-unit and the $\wedge$-counit.
(iv) The maps Δ and ∇ both preserve the $\vee$-multiplication and the $\wedge$-comultiplication.

[8] The conjecture is that it is not the case, but so far no counterexample could be constructed.

Condition (i) says in particular that the following diagram commutes

$$\begin{array}{ccc} & \mathsf{f} & \\ {\scriptstyle \amalg^A} \swarrow & & \searrow {\scriptstyle \Pi^{\mathsf{f}}} \\ A & \xrightarrow[\Pi^A]{} & \mathsf{t} \end{array} \qquad (10)$$

Consequently, every B1-category obeying (B2a) and (10) is not only single-mixed but also for every object A the composition $\mathsf{f} \stackrel{\amalg^A}{\to} A \stackrel{\Pi^A}{\to} \mathsf{t}$ yields the same result. In [LS05a] the equation (10) was used as basic axiom, and the mix map was constructed from that without the use proper units.

The next observation to make is that (ii) and (iii) of 5.1 are equivalent, provided (B3b) and (B3a) are present:

5.2 Proposition *In a* B2*-category with nullary medial and* (B3a) *the following are equivalent for every object A:*

(i) *The map Π^A preserves the $\vee$-multiplication.*
(ii) *The map ∇_A preserves the $\wedge$-counit.*
(iii) *The map $\amalg^{\bar{A}}$ preserves the $\wedge$-comultiplication.*
(iv) *The map $\Delta_{\bar{A}}$ preserves the $\vee$-unit.*

Condition 5.1 (iv) exhibits an example of a "creative tension" between algebra and proof theory. From the viewpoint of algebra, it makes perfectly sense to demand that the $\vee$-monoid structure and the $\wedge$-comonoid structure be compatible with each other, i.e., that 5.1 (i)–(iv) do all hold (see [Lam05]). However, from the proof theoretical point of view it is reasonable to make some fine distinctions: We have to keep in mind that in the sequent calculus it is the "contraction-contraction-case"

$$\mathsf{cut}\,\dfrac{\mathsf{cont}\,\dfrac{\overset{\pi_1}{\vdash \Gamma, A, A}}{\vdash \Gamma, A} \qquad \mathsf{cont}\,\dfrac{\overset{\pi_2}{\vdash \bar{A}, \bar{A}, \Delta}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta}$$

which spoils the confluence of cut elimination and which causes the exponential blow-up of the size of the proof. This questions 5.1 (iv), i.e., the commutativity of the diagram

$$\begin{array}{ccc} A \vee A & \xrightarrow{\nabla_A} & A \\ {\scriptstyle \Delta_A \vee \Delta_A} \downarrow & & \downarrow {\scriptstyle \Delta_A} \\ (A \wedge A) \vee (A \wedge A) & \xrightarrow[\nabla_{A \wedge A}]{} & A \wedge A \end{array} \qquad (11)$$

and motivates the distinction made in the following definition.

5.3 Definition We say a B1-category is *weakly smooth* if for every object A, the maps Π^A and $\amalg^A$ are strong and the maps Δ_A and ∇_A are quasientropies (i.e., 5.1 (i)–(iii) hold), and it is *smooth* if for every object A, the maps Π^A, $\amalg^A$, Δ_A and ∇_A are all strong (i.e., all of 5.1 (i)–(iv) do hold).

5.4 Corollary *A* B3*-category is weakly smooth, if and only if* Π^A *is a* $\vee$*-monoid morphism for every object* A*.*

To understand the next (and final) axiom of this paper, recall that in every *-autonomous category we have

$$\begin{array}{ccc} \mathsf{t} & \xrightarrow{\check{\imath}_A \wedge \check{\imath}_A} & (\bar{A} \vee A) \wedge (\bar{A} \vee A) \\ {\scriptstyle \check{\imath}_A}\downarrow & & \downarrow{\scriptstyle \hat{\mathsf{t}}} \\ \bar{A} \vee A & \xleftarrow[\bar{A} \vee \hat{\imath}_A \vee A]{} & \bar{A} \vee (A \wedge \bar{A}) \vee A \end{array} \tag{12}$$

and that this equation is the reason why the cut elimination for multiplicative linear logic (proof nets as well as sequent calculus) works so well. The motivation for the following definition is to obtain something similar for classical logic (cf. [LS05a]).

5.5 Definition A B1-category is *contractible* if the following diagram commutes for all objects A.

$$\begin{array}{ccc} \mathsf{t} & \xrightarrow{\check{\imath}_A} & \bar{A} \vee A \\ & & \downarrow{\scriptstyle \Delta_{\bar{A} \vee A}} \\ {\scriptstyle \check{\imath}_A}\downarrow & & (\bar{A} \vee A) \wedge (\bar{A} \vee A) \\ & & \downarrow{\scriptstyle \hat{\mathsf{t}}} \\ \bar{A} \vee A & \xleftarrow[\bar{A} \vee \hat{\imath}_A \vee A]{} & \bar{A} \vee (A \wedge \bar{A}) \vee A \end{array} \tag{13}$$

The following theorem states the main results of this paper. It explains the deep reasons why the cut elimination for the proof nets of [LS05b] is not confluent in the general case. It also shows that the combination of equations (11) and (13) leads to a certain collapse in a B4′-category, which can be compared to the collapse made by an LK-category, namely, that we are no longer able to count how often an axiom link is used in a proof. Consequently, it is in this setting not possible to speak of the size of a proof. Nonetheless, even with this collapse we can find reasonable models for proofs of Boolean logic, as it is shown in [LS05b,LS05a,Str05b,Str05c,Lam05].

5.6 Theorem *In a* B4′*-category that is smooth and contractible, we have*

$$1_A + 1_A = 1_A$$

for all objects A*.*

5.7 Corollary *Let $\mathscr{A}$ be a set of propositional variables and let $\mathscr{C}$ be the free smooth and contractible* B4′*-category generated by $\mathscr{A}$. Then $\mathscr{C}$ is idempotent.*

6 Conclusions

The results in this paper show that it is at the current state of the art not at all clear what a Boolean category could be. From the *normalisation-as-computation* point of view it is clearly desirable to keep the axioms of cartesian closed categories. But from the viewpoint of finding a *categorification* (in the sense of [BD01]) of Boolean algebras, one should certainly keep the symmetry between $\wedge$ and $\vee$. But even if we choose to go that way, it is not clear what axioms to add. There seems to be consensus up to the axioms of what I called here a B2-category. These axioms appear in the work of Führmann and Pym [FP04c] as well as in the work of Lamarche and Straßburger [LS05b,LS05a]. Adding the order enrichment as in [FP04c] seems to be too strong because it entails idempotency [FP04a]. Adding medial instead and going to a B3-category is an alternative. In [Str05b,Str05c], one can find examples (based on proof nets) of nonidempotent B3-categories (in fact of B4′-categories), which show that the two approaches (order enrichment vs. medial) are indeed different. Adding the equations for smooth and contractible also seems reasonable, but again entails (a weak form of) idempotency. The proof nets given in [LS05b,LS05a] are examples of smooth and contractible B4′-categories[9]. The models constructed by Lamarche in [Lam05], are not based on proof nets and are all B4-categories. Since not all of them are idempotent, we know that the axiom for B4-categories are strictly weaker.

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